Characterizing possible typical asymptotic behaviours of cellular automata

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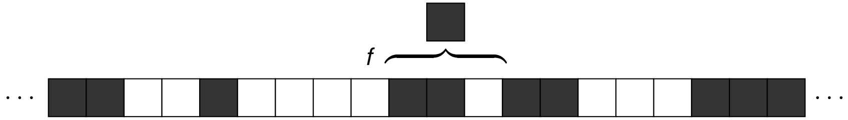
Definitions

- $\mathcal{A}, \mathcal{B}$ finite **alphabets**;
- $\mathcal{A}^*$ the (finite) **words**;
- $\mathcal{A}^{\mathbb{Z}}$ the **configurations**;
- σ the **shift action** $\sigma(a)_i = a_{i-1}$;

A **cellular automaton** is an action $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by a **local rule** $f : \mathcal{A}^{\mathbb{U}} \to \mathcal{A}$ on some neighbourhood $\mathbb{U}$.

For $\mathcal{A} = \{■, □\}$ and $\mathbb{U} = \{-1, 0, 1\}$:



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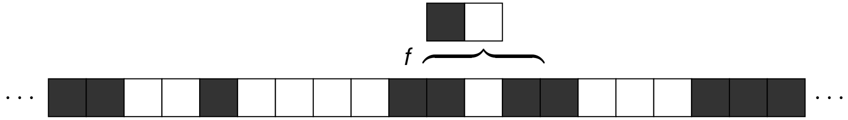

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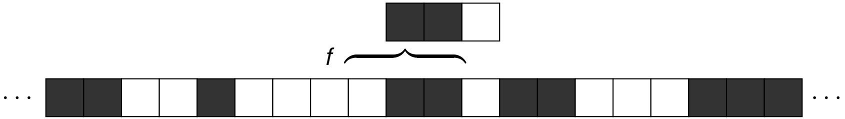

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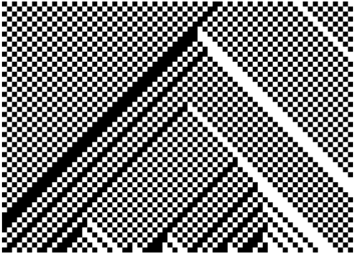
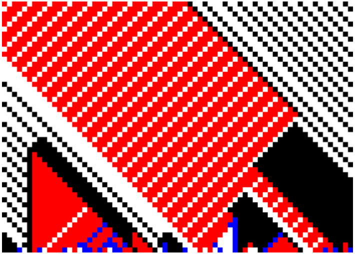
Simulations and typical asymptotic behaviour

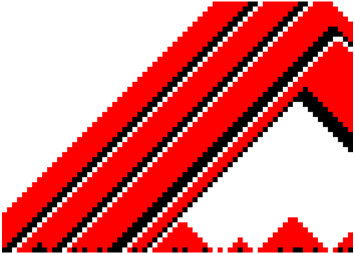
Traffic automaton



Captive automaton



3-state cyclic automaton



Additive automaton

Measure space

$\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ the **σ-invariant probability measures** on $\mathcal{A}^\mathbb{Z}$.

$\mu([u])$ the probability that a word $u \in \mathcal{A}^*$ appears, for $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$.

Examples

Bernoulli measure Let $(\lambda_a)_{a\in\mathcal{A}}$ such that $\sum \lambda_a = 1$. $\boxed{\forall u \in \mathcal{A}^*, \mu([u]) = \prod_{i=0}^{|u|-1} \lambda_{u_i}.}$

Dirac measure For $x \in \mathcal{A}^\mathbb{Z}$ and a borelian U, $\boxed{\delta_x(U) = \begin{cases} 1 \text{ if } x \in U \\ 0 \text{ otherwise.} \end{cases}}$

Measure supported by a periodic orbit For $w \in \mathcal{A}^*$, $\boxed{\widehat{\delta_w} = \frac{1}{|w|} \sum_{i=0}^{|w|-1} \delta_{\sigma^i({}^\infty w^\infty)}.}$

Markov measure With finite memory.

Action of an automaton on an initial measure

- F extends to an action $F_* : \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$:

$$\boxed{F_*\mu(U) = \mu(F^{-1}U)}$$

  for any borelian U.

- For an initial measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$, $F_*^t\mu$ represents the repartition at time t;
- Typical asymptotic behaviour is well described by the limit(s) of $(F_*^t\mu)_{t\in\mathbb{N}}$ in the **weak-* topology**:

$$\boxed{F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu \quad \Leftrightarrow \quad \forall u \in \mathcal{A}^*, F_*^t\mu([u]) \to \nu([u]).}$$

Examples of asymptotic behaviours

Proposition (left)

Let μ be the uniform Bernoulli measure on $\{0, 1, 2\}$ and F the 3-state cyclic automaton.

$$\boxed{F_*^t\mu \to \frac{1}{3}\widehat{\delta_0} + \frac{1}{3}\widehat{\delta_1} + \frac{1}{3}\widehat{\delta_2}.}$$

Examples of asymptotic behaviours

Proposition (right)

Let μ be a Bernoulli measure on $\{0,1\}$ and F the additive automaton.

$$\boxed{\frac{1}{n}\sum_{t=0}^{n} F_*^t\mu \to Ber\left(\frac{1}{2},\frac{1}{2}\right).}$$

Main question

Question

Which measures ν are reachable as the limit of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ for some cellular automaton F and initial measure μ?

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Which measures ν are reachable as the limit of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ for some cellular automaton *F* and initial measure μ?

Answer

All. (Take $F = Id$ and $\mu = \nu$)

Main question

Better question

Which measures ν are reachable as the limit of the sequence $(F_*^t\mu)_{t\in\mathbb{N}}$ for some cellular automaton F and **simple** initial measure μ (e.g. the uniform Bernoulli measure)?

In a sense, this would correspond to the “physically relevant” measure for F.

Section 2

Necessary conditions: computability obstructions

Topological obstructions

Topological obstruction

The accumulation points of $(F_*^t\mu)_{t\in\mathbb{N}}$ form a nonempty and **compact** set.

Measures and computability

$f : X \to Y$ is **computable** if there exists a Turing machine that, on any input $x \in X$, stops and outputs $f(x)$ (up to encoding).

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A probability measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is:

computable if there exists a computable function $f : \mathcal{A}^* \times \mathbb{N} \to \mathbb{Q}$ such that

$$\boxed{|\mu([u]) - f(u,n)| < 2^{-n}}$$

($\Leftrightarrow$ can be **simulated** by a probabilistic Turing machine)

Examples of computable measures

- Any periodic orbit measure;
- Any Bernoulli or Markov measure with computable parameters.

Measures and computability

$f : X \to Y$ is **computable** if there exists a Turing machine that, on any input $x \in X$, stops and outputs $f(x)$ (up to encoding).

A probability measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is:

semi-computable if there exists a computable function $f : \mathcal{A}^* \times \mathbb{N} \to \mathbb{Q}$ such that

$$\boxed{|\mu([u]) - f(u,n)| \underset{n\to\infty}{\longrightarrow} 0.}$$

($\Leftrightarrow$ **limit** of a computable sequence of measures)

Examples of computable measures

- Any periodic orbit measure;
- Any Bernoulli or Markov measure with computable parameters.

Computability obstruction

Action of an automaton on a computable measure

- If μ is computable, then $F_*^t\mu$ is **computable**;
- If μ is computable, and $F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu$, then ν is **semi-computable**.

Section 3

Sufficient conditions: construction of limit measures

State of the art

Theorem [Boyet, Poupet, Theyssier 06]

There is an automaton F such that the language of words u satisfying

$$F_*^t\mu([u]) \not\to 0$$

in **not computable** for any nondegenerate Bernoulli measure μ.

Theorem [Boyer, Delacourt, Sablik 10]

Let μ be the uniform Bernoulli measure.
For a large class of sets $U \subset \mathcal{A}^{\mathbb{Z}}$ (under computability conditions), there is an automaton F such that ***U* is the union of the supports of the limit points** of $(F_*^t\mu)_{t\in\mathbb{N}}$.

Main result

Action of an automaton on a computable measure

- If μ is computable, then $F_*^t\mu$ is **computable**;
- If μ is computable, and $F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu$,
  then ν is **semi-computable**.

Theorem

Let ν be a **semi-computable** measure. There exists:

- an alpabet $\mathcal{B} \supset \mathcal{A}$
- a cellular automaton $F : \mathcal{B} \to \mathcal{B}$

such that, for any **ergodic** and **full-support** measure $\mu \in \mathcal{M}_\sigma(\mathcal{B}^\mathbb{Z})$,

$$\boxed{F_*^t\mu \underset{t\to\infty}{\longrightarrow} \nu}$$

Approximation by periodic orbits

Proposition

Measures supported by periodic orbits are dense in $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$.

Example: Uniform Bernoulli measure

$w_0 = 01$
$w_1 = 0011$
$w_2 = 00010111$
$w_3 = 0000110100101111$

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Example: Uniform Bernoulli measure

$w_0 = 01$
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Proposition

If $\nu \in \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ is semi-computable, there is a **computable** sequence of words $(w_n)_{n\in\mathbb{N}}$ such that $\widehat{\delta_{w_n}} \to \nu$.

Our construction will compute each w_n in succession and approach the measure $\widehat{\delta_{w_n}}$ by writing concatenated copies of w_n on all the configuration.

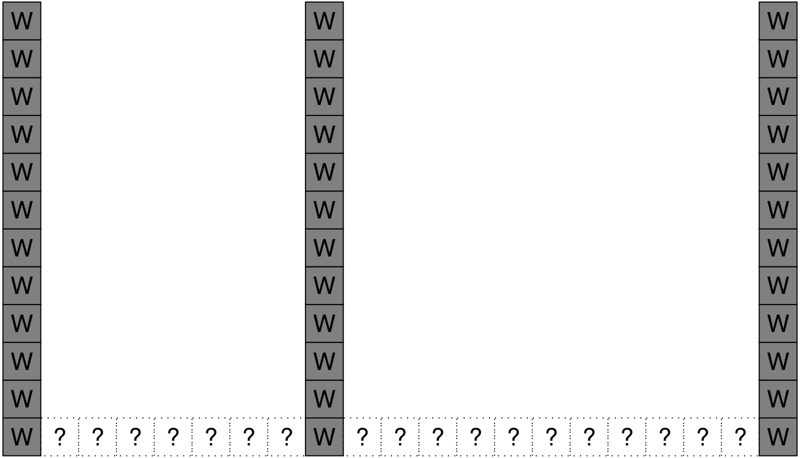
Computation in cellular automata

To simulate computation in a cellular automaton, **we use auxiliary states.**

- each cell contains the content of one tape cell;
- the cell where the head is located contains also the current state of the machine.

and the local rule corresponds to the rules of the machine.

Initialization



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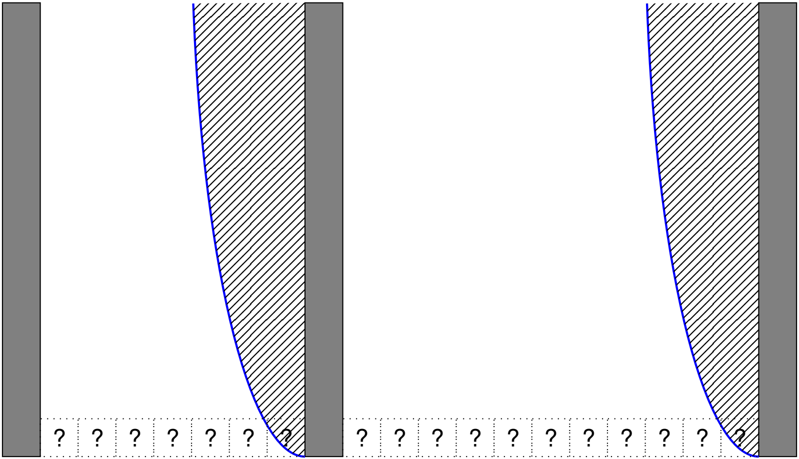

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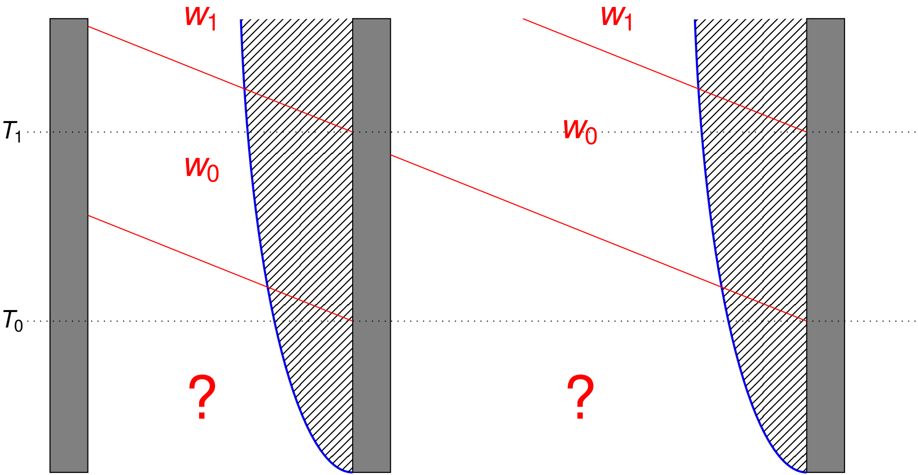

Time counters

?	?	?	?	#	1	0	2	W
?	?	?	?	?	#	2	1	W
?	?	?	?	?	#	1	2	W
?	?	?	?	?	#	1	1	W
?	?	?	?	?	?	#	2	W
?	?	?	?	?	?	#	1	W
?	?	?	?	?	?	#	0	W
?	?	?	?	?	?	?	#	I

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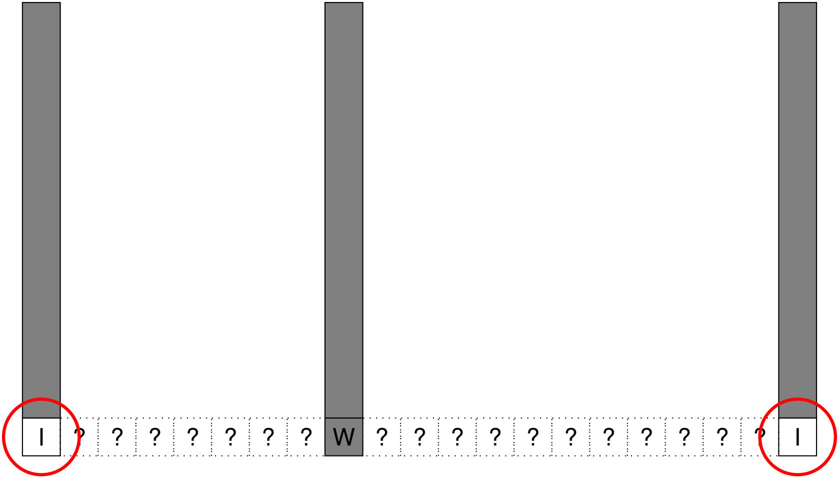

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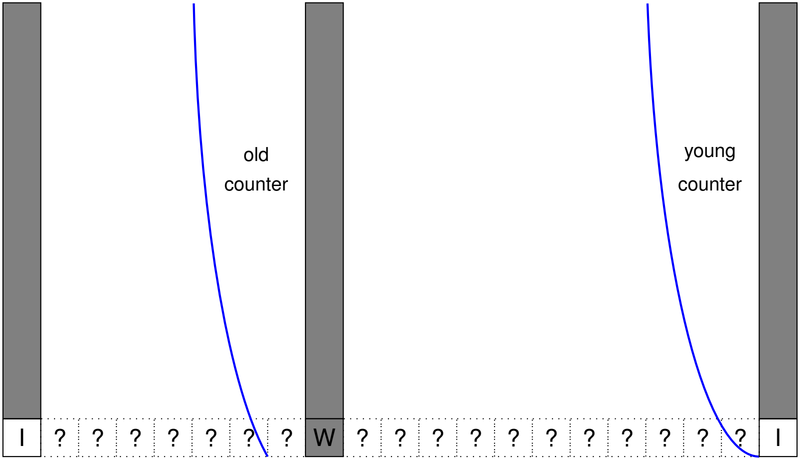
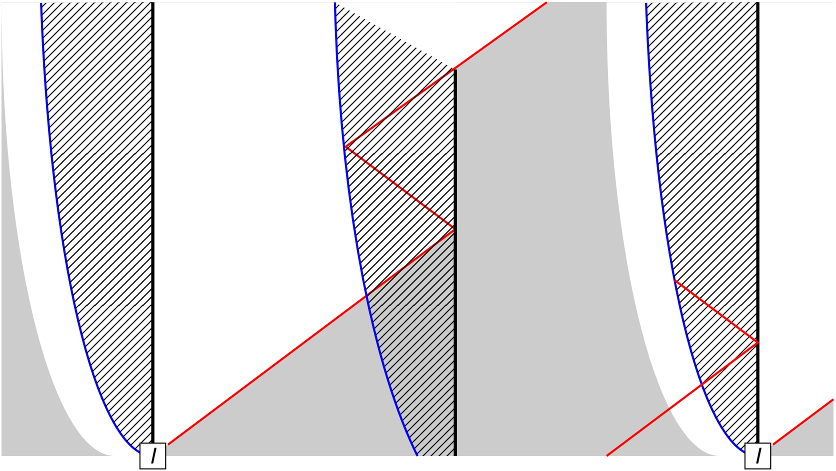
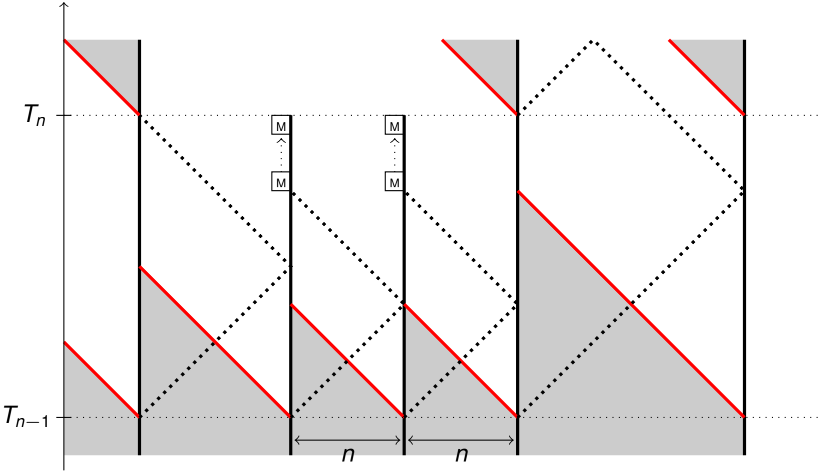


Sweeping counters



Merging and copying processes



Section 4

Extensions

General remarks

Implementation

- Non-trivial Turing machine satisfying space constraints;
- Large number of states;
  (for $|\mathcal{B}| = 2$, at least 2244 times more than the corresponding Turing machine)
- Speed of convergence $O\left(\frac{1}{\log t}\right)$ in the best case.

Extensions

- What about accumulation points?
- Can we remove auxiliary states?
- What about Cesaro mean convergence?
- Are properties of the limit measure decidable?
- Can we use the initial measure as an argument or an oracle?

Compact sets and computability

Consider the following distance function:

$$d_{\mathcal{M}}(\mu_1, \mu_2) = \sum_{n=0}^{\infty} \frac{1}{2^n} \max_{u \in \mathcal{A}^n} |\mu_1([u]) - \mu_2([u])|$$

Then the **computability of a compact set** $\mathcal{V}$ can be defined as the computability of the associated distance function $d_{\mathcal{V}} : \mu \to \min_{\nu \in \mathcal{V}} d_{\mathcal{M}}(\mu, \nu)$.

$\mathcal{V}$ computable if $d_{\mathcal{V}} : \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \mapsto \mathbb{R}$ is **computable**, that is:

$$\exists f : \mathcal{A}^* \times \mathbb{N} \to \mathbb{Q} \text{ computable}, |d_{\mathcal{V}}(\widehat{\delta_w}) - f(w, n)| \leq \frac{1}{2^n}$$

and $\exists b : \mathbb{N} \mapsto \mathbb{Q}$ computable,

$$d_{\mathcal{M}}(\mu_1, \mu_2) < b(m) \Rightarrow |f(\mu_1, n) - f(\mu_2, n)| \leq \frac{1}{2^m}$$

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Then the **computability of a compact set** $\mathcal{V}$ can be defined as the computability of the associated distance function $d_{\mathcal{V}} : \mu \to \min_{\nu \in \mathcal{V}} d_{\mathcal{M}}(\mu, \nu)$.

$\mathcal{V}$ Σ_2-computable if $\boxed{\boldsymbol{d_{\mathcal{V}} = \liminf d_i}}$, where d_i are elements in $\mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}}) \mapsto \mathbb{R}$, and:

$$\exists f : \mathbb{N} \times \mathcal{A}^* \times \mathbb{N} \mapsto \mathbb{Q} \text{ computable,}$$

$$|d_i(\widehat{\delta_w}) - f(i, w, n)| \leq \frac{1}{2^n} \textbf{ (sequential computability)}$$

$$\text{and } \exists b : \mathbb{N} \mapsto \mathbb{Q} \text{ computable,}$$

$$d_{\mathcal{M}}(\mu_1, \mu_2) < b(m) \Rightarrow |d_i(\mu_1) - d_i(\mu_2)| \leq \frac{1}{2^m} \textbf{ (effective uniform equicontinuity).}$$

Computability obstructions, again

Action of an automaton on a computable measure:

- If μ is computable, then $F_*^t\mu$ is **computable**;
- If μ is computable and the accumulation points of $(F_*^t\mu)_{t\in\mathbb{N}}$ are $\mathcal{V}$, then $\mathcal{V}$ is nonempty, compact and $\boldsymbol{\Sigma_2}$**-computable**.

Intuitively, $d_{\mathcal{V}} = \liminf d_{\mathcal{M}}(F_*^t\mu, .)$.

Proposition

If $\mathcal{V} \subset \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ is nonempty and Σ_2-computable, there is a **computable** sequence of words $(w_n)_{n\in\mathbb{N}}$ such that **$\mathcal{V}$ is the set of accumulation points of $(\widehat{\delta_{w_n}})_{n\in\mathbb{N}}$**.

Main result, again

Theorem

Let $\mathcal{V}$ be a nonempty, compact, **connected**, Σ_2-**computable** set of measures.
Then there exists an automaton $F : \mathcal{A} \to \mathcal{A}$ such that, for any measure $\mu \in \mathcal{M}_\sigma(\mathcal{A}^{\mathbb{Z}})$ $\boldsymbol{\sigma}$**-mixing** and **full-support**,

$$\boxed{\text{The set of accumulation points of } (F_*^t\mu)_{t\in\mathbb{N}} \text{ is } \mathcal{V}.}$$

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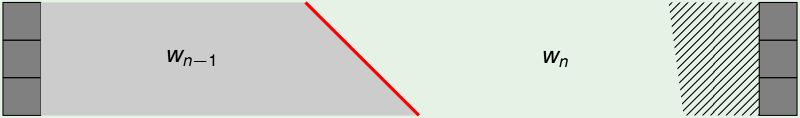
Hypothesis of connectedness



Removing the auxiliary states

Theorem

Let ν be a **non full-support**, **semi-computable** measure.
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Idea: use forbidden words to encode auxiliary states.

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Remark

If $F_*^t \mu \to \nu$ where ν is a full support measure, then F is a **surjective** automaton and **the uniform Bernoulli measure is invariant**.

Computation in the measure space

Let us consider the operator

$$\boxed{\mu \mapsto \text{accumulation points of } (F_*^t \mu)_{t\in\mathbb{R}}}$$

The previous construction gave us operators that were essentially **constant** (on a large domain).

Question

Which operators $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}) \to \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ (ou $\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}) \to \mathcal{P}(\mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z}))$) can be realized in this way?

Theorem

Let $\nu : \mathbb{R} \to \mathcal{M}_\sigma(\mathcal{A}^\mathbb{Z})$ be a **semi-computable** operator. There is:

- an alphabet $\mathcal{B} \supset \mathcal{A}$,
- an automaton $F : \mathcal{B}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z}$

such that, for any **full-support** and **exponentially σ-mixing** measure μ,

$$\boxed{F_*^t \mu \underset{t\to\infty}{\longrightarrow} \nu\left(\mu\left(\boxed{\mathtt{I}}\right)\right)}.$$

Some examples

Let $M \subset \mathcal{M}_\sigma(\mathcal{B}^\mathbb{Z})$ be the set of **full-support**, **exponentially σ-mixing** measures.

Example 1: Density classification

There exists an automaton $F : \mathcal{B}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z}$ realizing the operator:

$$
\begin{aligned}
M &\to \mathcal{M}_\sigma(\{0,1\}^\mathbb{Z}) \\
\mu &\mapsto \begin{cases} \widehat{\delta_0} & \text{if } \mu(\boxed{1}) < \frac{1}{2} \\ \widehat{\delta_1} & \text{otherwise.} \end{cases}
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Example 2: A simple oracle

There exists an automaton $F : \mathcal{B}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z}$ realizing the operator:

$$M \to \mathcal{M}_\sigma(\{0,1\}^\mathbb{Z})$$
$$\mu \mapsto Ber(\mu(\boxed{1}))$$

Implementation of a simple case

Fibonacci word

Consider the morphism :

$$\begin{array}{rccc} \varphi : & \mathcal{A}^* & \to & \mathcal{A}^* \\ & 0 & \mapsto & 01 \\ & 1 & \mapsto & 0 \end{array}$$

Then the sequence $\varphi^n(0)$ converges to an infinite word called **Fibonacci word**:

$$\varphi^\infty(0) = 0100101001001010010101\ldots$$

Proposition

The Fibonacci word is uniquely ergodic.