Deciding the finiteness of simple permutations contained in a wreath-closed class is polynomial.*

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1 Introduction

In [8], D. Knuth introduced pattern avoiding permutation classes. Theses classes present nice combinatorial properties, for example 231 avoiding permutations are in one-to-one correspondence with Dyck words. It is then natural to extend this enumerative result to all classes and compute, given B a set of permutations, the generating function $S(x) = \sum s_n x^n$ where s_n is the number of permutations of length n avoiding every permutation of B. From the work of Albert and Atkinson [3], many criteria arise giving sufficient conditions to decide the nature of the generating function: rational, algebraic, P-recursive ... In a series of three articles [6, 5, 7] the authors prove that it is decidable to know if a permutation class contains a finite number of simple permutations, and hence has an algebraic generating function. In this work, every algorithm involved is polynomial except the algorithm deciding if the class contains an arbitrarily long simple pin-permutation. In [4], the authors give a complete characterization of pin-permutations and we use this characterization to give a polynomial algorithm for the preceding question. We give a $\mathcal{O}(n \ln n)$ algorithm to decide if a finitely based wreath-closed class of permutations $Av(\pi_1, \pi_2, \ldots, \pi_k)$ contains a finite number of simple permutations where $n = \sum |\pi_i|$.

1.1 Definitions

We recall in this section a few definitions about permutations, pin representation and pinwords. More details can be found in [6, 7, 4]. A permutation $\sigma \in S_n$ is a bijective function from $\{1, \ldots, n\}$ onto $\{1, \ldots, n\}$. We either represent a permutation by a word $\sigma = 2\,3\,1\,4$ or its *diagram* (see Figure 1). A permutation $\pi = \pi_1\pi_2 \ldots \pi_k$ is a *pattern* of a permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ and we write $\pi \leq \sigma$ if and only if there exist $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that π is order isomorphic to $\sigma_{i_1} \ldots \sigma_{i_k}$. If π is not a pattern of σ we say that σ *avoids* π. A permutation class $Av(B)$ - where B is a finite or infinite set of permutations called the *basis*- is the set of all permutations avoiding every element of B. A permutation is called *simple* if it contains no mapping from $\{i, \ldots, (i + l)\}$ to $\{j, \ldots, (j + l)\}$ except the trivial ones $l = 0$ or $i = j = 1, l = n - 1$. When the basis B contains only simple permutations the permutation class $Av(B)$ is said to be *wreath-closed*. Wreath-closed classes are defined in [3] in a different way but the authors prove that this definition is equivalent. In this article, we study wreath-closed classes with finite basis.

A pin in the plane is a point at integer coordinates. A pin p *separates* - horizontally or vertically - the set of pins P from the set of pins Q iff a horizontal - resp. vertical - line drawn across p separates the plane in two parts, one of which contains P and the other one contains Q. A pin sequence is a sequence $(p_1, \ldots, p_k)$ of pins in the plane such that no two points lie in the same column or line and for all i, p_i lies outside the bounding box of $\{p_1, \ldots, p_{i-1}\}$ and respects one of the following conditions:

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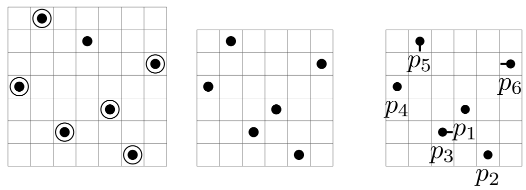

Figure 1: A permutation $\sigma = 4\,7\,2\,6\,1\,3\,5$, a pattern $\pi = 4\,6\,2\,1\,3\,5$ and a pin sequence of π. $14L2UR$ (if we place p_0 between p_3 and p_1) and $3DL2UR$ are pinwords corresponding to this pin sequence.

2	U	1
L		R
3	D	4

Figure 2: Encoding of pins by letters.

- p_i separates p_{i-1} from $\{p_1, \ldots, p_{i-2}\}$.
- p_i is independent from $\{p_1, \ldots, p_{i-1}\}$ i.e. it does not separate this set into two non empty sets.

A pin sequence represents a permutation σ iff it is order-isomorphic to its diagram. The pin-permutations are those that are represented by a pin sequence. Not all permutations are pin permutations (see Figure 1).

Pin sequences can be encoded by words on the alphabet $\{1, 2, 3, 4, U, D, L, R\}$ called *pinwords*. Consider a pin sequence $(p_1, \ldots, p_n)$ and choose an arbitrary origin in the plane (we can think the origin as extending the pin sequence to the sequence $(p_0, p_1, \ldots, p_n)$ where p_0 is the origin). Then every pin $p_1, \ldots, p_n$ is encoded by a letter with the following conditions (see Figure 2):

- The letter associated to p_i is U -resp. D, L, R- iff p_i separates p_{i-1} and $\{p_1, \ldots, p_{i-2}\}$ from the top. -resp. bottom, left, right-.
- The letter associated to p_i is 1 -resp. $2, 3, 4$- iff p_i is independent from $\{p_1, \ldots, p_{i-1}\}$ and is situated in the up-right -resp. up-left, bottom-left, bottom-right- corner of the bounding box of $\{p_1, \ldots, p_{i-1}\}$.

The region (see Figure 2) encoded by 1 is called the first *quadrant*. The same goes for $2, 3, 4$. The letters L, R, U, D are called *directions*.

To each pinword corresponds a unique pin sequence, hence a unique permutation but each pin-permutation has at least 4 pinwords associated -choice of the origin-. A *strict* pinword is a word that begins by a numeral followed only by letters. A *proper* pin permutation is a permutation that admits a strict pinword.

1.2 Background and known results

In [7], the authors studied conditions for a class to contain an infinite number of simple permutations. Introducing three new kinds of permutations they show that this problem is equivalent to looking for an infinite number of permutations of one of these three simpler kinds.

Theorem 1.1. *[7] A permutation class $Av(B)$ contains an infinite number of simple permutations if and only if it contains either:*

- *An infinite number of wedge simple permutations.*
- *An infinite number of parallel alternations.*
- *An infinite number of proper pin-permutations.*

Alternations and wedge simple permutations are well characterized in [7], where it is shown that it is easy to deal with the preceding problem using the three following lemmas.

Lemma 1.1. *[7] . The permutation class Av(B) contains only finitely many parallel alternations if and only if B contains an element of every symmetry of the class Av(123, 2413, 3412).*

Lemma 1.2. *[7] . The permutation class Av(B) contains only finitely many wedge simple permutations of type 1 if and only if B contains an element of every symmetry of the class Av(1243, 1324, 1423, 1432, 2431, 3124, 4123, 4132, 4231, 4312).*

Lemma 1.3. *[7] The permutation class Av(B) contains only finitely many wedge simple permutations of type 2 if and only if B contains an element of every symmetry of the class Av(2134, 2143, 3124, 3142, 3241, 3412, 4123, 4132, 4231, 4312).*

Then, deciding if a class contains an infinite number of wedge simple permutations or parallel alternations is easily computable, we only have to check if elements of the basis contains patterns of size at most 4. The exact complexity is given in Section 3.

For the last kind, i.e. proper pin-permutations, the authors prove that it is decidable to know if a class contains a infinite number of them by using language theory arguments. Analyzing their procedure, we can prove that it has an exponential complexity due to the resolution of a co-finiteness problem for a regular language. They also conjecture some results about the structure of proper pin permutations. In [4], we prove these conjectures and give a characterization of these permutations. We will use this characterization to give a polynomial algorithm for this last kind.

2 Pattern containment and pinwords

Let $Av(B)$ be a wreath-closed class of permutations, that is to say B contains only simple permutations. Let

$$\mathcal{S}P = \{1,2,3,4\}\left\{\{\epsilon, L, R\}\{U,D\}\left\{\{L,R\}\{U,D\}\right\}^*, \{\epsilon, U, D\}\{L,R\}\left\{\{U,D\}\{L,R\}\right\}^*\right\}$$

be the language of strict pinwords. We prove that the subset of strict pinwords corresponding to permutations that contain an element of B is defined as the set of all strict pin words that contain some factors.

We first recall the order on pinwords introduced in [7]. We define a partial order $\preceq$ on pinwords. Let u and w be two pinwords. We define a strong numeral-led factor to be a sequence of contiguous letters beginning with a numeral and followed by any number of directions (but no numerals). We decompose u in terms of its strong numeral-led factors as $u = u^{(1)} \dots u^{(j)}$. We then write $u \preceq w$ if w can be chopped into a sequence of factors $w = v^{(1)}w^{(1)} \dots v^{(j)}w^{(j)}v^{(j+1)}$ such that for all $i \in [j]$:

- if $w^{(i)}$ begins with a numeral then $w^{(i)} = u^{(i)}$, and
- if $w^{(i)}$ begins with a direction, then $v^{(i)}$ is nonempty, the first letter of $w^{(i)}$ corresponds to a point lying in the quadrant specified by the first letter of $u^{(i)}$, and all other letters in $u^{(i)}$ and $w^{(i)}$ agree.

This order is closely related to the pattern-containment order $\leq$ on permutations.

Lemma 2.1. *[7] If the pinword w corresponds to the permutation σ and $\pi \leq \sigma$ then there is a pinword u corresponding to π with $u \preceq w$. Conversely, if $u \preceq w$ then the permutation corresponding to u is contained in the permutation corresponding to w.*

As a consequence, if π and σ are permutations and w is a pinword corresponding to σ, then $\pi \leq \sigma$ if and only if there is a pinword u corresponding to π with $u \preceq w$. We extend this result to the special case of pinwords associated to simple permutations and show that in this case, we can associate to each pinword u a word $v = \phi(u)$ that does not contain numerals and such that the pattern involvement problem is equivalent to check if v is a factor of w.

Definition 2.1. *Let M be the set of words of length greater than or equal to 3 over the alphabet L, R, U, D such that R, L is followed by U, D and conversely. We define a bijection ϕ from $\mathcal{S}P$ to M by :*
For $u \in \mathcal{S}P$ s.t. $u = u'.u''$ with $|u'| = 2$ we set $\phi(u) = \varphi(u').u''$ where φ is defined by :

$1R \mapsto RUR$	$2R \mapsto LUR$	$3R \mapsto LDR$	$4R \mapsto RDR$
$1L \mapsto RUL$	$2L \mapsto LUL$	$3L \mapsto LDL$	$4L \mapsto RDL$
$1U \mapsto URU$	$2U \mapsto ULU$	$3U \mapsto DLU$	$4U \mapsto DRU$
$1D \mapsto URD$	$2D \mapsto ULD$	$3D \mapsto DLD$	$4D \mapsto DRD$

The map ϕ is a bijection from $\mathcal{SP}_n$ to M_{n+1} for any $n \geq 2$. For any $u \in \mathcal{SP}$,

- $u_i = \phi(u)_{i+1}$ for any $i \geq 2$,
- the two first letters of $\phi(u)$ are sufficient to determine the first letter of u (which is a numeral). Thus for a word u of $\{LU, LD, RU, RD, UL, UR, DL, DR\}$ we can by abuse of notation define $\phi^{-1}(u)$.

Notice that our bijection consists in replacing numbers in strict pinwords by letters. This implies that we know in which quadrant lies every pin of a pin representation corresponding to the pinword.

Lemma 2.2. *Let w be a strict pinword and p the pin representation corresponding to w. Then for any $i \geq 2$, w_{i-1} and w_i determine the quadrant in which lies p_i with respect to $\{p_0, \dots, p_{i-2}\}$:*

- *if $i \geq 3$, p_i lies in the quadrant $\phi^{-1}(w_{i-1}w_i)$.*
- *if $i = 2$, p_i lies in the quadrant $\phi^{-1}(BC)$ where $\phi(w_1w_2) = ABC$.*

Proof. (Sketch) If $i \geq 3$, w_{i-1} and w_i are letters. For example if $w_{i-1} = L$ and $w_i = U$, then p_i lies in the quadrant 2 and $\phi^{-1}(LU) = 2$. If $i = 2$, w_{i-1} is a numeral and w_i is a letter. For example if $w_{i-1} = 1$ and $w_i = L$, then p_i lies in the quadrant 2 and we have $\phi(1L) = RUL$ and $\phi^{-1}(UL) = 2$. $\square$

A pinword corresponding to a simple permutation is not always a strict pinword and can begin with two numerals followed only by letters, we call *quasi strict* pinwords these pinwords.

Lemma 2.3. *Let π be a simple permutation and u a pinword corresponding to π. Then u is a strict pinword or a quasi strict pinword.*

Proof. Let $p_1 \dots p_n$ be a pin representation of u and i such that u_i is a numeral. Then for any $j \geq i$, $p_1 \dots p_{i-1}$ are not separated by p_j so they form a block. But π is simple thus $i = 1$ or 2. Then u has either one numeral and is a strict pinword, or has exactly two numerals, u_1 and u_2, and is a quasi strict pinword. $\square$

We first study the case of strict pinwords.

Lemma 2.4. *For any strict pinwords u and w, $u \preceq w$ if and only if $\phi(u)$ is a factor of $\phi(w)$.*

Proof. If $u \preceq w$, as u is a strict pinword, writing u in terms of its strong numeral-led factors leads to $u = u^{(1)}$, thus w can be decomposed into a sequence of factors $w = v^{(1)}w^{(1)}v^{(2)}$.

If $v^{(1)}$ is empty, then $w^{(1)}$ begins with a numeral, $w^{(1)} = u^{(1)}$ and u is a prefix of w thus $\phi(u)$ is a prefix of $\phi(w)$.

Otherwise $w^{(1)}$ begins with a direction hence the first letter of $w^{(1)}$ corresponds to a point lying in the quadrant specified by u_1 the first letter of $u^{(1)}$, and all other letters (which are directions) in $u^{(1)}$ and $w^{(1)}$ agree : $u_2 \dots u_{|u|} = w_{i+2} \dots w_{i+|u|}$ where $i = |v^{(1)}|$.

If $|v^{(1)}| = i \geq 2$ then according to Lemma 2.2 we have $u_1 = \phi^{-1}(w_i.w_{i+1})$. We then have $\phi(u) = w_iw_{i+1} \dots w_{i+|u|}$ and $\phi(u)$ is a factor of w which do not contain numeral thus $\phi(u)$ is a factor of $\phi(w)$.

And if $|v^{(1)}| = 1$ then according to Lemma 2.2 we have $u_1 = \phi^{-1}(BC)$ where $\phi(w_1w_2) = ABC$ thus $\phi(u) = BCu_2 \dots u_{|u|}$ and $\phi(w) = ABCw_3 \dots w_{|w|}$. As $u_2 \dots u_{|u|} = w_3 \dots w_{|u|+1}$, $\phi(u)$ is a factor of $\phi(w)$.

Conversely if $\phi(u)$ is a factor of $\phi(w)$ then $\phi(w) = v.\phi(u).v'$. If v is empty then $\phi(u)$ is a prefix of $\phi(w)$ thus u is a prefix of w hence $u \preceq w$.

If $|v| = i \geq 2$ then $w = \phi^{-1}(v).\phi(u).v'$. Thus $\phi(u) = w_i \dots w_{i+|\phi(u)|-1} = w_i \dots w_{i+|u|}$ and $u_2 \dots u_{|u|} = w_{i+2} \dots w_{i+|u|}$, $u_1 = \phi^{-1}(w_i w_{i+1})$. u_1 is the quadrant in which w_{i+1} lies hence $u \preceq w$.

If $|v| = 1$ then $\phi(w_1 w_2) = v\,\phi(u)_1\phi(u)_2$ thus according to Lemma 2.2, w_2 lies in the quadrant $\phi^{-1}(\phi(u)_1\phi(u)_2) = u_1$. As $w_3 \dots w_{|u|+1} = \phi(w)_4 \dots \phi(w)_{|u|+2} = \phi(u)_3 \dots \phi(u)_{|u|+1} = u_2 \dots u_{|u|}$, $u \preceq w$, concluding the proof. □

The second possible structure for a pinword corresponding to a simple permutation is to begin with two numerals.

Lemma 2.5. *Let u be a quasi strict pinword corresponding to a permutation π and w be a strict pinword corresponding to a permutation σ. If $u \preceq w$ then $\phi(u_2 \dots u_{|u|})$ is a factor of $\phi(w)$ which begins at position $p \geq 3$.*

Proof. Decompose u into its strong numeral-led factors $u = u^{(1)}u^{(2)}$. Since $u \preceq w$, w can be decomposed into a sequence of factors $w = v^{(1)}w^{(1)}v^{(2)}w^{(2)}v^{(3)}$. Moreover $|w^{(1)}| = |u^{(1)}| = 1$ so $w^{(2)}$ contains no numerals thus $v^{(2)}$ is nonempty, the first letter of $w^{(2)}$ corresponds to a point lying in the quadrant specified by the first letter of $u^{(2)}$, and all other letters in $u^{(2)}$ and $w^{(2)}$ agree. Hence $w = v^{(1)}w^{(1)}v\phi(u^{(2)})v^{(3)}$ where v is the prefix of $v^{(2)}$ of length $|v^{(2)}| - 1$. Then $\phi(u^{(2)})$ is a factor of w which have no numeral thus $\phi(u^{(2)})$ is a factor of $\phi(w)$ which begin at position $p \geq 3$. □

Lemma 2.6. *Let u be a quasi strict pinword corresponding to a permutation π and w be a strict pinword corresponding to a permutation σ. If $\phi(u_2 \dots u_{|u|})$ is a factor of $\phi(w)$ which begins at position $p \geq 3$ then π is a pattern of σ.*

Proof. Since $\phi(u^{(2)})$ is a factor of $\phi(w)$ which begins at position $p \geq 3$ then by Lemma 2.4, $u^{(2)} \preceq w$. Let $p_1 \dots p_n$ be a pin representation of w (which corresponds to σ) and Γ be the subset of points corresponding to $u^{(2)}$, then $\Gamma \subseteq \{p_3 \dots p_n\}$. Let π' be the permutation corresponding to $\{p_1\} \cup \Gamma$, then $\pi' \leq \sigma$. We claim that $\pi' = \pi$. Let i be the quadrant in which lies p_1, and $v = i\,u^{(2)}$. Then v is a pinword corresponding to π'. As u begins with two numerals, there is $k \in \{1, \dots, 4\}$ such that $u = k\,u^{(2)}$ and u corresponds to π but $v = i\,u^{(2)}$ thus v corresponds to π hence $\pi' = \pi$. □

To any simple permutation π we associate $P(\pi)$ the set of its pin words. From Lemma 4.3, Fact 4.1 and Lemma 4.5 of [4], this set contains at most 64 elements. We define $E(\pi) = \{\phi(u)\,|u$ is a strict pinword corresponding to $\pi\} \cup \{v \in M \mid$ there is u a quasi strict pinword corresponding to π and $x \in \{LU, LD, RU, RD\} \uplus \{UL, UR, DL, DR\}$ such that $v = x\,\phi(u_2 \dots u_{|u|})\}$. For the second set, the first letter of $\phi(u_2 \dots u_{|u|})$ determines the set in which x lies.

It is then immediate to check that $|E(\pi)| \leq 320$.

Theorem 2.1. *Let π be a simple permutation and w be a strict pinword corresponding to a permutation σ. Then $\pi \not\leq \sigma$ if and only if $\phi(w)$ avoids the finite set of factors $E(\pi)$.*

Notice that it is enough to consider only one strict pinword corresponding to σ rather than all of them.

Proof. If $\pi \leq \sigma$, then by Lemma 2.1, there is a pinword u corresponding to π with $u \preceq w$. By Lemma 2.3, u is a strict pinword or a quasi strict pinword. If u is a strict pinword then, by Lemma 2.4, $\phi(u)$ is a factor of $\phi(w)$ and $\phi(w)$ has a factor in $E(\pi)$. If u is a quasi strict pinword then by Lemma 2.5, $\phi(u_2 \dots u_{|u|})$ is a factor of $\phi(w)$ which begins at position $p \geq 3$. Let x be the two letters preceding $\phi(u_2 \dots u_{|u|})$ in $\phi(w)$. As $\phi(w) \in M$, $x\phi(u_2 \dots u_{|u|})$ is a factor of $\phi(w)$ that belongs to $E(\pi)$.

Conversely suppose that $\phi(w)$ has a factor v in $E(\pi)$. If $v \in \{\phi(u)\,|u$ is a strict pinword corresponding to $\pi\}$ then by Lemma 2.4, there is u a pinword corresponding to π with $u \preceq w$ so by Lemma 2.1, $\pi \leq \sigma$. Otherwise there is u a quasi strict pin word corresponding to π and $x \in \{LU, LD, RU, RD, UL, UR, DL, DR\}$ such that $v = x\,\phi(u_2 \dots u_{|u|}) \in M$. Thus $\phi(u_2 \dots u_{|u|})$ is a factor of $\phi(w)$ which begin at position $p \geq 3$ and by Lemma 2.6, $\pi \leq \sigma$. □

3 An algorithm for wreath-closed classes

Lemma 3.1. *A wreath-closed class $Av(B)$ has arbitrarily long proper pin permutations if and only if there exists words of arbitrary length on the alphabet $\{L, R, U, D\}$ avoiding the set of factors $\cup_{\pi\in B}E(\pi) \cup \{LL, LR, RR, RL, UU, UD, DD, DU\}$.*

Proof. $Av(B)$ has arbitrarily long proper pin-permutations if and only if there exist arbitrarily long proper pin-permutations which have no pattern in B. That is, if and only if there exist arbitrarily long strict pin words w such that $\phi(w)$ avoids the set of factors $\cup_{\pi\in B}E(\pi)$ (by Lemma 2.1), or equivalently if and only if there exist words of arbitrary length on the alphabet $\{L, R, U, D\}$ which avoid the set of factors $\cup_{\pi\in B}E(\pi) \cup \{LL, LR, RR, RL, UU, UD, DD, DU\}$. □

Theorem 3.1. *Let $Av(B)$ be a finitely based wreath-closed class of permutations. Then there exists an algorithm to decide in time $n \ln n$ where $n = \sum_{\pi\in B} |\pi|$ whether this class contains finitely many simple permutations.*

Proof. From Theorem 1.1, we can look separately at alternations, wedge simple permutations and proper pin sequences. For alternations and wedge simple permutations, Lemmas 1.1, 1.2 and 1.3 express this problem as checking if there exists an element of B in every symmetry class of special pattern avoiding permutation classes where the basis contains only permutations of length at most 4. This can be done in $\mathcal{O}(n \ln n)$. [2].

The case of proper pin-permutations can be solved with Lemma 3.1. Checking if there exists an arbitrarily long word on $\{L, R, U, D\}$ which avoids a finite set of factors can be done in linear time - linear in the sum of the sizes of the factors - using Aho-Corasick algorithm [1] to build a deterministic automaton. Then it remains to test in the completed automaton whether there exists an accessible cycle which does not contain any final state. This is also linear. □

Conjecture We highly believe that Theorem 3.1 extends to all finitely based permutation classes (and not only wreath-closed classes). The methods used here seem to extend to the general case using the results of [4]. This indeed is a work in progress.

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