On the query complexity of real functionals

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computability $\longleftrightarrow$ continuity

complexity bound $\longleftrightarrow$ analytical properties

Computing with real numbers

$$(q_n)_{n\in\mathbb{N}} \rightsquigarrow x \text{ if } \forall n, |x - q_n| \leq 2^{-n}$$

$$\mathbb{R} \sim (\mathbb{N} \to \mathbb{N})$$

Model: **Oracle Turing Machines**

Finite-time computation $\longrightarrow$ finite number of queries
$\longrightarrow$ finite knowledge of the input
$\longrightarrow$ continuity

Bounding computation time $\iff$ bounding

- the computational power
- the number of queries

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Theorem

$f : \mathbb{R} \to \mathbb{R}$ computable w.r.t. an oracle
$\iff$ f is continuous.

Theorem

$f : \mathbb{R} \to \mathbb{R}$ polynomial time computable w.r.t. an oracle
$\iff$ its modulus of continuity is bounded by a polynomial.

$f \in \mathcal{C}[0,1] \iff \exists \mu, f_{\mathbb{Q}}$:
modulus of continuity : $\mu : \mathbb{N} \to \mathbb{N}$

$$|x - y| \leq 2^{-\mu(n)} \implies |f(x) - f(y)| \leq 2^{-n}$$

approximation function $f_{\mathbb{Q}} : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q}$

$$|f_{\mathbb{Q}}(q, n) - f(q)| \leq 2^{-n}$$

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$$\mathcal{C}[0,1] \sim \mathbb{N} \to \mathbb{N}$$

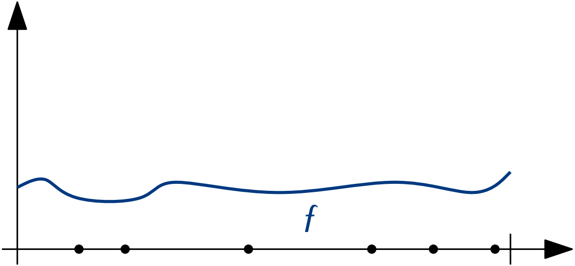
Theorem

$$F : \mathcal{C}[0,1] \to \mathbb{R} \textit{ computable w.r.t. an oracle} \iff F \textit{ is continuous.}$$

Theorem?

$$F : \mathcal{C}[0,1] \to \mathbb{R} \text{ polynomial time computable w.r.t. an oracle} \iff ?$$

The uniform norm is in $EXP \setminus P$
(even in $NP \setminus P$!)



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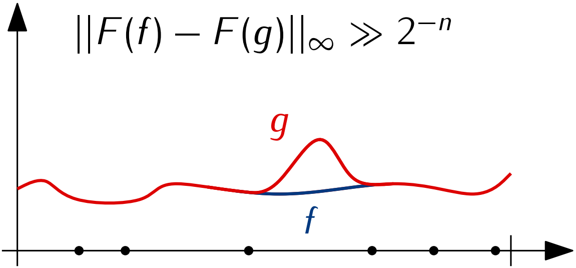
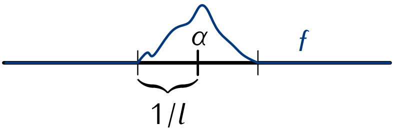


Dependence of a norm on a point

$$d_{F,\alpha}(n) = \sup\{l : \exists f \in Lip_1, Supp(f) \subseteq \mathcal{N}(\alpha, 1/l) \text{ and } F(f) > 2^{-n}\}$$



Example

$$F = ||.||_\infty \implies d_{F,\alpha}(n) \sim 2^n$$
$$F = ||.||_1 \implies d_{F,\alpha}(n) \sim 2^{\frac{n}{2}}$$

Relevant points

$$R_{n,l} = \{\alpha : d_{F,\alpha}(n) \geq l\}$$

Definition

α is relevant if $\exists c > 0, \forall n, d_{F,\alpha}(n) \geq c \cdot 2^{\frac{n}{2}}$

$$\mathcal{R} = \bigcup_k \underbrace{\bigcap_n R_{n,2^{\frac{n}{2}-k}}}_{\mathcal{R}_k}$$

Example

For $||.||_\infty$ and $||.||_1$, $\mathcal{R} = [0,1]$.

Theorem

$f = 0$ *on* $\mathcal{R}_{2\mu_f(k)} \implies F(f) \leq c.2^{-k}$.

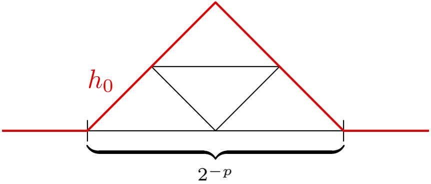
Corollary

$f = g$ *on* $\mathcal{R}_{\mu(k)} \implies |F(f) - F(g)| \leq 2^{-k}$.

Theorem

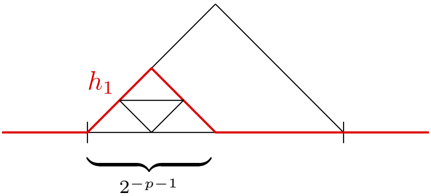
$\mathcal{R}$ *is dense.*

$\mathcal{R}$ is dense (proof)



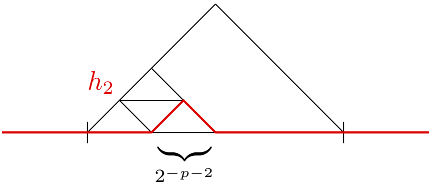
$$F(h_0) \geq 2^{-c}$$

$\mathcal{R}$ is dense (proof)



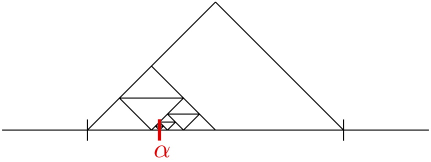
$$F(h_1) \geq 2^{-c-2}$$

$\mathcal{R}$ is dense (proof)



$$F(h_2) \geq 2^{-c-4}$$

$\mathcal{R}$ is dense (proof)



$$F(h_n) \geq 2^{-c-2n}$$

$$d_{F,\alpha}(2n+c) \geq 2^{n+p} \implies \alpha \in \mathcal{R}$$

Query complexity

Q_n : oracle calls of F on $x \mapsto 0$ with precision 2^{-n}.

Proposition

$$R_{n,l} \subseteq \mathcal{N}(Q_{n+1}, \frac{1}{l})$$

Definition

F has a polynomial query complexity if it is computable by a relativized OTM with $|Q_n| \leq P(n)$.

Theorem

If F has polynomial query complexity, then almost every point has a polynomial dependency.

Theorem

If F has polynomial query complexity, then $\mathcal{R}$ has Hausdorff dimension 0.

Proposition

$$
\begin{aligned}
& F \text{ has polynomial query complexity} \\
\implies & \exists \alpha, \frac{2^n}{d_{F,\alpha}(n)} \text{ is bounded by a polynomial.}
\end{aligned}
$$

Theorem

$$\begin{array}{ll} & F \textit{ is polynomial time computable w.r.t. an oracle} \\ \iff & F \textit{ has polynomial query complexity} \\ \iff & \mathcal{R}_k \textit{ can be polynomially covered (wrt. } k\textit{).} \end{array}$$

Open question

Can it be generalized for any $F : \mathcal{C}[0,1] \to \mathbb{R}$?

One oracle access case

Theorem

The following are equivalent:

- *F is computable by a polynomial time machine doing only one oracle query*
- $\forall f, F(f) = \phi(f(\alpha))$ *where:*
    - $\alpha \in Poly(\mathbb{R})$ *(but cannot be efficiently retrived from F!)*
    - $\phi \in Poly(\mathbb{R} \to \mathbb{R})$
    - *ϕ is uniformly continuous*

Open question

Generalization to any finite number of queries?