When a graph generates its own colorings

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Abstract

We investigate how graph colorings can be produced in a local way, where each vertex chooses its own color and only pairs of neighbors can coordinate their choices. This task can be achieved when the graph has bounded degree and the number of colors exceeds some threshold, and our goal is to find that threshold for any given graph. We identify its value for certain classes of graphs, such as the biregular ones, and give estimates for others.

Contents

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1 Introduction

When producing a coloring of a graph, the local constraints involving pairs of neighbors usually induce correlations between distant vertices. Intuitively, these indirect correlations should decrease as the number of available colors increases, and in certain cases they completely disappear when the number of colors exceeds a certain threshold, leaving only the correlations between direct neighbors. Our goal in this article is to find the threshold associated to any particular graph.

We make this informal problem precise by using the framework of local generation of languages developed in [Hoy25], whose purpose is to allow a sharp analysis of the informal notion of correlation, and which works as follows. If each vertex is given the task of choosing its own color, then some level of coordination is needed between them to make these choices compatible. Coordination is made possible by the vertices having access to common inputs, and the relevant object capturing this structure is a simplicial complex on the vertices, whose elements are the sets of vertices that have access to a common input. In this article, we focus on the case when the simplicial complex is the graph itself, i.e. when coordination only occurs between pairs of neighbors.

To be precise, we put arbitrary input values on the edges of the graph and each vertex has a local rule that, upon reading the values of its incident edges, produces a color for that vertex. We require that when the inputs vary, the possible outputs range across all the proper colorings of the graph. Whether this task is possible depends on the graph and on the number of available colors. For each graph G, we define $\Gamma(G)$ as the minimal number of colors for which this task can be done, and our general goal is to find the Γ value of any graph G. We analyze several classes of graphs (notably stars, paths, cycles, complete graphs and biregular graphs) and calculate or estimate their Γ values. These results are obtained by first deriving a purely combinatorial reformulation of the problem and then thoroughly investigating the properties of the involved combinatorial objects.

Generating colorings in a local way has been investigated in a probabilistic setting in [HL16, HSW17], using error-correcting cellular automata in [FMT22], and is a classical topic in distributed computing [Pel00, BE13, HS20]. Our model is reminiscent of various models from theoretical computer science, ranging from distributed computing [NS95, Rei15] to automata networks [GM90, GGPT21]. However, our model differs from these classical models in essentially two ways: the computation is made in one step, so no information can propagate through the graph; the task is to produce all the possible colorings using a fixed number of colors, while one usually starts from a coloring and reduce the number of colors. Therefore, the motivation and the model underlying this article are rather different from traditional topics in distributed computing, and to our knowledge the techniques in that field hardly apply to our setting. If precise connections exist, then they are yet to be discovered.

The article is organized as follows. In Section 2, we define the general setting and introduce the Γ value of a graph. In Section 3, we give a combinatorial reformulation of the problem and derive general properties, notably general bounds on the Γ value of graphs. In Section 5, we analyze various classes of graphs, calculating or estimating their Γ values. We devote Section 6 to a particular class of graphs, namely triple stars, whose simplicity contrasts with the complexity of their analysis. We end in Section 7 listing possible future directions.

2 Local generation of languages

We will be implicitly using the framework developed in [Hoy25]. If A and I are two sets and $L \subseteq A^I$, then the idea is to produce all the possible elements of L in a local way: each position $i \in I$ is seen as a cell whose task is to choose a value in A, and these cells need to coordinate their choices in order to collectively produce an element in L. We want moreover that every element of L can be produced. In the general setting, the communication between cells is represented by a simplicial complex K whose vertices are the elements of I, and a set $S \subseteq I$ belongs to K if all the elements of S collectively communicate. In this article, we do not need this framework in its full generality because we will only focus on simplicial complexes that are graphs, i.e. in which communication never involves more than two cells. Moreover, the framework developed in [Hoy25] is defined in the finite setting, where A and I, hence L, are all finite. We allow I and L to be infinite, which does not affect most of the theory.

2.1 Local generation by graphs

The next definitions formalize a computation model in which arbitrary input values are assigned to the edges of a graph, and each vertex has a local rule which, upon reading the input values of its incident edges, determines an output value for that vertex. The global output is then formed of these values, and the model is said to generate a given language if the set of all possible global outputs coincides with that language. We now make it precise.

In this article, a graph is a pair $G = (V, E)$ where V is a set (the vertices) and E is a set of unordered pairs $\{u, v\}$ with distinct $u, v \in V$ (the edges). For a vertex $u \in V$, its neighborhood is $N(u) = \{v \in V : \{u, v\} \in E\}$, and its degree is $\deg(u) = |N(u)|$. A graph is locally finite if every vertex has finite degree (although this degree may not be bounded).

Let G be a locally finite graph in which each vertex has degree at least 1. Let v be a vertex of G and $E(v)$ be the set of edges that are incident to v. Let A, B be finite sets, also called alphabets. A **local rule** at v is a function $\rho : B^{E(v)} \to A$. A family $(\rho_v)_{v\in V(G)}$ of local rules induces a global function $f : B^E \to A^V$, defined by $f(x)_v = \rho_v(x|_{E(v)})$, where $x|_{E(v)}$ is the restriction of x to $E(v)$. Note that contrary to cellular automata, no uniformity is required: each vertex has its own local rule.

Definition 2.1 (Language generation)**.** Let $G = (V, E)$ be a graph with no isolated point and $L \subseteq A^{V(G)}$. We say that G **generates** L if there exists B and a family $(\rho_v)_{v\in V(G)}$ of local rules $\rho_v : B^{E(v)} \to A$ inducing a function $f : B^E \to A^V$ such that $\mathrm{im}(f) = L$.

Let us make some observations. The data of the problem is usually given by the graph G, the alphabet A and the language L. One then investigate the existence of local rules, which require an input alphabet B, potentially unrelated to A and usually larger than A. We do not impose and restriction on the size of B, our only limitation is the coordination structure imposed by the graph.

The definitions contrasts with cellular automata and more generally automata networks, in which the inputs and outputs of local rules live in the same alphabet and are assigned to the same elements (the vertices), enabling one to study those models as dynamical systems that can be iterated. Our model does not allow any form of iteration, and the whole computation occurs in one step. In particular, vertices that are not direct neighbors have absolutely no way of exchanging information.

Moreover, in automata networks, a local rule updates a vertex color depending on the values of its neighbors, rather than its incident edges. It implies that coordination not only occurs between direct neighbors, but also between vertices at distance two from each other because they both read the color of their common neighbor. The model from [Hoy25] would represent this situation by a simplicial complex containing, for each vertex u, the simplex $u \cup N(u)$.

Remark 2.1. One could handle graphs with isolated points by using local rules of the form $\rho_v : B^{E(v)\cup\{v\}} \to A$. However, isolated points play no important role in graph coloring, so we prefer to trade full generality for simplicity.

It is often convenient to use a specific alphabet B_e for each edge e, which leads to an equivalent notion if there is a uniform bound on the sizes of these alphabets. Indeed, one can replace them by larger alphabet B and precompose the local rules with surjective functions from B to B_e.

In practice it is often more convenient to allow non-deterministic local rules $\rho_v : B^{E(v)} \to P^+(A)$, where $P^+(A)$ is the set of non-empty subsets of A. Such local rules induce a non-deterministic function $f : B^E \to P^+(A^V)$ defined by $y \in f(x)$ if for all $v \in V$, $y_v \in \rho_v(x|_{E(v)})$. We then let $\operatorname{im}(f) = \bigcup_{x\in B^E} f(x)$ and say that f generates L if $\operatorname{im}(f) = L$.

Proposition 2.1 (Non-determinism)**.** *Let G be a graph with no isolated point. The graph G generates a language L by non-deterministic local rules if and only if G generates L by deterministic ones.*

Proof. We show how to determinize the local rules $\rho_v : B^{E(v)} \to P^+(A)$. We fix an arbitrary orientation of the edges, which is a pair of functions $h, t : E \to V$ such that $e = \{h(e), t(e)\}$ for each edge e. For each v we choose an arbitrary edge $e_v \in E(v)$ (it will carry some additional value, used to make a deterministic choice). We fix for each $S \in P^+(A)$ a projection function $\pi_S : A \to S$ such that $\pi_S(A) = S$. Let $C = A \times A \times B$. We define deterministic local rules $\rho'_v : C^{E(v)} \to A$ as follows. Each input in $C^{E(v)}$ can be decomposed as a triplet (x, y, z) where $x, y \in A^{E(v)}$ and $z \in B^{E(v)}$. Let $\rho'_v(x, y, z) = \pi_S(a)$ where $S = \rho_v(z)$, $a = x(e_v)$ if v is the head of e_v and $a = y(e_v)$ if v is the tail of e_v. Observe that for any fixed z, $\{\rho'_v(x, y, z) : x, y \in A^{E(v)}\} = \rho_v(z)$.

We show that $\operatorname{im}(f') = \operatorname{im}(f)$, which follows from the fact that for each $z \in B^E$, $f(z) = \{f'(x, y, z) : x, y \in A^E\}$.

First, if $s = f'(x, y, z)$ then for each vertex v, $s_v \in \rho_v(z|_{E(v)})$ so $s \in f(z)$.

Conversely, let $s \in f(z)$. We define $x, y \in A^E$ such that $f'(x, y, z) = s$. Let v be a vertex and $S = \rho_v(z|_{E(v)})$. As $s_v \in S$, there exists $a \in A$ such that $s_v = \pi_S(a)$. If $v = h(e_v)$, then let $x(e_v) = a$, if $v = t(e_v)$ then let $y(e_v) = a$. We define x and y for each e_v in that way, and we extend them arbitrarily elsewhere. They are well-defined because for each edge e, we define $x(e)$ at most once (when handling the head of e) and similarly for $y(e)$, handling its tail. By construction, $f'(x, y, z) = s$. $\square$

2.2 When a graph generates its colorings

We now restrict our attention to the languages of proper colorings of a graph and investigate when the graph itself generates its colorings.

Let $G = (V, E)$ be a graph whose chromatic number $\chi(G)$ is finite. For $k \geq \chi(G)$, let $[k] = \{1, \dots, k\}$ and $\mathsf{Col}_k(G) \subseteq [k]^V$ be the set of proper k-colorings of G: each vertex is assigned a color in $[k]$ so that neighbors have distinct colors.

The main goal of this article is to investigate the following question: for which pairs (G, k) does G generate $\mathsf{Col}_k(G)$? Before developing the results, let us illustrate this problem on a simple example.

Example 2.1 (Stars). For $a \geq 1$, let S_a be the star made of one internal vertex of degree a surrounded by leaves. One has $\chi(S_a) = 2$. We claim that S_a generates $\mathsf{Col}_k(S_a)$ if and only if $k \geq a + 1$. We informally explain the argument and will give a precise formulation once we have developed the relevant tools (Proposition 4.1).

If $k \geq a + 1$, then S_a generates $\mathsf{Col}_k(S_a)$ as follows: each leaf takes its color from its unique incident edge, so the center of the star knows the color of each one of its a neighbors and can choose a distinct color among the k available colors.

If $k \leq a$ and S_a generates $\mathsf{Col}_k(S_a)$ then one can assign values to the edges so that the leaves take all the available colors, but then the center cannot make a valid choice, giving a contradiction.

3 Combinatorial reformulation

We first give a purely combinatorial reformulation of the problem, using classical combinatorial objects and concepts.

3.1 Completely separating systems

The next notion was introduced by Dickson [Dic69].

Definition 3.1 (Completely separating system)**.** Let X be a set. If $x, y \in X$ are distinct, then we say that $S \subseteq X$ **separates** x from y if $x \in S$ and $y \notin S$. A **completely separating system** over X is a set $\mathcal{S}$ of non-empty proper subsets of X such that for every pair of distinct elements $x, y \in X$, there exists $S \in \mathcal{S}$ that separates x from y and $T \in \mathcal{S}$ that separates y from x.

The **dual** of a completely separating system $\mathcal{S}$ is the completely separating system $\overline{\mathcal{S}} = \{\overline{S} : S \in \mathcal{S}\}$, where $\overline{S} = X \setminus S$.

Example 3.1 (Singletons). We will often use two particularly simple but important completely separating systems: the system of singletons over $[k]$, denoted by $\sigma_{[k]}$, and its dual $\overline{\sigma_{[k]}}$.

The definition of a completely separating system is redundant, because if for all distinct x, y there is $S \in \mathcal{S}$ separating x from y, then there is also $T \in \mathcal{S}$ separating y from x, by exchanging x and y. We phrase it this way anyway, to make the symmetry perfectly clear, and to distinguish it from the asymmetric *separating systems* studied by Rényi [Rén61] and Katona [Kat66].

The minimal size $C(k)$ of a completely separating system over $[k]$ is a known function of k, estimated by Dickson [Dic69] and calculated exactly by Spencer [Spe70] by relating it to Sperner's theorem [Spe28]:

$$\begin{aligned} C(k) &= \min\left\{n : \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq k\right\} \\ &= \log_2(k) + \frac{1}{2}\log_2(\log_2(k)) + O(1). \end{aligned}$$

We give its first values:

k	2	3	$4 \dots 6$	$7 \dots 10$	$11 \dots 20$	$21 \dots 35$	$36 \dots 70$
$C(k)$	2	3	4	5	6	7	8

Note that $C(k)$ is also the nondeterministic communication complexity of inequality over $[k]$, which is the minimal number n of rectangles $A_i \times B_i \subseteq [k]^2$ such that $\{(x, y) \in [k] : x \neq y\} = \{A_i \times B_i : i \in [n]\}$ [KN96].

3.2 Compatible families

We will need to use several completely separating systems and make sure that they are compatible in the following sense.

Definition 3.2 (t-compatible family)**.** Let X be a set and $t \geq 1$. A family $(\mathcal{S}_i)_{i\in I}$ of systems of subsets of X is **t-compatible** if for every family $(S_i)_{i\in I}$ where $S_i \in \mathcal{S}_i$ for each $i \in \mathcal{S}_i$, one has $|\bigcap_{i\in I} S_i| \geq t$. It is **compatible** if it is 1-compatible.

What we call t-compatible families are usually called r-cross t-intersecting families, where $r = |I|$. We use this shorter name for convenience and because in our setting r does not need to be spelled out as it is always the size of the family. Cross intersecting families have been thoroughly investigated in extremal combinatorics, from Erdős-Ko-Rado's theorem [EKR61] for the case $r = 1$ to more general cases $r \geq 2$ or $t \geq 2$ in [FT98, Tok10, FT11], the fully general case being more recently studied in [GMPS21, CLLW23].

We will only apply this notion to families $(\mathcal{S}_i)_{i\in I}$ where each $\mathcal{S}_i$ is a completely separating system over $[k]$. A family $(\mathcal{S}_i)_{i\in[1]}$ made of a single completely separating system is always compatible because each member of $\mathcal{S}_i$ is non-empty. Note that a family may contain several occurrences of some completely separating system.

We now reformulate the coloring generation problem in terms of compatible families of completely separating systems. In a graph G, the neighborhood of a vertex u is the set $N(u)$ of vertices v such that $\{u, v\}$ is an edge.

Proposition 3.1 (Combinatorial reformulation)**.** *Let G be a graph. For $k \geq \chi(G)$, the graph G generates $\mathsf{Col}_k(G)$ if and only if to each pair $(u, v) \in V^2$ with $\{u, v\} \in E$, one can associate a completely separating system $\mathcal{S}_{u,v}$ over $[k]$ such that $\mathcal{S}_{v,u} = \overline{\mathcal{S}_{u,v}}$ and for every vertex u, the family*

$$(\mathcal{S}_{u,v})_{v\in N(u)} \tag{1}$$

is compatible. We say that G generates $\mathsf{Col}_k(G)$ via $\mathcal{S}_{u,v}$.

Equivalently, one can fix an arbitrary orientation of the edges, assign a completely separating system $\mathcal{S}_e$ to each oriented edge e, and let $\mathcal{S}_{u,v} = \mathcal{S}_e$ and $\mathcal{S}_{v,u} = \overline{\mathcal{S}}_e$ if $e = (u, v)$.

Proof. Assume that such completely separating systems have been chosen and consider the following local rules: for each edge $\{u, v\}$, we choose some $S_{u,v} \in \mathcal{S}_{u,v}$ and $S_{v,u} = \overline{S_{u,v}}$, and a vertex u takes any value that belongs to $\bigcap_{v\in N(u)} S_{u,v}$, which is non-empty by compatibility. The output is always a valid coloring because for each edge $\{u, v\}$, u takes a value in $S_{u,v}$ and v takes a value in $\overline{S_{u,v}}$. Every valid coloring c can be obtained, because each $\mathcal{S}_{u,v}$ is completely separating: for each $\{u, v\} \in E$, choose an input $S_{u,v} \in \mathcal{S}_{u,v}$ such that $c_u \in S_{u,v}$ and $c_v \notin S_{u,v}$.

Conversely, assume that G generates $\mathsf{Col}_k(G)$ via a global function $f : B^E \to [k]^V$.

For each pair $(u, v) \in V^2$ such that $e = \{u, v\}$ is an edge and for each $b \in B$, let $S_{u,v}(b)$ be the set of values that u can take on some input x satisfying $x_e = b$. Note that $S_{u,v}(b)$ and $S_{v,u}(b)$ are disjoint: otherwise, let x, y be such $x_e = y_e = b$ and $f(x)_u = f(y)_v$; let z coincide with x on $E(u)$ and with z on $E(v)$, which is possible as x and y coincide on the intersection; one has $f(z)_u = f(z)_v$ so $f(z)$ is not a valid coloring, which is a contradiction. Let

$$\mathcal{S}_{u,v} = \{S \subseteq [k] : \exists b \in B, S_{u,v}(b) \subseteq S \subseteq \overline{S_{v,u}(b)}\}.$$

Note that $\mathcal{S}_{v,u} = \overline{\mathcal{S}_{u,v}}$.

We first show that each $\mathcal{S}_{u,v}$ is a completely separating system. If $c, d \in [k]$ are distinct colors, then there exists an input x producing a coloring assigning colors c and d to u and v respectively. Let $b = x_v$. One has $c \in S_{u,v}(b)$ and $d \in S_{v,u}(b) \subseteq \overline{S_{u,v}(b)}$, so $S_{u,v}(b)$ separates c from d.

We finally show that for each u, $(\mathcal{S}_{u,v})_{v \in N(u)}$ is compatible. Pick for each $v \in N(u)$ some set $S_v \in \mathcal{S}_{u,v}$ and let b_v be such that $S_{u,v}(b_v) \subseteq S_v$. Let x be an input taking value b_v on edge $\{u, v\}$ and let c be the color assigned to u in the corresponding output. By definition, one has $c \in S_{u,v}(b_v)$ for each v, so $c \in \bigcap_{v \in N(u)} S(v)$ which is therefore non-empty. $\square$

We give two simple techniques to build compatible families of completely separating systems.

Example 3.2 (Arithmetic example). Identify $[k]$ with $\mathbb{Z}/k\mathbb{Z}$. Say that $A \subsetneq \mathbb{Z}/k\mathbb{Z}$ is **unstable** if for every non-zero $i \in \mathbb{Z}/k\mathbb{Z}$, $A + i \neq A$ (it implies in particular that A is a non-empty proper subset of $\mathbb{Z}/k\mathbb{Z}$). Let A be unstable and let

$$\mathcal{S}_A = \{A + i : i \in \mathbb{Z}/k\mathbb{Z}\}.$$

It is a completely separating system because if $a, b \in \mathbb{Z}/k\mathbb{Z}$ are distinct, then there exists $j \in A$ such that $j + b - a \notin A$, so letting $i = a - j$, one has $a \in A + i$ but $b \notin A + i$.

Unstable sets are sometimes easy to find: if k is prime, then every non-empty proper subset of $\mathbb{Z}/k\mathbb{Z}$ is unstable.

If $A, B \subseteq \mathbb{Z}/k\mathbb{Z}$ are unstable, have the same size and are not translates of each other, then the family $(\mathcal{S}_A, \overline{\mathcal{S}}_B)$ is compatible. Indeed, for $i, j \in \mathbb{Z}/k\mathbb{Z}$, the sets $A+i$ and $B + j$ are distinct and have the same size, so $A + i$ intersects the complement of $B + j$.

Example 3.3 (Using permutations). Let π be a permutation of $[k]$. If $\mathcal{S}$ is a completely separating system over $[k]$, then so is $\pi(\mathcal{S}) = \{\pi(S) : S \in \mathcal{S}\}$. If $(\mathcal{S}_i)_{i \in I}$

is a compatible family of completely separating systems, then so is $(\pi(\mathcal{S}_i))_{i\in I}$. In particular, let $\mathcal{S}_n = \pi^n(\mathcal{S})$: if $(\overline{\mathcal{S}_0}, \mathcal{S}_1)$ is compatible, then for each n, $(\overline{\mathcal{S}_n}, \mathcal{S}_{n+1})$ is compatible.

3.3 The Γ value of a graph

We show that for each G, the set of values of k such that G generates $\mathsf{Col}_k(G)$ is an upper set.

Proposition 3.2. *If a complex generates* $\mathsf{Col}_k(G)$*, then it generates* $\mathsf{Col}_{k+1}(G)$*.*

Proof. Let $\mathcal{S}_{u,v}$ be completely separating systems over $[k]$ witnessing the fact that G generates $\mathsf{Col}_k(G)$. Let $\mathcal{T}_{u,v} = \mathcal{S}_{u,v} \cup \{S \sqcup \{k\} : S \in \mathcal{S}_{u,v}\}$. These systems are completely separating over $[k+1]$, they satisfy $\mathcal{T}_{v,u} = \overline{\mathcal{T}_{u,v}}$ and the compatibility conditions are met: if $T_{u,v} \in \mathcal{T}_{u,v}$ for each $v \in N(u)$, then $S_{u,v} := T_{u,v} \cap [k] \in \mathcal{S}_{u,v}$ so $\bigcap_{v\in N(u)} T_{u,v} \supseteq \bigcap_{v\in N(u)} S_{u,v} \neq \emptyset$. □

This observation naturally leads to the main definition of the article.

Definition 3.3. Let G be a graph. We define $\Gamma(G)$ as the minimal value of $k \geq 2$ such that G generates $\mathsf{Col}_k(G)$, and $\Gamma(G) = \infty$ if no such k exists.

We will see that $\Gamma(G) < \infty$ if and only if $\Delta(G) < \infty$, i.e. G has bounded degree, and we will calculate or estimate $\Gamma(G)$ for several classes of graphs.

In Example 2.1 we have identified the values of k such that the star S_a generates $\mathsf{Col}_k(S_a)$, which immediately gives the Γ value of stars.

Example 3.4 (Stars). For $a \geq 1$,

$$\Gamma(S_a) = a + 1.$$

4 Structural properties

In this section, we take leverage of the combinatorial reformulation of the problem to obtain general bounds on the Γ value of a graph.

4.1 Properties of compatible families

We now investigate general properties of compatible families of completely separating systems, and their consequences on the coloring generation problem.

The first observation is that the compatibility of a family imposes conditions on the size of the family and on the sizes of the members of the systems.

Lemma 4.1 (Size lemma). *Let $(\mathcal{S}_i)_{i\in I}$ be a t-compatible family of completely separating systems over $[k]$. For $J \subseteq I$, the subfamily $(\mathcal{S}_i)_{i\in J}$ is u-compatible, where $u = |I| - |J| + t$. In particular,*

- *The size $|I|$ of the family is at most $k - t$,*
- *The size $|S|$ of any member S of any $\mathcal{S}_i$ is at least $|I| + t - 1$.*

Proof. Choose some $S_j \in \mathcal{S}_j$ for each $j \in J$ and let $S = \bigcap_{j\in J} S_j$. Assume for a contradiction that $|S| \leq |I| - |J| + t - 1$. Express S as $S = A \sqcup B$ such that $|A| \leq |I| - |J|$ and $|B| \leq t - 1$. Pick some surjective function $f : I \setminus J \to A$ and for each $i \in I \setminus J$, choose $S_i \in \mathcal{S}_i$ that does not contain $f(i)$. One has $\bigcap_{i\in I} S_i \subseteq S \setminus A = B$ which has size at most $t - 1$, contradicting the t-compatibility of the family.

For $J = \emptyset$, the empty subfamily is $(|I| + t)$-compatible, and the "empty" intersection is $[k]$, so $k \geq |I| + t$.

For $J = \{i\}$, the singleton subfamily is $(|I| - 1 + t)$-compatible, so one has $|S| \geq |I| - 1 + t$ for any $S \in \mathcal{S}_i$. □

In particular, for $t = 1$, if $(\mathcal{S}_i)_{i\in I}$ is a compatible family of completely separating systems, then $|I| \leq k - 1$ and each member S of each $\mathcal{S}_i$ has size at least $|I|$.

We highlight a simple but important consequence of this fact.

Corollary 4.1 (Size vs degree). *Let G generate $\mathsf{Col}_k(G)$ and $\mathcal{S}_{u,v}$ be corresponding completely separating systems. If u and v are neighbors, then for each $S \in \mathcal{S}_{u,v}$,*

$$|S| \geq \deg(u) \text{ and } |\overline{S}| \geq \deg(v),$$

which implies that

$$\Gamma(G) \geq \max_{\{u,v\}\in E(G)} \deg(u) + \deg(v) \geq \Delta(G) + 1.$$

Proof. The family $(\mathcal{S}_{u,v})_{v\in N(u)}$ is compatible so its size $\deg(u)$ is a lower bound on the size of each S in each $\mathcal{S}_{u,v}$ by Lemma 4.1, and symmetrically for the family surrounding v. Therefore, $k = |S| + |\overline{S}| \geq \deg(u) + \deg(v)$ for all edges $\{u, v\}$. □

In particular, if $\Gamma(G)$ is finite, then G has bounded degree.

In Examples 2.1 and 3.4 we have identified the Γ value of stars. We show how the arguments can be formulated using the concepts developed so far.

Proposition 4.1 (Stars)**.** *For $a \geq 1$, let S_a be the star made of one internal vertex of degree a surrounded by leaves. One has*

$$\Gamma(S_a) = a + 1.$$

Moreover, a connected graph G satisfies $\Gamma(G) = \Delta(G) + 1$ if and only if G is a star.

Proof. Let $k = a + 1$. If l is a leaf and c is the center, then let $\mathcal{S}_{l,c} = \sigma_{[k]}$ be the system of singletons (introduced in Example 3.1), and $\mathcal{S}_{c,l} = \overline{\sigma_{[k]}}$. As $\deg(c) = a$, the family $(\mathcal{S}_{c,l})_{l \in N(c)}$ is indeed compatible, showing that S_a generates $\mathsf{Col}_k(S_a)$, i.e. $\Gamma(S_a) \leq a + 1$.

By Corollary 4.1, if l is any leaf then $\Gamma(S_a) \geq \deg(c) + \deg(l) = a + 1$.

If $\Gamma(G) = \Delta(G)+1$, then let u be a vertex of maximal degree. For every neighbor v of u, one has $\Delta(G) + 1 \geq \deg(u) + \deg(v)$ so $\deg(v) = 1$ by Corollary 4.1. As G is connected, it is a star centered at u. □

Leaves. In a generation procedure, a leaf takes its color from the value of its unique incident edge. It implies that its neighbor "knows" the color of the leaf, which simplifies its task of choosing a color. We make it precise by showing that if u is a leaf and v its neighbor, then one can always use the completely separating system $\mathcal{S}_{u,v} = \sigma_{[k]}$.

The idea that some completely separating system makes compatibility easier to achieve is expressed by the following pre-order. If $\mathcal{S}$ and $\mathcal{T}$ are completely separating systems over $[k]$, then we let

$$\mathcal{S} \preceq \mathcal{T} \iff \forall T \in \mathcal{T}, \exists S \in \mathcal{S}, S \subseteq T.$$

If $(\mathcal{S}_i)_{i \in I}$ and $(\mathcal{T}_i)_{i \in I}$ are families of completely separating systems such that $\mathcal{S}_i \preceq T_i$ for all i, then the compatibility of $(\mathcal{S}_i)_{i \in I}$ implies the compatibility of $(\mathcal{T}_i)_{i \in I}$ (so intuitively, the latter is "easier" to achieve than the former).

The simplest completely separating systems are the system $\sigma_{[k]}$ of singletons and its dual $\overline{\sigma_{[k]}}$. They are respectively least and greatest systems: for every completely separating system $\mathcal{S}$ over $[k]$, one has

$$\sigma_{[k]} \preceq \mathcal{S} \preceq \overline{\sigma_{[k]}}.$$

Proposition 4.2 (Leaves)**.** *If G generates $\mathsf{Col}_k(G)$ via $\mathcal{S}_{u,v}$, then we can assume that:*

- *If u is a leaf and v is its neighbor, then $\mathcal{S}_{u,v} = \sigma_{[k]}$,*

- *If u, v are neighbors and all the other neighbors of u are leaves, then the elements of $\mathcal{S}_{u,v}$ all have size* $\deg(u)$.

Proof. In the first case, one has $\mathcal{S}_{v,u} \preceq \overline{\sigma_{[k]}}$ so $(\mathcal{S}_{v,w})_{w\in N(v)}$ remains compatible when replacing $\mathcal{S}_{v,u}$ by $\overline{\sigma_{[k]}}$.

In the second case, we can assume by the first assertion that $\sigma_{[k]}$ is assigned to all the edges starting from a leaf. Let $\mathcal{T} = \{T \subseteq [k] : |T| = \deg(u), \exists S \in \mathcal{S}_{u,v}, T \subseteq S\}$. By Corollary 4.1, all the sets in $\mathcal{S}_{u,v}$ have size at least $\deg(u)$, so $\mathcal{T}$ is a completely separating system. One has $\mathcal{S}_{v,u} = \overline{\mathcal{S}_{u,v}} \preceq \overline{\mathcal{T}}$ by definition of $\mathcal{T}$, so $(\mathcal{S}_{v,w})_{w\in N(v)}$ remains compatible when replacing $\mathcal{S}_{v,u}$ by $\overline{\mathcal{T}}$. The family $(\mathcal{S}_{u,w})_{w\in N(u)}$ also remains compatible when replacing $\mathcal{S}_{u,v}$ by $\mathcal{T}$: if $T \in \mathcal{T}$ and $S_w \in \mathcal{S}_{u,w}$ for all $w \in N(u) \setminus \{v\}$, then $|T| = \deg(u)$, each S_w is the complement of a singleton and there are $\deg(u) - 1$ many of them, so $T \cap \bigcap_{w\in N(u)\setminus\{v\}} S_w \neq \emptyset$. □

Removing the leaves. We finally show how the analysis of a graph G can be reduced to the analysis of the subgraph obtained by removing the leaves of G. It relies on the following simple observation.

Proposition 4.3. *Let $(\mathcal{S}_i)_{i\in I}$ be a family of completely separating systems over X, and $(\mathcal{T}_j)_{j\in J}$ where $\mathcal{T}_j = \overline{\sigma_X}$ for all $j \in J$. The family $(\mathcal{S}_i)_{i\in I} \sqcup (\mathcal{T}_j)_{j\in J}$ is t-compatible if and only if $(\mathcal{S}_i)_{i\in I}$ is $(|J| + t)$-compatible.*

Proof. If the union is t-compatible, then the subfamily $(\mathcal{S}_i)_{i\in I}$ is u-compatible for $u = |I \sqcup J| - |I| + t = |J| + t$. Conversely, assume that $(\mathcal{S}_i)_{i\in I}$ is $(|J| + t)$-compatible, and let $S_i \in \mathcal{S}_i$ and $T_j \in \mathcal{T}_j$. As each $\mathcal{T}_j$ is the complement of a singleton, the complement of $\bigcap_j T_j$ has size at most $|J|$. The set $\bigcap_i S_i$ has size at least $|J| + t$ hence $\bigcap_i S_i \cap \bigcap_j T_j$ has size at least $(|J| + t) - |J| = t$. □

Therefore, one can essentially ignore the leaves of G, replacing compatibility conditions by t-compatibility conditions on other vertices.

Corollary 4.2. *Let G be a graph and H the subgraph obtained by removing the leaves of G. For any k, G generates $\mathsf{Col}_k(G)$ if and only if one can assign completely separating systems $\mathcal{S}_{u,v}$ to the edges of H so that*

- $\mathcal{S}_{v,u} = \overline{\mathcal{S}_{u,v}}$ *for* $\{u, v\} \in E(H)$,
- *For each $u \in V(H)$, $(\mathcal{S}_{u,v})_{v\in N(u)}$ is t-compatible, where t is 1 plus the number of neighboring leaves of u in G.*

Proof. When G generates $\mathsf{Col}_k(G)$ via $\mathcal{S}_{u,v}$, we can assume that $\mathcal{S}_{u,v} = \overline{\sigma_{[k]}}$ when v is a leaf by Proposition 4.2. Therefore, the compatibility condition at u in G is equivalent to the t-compatibility condition at u in H by Proposition 4.3. □

It does not seem possible to iterate this idea, removing the new leaves of H.

4.2 Combining graphs

We give several ways of comparing Γ values between graphs, that will prove useful to obtain lower and upper bounds.

We start by a straightforward observation.

Proposition 4.4 (Connected components). *Let $(G_i)_{i\in I}$ be the connected components of G. One has*

$$\Gamma(G) = \sup_{i\in I} \Gamma(G_i)$$

Proof. As each G_i is a subgraph of G, $\Gamma(G) \geq \Gamma(G_i)$. Conversely, if $k = \sup_i \Gamma(G_i)$, then the completely separating systems $\mathcal{S}_{u,v}$ witnessing that G_i generates $\mathsf{Col}_k(G_i)$ can be combined: for any vertex $u \in V(G_i)$ the compatibility condition in G_i and in G are the same. □

4.2.1 Locally injective homomorphisms

En easy way to derive a lower bound on the Γ value of a graph is to observe that if G is a subgraph of H, i.e. if there is an injective homomorphism from G to H, then $\Gamma(G) \leq \Gamma(H)$. More generally, this result holds for locally injective homomorphisms. They were introduced by Nešetřil [Neš71] and are defined as follows.

Definition 4.1. A graph homomorphism $f : G \to H$ is **locally injective** if for every $v \in V(G)$, the restriction of f to $N(v)$ is injective.

Proposition 4.5. *If a locally injective graph homomorphism $f : G \to H$ exists, then $\Gamma(G) \leq \Gamma(H)$.*

Proof. Let $k = \Gamma(H)$ and let $\mathcal{S}_{u,v}$ be separating systems in H witnessing that H generates $\mathsf{Col}_k(H)$. For each edge $\{u, v\}$ in G, let $\mathcal{T}_{u,v} = \mathcal{S}_{f(u),f(v)}$. One has $\mathcal{T}_{v,u} = \overline{\mathcal{T}_{u,v}}$ and for each $u \in V(G)$, $(\mathcal{T}_{u,v})_{v\in N(u)}$ is a subfamily of $(\mathcal{S}_{f(u),v})_{v\in N(f(u))}$ by local injectivity of f, so it is compatible. □

In particular, a covering map is locally injective so if G is a covering of H, then $\Gamma(G) \leq \Gamma(H)$.

Example 4.1. The $\mathbb{Z}$-graph is the graph on vertices $\mathbb{Z}$ with edges between consecutive numbers. There is a locally injective homomorphism from the $\mathbb{Z}$-graph to C_3, the cycle on 3 vertices. It is the map $n \mapsto n \bmod 3$. (see Figure 1). Therefore, $\Gamma(C_3) \leq \Gamma(\mathbb{Z})$ (we will see that they both equal 5).

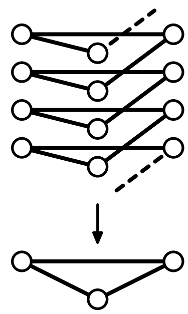


Figure 1: A locally injective homomorphism $\mathbb{Z} \to C_3$

We now explain how this technique enables one to isolate the substructures of H that prevents $\Gamma(H)$ to take a smaller value. For an infinite graph H, let $\mathcal{I}_H$ and $\mathcal{L}_H$ be the classes of *finite* graphs defined by

$$\begin{aligned} G \in \mathcal{I}_H &\iff \text{there exists an injective homomorphism } f : G \to H, \\ G \in \mathcal{L}_H &\iff \text{there exists a locally injective homomorphism } f : G \to H. \end{aligned}$$

Proposition 4.6 (Compactness)**.** *One has*

$$\Gamma(H) = \sup_{G \in \mathcal{I}_H} \Gamma(G) = \sup_{G \in \mathcal{L}_H} \Gamma(G).$$

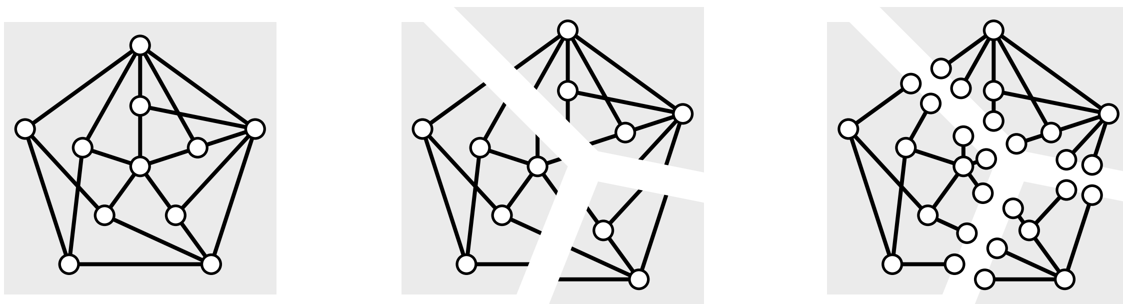
Proof. First, $\Gamma(H)$ is at least as large as these quantities by Proposition 4.5. As $\mathcal{I}_H \subseteq \mathcal{L}_H$, it is sufficient to show the first equality. Let k be such that every G in $\mathcal{I}_H$ generates $\mathsf{Col}_k(G)$. The set Σ of completely separating systems over $[k]$ is finite, so the set $\Sigma^{E(H)}$ endowed with the product topology is compact. When G ranges over $\mathcal{I}_H$, the elements of $\Sigma^{E(G)}$ witnessing the fact that G generates $\mathsf{Col}_k(G)$ have an accumulation point $\mathcal{S} \in \Sigma^{E(H)}$. The conditions $\mathcal{S}_{v,u} = \overline{\mathcal{S}_{u,v}}$ and the compatibility conditions hold, because they involve finitely many edges (as H is locally finite) so they are eventually satisfied as G grows, showing that H generates $\mathsf{Col}_k(H)$. $\square$

Of course when $\Gamma(H)$ is finite, each sup can be replaced by max.

In practice, for each H (even finite) one may try to find the smallest G in $\mathcal{I}_H$ or $\mathcal{L}_H$ such that $\Gamma(G) = \Gamma(H)$ (when this value is finite).

4.2.2 Fragmentations

In order to obtain an upper bound on $\Gamma(G)$, we show that if we decompose G into smaller graphs in a certain way, then $\Gamma(G)$ is at most the sum of the Γ values of these graphs. The suitable notion of decomposition can be explained using graphical intuition as follows. Draw the graph on a sheet of paper and cut the sheet into finitely many pieces, in such a way that the cuts do not meet the vertices. The part of the drawing lying on any of these pieces is still a graph, once new vertices have been added to the ends of the pending half-edges (see Figure 2 for an illustration).



(a) A graph (b) Cutting the graph (c) Adding new vertices

Figure 2: A graph fragmentation: partition the vertices, cut the crossing edges, add vertices at the ends of the half-edges

We call this process a fragmentation, which we formally define as follows. Let G be a graph and $U \subsetneq V(G)$ be non-empty. Let H_U be obtained by subdividing all the cut edges having one endpoint in U and one endpoint outside U, let W be the set of new vertices and let $G_U = H_U[U \cup W]$ be the subgraph induced in H_U by $U \cup W$.

Definition 4.2. A **fragmentation** of G is a family of graphs $(G_1, \ldots, G_p)$ such that $G_i = G_{V_i}$ for some partition $V(G) = V_1 \sqcup \ldots \sqcup V_p$.

We are going to combine the completely separating systems associated to the G_i's using the following join operator.

Definition 4.3. Let $\mathcal{S}, \mathcal{T}$ be two systems of subsets of A and B respectively. Their **join** is the system of subsets of $A \sqcup B$ defined by

$$\mathcal{S} * \mathcal{T} = \{S \sqcup T : S \in \mathcal{S}, T \in \mathcal{T}\}.$$

Let $(\mathcal{S}_i)_{i\in I}$ and $(\mathcal{T}_i)_{i\in I}$ be two families of systems of subsets of A and B respectively. Their join is defined as the family $(\mathcal{S}_i * \mathcal{T}_i)_{i\in I}$.

If $\mathcal{S}$ and $\mathcal{T}$ are completely separating systems over A and B respectively, then $\mathcal{S}*\mathcal{T}$ is completely separating over $A \sqcup B$.

If $(\mathcal{S}_i)_{i\in I}$ is compatible, then so is $(\mathcal{S}_i * \mathcal{T}_i)_{i\in I}$. Indeed, if $S_i \sqcup T_i \in \mathcal{S}_i * \mathcal{T}_i$, then $\bigcap_i (S_i \sqcup T_i)$ contains $\bigcap_i S_i \neq \emptyset$.

Proposition 4.7 (Fragmentation)**.** *If $(G_1, \ldots, G_p)$ is a fragmentation of G, then*

$$\Gamma(G) \leq \Gamma(G_1) + \ldots + \Gamma(G_p).$$

Proof. Let $V(G) = V_1 \sqcup \ldots \sqcup V_p$ be the partition. If an edge $\{u, v\}$ of G has its endpoints in two different sets $u \in V_i$ and $v \in V_j$ with $i \neq j$, then the construction of each of G_i and G_j involves a subdivision of $\{u, v\}$, hence the introduction of a new vertex. Let $w(u, v)$ be the new vertex in G_i and $w(v, u)$ the new vertex in G_j.

Let $k_i = \Gamma(G_i)$, and for $\{u, v\} \in E(G_i)$, let $\mathcal{S}^i_{u,v}$ be the completely separating systems witnessing that G_i generates $\mathsf{Col}_{k_i}(G_i)$. Let $k = k_1 + \ldots + k_p$. We first extend $\mathcal{S}^i_{u,v}$ to any $\{u, v\} \in E(G)$. There are three cases:

- If $u, v \in V_i$, then $\{u, v\} \in E(G_i)$ so $\mathcal{S}^i_{u,v}$ and $\mathcal{S}^i_{v,u} = \overline{\mathcal{S}^i}_{u,v}$ are already defined,
- If $u, v \notin V_i$, then let $\mathcal{S}^i_{u,v}$ be any completely separating system over $[k_i]$ (for instance the system of all non-empty proper subsets of $[k_i]$) and $\mathcal{S}^i_{v,u} = \overline{\mathcal{S}^i_{u,v}}$,
- If only one of u, v belongs to V_i, say $u \in V_i, v \in V_j$ with $j \neq i$, then let $\mathcal{S}^i_{u,v} = \mathcal{S}^i_{u,w(u,v)}$ and $\mathcal{S}^i_{v,u} = \overline{\mathcal{S}^i_{u,v}}$.

Let $u \in V(G)$ and let i be such that $u \in V_i$. The family $(\mathcal{S}^i_{u,v})_{\{u,v\}\in E(G)}$ coincides with the family $(\mathcal{S}^i_{u,v})_{\{u,v\}\in E(G_i)}$, so it is compatible.

Let now $\mathcal{S}_{u,v} = \mathcal{S}^1_{u,v} * \ldots * \mathcal{S}^p_{u,v}$. It is completely separating over $[k] \cong [k_1] \sqcup \ldots \sqcup [k_p]$ and for each u, $(\mathcal{S}_{u,v})_{v\in N(u)}$ is compatible because $(\mathcal{S}^i_{u,v})_{\{u,v\}\in E(G_i)}$ is, where i is such that $u \in V_i$. $\square$

We now prove apply the fragmentation technique to giving a general upper bound on $\Gamma(G)$. We have already seen that if $\Gamma(G) \geq \Delta(G) + 1$. In particular, if $\Gamma(G)$ is finite then so is $\Delta(G)$. We show that it is actually an equivalence.

We recall that the chromatic number $\chi(G)$ is the minimal k such that G is k-colorable, and that $\chi(G) \leq \Delta(G) + 1$. By Brooks' theorem, $\chi(G) \leq \Delta(G)$ if G is connected and is neither the complete graph nor an odd cycle, in particular if G is infinite and connected [BR76, Theorem 3].

Proposition 4.8 (Upper bound)**.** *One has*

$$\Gamma(G) \leq \chi(G)(\Delta(G)+1).$$

In particular, $\Gamma(G)$ *is finite if and only if* $\Delta(G)$ *is.*

Proof. We use the fragmentation technique. Let $c = \chi(G)$ and $\iota : V(G) \to [c]$ be a c-coloring of G. It induces a partition $V(G) = V_1 \sqcup \ldots \sqcup V_c$, where $V_i = \iota^{-1}(i)$. Let $(G_1, \ldots, G_c)$ be the corresponding fragmentation of G. As V_i does not contain neighbors, G_i is a disjoint union of stars S_j, with $j \leq \Delta(G)$, so $\Gamma(G_i) \leq \Delta(G)+1$. As a result, $\Gamma(G) \leq \sum_i \Gamma(G_i) \leq \chi(G)(\Delta(G)+1)$. □

5 Classes of graphs

We investigate the value of Γ for simple classes of graphs. We have identified the Γ value of stars, namely $\Gamma(S_a) = a+1$. We continue with double stars, whose analysis is also simple. We will then study paths, cycles and other families, involving arguments of increasing complexity. One of the main achievements will be the identification of the Γ values of biregular graphs.

5.1 Double stars

For $a, b \geq 1$, let $S_{a,b}$ be made of an edge $\{u, v\}$, $a-1$ other edges attached to u and $b-1$ other edges attached to v (see Figure 3). Double stars have been studied for instance in [GHK79].

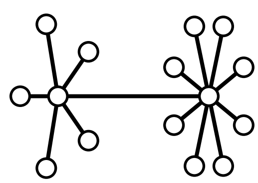


Figure 3: The double star $S_{6,9}$

Proposition 5.1 (Double stars)**.** *For* $a, b \geq 1$, *one has*

$$\Gamma(S_{a,b}) = a + b.$$

Proof. One has $\Gamma(S_{a,b}) \geq \deg(u) + \deg(v) = a + b$ by Corollary 4.1. Let $k = a + b$ and let $\mathcal{S}_{u,v}$ be the system of all subsets of $[k]$ of size a, so $\mathcal{S}_{v,u} = \overline{\mathcal{S}_{u,v}}$ is the system of all subsets of $[k]$ of size b. For a leaf l and its neighbor x, let $\mathcal{S}_{l,x} = \sigma_{[k]}$. The compatibility conditions are easily verified. $\square$

Note that when $a = 1$ or $b = 1$, $S_{a,b}$ is a star.

5.2 Paths

Let P_n be the path on n vertices.

Proposition 5.2 (Paths)**.** *For $n \geq 2$, one has*

$$\Gamma(P_n) = \min(n, 5).$$

Proof. For $n \leq 3$, $P_n = S_{n-1}$ so $\Gamma(P_n) = n$.

For $n = 4$, one has $\Gamma(P_n) \geq \max_{\{u,v\}\in E} \deg(u) + \deg(v) = 4$ by Corollary 4.1. We show that P_4 generates $\mathsf{Col}_4(P_4)$. The graph P_4 has vertex set $[4]$ and edges $\{1, 2\}$, $\{2, 3\}$ and $\{3, 4\}$. Let $\mathcal{S}_{1,2} = \mathcal{S}_{4,3} = \sigma_{[4]}$ and let $\mathcal{S}_{2,3}$ be the system of pairs. Each family $(\mathcal{S}_{2,1}, \mathcal{S}_{2,3})$ and $(\mathcal{S}_{3,2}, \mathcal{S}_{3,4})$ is compatible because on $[4]$, any triplet intersects any pair.

Let $n \geq 5$ and assume for a contradiction that P_n generates $\mathsf{Col}_4(P_n)$, and let $\mathcal{S}_{u,v}$ be the corresponding completely separating systems over $[4]$. The vertices $2, 3$ have degree 2, so any S in $\mathcal{S}_{2,3}$ is a pair, because $|S| \geq \deg(2) = 2$ and $|\overline{S}| \geq \deg(3) = 2$. Moreover, $\mathcal{S}_{2,3}$ contains at least four pairs, because $C(4) \geq 4$ (see the end of Section 3.1). The same holds for $\mathcal{S}_{3,4}$ because 4 has degree 2, so both $\mathcal{S}_{2,3}$ and $\mathcal{S}_{3,4}$ contain at least four pairs. As there are only six pairs in $[4]$, there exists $S \in \mathcal{S}_{2,3} \cap \mathcal{S}_{3,4}$. But then $\overline{S} \in \mathcal{S}_{3,2} = \overline{\mathcal{S}}_{2,3}$ is disjoint from $S \in \mathcal{S}_{3,4}$, which violates the compatibility of $(\mathcal{S}_{3,2}, \mathcal{S}_{3,4})$. We have reached a contradiction, so $\Gamma(P_n) \geq 5$.

We show that P_n generates $\mathsf{Col}_5(P_n)$. We identify $[5]$ with $\mathbb{Z}/5\mathbb{Z}$ and use completely separating systems from Example 3.2. Let $A = \{0, 1\}$ and $B = \{0, 2\}$, and $\mathcal{S}_A = \{\{i, i+1\} : i \in \mathbb{Z}/5\mathbb{Z}\}$ and $\mathcal{S}_B = \{\{j, j+2\} : j \in \mathbb{Z}/5\mathbb{Z}\}$. Graphically, $\mathcal{S}_A$ and $\mathcal{S}_B$ are the sets of yellow and gray edges respectively in Figure 4. As A and B are not translates of each other, the family $(\mathcal{S}_A, \overline{\mathcal{S}}_B)$ is compatible, as well as the family $(\mathcal{S}_B, \overline{\mathcal{S}}_A)$.

For each edge $\{i, i+1\}$ of P_n, we let $\mathcal{S}_{i,i+1} = \mathcal{S}_A$ if i is even and $\mathcal{S}_{i,i+1} = \mathcal{S}_B$ if i is odd. The compatibility conditions are verified, so P_n generates $\mathsf{Col}_5(P_n)$. $\square$

The argument for $n \geq 5$ applies to the infinite path $P_{\mathbb{N}}$ and the bi-infinite path $P_{\mathbb{Z}}$, so

$$\Gamma(P_{\mathbb{N}}) = \Gamma(P_{\mathbb{Z}}) = 5.$$

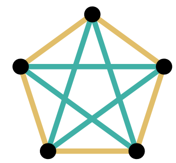


Figure 4: Two separating systems over [5] made of pairs

5.3 Cycles

For $n \geq 3$, let C_n be the cycle graph on n vertices. Its analysis is slightly more complex.

Proposition 5.3 (Cycles)**.** *For $n \geq 3$, one has*

$$\Gamma(C_n) = \begin{cases} 5 & \text{if } n \text{ is a multiple of } 2 \text{ or } 3, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. We first show the upper bounds.

When n is even, we can use the completely separating systems over [5] from Proposition 5.2, assigning them in alternation to the edges of C_n.

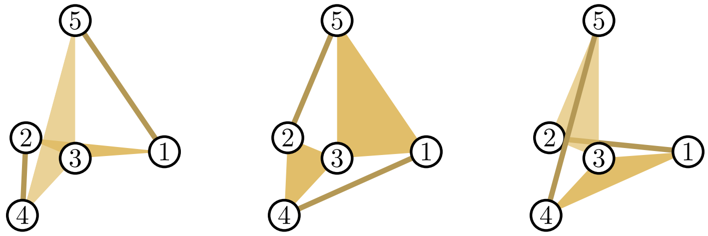
When n is a multiple of 3, we define three completely separating systems over [5] using Example 3.3. Let

$$\mathcal{S} = \{123, 345, 15, 24\},$$

which is illustrated in Figure 5. Let $\pi = (124)$ and $\mathcal{S}_{i,i+1} = \pi^i(\mathcal{S})$ for each $i \in \mathbb{Z}/n\mathbb{Z}$. One has $\pi(\mathcal{S}) = \{135, 234, 25, 14\}$, and the family $(\overline{\mathcal{S}}, \pi(S))$ is compatible because no element of $\pi(\mathcal{S})$ is contained in any element of $\mathcal{S}$. Therefore, each family $(\overline{\mathcal{S}}_{i,i+1}, \mathcal{S}_{i+1,i+2})$ is compatible, in particular when $i = n-1$ as n is a multiple of 3. We assign $\pi^i(\mathcal{S})$ to the ith edge of C_n. The compatibility conditions are met, so $\Gamma(C_n) \leq 5$.

Finally let n be odd. We identify [6] with $\mathbb{Z}/6\mathbb{Z}$ and consider three completely separating systems over [6] using Example 3.2 again. Let $A = \{0,1,2\}$, $B = \{0,2,3\}$ and $C = \{0,3,4\}$. These sets are unstable so the systems $\mathcal{S}_A, \mathcal{S}_B$ and $\mathcal{S}_C$ are completely separating. Moreover, the sets A, B, C are not translates of each other, so the families $(\mathcal{S}_X, \overline{\mathcal{S}}_Y)$ are compatible for distinct $X, Y \in \{A, B, C\}$. We can for instance assign $\mathcal{S}_A$ to an edge of C_n, and then alternate between $\mathcal{S}_B$ and $\mathcal{S}_C$. The compatibility conditions hold, showing that $\Gamma(C_n) \leq 6$.

We now prove the lower bounds. For every $n \geq 3$, there is a locally injective homomorphism $f : P_\mathbb{Z} \to C_n$, so $\Gamma(C_n) \geq \Gamma(P_\mathbb{Z}) = 5$.



(a) $\mathcal{S}$ (b) $\pi(\mathcal{S})$ (c) $\pi^2(\mathcal{S})$

Figure 5: $\pi(\mathcal{S})$ is obtained by rotating $\mathcal{S}$ by 120° around the vertical axis. The pair $(\overline{\mathcal{S}}, \pi(\mathcal{S}))$ is compatible, because $T \nsubseteq S$ for $S \in \mathcal{S}, T \in \pi(\mathcal{S})$

Let $n \geq 3$ and assume that C_n generates $\mathsf{Col}_5(C_n)$. The idea is to show that the only "cycles" of completely separating systems are the ones that are used above, of lengths 2 and 3 respectively, so n must be a multiple of 2 or 3.

Assume that $n \geq 3$ is odd, let $\mathcal{S}_0, \ldots, \mathcal{S}_{n-1}$ be the completely separating systems over $[5]$ that are assigned to the edges of C_n and let $\mathcal{S}_n = \mathcal{S}_0$. For each $i < n$, the family $(\overline{\mathcal{S}_i}, \mathcal{S}_{i+1})$ is compatible. As each vertex has degree 2, each element of each $\mathcal{S}_i$ is either a pair or a triplet.

Claim 1. In any $\mathcal{S}_i$ and any $\overline{\mathcal{S}}_i$, each $j \in [5]$ belongs to at most one pair.

Proof of Claim 1. By symmetry, it is sufficient to prove it for $\mathcal{S}_n$ and $j = 1$. Assume for a contradiction that $\mathcal{S}_n$ contains two pairs containing 1, say $\{1,2\}$ and $\{1,3\}$. We show that $\mathcal{S}_{n-1}$ contains $\{1,4\}$ and $\{1,5\}$. Let $S \in \mathcal{S}_{n-1}$ separate 1 from 5; as $\overline{S}$ does not contain 1 and intersects both $\{1,2\}$ and $\{1,3\}$, it contains 2 and 3, so $\overline{S} = \{2,3,5\}$ hence $S = \{1,4\}$. By symmetry, $\mathcal{S}_{n-1}$ contains $\{1,5\}$ as well. By the same argument, $\mathcal{S}_{n-2}$ contains $\{1,2\}$ and $\{1,3\}$, and we can iterate this. As n is odd, $\mathcal{S}_0$ contains $\{1,4\}$ and $\{1,5\}$. As $\mathcal{S}_0 = \mathcal{S}_n$, it contains each pair containing 1. Let $S \in \mathcal{S}_{n-1}$ separate 1 from 2, so $\overline{S}$ does not contain 1 but intersects each pair containing 1, implying $\overline{S} = \{2,3,4,5\}$, contradicting the fact that $\overline{\mathcal{S}_{n-1}}$ is made of pairs and triplets. Therefore, $\mathcal{S}_n$ contains at most one pair containing 1, which concludes the proof. ⌟

Therefore, each $\mathcal{S}_i$ contains at most two pairs (otherwise two of them intersect) and at most two triplets (because $\overline{\mathcal{S}_i}$ contains at most two pairs).

Claim 2. Each $\mathcal{S}_k$ is $\{123, 345, 15, 24\}$, up to some permutation of $[5]$.

Proof of Claim 2. First, $\mathcal{S}_k$ is made of two pairs and two triplets. Indeed, each $i \in [5]$ belongs to at least one triplet, otherwise it belongs to only one pair $\{i,j\}$, so it cannot be separated from j. Therefore, $\mathcal{S}_k$ contains two triplets that intersect in one point, say 123 and 345. For each $j \in \{1,2,4,5\}$ there is a pair in $\mathcal{S}_k$ separating j from 3.

The two pairs cannot be 12 and 45 because 1 would not be separated from 2, so they are 14 and 25 or 15 and 24, and these two cases are images of each other by the permutation (45). ⌟

Claim 3. If $\mathcal{S}_0 = \{123, 345, 15, 24\}$, then $\mathcal{S}_1 = \{135, 234, 25, 14\} = \pi(\mathcal{S}_0)$ where $\pi = (124)$.

Proof of Claim 3. $\mathcal{S}_1$ contains two pairs, each one of these pairs cannot be contained in any $S \in \mathcal{S}_0$ as it intersects $\overline{S}$. The only pairs satisfying this condition are 14 and 25. By the previous claim, $\mathcal{S}_1$ equals $\mathcal{S}_0$ up to some permutation of [5], so it can only be $\{135, 234, 14, 25\}$ or $\{123, 345, 14, 25\}$. The latter is impossible because $123 \in \mathcal{S}_1$ does not intersect the complement of $123 \in \mathcal{S}_0$. ⌟

As a result, one inductively has $\mathcal{S}_i = \pi^i(\mathcal{S}_0)$ for all $i \leq n$. As $\mathcal{S}_0 = \mathcal{S}_n = \pi^n(\mathcal{S}_0)$, we conclude that n is a multiple of 3. □

5.4 Complete graphs

Let K_n be the complete graph over n vertices. It is a regular graph of degree $\Delta(K_n) = n-1$ and of chromatic number $\chi(K_n) = n$, so Corollary 4.1 and Proposition 4.8 imply that

$$2n - 2 \leq \Gamma(K_n) \leq n^2.$$

We improve these bounds, showing that $\Gamma(K_n)$ is of the order of n^2, roughly between $\frac{n^2}{4}$ and n^2.

Theorem 5.1 (Complete graph)**.** *One has*

$$\left\lfloor \frac{(n+1)^2}{4} \right\rfloor \leq \Gamma(K_n) \leq n(n-1) + (n \bmod 2).$$

Proof. We first prove the upper bound. We partition $[n]$ into pairs as well as a singleton if n is odd. It induces a fragmentation of K_n by double stars $S_{n-1,n-1}$ and possibly a single star S_{n-1}, so $\Gamma(K_n)$ is bounded by the sum of their Γ's by Proposition 4.7. If $n = 2m$ is even, then $\Gamma(K_n) \leq m \cdot 2(n-1) = n(n-1)$. If $n = 2m+1$ is odd, then $\Gamma(K_n) \leq m \cdot 2(n-1) + n = n(n-1) + 1$.

We now prove the lower bound. Assume that K_n generates $\mathsf{Col}_k(K_n)$ via $\mathcal{S}_{u,v}$. For each edge $\{u, v\}$, choose a set $A_{u,v} \in \mathcal{S}_{u,v}$ and its complement $A_{v,u} = \overline{A_{u,v}} \in \mathcal{S}_{v,u}$.

We fix some $j \leq n$. For $u \in [j]$, let $N_j(u) = \{v \leq j, v \neq u\}$ be the set of neighbors of u in $[j]$, and let

$$B_u = \bigcap_{v \in N_j(u)} A_{u,v}.$$

As the family $(\mathcal{S}_{u,v})_{v\in N(v)}$ is compatible of size $n-1$ and $N_j(u)$ has size $j-1$, its subfamily $(\mathcal{S}_{u,v})_{v\in N_j(v)}$ of size $j-1$ is $(n-j+1)$-compatible by Lemma 4.1, so

$$|B_u| \geq n-j+1.$$

For distinct $u, v \in [j]$, the sets B_u and B_v are disjoint because $B_u \subseteq A_{u,v}$ and $B_v \subseteq A_{v,u} = \overline{A}_{u,v}$. As a result,

$$k \geq \sum_{u\in[j]} |B_u| \geq j(n-j+1).$$

As j ranges over $[n]$, the maximal value of the right-hand side is $\lfloor \frac{(n+1)^2}{4} \rfloor$. Indeed, consider the change of variable $i = 2j-n-1$ and observe that $j(n-j+1) = \frac{(n+1)^2-i^2}{4}$, which is maximal when $i = 0$ or 1, depending on whether $n+1$ is even or odd respectively. □

5.5 Biregular graphs and caterpillars

We derive the Γ-value of biregular graphs by analyzing a particular class of graphs, which are themselves a subclass of the so-called *caterpillars* [HS73]. We recall that a graph is biregular if it is bipartite and any two vertices lying on one side of the bipartition have the same degree.

Theorem 5.2 (Biregular graphs)**.** *Let $d \geq e \geq 1$ and G be a biregular graph of degrees d and e. One has*

$$\Gamma(G) = \begin{cases} d+e & \text{if } e = 1, \\ d+e+1 & \text{if } d = e = 2, \\ d+e+2 & \text{otherwise.} \end{cases}$$

Proof. If $e = 1$, then G is the star S_d so $\Gamma(G) = d+1 = d+e$. If $d = e = 2$, then G is a cycle of even length, so $\Gamma(G) = 5 = d+e+1$. In any other case, $d \geq 3$ and $e \geq 2$. The bipartition induces a fragmentation of G by two graphs, a disjoint union of S_d on one side and a disjoint union of S_e on the other side, so $\Gamma(G) \leq (d+1)+(e+1)$. We show in Proposition 5.4 below that the Γ value of the caterpillar graph $C_{d,e}$ is $d+e+2$. There is a locally injective homomorphism $f : C_{d,e} \to G$, so $\Gamma(G) \geq \Gamma(C_{d,e}) = d+e+2$. □

Let $d, e \geq 2$ and let $C_{d,e}$ be the caterpillar graph obtained from the bi-infinite path $P_{\mathbb{Z}}$ by attaching alternatively $d-2$ and $e-2$ edges, so that the odd vertices in $P_{\mathbb{Z}}$ have degree d and the even vertices have degree e (see Figure 6).

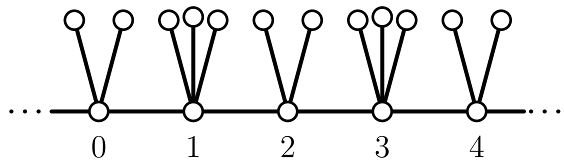


Figure 6: The caterpillar graph $C_{5,4}$

Proposition 5.4. *Let $d, e \geq 2$. One has*

$$\Gamma(C_{d,e}) = \begin{cases} d+e+1 & \text{if } d = e = 2, \\ d+e+2 & \text{otherwise.} \end{cases}$$

Proof. As $C_{d,e} \cong C_{e,d}$, we can assume that $d \geq e$. If $d = e = 2$, then $C_{2,2} = P_{\mathbb{Z}}$ so $\Gamma(C_{2,2}) = 5$ by Proposition 5.2.

Now assume that $d \geq 3$. Partition $V(C_{d,e})$ into the odd vertices of $P_{\mathbb{Z}}$ and their neighboring leaves on one side, and the even vertices and their neighboring leaves on the other side. We obtain a fragmentation of $C_{d,e}$ by a disjoint union of S_d and a disjoint union of S_e, so $\Gamma(C_{d,e}) \leq (d+1)+(e+1) = d+e+2$. Let $k = d+e+1$ and assume for a contradiction that $C_{d,e}$ generates $\mathsf{Col}_k(C_{d,e})$ via $\mathcal{S}_{u,v}$.

The rest of the proof is a combination of the technical lemmas 5.1, 5.2 and 5.3. Let $k = d+e+1$ and assume that $C_{d,e}$ generates $\mathsf{Col}_k(C_{d,e})$ via $\mathcal{S}_{u,v}$.

Let $n \in \mathbb{Z}$ and let $a = d$ if n is odd, $a = e$ if n is even. One has $\deg(n) = a$ and $\deg(n \pm 1) = k - a - 1$ so $\mathcal{S}_{n,n\pm 1}$ is made of a-sets and $(a+1)$-sets, and the family $(\mathcal{S}_{n,n-1}, \mathcal{S}_{n,n+1})$ is $(a-1)$-compatible.

If $d \geq 4$ or $e \geq 3$, then we apply Lemma 5.1 to $(d, \mathcal{S}_{1,0}, \mathcal{S}_{1,2})$ and $(e, \mathcal{S}_{2,3}, \mathcal{S}_{2,1})$. As $k \geq 7$, the first statement of the lemma holds. In particular, the d-sets in $\mathcal{S}_{1,2}$ do not contain some $x \in [k]$ and the e-sets in $\mathcal{S}_{2,1}$ do not contain some $y \in [k]$. Let $T \in \mathcal{S}_{1,2}$ separate x from y. One must have $|T| = d+1$ and $|\overline{T}| = e+1$, so $k = |T| + |\overline{T}| = d+e+2$, which is a contradiction.

Now assume that $d = 3$ and $e = 2$, so $k = 6$. We apply Lemma 5.2 to $(d, \mathcal{S}_{1,0}, \mathcal{S}_{1,2})$ and $(d, \mathcal{S}_{3,4}, \mathcal{S}_{3,2})$. Every $x \in [k]$ is outside some 4-set in $\mathcal{S}_{1,2}$ and some 4-set in $\mathcal{S}_{3,2}$, in other words the pairs in $\mathcal{S}_{2,1}$ and $\mathcal{S}_{2,3}$ cover $[k]$. We can apply Lemma 5.3 to $\mathcal{S}_{2,1}$ and $\mathcal{S}_{2,3}$ which are compatible, giving a contradiction. $\square$

Lemma 5.1. *Let $d \geq 3$ and $k \geq d+3$. Let $\mathcal{S}, \mathcal{T}$ be completely separating systems over $[k]$ made of sets of size at most $d+1$, such that $(\mathcal{S}, \mathcal{T})$ is $(d-1)$-compatible. One of the following two conditions is met:*

- *Either the union of the d-sets in $\mathcal{T}$ has size at most $d+1$,*

- *Or* $(d,k)=(3,6)$ *and* $\mathcal{T}$ *contains exactly two* d*-sets, and these sets are disjoint.*

Proof. If $\mathcal{T}$ contains at most one d-set, then the first statement holds.

We first show that if distinct d-sets $T,T'\in\mathcal{T}$ intersect, then $|T\cap T'|=d-1$. Let $x\in T\cap T'$ and let $S\in\mathcal{S}$ that does not contain x. As S intersects both T and T' in at least $d-1$ points, $T\cup T'$ is contained in $S\sqcup\{x\}$, implying $|T\cup T'|\leq|S|+1\leq d+2$. As $k\geq d+2$, there exists $y\notin T\cup T'$. Let $S\in\mathcal{S}$ separate y from x. $T\cup T'$ is contained in $(S\setminus\{y\})\sqcup\{x\}$ so $|T\cup T'|\leq|S|\leq d+1$, hence $|T\cap T'|=|T|+|T'|-|T\cup T'|\geq 2d-(d+1)=d-1$.

We next show that if $(d,k)\neq(3,6)$, then every pair of d-sets $T,T'\in\mathcal{T}$ intersect. Assume that some $T,T'\in\mathcal{T}$ are disjoint d-sets and take any $S\in\mathcal{S}$, it intersects both T and T' in at least $d-1$ points, so $2(d-1)\leq|S|\leq d+1$ which is impossible if $d\geq 4$. If $d=3$ and $k\geq 7$, then $|T\cup T'|=2d<k$ so there exists $y\notin T\cup T'$. Let $S\in\mathcal{S}$ contain y, it intersects both T and T' in at least $d-1$ points, $2(d-1)+1\leq|S|\leq d+1$, which is impossible.

Therefore, if $\mathcal{T}$ contains two disjoint d-sets T,T', then $(d,k)=(3,6)$ and $\mathcal{T}$ cannot contain another d-set, because it would intersect both T and T' but it would intersect one of them in 1 point only, contradicting the first assertion. For the rest of the proof, we assume that $(d,k)\neq(3,6)$, so every pair of distinct d-sets in T intersect in $d-1$ points. Fix two of them, T and T'.

Let $T''\in\mathcal{T}$ be a d-set and assume for a contradiction that T'' contains an element $z\notin T\cup T'$. As T'' intersects both T and T' in $d-1$ points, one has $T''\setminus\{z\}\subseteq T\cap T'$. The set $T\cup T'\cup T''$ has size $d+2$ so there exists y outside that set. Let $x\in T\cap T'\cap T''$ and let $S\in\mathcal{S}$ separate y from x. $T\cup T'\cup T''$ is contained in $(S\setminus\{y\})\sqcup\{x\}$, so $|T\cup T'\cup T''|\leq|S|\leq d+1$ which is a contradiction. Therefore, $T''\subseteq T\cup T'$ for every d-set $T''\in\mathcal{T}$, so their union is $T\cup T'$ whose size is $d+1$. ☐

Lemma 5.2. *Let* $d\geq 3$ *and* $k\geq d+3$*. Let* $\mathcal{S},\mathcal{T}$ *be completely separating systems over* $[k]$*, made of sets of size at most* $d+1$ *and such that* $(\mathcal{S},\mathcal{T})$ *is* $(d-1)$*-compatible. For every* $x\in[k]$ *there exists a* $(d+1)$*-set* $T\in\mathcal{T}$ *such that* $x\notin T$*.*

Proof. We apply Lemma 5.1. If the union of the d-sets in $\mathcal{T}$ has size at most $d+1$, then let $y,z\in[k]$ be distinct elements that are not covered by the d-sets of $\mathcal{T}$. For every $x\in[k]$, there is a set $T\in\mathcal{T}$ separating y or z from x (when $x=y$, take T separating z from x, when $x\neq y$, take T separating y from x). As T contains y or z, it is a $(d+1)$-set that does not contain x.

Now assume that $(d,k)=(3,6)$ and $\mathcal{T}$ has exactly two d-sets T,T', which are disjoint. For $x\in[k]$, let $y\neq x$ lie on the same side of the partition $[k]=T\sqcup T'$. There is $T\in\mathcal{T}$ separating y from x and T must be a $(d+1)$-set. ☐

Lemma 5.3. *Let $k \geq 5$. Let $\mathcal{S}, \mathcal{T}$ be completely separating systems over $[k]$ made of sets of size at most 3, such that $(\mathcal{S}, \mathcal{T})$ is compatible. It is not possible that both the pairs in $\mathcal{S}$ cover $[k]$ and the pairs in $\mathcal{T}$ cover $[k]$.*

Proof. Assume the existence of such systems. First, two pairs in $\mathcal{S}$ must intersect: otherwise, assume for instance that $\mathcal{S}$ contains 12 and 34, a pair in $\mathcal{T}$ containing 5 cannot intersect both. As $k > 3$, the Helly-type theorem implies that the pairs in $\mathcal{S}$ all intersect in one point. Therefore, up to some permutation, $\mathcal{S}$ contains the pairs $12, 13, 14$. Let $T \in \mathcal{T}$ separate 5 from 1. As T intersects every element of $\mathcal{S}$, T contains 2345, contradicting the assumption that elements of $\mathcal{T}$ have size at most 3. □

Remark 5.1. From the proof, we can actually derive that when $(d, e) \neq (3, 2)$, if G is the subgraph of $C_{d,e}$ induced by vertices $0, 1, 2, 3$ and their neighbors, then $\Gamma(G) = d + e + 2$. For $(d, e) = (3, 2)$, one can show that $\Gamma(G) = 6 = d + e + 1$, so this particular case is inevitable and the proof shows that if H is the subgraph of $C_{d,e}$ induced by the vertices $0, 1, 2, 3, 4$ and their neighbors, then $\Gamma(H) = d + e + 2$.

Theorem 5.2 gives the Γ value of the bipartite complete graph $K_{d,e}$. We explicitly state other particular instances of Theorem 5.2.

Corollary 5.1. *Let $d \geq 1$ and G be a bipartite regular graph of degree d. One has*

$$\Gamma(G) = \begin{cases} 2d & \textit{if } d = 1, \\ 2d + 1 & \textit{if } d = 2, \\ 2d + 2 & \textit{if } d \geq 3. \end{cases}$$

The case $d = 1$ is a disjoint union of edges, and $d = 2$ is a cycle of even length.

A particularly important example is the case of the $\mathbb{Z}^d$-graph, the graph on vertices $\mathbb{Z}^d$ with an edge between vertices whose Manhattan distance is 1. It is bipartite and regular of degree $2d$.

Corollary 5.2. *For $d \geq 1$, one has*

$$\Gamma(\mathbb{Z}^d) = \begin{cases} 4d + 1 & \textit{if } d = 1, \\ 4d + 2 & \textit{if } d \geq 2. \end{cases}$$

Example 5.1 (Triangular tessellation). Consider the regular tesselation of the plane by triangles. The neighbor relation is given by the honeycomb graph H (see Figure 7), which is a bipartite regular graph of degree 3, so

$$\Gamma(H) = 8.$$

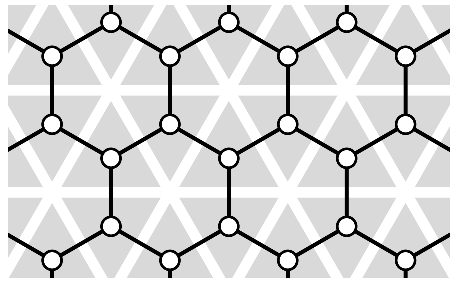


Figure 7: Triangular tesselation and its dual graph

5.6 Small numbers of colors

We end this section by identifying the connected graphs G such that $\Gamma(G)$ is small.

Proposition 5.5. *For a non-degenerate connected graph G, one has*

- $\Gamma(G) = 2 \iff G = P_2 = S_1$,
- $\Gamma(G) = 3 \iff G = P_3 = S_2$,
- $\Gamma(G) = 4 \iff G = P_4$ *or* S_3.

In other words, for $k \leq 4$, a connected graph G satisfies $\Gamma(G) = k$ if and only if G has k edges. This pattern breaks for $k \geq 5$, because for instance $\Gamma(\mathbb{Z}) = 5$.

Proof. We already know that these graphs have these Γ values. Conversely, we use $\Delta(G) \leq \Gamma(G) - 1$. If $\Gamma(G) = 2$, then every vertex has degree 1, so $G = P_2$. If $\Gamma(G) = 3$, then every vertex has degree at most 2, so G is a cycle or a path, and we know from Propositions 5.2 and 5.3 that the only possibility is P_3. If $\Gamma(G) = 4$, then either G is a path or a cycle, so $G = P_4$, or G contains a vertex of degree 3. In the latter case, its neighbors must have degree 1 by Corollary 4.1, so $G = S_4$. $\square$

6 Triple stars

We devote a whole section to another class of graphs which illustrate the difficulty of evaluating Γ exactly. We have seen that the Γ value of stars and double stars can be easily found. It turns out that the behavior of triple stars is much more complicated.

A triple star is obtained from P_3, the path with three vertices u, v, w, by attaching edges to each vertex. Let a, b, c be the degrees of u, v, w respectively, they

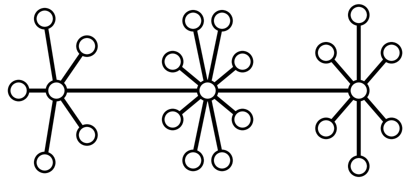


Figure 8: The triple star $S_{6,10,7}$

satisfy $a, c \geq 1$ and $b \geq 2$. We denote by $S_{a,b,c}$ the resulting graph, illustrated in Figure 8.

The Γ value of triple stars can be found up to 1 unit.

Proposition 6.1 (Estimate)**.** *One has*

$$\max(a+b, b+c) \leq \Gamma(S_{a,b,c}) \leq \max(a+b, b+c) + 1.$$

Proof. The triple star $\mathcal{S}_{a,b,c}$ contains the double stars $\mathcal{S}_{a,b}$ and $\mathcal{S}_{b,c}$, so $\Gamma(\mathcal{S}_{a,b,c}) \geq \max(a+b, b+c)$ by Proposition 4.5. Let $M = \max(a, c)$ and $k = b + M + 1$. For any leaf x and its neighbor y, let $\mathcal{S}_{x,y} = \sigma_{[k]}$. Let $\mathcal{S}_{v,u} = \mathcal{S}_{v,w}$ be the separating system $\sigma_{[M]} * \overline{\sigma_{[b+1]}}$ over $[k] \cong [M] \sqcup [b+1]$. At u (and symmetrically w), the compatibility comes from the fact that $\mathcal{S}_{u,v}$ is made of sets of size $M \geq a$. At v, the compatibility comes from the fact that if $S, T \in \overline{\sigma_{[b+1]}}$, then $|S \cap T| \geq y - 1$. $\square$

Remark 6.1. If G is biregular of degrees $d, e \geq 2$, then there is a locally injective homomorphism from $S_{d,e,d}$ to G, so $\Gamma(G) \geq d + e + 1$. However, we have seen that this lower bound is not optimal unless $d = e = 2$, so triple stars are not enough to find $\Gamma(G)$.

The lower bound in Proposition 6.1 is reached for instance when $a = 1$ or $c = 1$, because $S_{a,b,c}$ is then a double star. The results of the next section imply in particular that upper bound is also reached.

6.1 On the quest for the exact value

Although Proposition 6.1 already gives a good estimate of $\Gamma(S_{a,b,c})$, we push the investigation further in order to illustrate the complexity of the problem even for simple graphs. We are trying to find the exact profile

$$\mathcal{P} = \{(a, b, c) : \Gamma(S_{a,b,c}) = \max(a+b, b+c)\}.$$

Assuming by symmetry that $a \le c$, we will see that for each values of a and b, there is threshold $\mu(a,b) \ge a$ such that

$$\Gamma(\mathcal{S}_{a,b,c}) = \begin{cases} b+c+1 & \text{if } a \le c < \mu(a,b), \\ b+c & \text{if } c \ge \mu(a,b). \end{cases}$$

We are going to estimate the value of $\mu(a,b)$. We first reformulate the problem in terms of the existence of graphs having certain properties.

Theorem 6.1 (Reformulation)**.** *Let $a, c \ge 1$, $b \ge 2$ and $k = b+c$. One has $\Gamma(S_{a,b,c}) = b+c$ if and only if there exists a graph on vertices $V = [k]$ whose complete subgraphs of size a form a completely separating system over $[k]$ and whose independent sets of size b form a completely separating system over $[k]$.*

Proof. Assume that such a graph H exists. Let $\mathcal{S}_{u,v}$ be the set of complete subgraphs of H of order a and $\mathcal{S}_{v,w}$ be the set of independent sets of H of size b. As usual, for each leaf x and its neighbor y, let $\mathcal{S}_{x,y} = \sigma_{[k]}$. Let $S \in \mathcal{S}_{u,v}$ and $T \in \mathcal{S}_{v,w}$. One has

- $|S| = a = \deg(u)$,
- $|S \cap T| \le 1$ so $|\overline{S} \cap T| = |T| - |S \cap T| \ge b - 1 = \deg(v) - 1$,
- $|\overline{T}| = c = \deg(w)$.

Therefore, all the compatibility conditions are met, so G generates $\mathsf{Col}_k(G)$.

Conversely, assume that G generates $\mathsf{Col}_k(G)$ for $k = b + c$, and let $\mathcal{S}$ be the corresponding completely separating systems. Observe that each $T \in \mathcal{S}_{v,w}$ has size b, because $|T| \ge \deg(v) = b$ and $\overline{T} \ge \deg(w) = k - b$. Each $S \in \mathcal{S}_{u,v}$ has size at least a, and $\overline{S} \cap T \ge b - 1$ by Lemma 4.1, so $|S \cap T| \le 1$. Let H be the graph over vertices $[k]$, where $\{i, j\}$ is an edge of H if there exists $S \in \mathcal{S}_{u,v}$ containing i and j. By definition, each subset of each $S \in \mathcal{S}_{u,v}$ is a complete subgraph of H. Therefore, the complete subgraphs of H of size a form a completely separating system: if c, d are distinct colors, then some $S \in \mathcal{S}_{u,v}$ separates c from d, so any subset of S of size a containing c also separates c from d. Let $T \in \mathcal{S}_{v,w}$. It cannot contain an edge $\{i, j\}$, otherwise $|S \cap T| \ge 2$, contradicting the above observation. Therefore, T is an independent set of size b. As $\mathcal{S}_{v,w}$ is completely separating, so is the system of independent sets of size b. $\square$

Using this characterization, we can already obtain triplets (a, b, c) belonging to $\mathcal{P}$.

Example 6.1. If $a = 2$ and $c \geq b + 1$, then $(a, b, c) \in \mathcal{P}$.

Let $k = b + c$ and H be the cyclic graph over $\mathbb{Z}/k\mathbb{Z}$. The edges form a completely separating system. The set $\{0, 2, \ldots, 2b-2\}$ is independent because $2b-2 \leq k-2$, it is unstable (see Example 3.2), so its translates form a completely separating system of independent sets of size b.

Remark 6.2. One easily sees that for $c = a \geq 2$, $\Gamma(S_{a,b,a}) = a + b + 1$. Otherwise, let H be a graph satisfying the conditions of Theorem 6.1, each vertex has degree at least a and degree at least b in the complement graph $\overline{H}$, so $a + b + 1 \leq k = b + c$, which is a contradiction.

In order to understand $\mathcal{P}$ we identify, for each pair (a, b), the set of values of c such that $(a, b, c) \in \mathcal{P}$. It turns out that it is always an upper set.

Proposition 6.2. *If* $(a, b, c) \in \mathcal{P}$*, then* $(a, b, c + 1) \in \mathcal{P}$*.*

Proof. Let H be a graph witnessing that $(a, b, c) \in \mathcal{P}$. Choose some arbitrary complete subgraph K of order a, and let G be obtained from H by adding a new vertex v and edges between v and each vertex of K. We claim that G witnesses that $(a, b, c + 1) \in \mathcal{P}$. Let $a, b \in V(G)$ be distinct. If $a \neq v$, then the complete subgraphs of H of order a and the independent sets of H of size b separating a from b are still complete in G and independent in G respectively. If $a = v$, then for $u \in K$, the sets $K \setminus \{u\} \cup \{v\}$ are complete of size a and separate a from every other vertex. Let $I \subseteq V(H)$ be independent in H of size b. Let $u \in I$ be the unique element of $I \cap K$ if it exists, or any element of I otherwise. The set $I \setminus \{u\} \cup \{v\}$ is independent in G and has size b, and the sets obtained this way separate v from any other vertex. $\square$

Therefore, $\mathcal{P}$ can be described by the function

$$\mu(a, b) = \min\{c \geq a : (a, b, c) \in \mathcal{P}\}.$$

One has $\mu(1, b) = 1$ because $S_{1,b,c}$ is the double star $S_{b,c}$, and $\mu(a, b) \geq a+1$ for $a \geq 2$ by Remark 6.2. Example 6.1 shows that $\mu(2, b) \leq b + 1$.

6.2 Lower bounds

We give two lower bounds on $\mu(a, b)$. The first one is better when $a \gg b$, the second one is better when $b \gg a$ (calculations show that the second lower bound is larger than the first one precisely when $b > (\frac{4+2\sqrt{3}}{3})a \simeq 2.49a$).

Proposition 6.3 (First lower bound)**.** *For $a \geq 3$ and $b \geq 2$, one has*

$$\mu(a,b) \geq \frac{1}{2}\left(a + \sqrt{4ab + a^2}\right).$$

Proof. Let $(a,b,c) \in \mathcal{P}$, witnessed by a graph H. Let I be an independent b-set. There are are least ab edges going out of I and there are c vertices outside I, so some vertex $u \notin I$ is connected to at least $\frac{ab}{c}$ vertices in I. Let $v \neq u$ which is not a neighbor of u (it exists as u belongs to an independent b-set and $b \geq 2$), and let $w \in N(u) \setminus I$ (it exists as u belongs to a complete subgraph of order $a \geq 3$, which intersects I in at most one vertex). Let J be an independent b-set separating u from v and K a complete subgraph of order a separating u from w. The set $N(u) \sqcup \{v\}$ is disjoint from J, so $|N(u)| + 1 \leq |\overline{J}| = c$. For $S \subseteq [k]$, we denote by $N_S(u)$ the set of neighbors of u in S. One has $N(u) = \{w\} \sqcup N_K(u) \sqcup N_I(u) \setminus N_K(u)$ and K intersects I in at most one vertex, so

$$\begin{aligned} c - 1 \geq |N(u)| &\geq 1 + |N_K(u)| + |N_I(u)| - 1 \\ &\geq a + \frac{ab}{c} - 1. \end{aligned}$$

Therefore, $c^2 \geq ac + ab$ which is equivalent, by Lemma 6.1 below, to $c \geq \frac{a}{2} + \sqrt{\frac{a^2}{4} + ab}$. □

Lemma 6.1. *Let $b \geq 0$. For $x > 0$, one has*

$$x^2 \geq 2ax + b \iff x \geq a + \sqrt{a^2 + b}.$$

Proof. The first inequality can be rewritten as $(x-a)^2 \geq a^2 + b$. If $0 < x \leq a$, then $(x-a)^2 < a^2 \leq a^2 + b$, so the inequality can only be satisfied for $x > a$. It is then equivalent to $x \geq a + \sqrt{a^2 + b}$. □

We now derive the second lower bound.

Proposition 6.4 (Second lower bound)**.** *For $a, b \geq 2$, one has*

$$\mu(a,b) \geq 2\sqrt{ab} - a.$$

Proof. Let $(a,b,c) \in \mathcal{P}$, witnessed by H. Note that $b \geq 2\sqrt{ab} - a$, so if $c \geq b$ then we are done. We now assume that $c \leq b$.

Let I be an independent b-set. Let $u \notin I$ maximize $|N_I(u)|$ and let $d = |N_I(u)|$. As in the proof of Proposition 6.3, one has $d \geq \frac{ab}{c} \geq a$.

Let J be an independent b-set containing u and let $y = |I \cap J|$.

As J is independent and $u \in J$, $N(u)$ is disjoint from J. Therefore, $N_I(u)$ and $I \cap J$ are disjoint and contained in I, so $b = |I| \geq d + y$, hence

$$y \leq b - d. \tag{2}$$

Let S be the set of neighbors of vertices in $I \cap J$. As I and J are independent, S is disjoint from both of them. The number e of edges between $I \cap J$ and S can be estimated in two ways. As each vertex has degree at least a, one has $e \geq a|I \cap J| = ay$. As each vertex outside I is connected to at most d vertices in I, one has $e \leq d|S|$. Therefore, $|S| \geq \frac{ay}{d}$. Finally,

$$\begin{aligned}
k &\geq |I \cup J| + |S| \\
&\geq 2b - y + \frac{ay}{d} \\
&= 2b + \frac{a-d}{d} y \\
&\geq 2b + \frac{a-d}{d}(b-d) && \text{as } d \geq a \text{ and } y \leq b - d \text{ by (2)} \\
&= b - a + \frac{ab}{d} + d \\
&\geq b - a + 2\sqrt{ab} && \text{as } \min_{x>0} \left(\frac{a}{x} + x\right) = 2\sqrt{a}.
\end{aligned}$$

As $k = b + c$, we conclude that $c \geq 2\sqrt{ab} - a$. □

6.3 Upper bound

We give an upper bound on $\mu(a, b)$, which is twice the lower bound from Proposition 6.3.

Theorem 6.2 (Upper bound). *For $a, b \geq 2$, one has*

$$\mu(a, b) \leq a + \left\lceil \sqrt{4ab + a^2} \right\rceil.$$

The result will immediately follow from the next two technical lemmas.

Lemma 6.2. *Let $b \geq 2$ and $c, a \geq 1$. If there exists $p \in \mathbb{N}$ such that*

$$2 \leq p \leq b + c - pa \leq p(c - pa),$$

then there exists a graph H satisfying the conditions of Theorem 6.1.

Proof. The inequalities imply that we can choose p numbers $1 \le A_1, \dots, A_p \le c - pa$ whose sum is $b + c - pa$. For any $i \in [p]$, the set $\bigsqcup_{j \ne i} [A_j]$ contains at least $b + c - pa - (c - pa) = b$ elements.

For $i \in [p]$, let K_a^i be a complete graph on a vertices and I_{A_i} be the graph with A_i vertices and no edge. We define the graph

$$H = (K_a^1 * I_{A_1}) \sqcup \ldots \sqcup (K_a^p * I_{A_p}),$$

where the join $G_1 * G_2$ of two graphs is obtained from their disjoint union by adding all the edges connecting a vertex of G_1 to a vertex of G_2.

This graph has $pa + \sum_i A_i = b + c$ vertices. For each choice of $i \in [p]$, $u \in K_a^i$ and $v \in I_{A_i}$, the subgraph $(K_a^i \setminus \{u\}) * \{v\}$ is complete of size a, and these graphs can separate any pair of vertices. For distinct $i, j \in [p]$, $u \in K_a^i$ and $v \in I_{A_j}$, the set $\{u\} \sqcup (\bigsqcup_{k \ne i} I_{A_k}) \setminus \{v\}$ is an independent set of size at least b, and these sets can separate any pair of vertices. □

We give a sufficient condition for the assumption of Lemma 6.2 to be satisfied.

Lemma 6.3. *Let $a, c \ge 1$ and $b \ge 2$. If*

$$c \ge a + \sqrt{4ab + a^2},$$

then there exists $p \in \mathbb{N}$ such that $2 \le p \le b + c - pa \le p(c - pa)$.

Proof. Let $p = 1 + \lfloor \frac{c}{2a} \rfloor$.

- First, we choose this value of p because it is a natural number that is closest to $\frac{c+a}{2a}$, i.e. $|p - \frac{c+a}{2a}| \le \frac{1}{2}$, which follows from $\frac{c}{2a} < p \le 1 + \frac{c}{2a}$,
- The assumption implies that $c \ge 2a$, so $p \ge 2$,
- The inequality $p \le b + c - pa$ can be rewritten as $p \le \frac{b+c}{1+a}$, which we now prove. The function $f(x) = \frac{2x+c}{x+a}$ is non-increasing as $\det \left(\begin{smallmatrix} 2 & c \\ 1 & a \end{smallmatrix}\right) = 2a - c \le 0$, so $f(a) \le f(1)$, hence

$$p = \left\lfloor \frac{2a + c}{a + a} \right\rfloor \le \frac{2a + c}{a + a} \le \frac{2 + c}{1 + a} \le \frac{b + c}{1 + a},$$

- The last inequality $b + c - pa \le p(c - pa)$ can be rewritten as

$$\begin{aligned} p^2 - \frac{c + a}{a} p &\le -\frac{b + c}{a} \\ \iff \left(p - \frac{c + a}{2a}\right)^2 &\le \left(\frac{c + a}{2a}\right)^2 - \frac{b + c}{a}. \end{aligned}$$

The left-hand side is at most $\frac{1}{4}$, so it is sufficient to show that the right-hand side is at least $\frac{1}{4}$. It can be derived from the assumption $c - a \geq \sqrt{4ab + a^2}$ as follows:

$$\begin{aligned}\left(\frac{c+a}{2a}\right)^2 - \frac{b+c}{a} &= \frac{1}{4a^2}\big(c^2 + 2ac + a^2 - 4a(b+c)\big)\\ &= \frac{1}{4a^2}\big((c-a)^2 - 4ab\big)\\ &\geq \frac{1}{4}.\end{aligned}$$

□

6.4 Asymptotic behavior

All these bounds on $\mu(a, b)$ enable us to obtain the asymptotic behavior of $\mu(a, b)$ when one of its parameters is fixed.

Corollary 6.1. *The asymptotic values of $\mu(a, b)$ are as follows:*

- *For any fixed value of $a \geq 2$,*

$$\mu(a, b) \sim 2\sqrt{ab} \quad (b \to \infty).$$

- *For any fixed value of $b \geq 2$,*

$$\mu(a, b) = \Theta(a) \quad (a \to \infty).$$

For $a = 2$ and $b \geq 2$,

$$\left|\mu(2, b) - \sqrt{8b}\right| < 4.$$

Proof. The first and thid estimates come from Proposition 6.4 and Theorem 6.2. One only needs to check that $\sqrt{8b + 4} < \sqrt{8b} + 1$, which is easily obtained by squaring this inequality. The second estimate comes from Proposition 6.3 and Theorem 6.2. □

7 Future directions

As we have seen, finding the Γ value of a graph is sometimes a difficult problem. We ask whether general techniques can be developed to help establishing new results or to revisit results of this article using simpler arguments. We have seen for instance inequalities relating $\Gamma(G)$ to $\Delta(G)$ and $\chi(G)$, and we ask whether $\Gamma(G)$ is related to other graph invariants.

When the number k of colors is smaller than $\Gamma(G)$, distant vertices need coordination when producing a k-coloring, which raises the problem of finding the exact coordination structures for each value of k. These structures are simplicial complexes, in which for instance three vertices may coordinate collectively by having access to a common input value. Using the framework from [Hoy25], one may identify the simplicial complexes generating $\mathsf{Col}_k(G)$ for any graph G and any k between $\chi(G)$ and $\Gamma(G)$.

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