Absolute continuity of measures and preservation of Randomness

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Abstract. We continue to explore the relationship between notions of effective measure theory and properties of algorithmically random elements, as started in [1]. We prove a tight quantitative equivalence between effective versions of absolute continuity and preservation of randomness. We then relate these to the computability of the Radon-Nikodym derivative, as an element of L^1. In doing this we show how algorithmic randomness enables direct and short proofs when concerned with L^1-computability.

Keywords: Martin-Löf randomness, randomness deficiency, computable analysis

1 Introduction

This paper is concerned with the theory of algorithmic randomness in connection with Computable Analysis and Measure Theory. Theorems about random points are usually presented as strengthened versions of results from ordinary probability theory. On the other hand, computable analysis generally provides effective versions of notions from probability theory, that in turn imply results about random points.

For example, if f_n is a sequence of computable random variables, then

$$\boxed{\text{Effective } \mu\text{-a.e. convergence} \implies \text{convergence on } \mathsf{ML}^\mu \implies \mu\text{-a.e. convergence,}}$$

and no implication can be reversed. Indeed, there are examples (convergence of martingales, or ergodic theorems) where the convergence is not effective and yet it holds at each random element.

Thus, the result for random elements stands somewhere between the computable world and the ordinary one, and there is, in general, no exact correspondence. Nevertheless, the theory of algorithmic randomness provides additional structure which allows to quantify the degree at which a given point is random. In this way, random points can be stratified via their *randomness deficiency* (see

Section 1.1). When this information is taken into account, much sharper results are possible. In our example above, if $d(x)$ denotes the randomness deficiency of x, an exact correspondence can be obtained as follows (see [1]):

$$\boxed{\text{Effective } \mu\text{-a.e. convergence} \iff \begin{array}{r}\text{convergence on each } x \in \mathsf{ML}^\mu, \\ \text{at a rate computable from } d(x).\end{array}}$$

In this paper we investigate from the above point of view the relationship between absolute continuity of measures and preservation of randomness. It can be seen as a continuation of [2] where the authors showed that preservation of randomness implies absolute continuity, and that this implication cannot be reversed.

Our main result is a tight quantitative relationship between the modulus of absolute continuity of a measure μ w.r.t. a measure λ and the change in the randomness deficiencies from μ-random to λ-random points. In particular, this gives a characterization of *effective absolute continuity* in terms of randomness deficiency, showing that the above picture also holds in this case.

We then apply this result and obtain various connections between the computability of the Radon-Nikodym derivative, the effectivity of the absolute continuity and randomness preservation. In particular, we prove a converse to the second main result from [3] which asserts that computable normability of a measure μ relative to a measure λ (see Section 3) is a sufficient condition for the computability of the density, as an element of L^1. We show that this condition is also necessary. We also provide an alternative counter-example to show that even if the absolute continuity is effective, the Radon-Nikodym derivative need not be computable, as an element of L^1. In all the proofs we present, randomness deficiency and layerwise computability play an essential role to capture and manipulate integrable functions.

The paper is organized as follows. In the rest of this section we recall the relevant definitions and properties that we will need. In Section 2 we introduce the notion of *effective absolute continuity* and prove our main result. In Section 3 we present the already mentioned applications.

1.1 Background

Computability. Let us first recall some basic results established in [4, 5]. We work on the well-studied computable metric spaces (see [6–10]).

Definition 1. *A* **computable metric space** *is a triple* $(X, d, \mathcal{S})$ *where:*

1. (X, d) *is a separable metric space,*
2. $\mathcal{S} = \{s_i : i \in \mathbb{N}\}$ *is a countable dense subset of* X *with a fixed numbering,*
3. $d(s_i, s_j)$ *are uniformly computable real numbers.*

$\mathcal{S}$ is called the set of ***simple*** points. If $x \in X$ and $r > 0$, the metric ball $B(x, r)$ is defined as $\{y \in X : d(x, y) < r\}$. The set $\mathcal{B} := \{B(s, q) : s \in \mathcal{S}, q \in$

$\mathbb{Q}, q > 0\}$ of ***simple balls***, which is a basis of the topology, has a canonical numbering $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$. An ***effective open set*** is an open set U such that there is a r.e. set $E \subseteq \mathbb{N}$ with $U = \bigcup_{i \in E} B_i$. If X' is another computable metric space, a function $f : X \to X'$ is ***computable on a set*** D if there are uniformly effective open sets $U_i \subseteq X$ such that $f^{-1}(B'_i) \cap D = U_i \cap D$ for every i. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. A function $f : X \to \overline{\mathbb{R}}$ is ***lower semi-computable on a set*** D if there are uniformly effective open sets $U_i \subseteq X$ such that $f^{-1}(q_i, +\infty] \cap D = U_i \cap D$ for every i (where $q_0, q_1, \dots$ is a fixed effective enumeration of the set of rational numbers $\mathbb{Q}$).

A study of effective version of general measurable spaces was carried on in [11]. In the present paper, we restrict our attention to metric spaces endowed with the Borel σ-field (the σ-field generated by the open sets) over which we consider probability measures. Let $(X, d, \mathcal{S})$ be a computable metric space.

Definition 2 (from [5, 12, 13]). *A Borel probability measure is computable if for every effective open set U, $\mu(U)$ is lower semi-computable, uniformly in an index of U. Equivalently, μ is computable if for every lower semi-computable function $f : X \to [0, +\infty]$, $\int f \,\mathrm{d}\mu$ is lower semi-computable, uniformly in an index of f.*

Given two computable Borel probability measures μ and λ over X, it can be proved that there is a computable sequence r_n of real numbers that is dense in $(0, +\infty)$ such that the balls $B(s, r_n)$ are all sets of ***μ-continuity***[3] and of λ-continuity, for all $s \in \mathcal{S}$ (see [14]). The notion of effective open set could be defined using this alternative basis, giving the same notions. While the measures of simple balls are in general only lower semi-computable, the numbers $\mu(B(s, r_n))$ are all computable, uniformly in s, n.

Algorithmic Randomness. If μ is a computable measure, the set ML^μ of ***Martin-Löf random*** points is defined as the maximal set of measure one that is *effective* in some particular sense (see [1] for more details). The set of random points ML^μ comes with a canonical decomposition into ***layers***: $\mathsf{ML}^\mu = \bigcup_n \mathsf{ML}^\mu_n$. The layers ML^μ_n have the properties:

(i) $\mathsf{ML}^\mu_n \subseteq \mathsf{ML}^\mu_{n+1}$,
(ii) $\mu(\mathsf{ML}^\mu_n) > 1 - 2^{-n}$ and
(iii) $X \setminus \mathsf{ML}^\mu_n$ is an effective open set, uniformly in n.

This decomposition is ***universal*** in the sense that, for any other sequence of sets K_n satisfying the above three conditions, there exists a constant c (computable from an index of the sequence K_n) for which

$$\mathsf{ML}^\mu_n \subseteq K_{n+c} \qquad \text{for all } n. \tag{1}$$

For such a sequence K_n, the sets $X \setminus K_n$ form what is known as a ***Martin-Löf test***. Another classical way of characterizing Martin-Löf randomness is using the

[3] A set is of μ-continuity if the measure of its boundary is zero.

notion of integrable test: an ***integrable test*** is a lower semi-computable function $t : X \to [0, +\infty]$ such that $\int t \, \mathrm{d}\mu \leq 1$. For every computable measure μ there exists a test, denoted t_μ, which is ***universal*** in the sense that for any other test t' there exists a constant c (computable from an index of t') such that $t' \leq c \cdot t_\mu$. The universal test t_μ characterizes randomness as follows: a point x is μ-random if and only if $t_\mu(x)$ is finite. Observe that large values of $t_\mu(x)$ are unlikely, by Markov's inequality. For a random point x, the ***randomness deficiency*** $\mathbf{d}_{\boldsymbol{\mu}}\boldsymbol{(x)}$ is usually defined by $d_\mu := \log(t_\mu(x))$. A slightly different way of measuring the randomness deficiency of a random point x, is by considering the minimal n such that $x \in \mathsf{ML}_n^\mu$. Denoting this quantity by $\boldsymbol{l_\mu(x)}$, the relation between the two is given by

$$\mathrm{d}_\mu(x) \leq l_\mu(x) + c \leq \mathrm{d}_\mu(x) + 2\log(\mathrm{d}_\mu(x)) + 2c,$$

where c is a constant independent of x.

Once the layers have been fixed, virtually every computability notion has a natural (weaker) *layerwise* counterpart, by restricting the requirements of the definition to hold on every layer, in a uniform way. We will work with the following two instances.

Definition 3. *A function $f : X \to Y$ is said to be* **$\boldsymbol{\mu}$-layerwise computable** *if it is computable on every layer ML_n^μ, uniformly in n.*

In the language of representations, f is μ-layerwise computable if there is a machine which takes n and a Cauchy representation of $x \in \mathsf{ML}_n$, and outputs a Cauchy representation of $f(x)$. Observe that a layerwise computable function is *computable in probability*, in the sense that there is a machine which, upon input n, computes the function f with a probability of error less than 2^{-n}.

In the same vain, for real-valued functions, we define the layerwise version of the notion of lower semi-computability.

Definition 4. *A function $f : X \to \overline{\mathbb{R}}$ is said to be* **$\boldsymbol{\mu}$-layerwise lower semi-computable** *if it is lower-semi-computable on every layer ML_n^μ, uniformly in n.*

A nice phenomenon is that the layerwise versions of many computability notions inherit some of their properties. The integral of a lower semi-computable nonnegative function is lower semi-computable, and it remains true for *layerwise* lower semi-computable functions:

Proposition 1. *Let μ be a computable measure and $f : X \to [0, +\infty]$ a μ-layerwise lower semi-computable function. The number $\int f \, \mathrm{d}\mu$ is lower semi-computable.*

In [1], it is shown that there is a sharp correspondence between layerwise computability notions and effective versions of measurability or integrability. Among them we will use the following.

Theorem 1 (from [1]). *A function $f : X \to \overline{\mathbb{R}}$ is $L^1(\mu)$-computable if and only if it is equivalent to a μ-layerwise computable function and $\int f \, \mathrm{d}\mu$ is a computable number.*

We recall that when μ is a computable Borel probability measure over X, the spaces $L^p(\mu)$ (we will only use $p = 1, 2$) can be made computable metric spaces using any computable dense sequence $\mathcal{F}_0 = \{g_0, g_1, \dots\}$ of bounded computable functions (such that a bound for g_i can be computed from i) as ideal elements. We say then that a function $f \in L^p(\mu)$ is $\boldsymbol{L^p(\mu)}$***-computable*** if its equivalence class is a computable element of $L^p(\mu)$. The above theorem says that the computable elements (equivalent classes) of $L^1(\mu)$ are exactly those that contain a μ-layerwise computable representative whose integral is a computable number.

2 Absolute continuity

Let μ and λ be computable Borel probability measures. Classically, measure μ is said to be ***absolutely continuous w.r.t.*** λ (denoted $\mu \ll \lambda$) if $\mu(A) = 0$, whenever A is a Borel set and $\lambda(A) = 0$. In [2] it is proved that $\mathsf{ML}^\mu \subseteq \mathsf{ML}^\lambda$ (preservation of randomness), implies $\mu \ll \lambda$. It is also proved that the converse does not hold in general. In this Section we show that an equivalence can be obtained under a natural effectivity assumption. It is easy to see that absolute continuity of μ w.r.t. λ is equivalent to the following: there is a function $\phi : \mathbb{N} \to \mathbb{N}$ such that for all Borel sets A and all $n \in \mathbb{N}$, if $\lambda(A) < 2^{-\varphi(n)}$ then $\mu(A) < 2^{-n}$. The obvious effective version is therefore as follows.

Definition 5. *Let μ, λ be two Borel probability measures. We say that μ is* **absolutely continuous w.r.t. λ, effectively**, *if there is a computable function $\varphi : \mathbb{N} \to \mathbb{N}$ such that for all Borel sets A and all $n \in \mathbb{N}$, if $\lambda(A) < 2^{-\varphi(n)}$ then $\mu(A) < 2^{-n}$. This is denoted $\mu \ll_{\text{eff}} \lambda$.*

It is straightforward to see that if μ and λ are computable, $\mu \ll_{\text{eff}} \lambda$ implies $\mathsf{ML}^\mu \subseteq \mathsf{ML}^\lambda$, as $\mu(U_n^\lambda)$ converges effectively to 0, where U_n^λ is a universal λ-test. One obtains the following picture:

$$\boxed{\mu \ll_{\text{eff}} \lambda \implies \mathsf{ML}^\mu \subseteq \mathsf{ML}^\lambda \implies \mu \ll \lambda.}$$

Remark 1. There is a weaker notion of randomness, due to Schnorr. It is also straightforward to see that $\mu \ll_{\text{eff}} \lambda$ implies that μ-Schnorr random points are λ-Schnorr random. In [2], it is proved that there exist computable measures μ, λ such that $\mathsf{ML}^\mu \subseteq \mathsf{ML}^\lambda$ but $\mathsf{Sch}_\mu \not\subseteq \mathsf{Sch}_\lambda$. From this, $\mathsf{ML}^\mu \subseteq \mathsf{ML}^\lambda$ does not imply $\mu \ll_{\text{eff}} \lambda$.

With the next theorem we show that an equivalence can be obtained when the randomness deficiency is taken into account:

$$\boxed{\mu \ll_{\text{eff}} \lambda \iff \mathsf{ML}_n^\mu \subseteq \mathsf{ML}_{\psi(n)}^\lambda \text{ for some computable } \psi : \mathbb{N} \to \mathbb{N}.}$$

Moreover, the modulus of absolute continuity and the variation in randomness deficiency are sharply related.

Theorem 2. *Let μ, λ be two computable Borel probability measures. The following are equivalent:*

1. *$\mu \ll_{\mathit{eff}} \lambda$,*
2. *there is a computable $\psi : \mathbb{N} \to \mathbb{N}$ such that $\mathsf{ML}^{\mu}_{n} \subseteq \mathsf{ML}^{\lambda}_{\psi(n)}$ for all n.*

Moreover, given φ in 1, one can compute c such that $\psi(n) := \varphi(n+c)$ satisfies 2, and given ψ in 2, one can compute c such that $\varphi(n) := \psi(n) + c$ satisfies 1.

Proof. 1 $\Rightarrow$ 2. Suppose $\mu \ll_{\text{eff}} \lambda$ with a function φ and put $K_n = \mathsf{ML}^{\lambda}_{\varphi(n)}$. Then, since $\lambda(K_n) > 1 - 2^{\varphi(n)}$, the effective absolute continuity implies $\mu(K_n) > 1 - 2^{-n}$. Therefore, by the universality property (1) of ML^{μ}_{n}, one can compute c such that $\mathsf{ML}^{\mu}_{n} \subseteq K_{n+c}$ for all n, which means that $\mathsf{ML}^{\mu}_{n} \subseteq \mathsf{ML}^{\lambda}_{\varphi(n+c)}$ for all n. Thus, the function $\psi(n) = \varphi(n+c)$ satisfies the requirements of 2.

2 $\Rightarrow$ 1. We will need the following lemma.

Lemma 1. *Let μ, λ be computable measures. Let $\alpha < \beta$ be two rational numbers. If there is a Borel set A such that $\mu(A) > \alpha$ and $\lambda(A) < \beta$, then there exists a finite union of μ-continuity and λ-continuity balls $A' = B_{n_1} \cup \ldots \cup B_{n_k}$ satisfying $\mu(A') > \alpha$ and $\lambda(A') < \beta$.*

Proof. Let A be a Borel set satisfying $\lambda(A) < \beta$ and $\mu(A) > \alpha$. By regularity of Borel measures, there exists an open set $U \supseteq A$ such that $\lambda(U) < \beta$. Let $B_1, B_2, \ldots$ be an enumeration of the (μ and λ)-continuity balls introduced after Definition 2. Since U is open, there is a sequence $n_1, n_2, \ldots$ such that $U = \cup_i B_{n_i}$. Moreover, since

$$\mu(\cup_{i=1}^{k} B_{n_i}) \nearrow \mu(U) \quad \text{as} \quad k \to \infty \qquad \text{and} \qquad \mu(U) \geq \mu(A) > \alpha,$$

there must be k such that $A' = \cup_{i=1}^{k} B_{n_i}$ satisfies $\mu(A') > \alpha$. As $A' \subseteq U$, $\lambda(A') < \beta$. The Lemma is proved.

We proceed with the proof of the theorem. Suppose $\mathsf{ML}^{\mu}_{n} \subseteq \mathsf{ML}^{\lambda}_{\psi(n)}$ for all $n \in \mathbb{N}$. We consider the following procedure: given c as input, search for a finite union of (μ and λ)-continuity balls $A = B_{n_1} \cup \ldots \cup B_{n_k}$ and some $n \in \mathbb{N}$ such that

$$\lambda(A) < 2^{-\psi(n)-c} \quad \text{but} \quad \mu(A) > 2^{-n}. \tag{2}$$

Since the measures of finite unions of continuity balls are computable, one can effectively find such an A and n, if they exist. Now, for each c, define A_c and n_c by

$$(A_c, n_c) = \begin{cases} (A, n) & \text{if there exists } A \text{ and } n \text{ satisfying } (2) \\ (\emptyset,\ \text{not defined}) & \text{otherwise.} \end{cases}$$

We have that the sets A_c are effectively open, uniformly in c. Moreover, whatever the case one always has

$$\lambda(A_c) < 2^{-\psi(n_c)-c} < 2^{-c},$$

so that the sequence A_c is a Martin-Löf test w.r.t. λ. We now modify the test A_c by stretching it as much as possible in a way that it remains a λ-test. When n_c is defined, let

$$u_c := \psi(n_c) + c.$$

We first extract a nondecreasing subsequence u_{c_k} by defining c_k as follows:

- put $c_0 = 0$;
- if c_k has been defined, search for $c > c_k$ such that u_c is defined and $u_c > u_{c_k}$. If such a c exists, let c_{k+1} be the first one that is enumerated, otherwise c_{k+1} is not defined.

The modified test U_i is defined by

$$U_i = \begin{cases} A_{c_k} & \text{if there is } k \text{ such that } c_{k-1} \text{ and } c_k \text{ are defined, and } u_{c_{k-1}} < i \leq u_{c_k}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Again U_i is an effective open set, uniformly in i, and $\lambda(U_i) < 2^{-i}$ for all i: indeed,

$$\text{if} \quad u_{c_{k-1}} < i \leq u_{c_k} \quad \text{then} \quad \lambda(U_i) = \lambda(A_{c_k}) < 2^{-u_{c_k}} \leq 2^{-i}.$$

We now modify it a little further and define

$$V_i = \bigcup_{j \geq i+1} U_j.$$

Clearly, $\lambda(V_i) < 2^{-i}$ so that V_i is a λ-test as well. By universality of ML_n^λ, one can therefore compute $b \in \mathbb{N}$ such that

$$\mathsf{ML}_i^\lambda \cap V_{i+b} = \emptyset \quad \text{for all } i. \tag{3}$$

Observe that the above relation remains true for any $b' \geq b$ since $V_{i+b'} \subseteq V_{i+b}$. In particular, assume there is $k \in \mathbb{N}$ such that c_k is defined and $c_k \geq b$. First, we see that

$$\mathsf{ML}_i^\lambda \cap V_{i+c_k} = \emptyset \quad \text{for all } i \in \mathbb{N},$$

so that for $i = \psi(n_{c_k})$ one has

$$\mathsf{ML}_{\psi(n_{c_k})}^\lambda \cap V_{u_{c_k}} = \emptyset \implies \mathsf{ML}_{\psi(n_{c_k})}^\lambda \cap A_{c_k} = \emptyset.$$

We now use the hypothesis: $\mathsf{ML}_{n_{c_k}}^\mu \subset \mathsf{ML}_{\psi(n_{c_k})}^\lambda$, which implies

$$\mathsf{ML}_{n_{c_k}}^\mu \cap A_{c_k} = \emptyset$$

and thus $\mu(A_{c_k}) < 2^{-n_{c_k}}$, which contradicts the construction of A_{c_k}. The conclusion is that there cannot be a k such that c_k is defined and $c_k \geq b$. Hence, $\mu(A) < 2^{-n}$ whenever $\lambda(A) < 2^{-\psi(n)-b}$, as was to be shown.

The implication 1 ⇒ 2 has a useful consequence.

Corollary 1. *If μ, λ are computable measures satisfying $\mu \ll_{\mathit{eff}} \lambda$ then every λ-layerwise computable function is also μ-layerwise computable.*

Proof. Let ψ be as in Theorem 2. Let f be λ-layerwise computable. Let M be a machine that on input n and oracle $x \in \mathsf{ML}_n^\lambda$ computes $f(x)$. Let M' be the machine that given n as input and x as oracle, simulates M on input $\psi(n)$ and oracle x. If $x \in \mathsf{ML}_n^\mu$ then $x \in \mathsf{ML}_{\psi(n)}^\lambda$ so M' will compute $f(x)$.

In the particular case when μ is bounded by λ (up to a constant), one obtains the following characterizations:

Proposition 2. *Let μ, λ be computable Borel probability measures. The following are equivalent:*

1. $L^1(\lambda) \subseteq L^1(\mu)$,
2. $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \in L^\infty(\lambda)$,
3. $\mu \leq c\,\lambda$ *for some constant* c,
4. $\mathsf{ML}_n^\mu \subseteq \mathsf{ML}_{n+c}^\lambda$ *for some constant* c,
5. $t_\lambda \leq c\,t_\mu$ *for some constant* c,
6. $t_\lambda \in L^1(\mu)$.

Proof. 1 ⇔ 2 ⇔ 3 is classical, 3 ⇔ 4 follows from Theorem 2, 5 ⇔ 6 is trivial, as well as 1 ⇒ 6. We now prove 5 ⇒ 3. By contradiction, suppose for every n there is a Borel set A_n such that $\mu(A_n) > 2^n \lambda(A_n)$. Once again, by Lemma 1, the sets A_n can be assumed to be a finite union of balls that are μ-continuity and λ-continuity sets. Hence, the sets A_n can be effectively constructed. The function $f = \sum_n^\infty \mu(A_n)^{-1} \mathbf{1}_{A_n}$ is therefore lower semi-computable and $\int f \,\mathrm{d}\lambda = \sum_n^\infty \mu(A_n)^{-1}\lambda(A_n) \leq \sum_n^\infty 2^{-n} \leq 1$, so there is c such that $f \leq c\,t_\lambda$. If $t_\lambda \leq c\,t_\mu$ then $f \leq c\,t_\mu$ which is impossible as $\int f\,\mathrm{d}\mu = \sum_n^\infty 1 = +\infty$. The proof is complete.

3 Application: The Radon-Nikodym derivative

We now apply the Layerwise machinery to study from a computability point of view the Radon-Nikodym theorem. In [3], it is shown that the Radon-Nikodym operator has the degree of the (non-computable) operator EC. Moreover, an explicit condition is given on the measures which implies computability of the derivative. Namely, it is shown that

Theorem 3 ([3]). *For measures $\mu \ll \lambda$, if μ is computably normable relative to λ, then $h = \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ is $L^1(\lambda)$-computable, from λ and μ.*

We now explain what it means for a measure to be *computably normable* relative to another. Let μ and λ be Borel measures. Consider the linear operator $L_\mu : L^2(\mu+\lambda) \to \mathbb{R}$ be defined by

$$L_\mu(f) := \int f\,\mathrm{d}\mu.$$

This is a bounded linear operator and it is easy to see that from μ, λ and f, one can compute the value $L_\mu(f)$. We recall that for a linear operator u acting on the Hilbert space L^2, the norm $\|u\|$ is defined by

$$\|u\| := \sup\{c \in \mathbb{R} : |u(f)| \leq c\|f\|_2\} = \sup_{\|f\|_2=1} |u(f)|.$$

Definition 6. *A computable measure μ is said to be* **computably normable relative to** *some other computable measure λ, if the operator L_μ (as defined above) has a computable norm.*

The Radon Nikodym theorem has also been studied in Bishop's style constructive mathematics. A theorem very similar to Theorem 3 was presented for example in [15]. In what follows we prove the following converse to Theorem 3, for computable measures. Observe that the statement does not involve algorithmic randomness at all. The proof, however, relies heavily on the notion of randomness deficiency via layerwise computability (especially point 3) to capture and manipulate integrable functions in a simple way.

Theorem 4. *Let λ be a computable probability measure. Let $h : X \to [0, +\infty]$ be a $L^1(\lambda)$-computable function such that $\int h\lambda = 1$ and denote by μ the measure with density h. Then*

1. *μ is computable,*
2. *not only $\mu \ll \lambda$ but $\mu \ll_{\mathit{eff}} \lambda$, and*
3. *μ is computably normable relative to λ.*

Proof. By Theorem 1, we can assume without loss of generality that h is the λ-layerwise computable representative of its equivalence class.

1. We have to prove that for effective open sets U, the values $\mu(U)$ are lower semi-computable real numbers, uniformly in U. By definition, $\mu(U) = \int h\mathbf{1}_U \,\mathrm{d}\lambda$ and $h\mathbf{1}_U$ is clearly λ-layerwise lower semi-computable, which implies the lower semi-computability of $\mu(U)$ by Proposition 1.

2. As h is λ-layerwise computable, the functions $h\mathbf{1}_{\{h<n\}}$ are uniformly λ-layerwise lower semi-computable, so their λ-integrals are lower semi-computable numbers which increase to 1. We can then compute a subsequence n_i such that

$$\int h\mathbf{1}_{\{h\geq n_i\}} \,\mathrm{d}\lambda < 2^{-i-1}.$$

Now, for each i, put $\varphi(i) = i + 1 + \lceil\log(n_i)\rceil$. Let A be a Borel set such that $\lambda(A) < 2^{-\varphi(i)}$. Then

$$\begin{aligned}\mu(A) = \int h\mathbf{1}_A \,\mathrm{d}\lambda = \int\limits_{\{h\geq n_i\}} h\mathbf{1}_A \,\mathrm{d}\lambda + \int\limits_{\{h<n_i\}} h\mathbf{1}_A \,\mathrm{d}\lambda \\ \leq 2^{-i-1} + n_i\lambda(A) \\ < 2^{-i-1} + n_i 2^{-\varphi(i)} \leq 2^{-i}.\end{aligned}$$

3. The Radon-Nikodym derivative of μ with respect to $(\mu+\lambda)$ is given by the function

$$g := \frac{h}{h+1}.$$

Indeed, for $f \in L^1(\mu)$,

$$\int f \,\mathrm{d}\mu = \int hf \,\mathrm{d}\lambda = \int g(h+1)f \,\mathrm{d}\lambda = \int gf \,\mathrm{d}(\mu+\lambda).$$

The norm of L_λ is $\|g\|_{L^2(\mu+\lambda)} = \sqrt{\int g^2 \,\mathrm{d}(\mu+\lambda)} = \sqrt{\int g \,\mathrm{d}\mu}$. We prove that $\int g \,\mathrm{d}\mu$ is computable.

Since h is λ-layerwise computable, for any $x \in \mathsf{ML}_n^\lambda$ one can compute $h(x)$ from x and n. Therefore, we can also compute $\frac{h(x)}{1+h(x)}$, which means that the function g is λ-layerwise computable too. By part 2, we know that $\mu \ll_{\text{eff}} \lambda$. By Corollary 1, g is μ-layerwise computable too. Let $\nu = (\mu+\lambda)/2$. One easily obtains $\mathsf{ML}_n^\nu = \mathsf{ML}_n^\mu \cup \mathsf{ML}_n^\lambda$, which implies that g is even ν-layerwise computable. Now, since g is the density of μ w.r.t. $(\mu+\lambda)$, we get that $\int g \,\mathrm{d}\nu = \frac{1}{2}\int g \,\mathrm{d}(\mu+\lambda) = \frac{1}{2}\int 1 \,\mathrm{d}\mu = \frac{1}{2}$ which is a computable number, so g is $L^1(\mu+\lambda)$-computable by Theorem 1. As $\mu \leq \mu+\lambda$, it follows that g is also $L^1(\mu)$-computable (the $L^1(\mu)$-norm is dominated by the $L^1(\mu+\lambda)$-norm, so a computable sequence of simple functions converging rapidly to g in $L^1(\mu+\lambda)$ also converge rapidly to g in $L^1(\mu)$). As a result, $\int g \,\mathrm{d}\mu$ is computable, as was to be shown.

The following corollary states that in case the measures are equivalent, computability of one of the densities entails the computability of the other.

Corollary 2. *Let μ, λ be computable probability measures such that $\mu \sim \lambda$ and $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ is $L^1(\lambda)$-computable. Then $\frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$ is $L^1(\mu)$-computable and therefore $\mu \sim_{\text{eff}} \lambda$.*

Moreover, if we assume $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ to be the λ-layerwise computable representative of its class, then there are constants c_1, c_2 such that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x)\, c_1 \leq \frac{t_\lambda}{t_\mu}(x) \leq \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(x)\, c_2 \qquad \textit{for every} \quad x \in \mathsf{ML}^\lambda = \mathsf{ML}^\mu. \tag{4}$$

Proof. Let h be a λ-layerwise computable representative of $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$. h^{-1} is λ-layerwise computable. As $\mu \ll_{\text{eff}} \lambda$ by Theorem 4, h^{-1} is also μ-layerwise computable by Corollary 1. As $\int h^{-1} \,\mathrm{d}\mu = 1$ is a computable number, $h^{-1} = \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$ is $L^1(\mu)$-computable by Theorem 1. Now, ht_μ is λ-layerwise lower semicomputable and $\int ht_\mu \,\mathrm{d}\lambda = \int t_\mu \,\mathrm{d}\mu \leq 1$, so there is a constant c such that $ht_\mu \leq t_\lambda c$. Let then $c_1 = c^{-1}$. By symmetry, switching μ and λ gives $h^{-1}t_\lambda \leq t_\mu c_2$ for some constant c_2. The two relations put together give (4), which finishes the proof.

Now, if μ, λ are computable and $\mu \ll \lambda$, it was proved in [16] that the Radon Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ is not necessarily $L^1(\lambda)$-computable. Here we prove that even under the stronger assumption $\mu \ll_{\text{eff}} \lambda$, the derivative need not be computable. In particular, effective absolute continuity does not imply computable normability.

Proposition 3. *There exist computable measures μ, λ such that $\mu \leq c \cdot \lambda$ for some c, hence $\mu \ll_{\mathit{eff}} \lambda$, but $\frac{d\mu}{d\lambda}$ is not $L^1(\lambda)$-computable.*

Proof. Let $\Omega = [0,1]$, let λ be the Lebesgue measure. We construct a fat Cantor set K such that the restriction of μ to K (denoted $\mu := \lambda(.|K)$) is computable, but $\lambda(K)$ is not computable. Let a_n be a computable sequence of positive real numbers such that $\sum_n a_n$ is finite and non-computable. Let $b_n = 2^{-a_n} < 1$: $\prod_n b_n = 2^{-\sum_n a_n}$ is positive and non-computable. We construct, for every word $w \in \{0,1\}^*$, a closed interval I_w, by induction on $|w|$. First $I_\varepsilon = [0,1]$ (ε is the empty word). Then, if I_w is defined, I_{w0} and I_{w1} are constructed removing an open segment centered at the middle of I_w, such that $|I_{w0}| + |I_{w1}| = b_{|w|}|I_w|$, as depicted in figure 1.

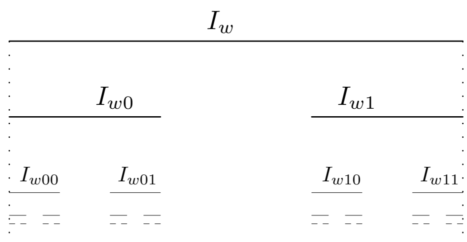

Fig. 1. Construction of the building intervals of the set K.

Let $K = \bigcap_n \bigcup_{|w|=n} I_w$. By construction, $\lambda(K) = \prod_n b_n > 0$. The bounds of I_i^n are computable real numbers, uniformly in i, n. Let $\phi : \{0,1\}^{\mathbb{N}} \to K$ be the homeomorphism mapping a binary sequence ω to the single point lying in the intersection of the intervals I_w with w prefix of ω. ϕ is clearly computable. The measure $\mu = \lambda(.|K) = \frac{\lambda(.\cap K)}{\lambda(K)}$ is the push-forward of the uniform measure over the Cantor space through ϕ, so μ is computable. The Radon-Nikodym derivative $f = \frac{d\mu}{d\lambda} = \lambda(K)^{-1}\mathbf{1}_K$ is bounded but is not $L^1(\lambda)$-computable. For we could assume f to be the λ-layerwise computable representative and consider $\lambda(f^{-1}(1, +\infty)) = \lambda(K)$ which, as $(1, +\infty)$ is effectively open, should be a lower semi-computable number. But $\lambda(K) = 2^{-\sum_n a_n}$, which is only upper semi-computable.

References

1. Hoyrup, M., Rojas, C.: Applications of effective probability theory to Martin-Löf randomness. In: ICALP 2009. Volume 5555 of Lecture Notes in Computer Science., Springer (2009) 549–561 1, 2, 3, 4
2. Bienvenu, L., Merkle, W.: Effective randomness for computable probability measures. Electron. Notes Theor. Comput. Sci. **167** (2007) 117–130 2, 5
3. Hoyrup, M., Rojas, C., Weihrauch, K.: Computability of the Radon-Nikodym derivative. Submitted. (2011) 2, 8

4. Gács, P.: Uniform test of algorithmic randomness over a general space. Theoretical Computer Science **341** (2005) 91–137 2
5. Hoyrup, M., Rojas, C.: Computability of probability measures and Martin-Löf randomness over metric spaces. Inf. Comput. **207**(7) (2009) 830–847 2, 3
6. Edalat, A., Heckmann, R.: A computational model for metric spaces. Theoretical Computer Science **193** (1998) 53–73 2
7. Yasugi, M., Mori, T., Tsujii, Y.: Effective properties of sets and functions in metric spaces with computability structure. Theoretical Computer Science **219**(1-2) (1999) 467–486 2
8. Weihrauch, K.: Computable Analysis. Springer, Berlin (2000) 2
9. Hemmerling, A.: Effective metric spaces and representations of the reals. Theor. Comput. Sci. **284**(2) (2002) 347–372 2
10. Brattka, V., Presser, G.: Computability on subsets of metric spaces. Theoretical Computer Science **305**(1-3) (2003) 43–76 2
11. Wu, Y., Ding, D.: Computability of measurable sets via effective topologies. Archive for Mathematical Logic **45** (2006) 365–379 3
12. Edalat, A.: The Scott topology induces the weak topology. In: LICS '96: Proceedings of the 11th Annual IEEE Symposium on Logic in Computer Science, Washington, DC, USA, IEEE Computer Society (1996) 372 3
13. Schröder, M.: Admissible representations of probability measures. Electronic Notes in Theoretical Computer Science **167** (2007) 61–78 3
14. Gács, P., Hoyrup, M., Rojas, C.: Randomness on computable probability spaces - A dynamical point of view. In: STACS. (2009) 469–480 3
15. Bishop, E., Bridges, D.: Constructive Mathematics. Springer-Verlag New York, Inc. (1985) 9
16. Hoyrup, M., Rojas, C., Weihrauch, K.: The Radon-Nikodym operator is not computable. Computability and Complexity in Analysis (CCA) (2011) 10