The typical constructible object

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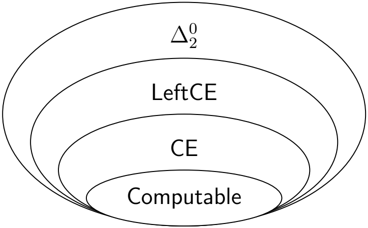
- In computability theory and computable analysis, one studies "constructible" objects:



Some classes of constructible subsets of $\mathbb{N}$

- Applying tools and technics from ordinary mathematics is not always possible: these classes of objects do not have ordinary structures.
- Goal: adapt mathematics to these spaces.
- Here: Baire Category.

Baire Category

In a complete metric space (X, d), gives a notion of **typical point**. Let $P(x)$ be a property of points $x \in X$. If

$$\{x \in X : \neg P(x)\}$$

is **small** then a **typical point** satisfies P.

Baire Category

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Example (Banach, Mazurkiewicz, 1932)

Let $X = \mathscr{C}[0,1]$ with $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$. The typical continuous function is **not differentiable at any point**.

Example (Weil, 1976)

For a suitable choice of $X \subseteq \mathscr{C}[0,1]$ and $d(f,g) = \sup_{[0,1]} |f' - g'|$, the typical element of X is a **differentiable nowhere monotonic** function.

Baire Category on classes of constructible objects?

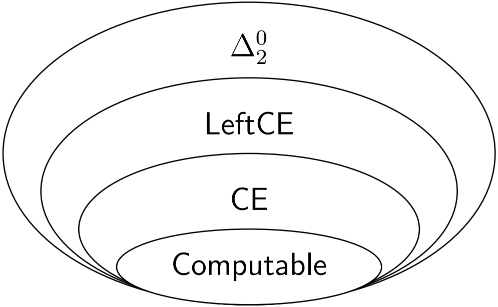


Figure: Classes of constructible subsets of $\mathbb{N}$

What does the typical object of each class look like?

Baire Category on classes of constructible objects?

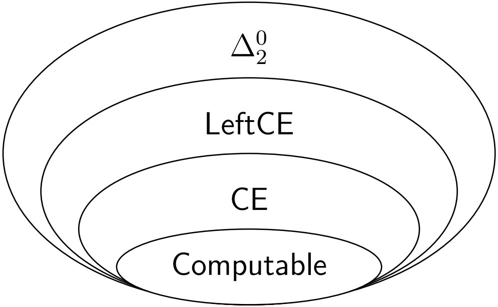


Figure: Classes of constructible subsets of $\mathbb{N}$

What does the typical object of each class look like?

Problem

These spaces are **not** complete metric spaces. Baire Category does not work there. We have to adapt it.

Introduction

Baire Category

Typical constructible objects

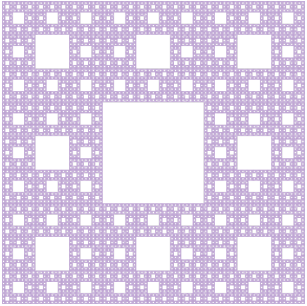
Limitations

Baire Category

Provides notions of **small** and **large** sets.

Definition

A set is **nowhere dense** if it is contained in the complement of a dense open set.



The Sierpiński Carpet is nowhere dense

Baire Category

Definition

Small sets:

- Nowhere dense sets,
- Their countable unions.

Large sets: complements of **small** sets.

Baire Category

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Does it make sense? Can a small set contain everything?

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Baire Category Theorem (Baire, 1899)

In a complete metric space, these notions make sense: large sets are non-empty (and even dense).

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A space where it fails

Let $X = \mathbb{Q}$ with the usual metric. Each singleton $\{q\}$ is small so $X = \bigcup_{q\in\mathbb{Q}}\{q\}$ is small. But it covers X!

Baire Category

If $A \subseteq X$ is small then a **typical element** of X lies outside A.

Example

Let $X = \mathscr{C}[0,1]$ with $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$. The typical continuous function is not differentiable at any point.

Proof.

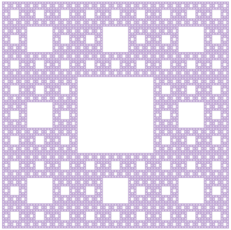
If f is differentiable at some x then $f \in \bigcup_n E_n$, where

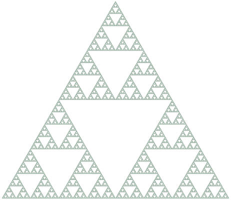
$$E_n = \left\{ f : \exists x \in [0, 1 - \tfrac{1}{n}], \forall h \in [0, 1 - x], \frac{|f(x+h) - f(x)|}{h} \leq n \right\}$$

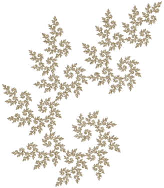
is nowhere dense. □

Proof of Baire Category Theorem

The following sets are nowhere dense:



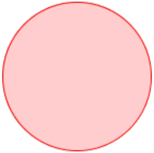




...

Let's build a point avoiding them.

Proof of Baire Category Theorem



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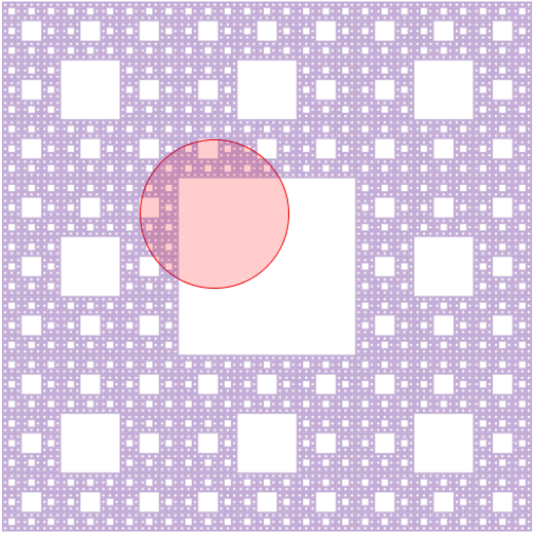

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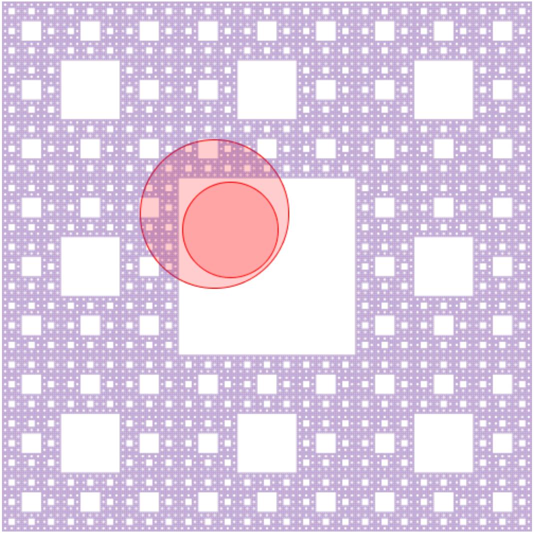

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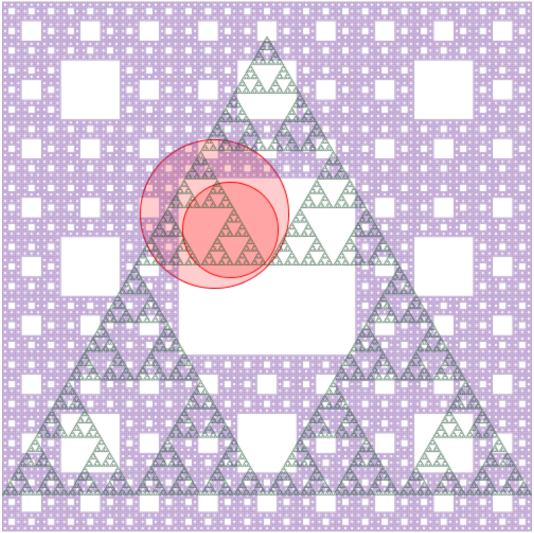

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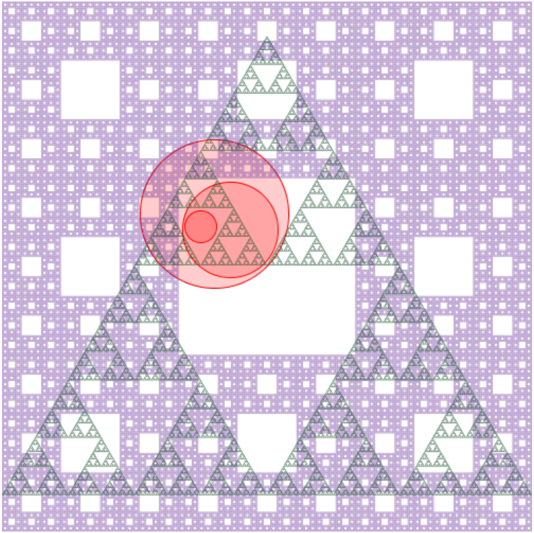

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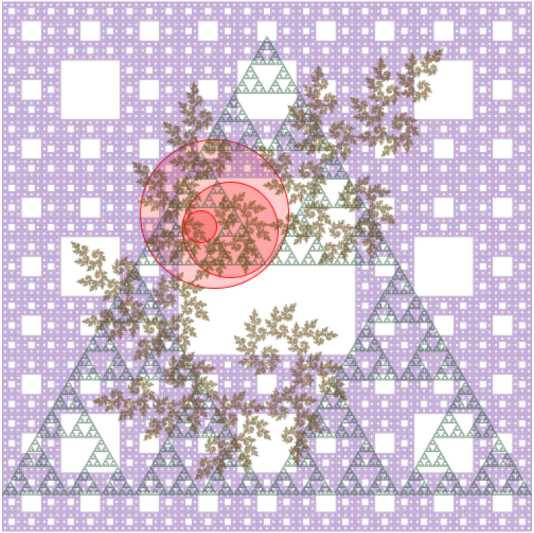

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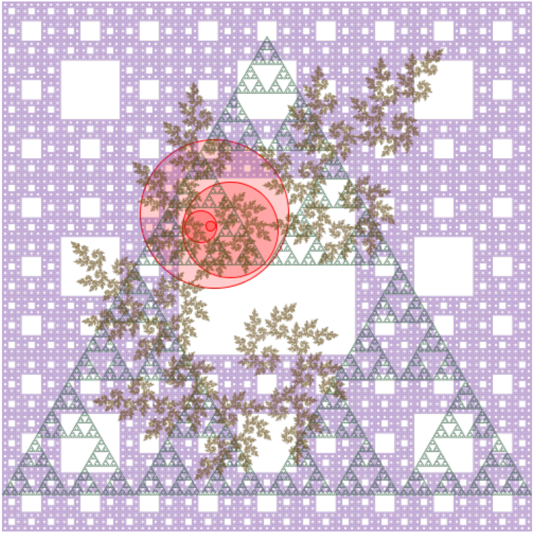

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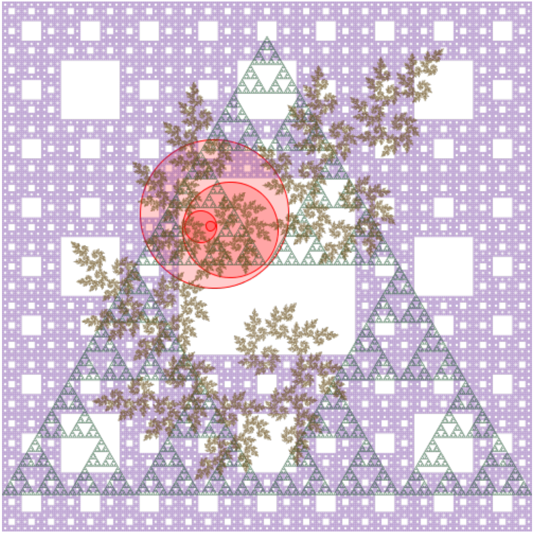


and so on...

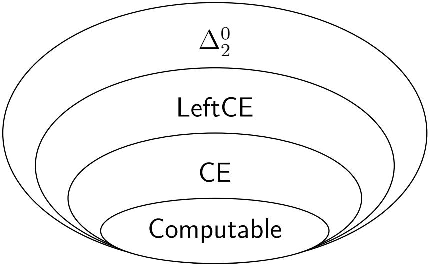
Introduction

Baire Category

Typical constructible objects

Limitations

Baire Category on classes of constructible objects?

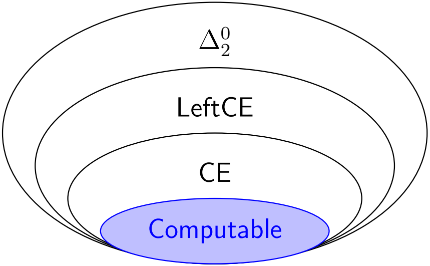


Some classes of constructible subsets of $\mathbb{N}$

What does the typical object of each class look like?

For each class we adapt the notion of **nowhere dense set** and prove a Baire Category theorem.

Baire Category on classes of constructible objects?



Some classes of constructible subsets of $\mathbb{N}$

What does the typical object of each class look like?

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Baire Category on COMPUTABLE

Reminder

Small and large sets

Small sets:

- Complements of dense open sets,
- Their countable unions.

Large sets: complements of **small** sets.

Baire Category Theorem (Baire, 1899)

In a complete metric space, large sets are non-empty (and even dense).

Baire Category on Computable

Small and large sets in Computable

Small sets:

- Complements of dense effective open sets,
- Their effective unions.

Large sets: complements of **small** sets.

Baire Category Theorem on Computable

In Computable, large sets are non-empty (and even dense).

Yasugi, Mori, Tsujii (1999)

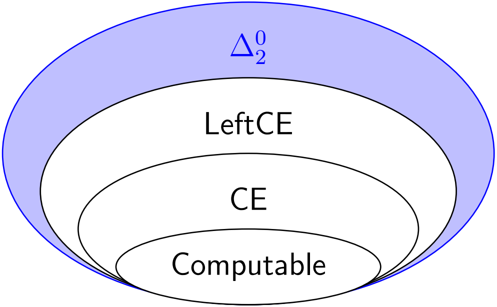
Baire Category on COMPUTABLE

Example

The typical computable function $f \in \mathscr{C}[0,1]$ is nowhere differentiable.

Can also be developed on the class of polytime computable functions. [Breutzmann, Juedes, Lutz, 2001]

Baire Category on Δ^0_2



Baire Category on Δ_2^0

Just relativize...

Small and large sets in Δ_2^0

Small sets:

- Complements of dense $\emptyset'$-effective open sets,
- Their effective unions.

Large sets: complements of **small** sets.

Baire Category Theorem on Δ_2^0

In Δ_2^0, large sets are non-empty (and even dense).

Baire Category on Δ_2^0

The **boundary** of an effective open set is the complement of a dense $\emptyset'$**-**effective open set.

Corollary

The class Δ_2^0 is not covered by the boundaries of effective open sets.

The uncovered elements are called 1**-generic** [Jockush, 1977].

Baire Category on Δ^0_2

Example

If (A, B) is 1-generic pair then A and B are Turing incomparable.

Proof.

Given a Turing functional Φ,

$$U_\Phi := \{(A, B) : \exists n, \Phi^A(n) = 0 \text{ but } B(n) = 1\}$$

is an effective open set. If $\Phi^A = B$ then (A, B) belongs to the boundary of U_Φ. □

Baire Category on Δ_2^0

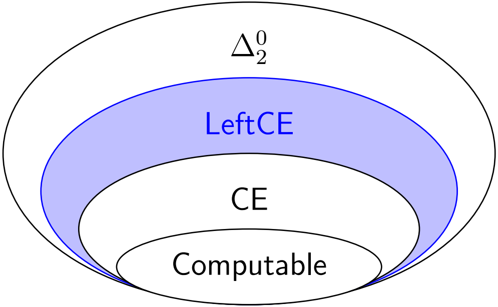
Fact

LeftCE is small in Δ_2^0: a 1-generic real is never left-c.e.

Proof.

If x is left-c.e. then x belongs to the boundary of the effective open set $U = [0, x)$. □

Baire Category on LeftCE

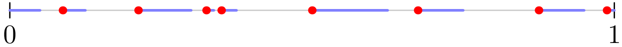


Baire Category on LeftCE

An open set:

its boundary:

and we define its left-boundary:



Baire Category on LeftCE

Small and large sets in LeftCE

Small sets:

- Left-boundaries of effective open sets,
- Their effective unions.

Large sets: complements of **small** sets.

Baire Category Theorem on LeftCE

In LeftCE, *large sets are non-empty (and even dense).*

Baire Category on LEFTCE

Definition

A left-c.e. real is **generic from the right** if it avoids the left-boundary of every effective open set.

Baire Category on LeftCE

If $x \in [0, 1]$ then its binary representation $\mathrm{bin}(x) \in \{0, 1\}^{\mathbb{N}}$ is always computable relative to x.
However,

Theorem

If $x \in [0, 1]$ is generic from the right then $\mathit{bin}(x) \not\leq_{computable\ modulus} x$.

Proof.

Effectivization of the following fact: if the restriction of bin to a set $C \subseteq [0, 1]$ is uniformly continuous then C is nowhere dense. □

Baire Category on LeftCE

Corollary (Downey, Hirschfeldt and LaForte, 2004)

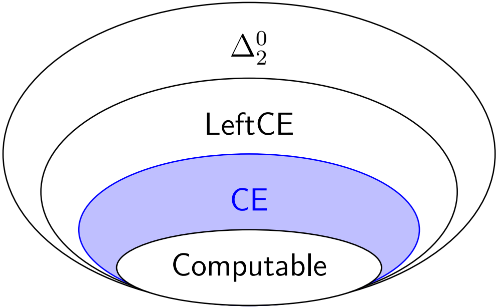
There exist left-c.e. reals x, y such that

$$bin(x) \leq_{cm} bin(y) \quad \textit{but} \quad x \not\leq_{cm} y.$$

Proof.

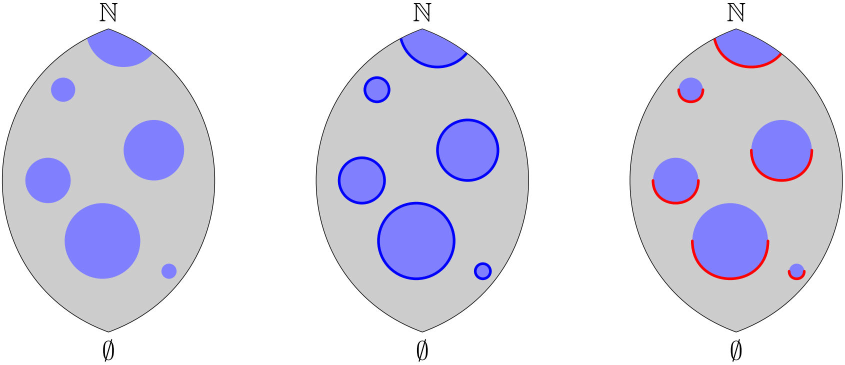
Let y be generic from the right and $\mathrm{bin}(x)_n = 1 \iff d_n < y$, where $(d_n)_{n\in\mathbb{N}}$ is an enumeration of the dyadic rationals. □

Baire Category on CE



Investigated by Lachlan, Ingrassia, Maass, Jockush and others (1970's and early 1980's).

Baire Category on CE



An open set, its boundary and its down-boundary.

Baire Category on CE

Let $A \in 2^{\mathbb{N}}$ and $U \subseteq 2^{\mathbb{N}}$.

Reminder

A belongs to the **boundary** of U if $A \notin U$ and

$$\exists A_n \in U \text{ such that } \lim_{n \to \infty} A_n = A.$$

Definition

A belongs to the **down-boundary** of U if $A \notin U$ and

$$\exists A_n \in U \text{ such that } \lim_{n \to \infty} A_n = A \text{ and } A \subseteq A_n.$$

Baire Category on CE

Small and large sets in CE

Small sets:

- Down-boundaries of effective open sets,
- Their effective unions.

Large sets: complements of **small** sets.

Baire Category theorem on CE

In CE*, large sets are non-empty (and even dense).*

Baire Category on CE

Definition

A c.e. set is **generic from above** if it avoids the down-boundary of every effective open set.

Coincides with Ingrassia's p-generic sets (1981).

Baire Category on CE

Theorem

For a typical pair of c.e. sets (A, B), A and B are Turing incomparable.

Same proof as for 1-generics.

Given a Turing functional Φ,

$$U_\Phi := \{(A, B) : \exists n, \Phi^A(n) = 0 \text{ but } B(n) = 1\}$$

is an effective open set. If $\Phi^A = B$ then (A, B) belongs to the **down-boundary** of U_Φ. □

Corollary (Friedberg-Muchnik, 1957-1956)

There exists a pair of Turing incomparable c.e. sets.

Baire Category and ergodic decomposition

- Ergodic measures are a special type of measures.
- Ergodic measures μ, ν are **uniquely determined** by their sum: if μ', ν' are ergodic and $\mu' + \nu' = \mu + \nu$ then $\{\mu', \nu'\} = \{\mu, \nu\}$.
- Are they **computably determined** by their sum?

Baire Category and ergodic decomposition

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- Are they **computably determined** by their sum?

No!

Theorem (H., 2012)

There exist non-computable ergodic measures μ and ν such that $\mu + \nu$ is computable.

Baire Category and ergodic decomposition

Definition

(μ, ν) belongs to the **?-boundary** of U if $(\mu, \nu) \notin U$ and

$$\exists (\mu_n, \nu_n) \in U \text{ such that } \lim_{n \to \infty} (\mu_n, \nu_n) = (\mu, \nu) \text{ and } \mu_n + \nu_n = \mu + \nu.$$

Let ComputableSum $= \{(\mu, \nu) : \mu + \nu \text{ is computable}\}$.

Baire Category theorem on ComputableSum

ComputableSum *is not covered by ?-boundaries of effective open sets.*

Baire Category and ergodic decomposition

Theorem

The typical element (μ, ν) *of* COMPUTABLESUM *satisfies:*

- μ *and* ν *are ergodic,*
- μ *and* ν *are not computable,*
- *For each string* w, *if* $\mu([w]) < \nu([w])$ *then*
  - $\mu([w])$ *is left-c.e. and generic from the right,*
  - $\nu([w])$ *is right-c.e. and generic from the left.*

The general result

Let (X, τ) be a Polish space and τ' a weaker topology.

Definition (Specialization pre-order)

Define $x \leq_{\tau'} y$ if every neighborhood of x is a neighborhood of y.

Definition

x belongs to the **down-boundary** of U if $x \notin U$ and

$$\exists x_n \in U \text{ such that } \lim_{n\to\infty} x_n = x \text{ and } x \leq_{\tau'} x_n.$$

Under reasonable computability assumptions on τ and τ',

Baire Category on the τ'-computable points (H., 2014)

There exists τ'-computable points that do not belong to the down-boundary of any τ-effective open set.

Introduction

Baire Category

Typical constructible objects

Limitations

Limitations

- These versions of the Baire Category theorem only capture simple constructions (simplest form of priority method with finite injury).
- One should find weaker notions of small sets and prove stronger versions of Baire Category theorem.

An example

The class CE, identified to

$$\mathcal{S} = \left\{ \sum_{n \in A} \frac{1}{2^n} : A \text{ is a c.e. subset of } \mathbb{N} \right\},$$

is small in LeftCE.

Is this one small too?

$$\mathcal{S}' = \left\{ \sum_{n \in A} \frac{1}{n^2} : A \text{ is a c.e. subset of } \mathbb{N} \right\}$$

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Theorem

There exists a c.e. set A such that $\sum_{n \in A} \frac{1}{n^2}$ is generic from the right.

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Proof idea.

This is an existence result: instead of building A, we take it generic from above, in a suitable topology.

Theorem

There exists a c.e. set A such that $\sum_{n\in A}\frac{1}{n^2}$ is generic from the right.

Proof idea.

This is an existence result: instead of building A, we take it generic from above, in a suitable topology.

Declare the following classes as open:

$$\mathcal{E}_n := \{A \subseteq \mathbb{N} : A \subseteq A_n\}$$

where

$$A_n = \{2^n(2k+1) : k \in \mathbb{N}\}.$$

Lemma

If $A \subseteq \mathbb{N}$ is generic from above ***in the new topology*** *then $\sum_{n\in A}\frac{1}{n^2}$* ***is*** *generic from the right.*

□

Hence the class

$$\mathcal{S}' = \left\{ \sum_{n \in A} \frac{1}{n^2} : A \text{ is a c.e. subset of } \mathbb{N} \right\}$$

is not small in LeftCE, but it should be!

Possible future directions

- Define weaker notions of small set,
- Prove stronger version of the Baire Category theorem to capture more involved constructions.
- We have a notion of **typical/generic** c.e. set. What is a **random** c.e. set?

Sara H. Jones. Applications of the Baire Category Theorem. *Real Analysis Exchange*, 23(2):363–394, 1999.