Borel complexity of continua

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Abstract

In this article, we investigate the complexity of topological properties of compact metrizable spaces, with particular emphasis on the complexity of being homeomorphic to a given space. Among other results, we relate the complexity of separation of spaces with the classical notion of likeness and identify the complexity levels of several homeomorphism classes.

Contents

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1 Introduction

A common topic in general topology is to identify topological invariants to distinguish between non-homeomorphic spaces, or to provide characterizations of particular spaces as the only spaces satisfying certain topological properties. We address these problems from the viewpoint of Descriptive Set Theory, by investigating the complexity of topological properties of spaces, with a particular emphasis on the property of being homeomorphic to a particular space. We only work with continua, which are connected compact metrizable spaces. These spaces embed in the Hilbert cube $Q = [0,1]^{\mathbb{N}}$ with the product of the Euclidean topology, therefore they can be seen as elements of the hyperspace $\mathcal{K}(Q)$ of compact subsets of Q, endowed with the Vietoris topology.

On $\mathcal{K}(Q)$, the homeomorphism equivalence relation is known to be non-Borel. However, by a result of Ryll-Nardewski, for any fixed space, the property of being homeomorphic to that space is always Borel, which calls for the identification of the Borel complexity associated to any particular space.

A first interest in identifying the exact complexity of a homeomorphism class is to show that the underlying characterization of this class is in a sense as simple as it can be. We show how classical characterizations of the disk and closed surfaces are optimal in this sense.

A second interest of a complexity-sensitive study of topological properties is to give a precise measure of proximity between spaces: if two spaces X and Y are difficult to separate, in the sense that any topological property separating them has a high complexity level, then these spaces are similar in the sense that they share all the properties of low complexity. We illustrate this phenomenon at the Π^0_2 level, showing that the impossibility of separating spaces by Π^0_2 invariant properties is closely related to the classical notion of likeness, which is a purely topological notion of similarity.

The complexity of topological invariants has been investigated in many works ranging from general topology to descriptive set theory, among which:

- Kuratowski [Kur31] and Mazurkiewicz [Maz31] proved that the set of Peano continua is Π^0_3-complete,
- Dobrowolski and Rubin [DR94] proved that being an ANR is Π^0_4-complete,
- Camerlo, Darji and Marcone [CDM05] proved among many other results that for every finite graph G, its homeomorphism class $\mathcal{H}(G)$ is Π^0_3-complete [CDM05], that if A is the pseudoarc, then $\mathcal{H}(A)$ is Π^0_2-complete, and that if X is a non'degenerate finitely triangulable continuum, then being X-like is a Π^0_2-complete property,
- Debs and Saint Raymond studied the descriptive complexity of connectedness properties [DS20].

A survey on these questions was written by Marcone [Mar06].

The main results of the present article are:

- An identification of the possible complexity levels of orbits in Polish group actions,
- For any finitely triangulable continuum Y, being Y-like is the minimal Π^0_2 invariant property satisfied by Y,
- The homeomorphism class of any closed surface is Π^0_3-complete, and $D_2(\Sigma^0_2)$-complete among the Peano continua,

- The properties of being cyclic and of being homeomorphic to the disk are both Π^0_3-complete, even among the Peano continua,
- The homeomorphism classes of the cylinder of the triod and of the 3-dimensional ball are Π^0_4-hard.

To the best of our knowledge, it is conceivable that the homeomorphism class of every finitely triangulable continuum is Π^0_4, and we leave it as an open question.

The article is organized as follows. In Section 2 we recall definitions from invariant Descriptive Set Theory and prove general results about the complexity of orbits of Polish group actions. In Section 3 we investigate the expressiveness of low complexity invariant properties of continua. In Section 4 we establish the complexity of homeomorphism classes of some two-dimensional continua. In Section 5 we identify finitely triangulable spaces whose complexity level is at least Π^0_4.

2 Descriptive set-theoretic tools for invariant classification

We shall not recall here the definition of the Borel hierarchy. For a thorough account, we refer the reader to the monograph [Kec95] by Kechris on Descriptive Set Theory. The descriptive set theory aspects of Polish group actions can be found in [BK96, Gao08]. Our aim in this section is rather to clarify certain terminological points and to highlight a few properties that will be useful to us. More precisely, we make explicit the folklore notions of relative complexity and separation complexity, we recall the main facts concerning Vaught transforms, and we examine to what extent invariance induces rigidity in the difference hierarchy.

2.1 Relative complexity and complexity of separation

A complexity class Γ consists, for each topological space X, of a set $\Gamma(X)$ of subsets of X. Important examples are the classes of the Borel hierarchy Σ^0_α, Π^0_α and Δ^0_α for any countable ordinal α, and the classes of the Hausdorff difference hierarchy $D_\alpha(\Sigma^0_\beta)$ and $\check{D}_\alpha(\Sigma^0_\beta)$ for countable ordinals α and β.

These complexity classes intuitively measure the difficulty of testing a property of points of a topological space X. The complexity of a property may decrease when assuming that the points already satisfy some other property. This is captured by the notion of relative complexity defined below. A subset of a topological space can be endowed with the subspace topology.

Definition 2.1 (Relative complexity)**.** Let X be a topological space, $A \subseteq P \subseteq X$ and Γ be a complexity class. We say that A has complexity Γ relative to P if A has complexity Γ in the subspace P, i.e. $A \in \Gamma(P)$. We say that A is Γ-hard relative to P if it is Γ-hard in the subspace P.

When the class Γ is Σ^0_α, $D_\alpha(\Sigma^0_\beta)$ or their dual classes, A has complexity Γ relative to P if and only if there exists a set $E \in \Gamma(X)$ such that $A = E \cap P$. This equivalence does not hold for the classes Δ^0_α in general.

The famous Wadge's lemma holds relative to a Borel set P. As is usual, we denote by $\check{\Gamma}$ the class of complements of sets in Γ.

Lemma 2.1 (Wadge's lemma)**.** *Let X be a Polish space, Γ be a complexity class $D_\alpha(\Sigma^0_\beta)$ or its dual. Let $A \subseteq P \subseteq X$ where A, P are Borel. Then*

$$A \text{ is } \check{\Gamma}\text{-hard relative to } P$$

if and only if

$$A \text{ does not have complexity } \Gamma \text{ relative to } P.$$

For A, B disjoint subsets of X, we define the complexity of separating A from B as the complexity of A relative to $A \cup B$. More precisely, we say that A is Γ-separable from B if A has complexity Γ relative to $A \cup B$ and that A is Γ-hard to separate from B if A is Γ-hard relative to $A \cup B$. These notions also enjoy the corresponding version of Wadge's lemma when A and B are Borel.

2.2 Polish group actions

Throughout this section, G denotes a Polish group and X a Polish space, equipped with a continuous action of G, which we denote by

$$(g, x) \longmapsto g \cdot x.$$

Details about such actions can be found in [Gao08].

In this article, we will exclusively work with the following action. Let $Q = [0,1]^{\mathbb{N}}$ be the Hilbert cube endowed with the product of the Euclidean topology. It is a Polish space. Let $\mathcal{K}(Q)$ be the hyperspace of non-empty compact subsets of Q. The Vietoris topology, or equivalently the Hausdorff metric makes it a Polish space. Let $\mathcal{H}(Q)$ be the Polish group of self-homeomorphisms of Q endowed with the compact-open topology. More details on these spaces can be found in [vM01]. Then $\mathcal{H}(Q)$ continuously acts on $\mathcal{K}(Q)$: for $h \in \mathcal{H}(Q)$ and $K \in \mathcal{K}(Q)$, let $h \cdot K = h(K)$.

2.2.1 Vaught transforms

A set $A \subseteq X$ is G-invariant if $g \cdot A = A$ for all $g \in G$. There are several ways of turning a set A into a G-invariant one. The most obvious ones are its *saturation* $[A]$, which is the largest invariant set containing A, and its *hull* (A), which is the smallest invariant set contained in A. Thus,

$$[A] = \{x \in X : \exists g \in G,\ g \cdot x \in A\}, \qquad (A) = \{x \in X : \forall g \in G,\ g \cdot x \in A\},$$

and

$$(A) \subseteq A \subseteq [A].$$

However, the descriptive complexity of these sets are usually much higher than the complexity of A, and they are typically not Borel even when A is. The Vaught transforms are a clever way of turning a set into a G-invariant one while preserving the Borel complexity.

Definition 2.2. For $A \subseteq X$, the Vaught transforms of A are

$$A^* = \{x : \{g \in G : g \cdot x \in A\} \text{ is comeagre in } G\},$$
$$A^\Delta = \{x : \{g \in G : g \cdot x \in A\} \text{ is non-meagre in } G\}.$$

Intuitively, A^* is the set of elements x whose orbit is almost entirely contained in A, in the sense that most elements of the group send x into A. Similarly, A^Δ is the set of elements x whose orbit intersects A in a thick manner, in the sense that a non-meagre set of elements of the group send x into A. The sets A^* and A^Δ are indeed G-invariant, and satisfy the following inclusions:

$$(A) \subseteq A^* \subseteq A^\Delta \subseteq [A].$$

The reason why the Vaught transforms preserve the Borel complexity is that they commute with boolean operations in the following ways (see [Kec95, Exercise 16.4]):

$$A^\Delta = X \setminus (X \setminus A)^*, \qquad \Big(\bigcap_n A_n\Big)^* = \bigcap_n A_n^*, \qquad \Big(\bigcup_n A_n\Big)^\Delta = \bigcup_n A_n^\Delta.$$

It can be shown that if A is open, then $A^\Delta = [A]$ is open as well. The previous equalities then imply by induction that if A is Σ^0_α, then A^Δ is also Σ^0_α, and if A is Π^0_α, then A^* is also Π^0_α ([Gao08, Theorem 3.2.7]).

It is also possible to make a set G-invariant preserving the intermediate complexity levels of the Hausdorff difference hierarchy. We recall that if $(A_\eta)_{\eta<\alpha}$ is an increasing sequence of sets and $p(\eta)$ is the parity of the ordinal η, then

$$D_\alpha((A_\eta)_{\eta<\alpha}) = \bigcup_{\substack{\eta<\alpha \\ p(\eta)\neq p(\alpha)}} A_\eta \setminus \bigcup_{\delta<\eta} A_\delta.$$

The complexity class $D_\alpha(\Sigma^0_\beta)$ is then the class of sets $D_\alpha((A_\eta)_{\eta<\alpha})$ where $(A_\eta)_{\eta<\alpha}$ is a growing sequence of sets in Σ^0_β. Hausdorff-Kuratowski's theorem states that in a Polish space, the class $\Delta^0_{\beta+1}$ is the union of the classes $D_\alpha(\Sigma^0_\beta)$, for countable ordinals α [Kec95, Theorem 22.27].

In the next result, we do not need X to be Polish.

Lemma 2.2. *Let X be metrizable and G be a Polish group acting continuously on X. Let α, β be countable ordinals. If $A \subseteq X$ belongs to the class $A \in D_\alpha(\Sigma^0_\beta)$, then there exists a G-invariant set $A' \in D_\alpha(\Sigma^0_\beta)$ such that*

$$A^* \subseteq A' \subseteq A^\Delta.$$

Moreover, $A' = D_\alpha((A_\eta)_{\eta<\alpha})$ where each $A_\eta \in \Sigma^0_\beta$ is G-invariant.

Proof. The argument relies on two observations. Let $B = D_\alpha((B_\eta)_{\eta<\alpha})$ where $B_\eta \in \Sigma^0_\beta$. The first observation [Kec95, Exercise 22.26.iv] is that if we let $B_\alpha = X$, then

$$X \setminus B = D_{\alpha+1}((B_\eta)_{\eta<\alpha+1}).$$

The second observation is that $C^\Delta \setminus D^\Delta \subseteq (C \setminus D)^\Delta$, which follows from $C^\Delta = ((C \setminus D) \cup D)^\Delta = (C \setminus D)^\Delta \cup D^\Delta$. It implies in particular that

$$D_\alpha((B_\eta^\Delta)_{\eta<\alpha}) \subseteq B^\Delta.$$

Let now $A = D_\alpha((A_\eta)_{\eta<\alpha})$ where $A_\eta \in \Sigma^0_\beta$ and define $A' = D_\alpha((A_\eta^\Delta)_{\eta<\alpha})$. By the first observation applied to A and A', if we let $A_\alpha := X$, hence $A_\alpha^\Delta = X$, then

$$X \setminus A = D_{\alpha+1}((A_\eta)_{\eta<\alpha+1}),$$
$$X \setminus A' = D_{\alpha+1}((A_\eta^\Delta)_{\eta<\alpha+1}).$$

By the second observation applied to A and $X \setminus A$ (and $\alpha+1$), one has

$$A' \subseteq A^\Delta,$$
$$X \setminus A' \subseteq (X \setminus A)^\Delta.$$

As $(X \setminus A)^\Delta = X \setminus A^*$, we obtain $A^* \subseteq A' \subseteq A^\Delta$. ∎

A version of Lemma 2.2 in the context of admissible representations appears in [dB13].

This result implies in particular that the levels of the difference hierarchy are no more expressive than the base level Σ^0_β in the sense that all their G-invariant sets are combinations of G-invariant sets of complexity Σ^0_β.

2.2.2 Complexity of orbits

A G-action induces an equivalence relation $x \sim_G y \iff \exists g \in G, x = g \cdot y$. Although this relation is not Borel in general, Ryll-Nardewski's theorem [RN65] states that every orbit is. In the case of $\mathcal{H}(Q)$ acting on $\mathcal{K}(Q)$, the equivalence relation is not Borel and even universal among the equivalence relations induced by Polish group actions [Zie16], and the orbits can have arbitrarily high Borel complexity [CDM05, Theorem 1.3].

We show however that orbits cannot occupy all the complexity levels.

Theorem 2.1. *Let X be a Polish space endowed with an action of Polish group G.*

- *Every orbit is complete for one of the following complexity classes:*

$$\Sigma^0_{\alpha+1}, \Pi^0_\alpha \text{ and } D_2(\Sigma^0_\alpha).$$

- *If $P \subseteq X$ is a Borel G-invariant set, then every orbit that is contained in P is complete for one of the following complexity classes relative to P:*

$$\Sigma^0_{\alpha+1}, \Delta^0_{\alpha+1}, \Pi^0_\alpha \text{ and } D_2(\Sigma^0_\alpha).$$

- *If x and y are not $\sim_G$-equivalent, then the complexity of separating* $\operatorname{Orb}(x)$ *from* $\operatorname{Orb}(y)$ *is complete for one of the following complexity classes:*

$$\Sigma^0_{\alpha+1}, \Delta^0_{\alpha+1}, \Pi^0_{\alpha+1}.$$

Proof. Let A be an orbit. A has the particular property of being minimal, i.e. it does not properly contain a non-empty G-invariant set.

First, by Ryll-Nardzewski's theorem, A is Borel, let β be minimal such that $A \in \Sigma^0_\beta$. We first show that β is a successor ordinal. Let $A = \bigcup_{n\in\mathbb{N}} A_n$ where $A_n \in \Pi^0_{\beta_n}$ and $\beta_n < \beta$. As A is invariant, $A = A^\Delta = \bigcup_{n\in\mathbb{N}} A_n^\Delta$. As each A_n^Δ is invariant, one of them must equal A by minimality of A, implying that $A \in \Sigma^0_{\beta_n+1}$, hence $\beta = \beta_n + 1$ by minimality of β.

Let α be such that $\beta = \alpha + 1$. If $A \notin \Pi^0_{\alpha+1}$, then A is $\Sigma^0_{\alpha+1}$-complete by Wadge's lemma 2.1. If $A \in \Pi^0_{\alpha+1}$, then $A \in \Delta_{\alpha+1}$ so by Hausdorff-Kuratowski's theorem, $A \in D_\beta(\Sigma^0_\alpha)$ for some β.

Let $(A_\eta)_{\eta<\beta}$ be an increasing sequence of Σ^0_α sets such that $A = D_\beta((A_\eta)_{\eta<\beta})$. By Lemma 2.2, $A = A' = D_\beta((A^\Delta_\eta)_{\eta<\beta})$. Again by minimality of A, there exists $\eta < \beta$ whose parity is opposite to that of β, and such that $A^\Delta_\eta \setminus \bigcup_{\delta<\eta} A^\Delta_\delta = A$, implying that $A \in D_2(\Sigma^0_\alpha)$.

If $A \notin \check{D}_2(\Sigma^0_\alpha)$, then A is $D_2(\Sigma^0_\alpha)$-complete. If $A \in \check{D}_2(\Sigma^0_\alpha)$, then by Lemma 2.2 applied to $X \setminus A$, $A = B \cup C$ where $B \in \Sigma^0_\alpha$ and $C = \Pi^0_\alpha$ are G-invariant. By minimality of A, $A = B$ or $A = C$. The first alternative is not possible because $B \in \Sigma^0_\alpha$ but $A \notin \Sigma^0_\alpha$ by minimality of $\beta = \alpha + 1$. Therefore, $A = C \in \Pi^0_\alpha$. As $A \notin \Sigma^0_\alpha$, A is Π^0_α-complete.

In the relative version, Wadge's lemma still holds but Hausdorff-Kuratowski's theorem may fail. Therefore, in the previous argument there is one case left: $A \in \Delta^0_{\alpha+1}$ but $A \notin D_\beta(\Sigma^0_\alpha)$ for any β. By Wadge's lemma, A is $\check{D}_\beta(\Sigma^0_\alpha)$-hard relative to P for each β, hence A is $\Delta^0_{\alpha+1}$-hard (by Hausdorff-Kuratowski's theorem on $2^\mathbb{N}$, every $C \in \Delta^0_{\alpha+1}(2^\mathbb{N})$ belongs to some level of the difference hierarchy).

In particular, if $P = \mathrm{Orb}(x) \cup \mathrm{Orb}(y)$, then the complexity of separating $\mathrm{Orb}(x)$ from $\mathrm{Orb}(y)$ is complete for a class Γ which has one of the four types from the previous item, and by symmetry $\check{\Gamma}$ has one of these types as well. Therefore, Γ must be $\Sigma^0_{\alpha+1}$, $\Delta^0_{\alpha+1}$ or $\Pi^0_{\alpha+1}$. □

We suspect that the homeomorphism class of a compact metrizable space cannot actually be complete for a class $\Sigma^0_{\alpha+1}$. We will show in Section 3.2 below that it cannot be Σ^0_2-complete.

3 Low complexity levels

From now on, we work in the space $\mathcal{K}(Q)$ endowed with the action of $\mathcal{H}(Q)$. By an "invariant property", we always mean a subset $\mathcal{P}$ of $\mathcal{K}(Q)$ which is invariant for this group action, i.e. such that if $X \in \mathcal{P}$ and $h \in \mathcal{H}(Q)$, then $h(X) \in \mathcal{P}$.

In this section, we investigate the expressiveness of invariant properties whose complexity lies in one of the levels Σ^0_2 or Π^0_2.

3.1 Invariant properties

In [AH25], a complete description of the Π^0_1 invariant properties was given. If $0 \leq p \leq n$, then let $\mathscr{C}_{n,p}$ be the set of spaces having at most n connected components, among which at most p are not singletons. $\mathscr{C}_{n,p}$ is Π^0_1 and the Π^0_1 invariant properties are the finite unions of these basic properties. In particular, non-degenerate continua can never be Σ^0_1-separated or Π^0_1-separated.

Obtaining a complete description of the Π^0_2 invariant properties seems out of reach. However, we show that for every finitely triangulable space X, the classical invariant property of being X-like, which was proved to be Π^0_2 in [CDM05], is actually the strongest invariant property of complexity Π^0_2 satisfied by X. In other words, X can be Π^0_2-separated from a space Y if and only if Y is not X-like.

This result shows that the notion of likeness, which expresses that a space resembles another one from a topological viewpoint, also means that these spaces are indistinguishable by low complexity invariant properties.

3.1.1 Being like another space

Let $\epsilon > 0$ be a real number, let X be a metric space, and let Y be a topological space. A continuous surjection $f : X \to Y$ is called an *ϵ-map* if, for every point $y \in Y$, the diameter of $f^{-1}(y)$ is strictly less than ϵ. This notion was introduced by Alexandroff [Ale28].

Definition 3.1. Let $\mathscr{C}$ be a class of compact metrizable spaces and let X be a compact metrizable space. We say that X is a *$\mathscr{C}$-like* if for every real number $\epsilon > 0$, there exists an ϵ-map from X onto one of the spaces in $\mathscr{C}$.

This notion does not depend on the compatible metric put on X. When $\mathscr{C} = \{Y\}$, we say that X is Y-like and denote it by $X \preceq Y$ (this pre-order was introduced by Maxwell [Max61]). Two compact metrizable spaces X and Y are said to be *quasi-homeomorphic* if $X \preceq Y$ and $Y \preceq X$, denoted $X \equiv Y$. Of course, if X and Y are homeomorphic, then they are quasi-homeomorphic.

Let $\mathscr{C}$ be a class of continua. We denote by $\mathcal{L}_\mathscr{C}$ the class of continua that are $\mathscr{C}$-like.

It is known that if $\mathscr{C}$ denotes the class of all finitely triangulable compact metric spaces, then $\mathcal{L}_{\mathscr{C}}$ coincides with the class of all compact metric spaces. Similarly, when $\mathscr{C}$ represents the class of all triangulable continua, $\mathcal{L}_{\mathscr{C}}$ coincides with the class of continua themselves.

One of the most significant results of Mardešić and Segal in [MS63] is the establishment of the connection between the notion of likeness and that of inverse limits.

In our context, an inverse limit is defined as the limit in the category of topological spaces of a diagram of the following form:

$$\cdots \xrightarrow{d_4} X_4 \xrightarrow{d_3} X_3 \xrightarrow{d_2} X_2 \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0$$

where each continuous map d_i is surjective.

Theorem 3.1. *If $\mathscr{C}$ denotes a class of triangulable continua, then $\mathcal{L}_{\mathscr{C}}$ turns out to be exactly the class of inverse limits of the form $\varprojlim(X_i, d_i)_{i\in\mathbb{N}}$, where each space X_i belongs to $\mathscr{C}$ and $d_i : X_{i+1} \to X_i$ is surjective continuous.*

In our context, the interest in the like relation is motivated by the low complexity of this property, as shown by Camerlo, Darji and Marcone in [CDM05]. More precisely, they obtained some of the following results, which will be useful to us later on.

Theorem 3.2. *If $\mathscr{C}$ is a class of finitely triangulable continua, then $\mathcal{L}_{\mathscr{C}}$ is Π^0_2.*

Theorem 3.3. *If C is a non-degenerate finitely triangulable continuum, then $\mathcal{L}_C$ is Π^0_2-complete.*

In [AH25], the pre-order $\sqsubseteq$ is introduced: $X \sqsubseteq Y$ if every Π^0_2 invariant satisfied by Y is satisfied by X (this preorder was denoted by $\preceq$ in [AH25]). We show that $\sqsubseteq$ is closely related to $\preceq$, and sometimes coincides.

Theorem 3.4. *Let X, Y be continua and consider the following conditions:*

1. *$X \cong \varprojlim(Y, f_i)$ where $f_i : Y \to Y$ are surjective continuous,*
2. *$X \preceq Y$ (X is Y-like),*
3. *$X \sqsubseteq Y$.*

One has

$$1. \implies 2. \implies 3.$$

and they are equivalences when Y is finitely triangulable.

In order to prove Theorem 3.4, we need the following result from [AH25].

Theorem 3.5. *$X \sqsubseteq Y$ if and only if there exists a copy $X_0 \subseteq Q$ of X such that for every $\epsilon > 0$, there exists a sequence of functions $f_n : X_0 \to Q$ satisfying $d(f_n(x), x) < \epsilon$ for all $x \in X_0$, and such that $f_n(X)$ converge in the Hausdorff metric to a copy of Y.*

Proof of Theorem 3.4. The first implication was proved by Mardešić and Segal [MS63, Lemma 1] when Y is triangulable. More generally, assume that Y is embedded in Q, and embed X in $\prod_i Y \subseteq \prod_i Q \cong Q$. For $k \in \mathbb{N}$, define the map $h_k : \prod_i Q \to Q$ by $h_k(y_0, y_1, \ldots) = (y_0, \ldots, y_k, 0, 0, \ldots)$. One has $h_k(X) \cong Y$ and for each $\epsilon > 0$, if k is sufficiently large then h_k is an ϵ-map. Therefore, $X \preceq Y$.

Now assume that $X \preceq Y$. Eilenberg [Eil35] proved that ϵ-maps can be transformed into functions that move points by small amounts. More precisely, for each $i \in \mathbb{N}$, let $\epsilon_i = \frac{1}{i}$ and $f_i : X \to Y$ be a surjective ϵ_i-map. There exists a compact metrizable space Z containing a homeomorphic copy X' of X and homeomorphic copies Y_i of Y such that the corresponding maps $f_i : X' \to Y_i$ are closer and closer to the identity on X'. As Z embeds in Q, it implies that $X \sqsubseteq Y$ by Theorem 3.5.

When Y is triangulable, 1. and 3. are equivalent by Mardešić and Segal's Theorem 3.1, so they are equivalent to 2. □

This result has the following consequence.

Corollary 3.1. *Let Y be a finitely triangulable continuum. The invariant property*

$$\{X \in \mathcal{K}(Q) : X \text{ is } Y\text{-like}\}$$

is the minimal Π^0_2 invariant property containing Y.

In particular, if X is Y-like but not homeomorphic to Y, then X cannot be Σ^0_2-separated from Y, equivalently Y cannot be Π^0_2-separated from X. In particular, $\mathcal{H}(X) \notin \Sigma^0_2$ and $\mathcal{H}(Y) \notin \Pi^0_2$.

Example 3.1. Let I be the line segment, S be the circle and $I \vee S$ be obtained by attaching one of the endpoints of I to S. One has

$$I \preceq I \vee S \text{ and } S \preceq I \vee S$$

and no other comparison holds (it follows from the fact that a graph G is like a graph H if and only if G can be obtained by contracting edges in some subdivision of H, see [KY00, AH25]). Therefore, the complexity of separating I (resp. S) from $I \vee S$ is Π^0_2-complete, and the complexity of separating I from S is Δ^0_2-complete (it cannot be Σ^0_1 or Π^0_1 as they are both connected).

Remark. The other implications in Theorem 3.4 fail in general. For instance, if I is the line segment and S is the topologist's sine curve, then $I \sqsubseteq S$ [AH25, Proposition 3.5] but $I \not\preceq S$ (there is no surjective continuous function $f : I \to S$).

Many results stating that certain properties are preserved by the relation $\preceq$, or by inverse limits with surjective bonding maps, can be revisited using Theorem 3.4, by observing that those properties are Π^0_2. We list a few examples of such properties:

- Having dimension at most n is preserved by $\preceq$ [Ale28] and indeed it is Π^0_2 [AH25],
- Indecomposability is preserved by inverse limits with surjective bonding maps [Nad17, Theorem 2.7] and indeed, indecomposability is a Π^0_2 invariant property [Nad17, Exercise 1.17].
- Nadler [Nad71] proved that unicoherence is preserved by inverse limits with surjective bonding maps, which can be explained by the fact that it is a Π^0_2 property [CDM05, Theorem 8.1],
- Ganea [Gan59] proved that having trivial Čech cohomology groups is preserved by $\preceq$ (at least when Y is a closed manifold), and indeed it is a Π^0_2 invariant property. This is a consequence of a result of Lupini, Melnikov and Nies [LMN23] and Downey, Melnikov [DM23], who proved that a presentation of $\check{H}^n(X)$ can be continusouly (in fact, computably) derived from X, which we discuss in the next result.

The latter example can be generalized as follows.

Proposition 3.1. *Let G be a finitely generated abelian group and $n \in \mathbb{N}$. The property*

$$G \text{ is a subgroup of } \check{H}^n(X)$$

is Σ^0_2.

Proof. We use the result from proved by Lupini, Melnikov and Nies [LMN23] and Downey, Melnikov [DM23], who showed that the Čech cohomology groups are computable. It means that there is a program or a Turing machine that, given a compact set $X \subseteq Q$ represented by an infinite sequence of natural numbers, produces a presentation of $\check{H}^n(X)$. We do not need the fact that this procedure is computable, but only that it is continuous, and we need to use the language of computable analysis to state and use this continuity result. Indeed, it does not seem possible to endow the space of groups with a topology such that the function $X \mapsto \check{H}^n(X)$ is continuous. However, one can represent or encode the compact subsets of Q and the countably generated groups by elements of $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$, which is endowed with the product of the discrete topology, and then show that there is a continuous function sending a code of X to a code of $\check{H}^n(X)$.

As $\mathcal{K}(Q)$ is a Polish space, there exists a continuous open surjective function $\kappa : \mathcal{N} \to \mathcal{K}(Q)$ [dB13, Theorem 41]. An element p is seen as a code of $\kappa(p)$. We will only use the important property that for every set $\mathcal{P} \subseteq \mathcal{K}(Q)$, the descriptive complexity of $\mathcal{P}$ is identical to the descriptive complexity of $\kappa^{-1}(\mathcal{P})$ [dB13, Theorem 68]. In particular, we will use the fact that if $\kappa^{-1}(\mathcal{P})$ is Σ^0_2, then so is $\mathcal{P}$.

On the other hand, we define a representation of the countably generated groups in the following way. We fix a countable alphabet $A = \{a_0, a_1, \ldots\}$ and a bijection $\nu : \mathbb{N} \to F_A$, where F_A is the free group over A. An element $q \in \mathcal{N}$ is a name of a group G if $N = \mathrm{im}(\nu \circ q)$ is a normal subgroup of F_A and G is isomorphic to F_A/N, we then write $G = \gamma(q)$. Note that this representation is related to the usual representation of marked groups, but is not topologically equivalent (q is not the characteristic function of N, but an enumeration of N).

It was proved in [LMN23] and [DM23] that for every $n \in \mathbb{N}$, there exists a continuous (in fact, computable) function $F_n : \mathcal{N} \to \mathcal{N}$ such that $\gamma(F_n(p))$ is isomorphic to $\check{H}^n(\kappa(p))$. In other words, there is a continuous function sending a code of $X \in \mathcal{K}(Q)$ to a code of $\check{H}^n(X)$.

We now proceed with the proof of the proposition. Let $\mathcal{P} \subseteq \mathcal{K}(Q)$ be the set of compact sets $X \subseteq Q$ such that G embeds in $\check{H}^n(X)$. Let $\langle S|R\rangle$ be a finite presentation of G, and let H be the normal subgroup of F_S generated by R, so $G \cong F_S/H$.

Let $X \in \mathcal{K}(Q)$ and p be a code of X (i.e., $\kappa(p) = X$). Therefore, $q = F_2(p)$ is a code of $\check{H}^2(X)$, more precisely $\check{H}^2(X) \cong F_A/N$ where $N = \mathrm{im}(\nu \circ q)$.

One has $X \in \mathcal{P}$ if and only if there exists a group homomorphism $f : F_S \to F_A$ such that $f(H) = N$, which is equivalent to the existence of a function $f : S \to F_A$, inducing a function $f^* : F_S \to F_A$, such that for all $x \in R$, $f^*(x) \in N$, and for all $x \in F_S \setminus H$, $f^*(x) \notin N$. The logical complexity of this sentence is Σ^0_2, let us give the details.

Let $(f_i)_{i\in\mathbb{N}}$ be an enumeration of the functions from S to F_A. For each $i \in \mathbb{N}$, the set

$$B_i = \{q \in \mathcal{N} : f_i^*(R) \subseteq \mathrm{im}(\nu \circ q)\}$$

is open or Σ^0_1 because R is finite, and the set

$$C_i = \{q \in \mathcal{N} : f_i^*(F_S \setminus H) \cap \mathrm{im}(\nu \circ q) = \emptyset\}$$

is closed or Π^0_1. Therefore,

$$\kappa^{-1}(\mathcal{P}) = \bigcup_{i\in\mathbb{N}} F_2^{-1}(B_i) \cap F_2^{-1}(C_i)$$

is Σ^0_2, so $\mathcal{P}$ is Σ^0_2 as well.

We finally note that this argument is effective: F_2 is computable and we can assume that H is a decidable subset of F_S as G is finitely generated abelian (said differently, its word problem is decidable), so the property $\mathcal{P}$ is effectively Σ^0_2. □

3.2 Homeomorphism classes of low complexity

We briefly discuss which spaces have homeomorphism classes in the very first levels of the Borel hierarchy.

The only space X such that $\mathcal{H}(X)$ is Π^0_1-complete is the singleton, and the only spaces whose homeomorphism classes are $D_2(\Pi^0_1)$ are the finite spaces which are not singletons.

The homeomorphism classes of the Cantor space and the pseudo-arc are Π^0_2-complete.

Proposition 3.2. *There is no compact metrizable space X such that $\mathcal{H}(X)$ is Σ^0_2-complete.*

Proof. If X is a singleton, then $\mathcal{H}(X)$ is Π^0_1-complete. If X is finite but not a singleton, then $\mathcal{H}(X)$ is $D_2(\Sigma^0_1)$-complete.

Let X be infinite. We show that X is Y-like for some Y that is not homeomorphic to X. As X is infinite, it contains a non-isolated point x. We are going to define $Y = (X, x) \vee (Z, z)$ for some continuum Z. As proved in [AH25, Theorem 3.3], X is Y-like. We need to choose Z so that Y is not homeomorphic to X.

Let us present one possible way to ensure this property.

A true cyclic element of X is a maximal cyclic Peano continuum contained in X which is not a singleton. It is known that X contains at most countably many true cyclic elements [McA66]. The cardinality of the set of homeomorphism classes of cyclic Peano continua is the cardinality of the continuum. Indeed, one can associate to any element x of the Cantor space a cyclic Peano continuum C_x such that distinct elements are sent to non-homeomorphic spaces: in the plane, take for each n a geometric hollow square whose lower side is delimited by the points $(2^{-n-1}, 0)$ and $(2^{-n}, 0)$ and fill the square if $x_n = 1$, and add the limit point $(0, 0)$. The spaces C_x and $C_{x'}$ are homeomorphic if and only if $x = x'$.

Therefore, by cardinality we can take a non-degenerate continuum Z which is not homeomorphic to any true cyclic element of X. Let $z \in Z$ be any distinguished point. Let $Y = (X, x) \vee (Z, z)$. The point z is a cut-point of Y and Z is a true cyclic element of Y, so Y is not homeomorphic to X. □

4 Complexity of certain two-dimensional continua

The aim of this section is to investigate the complexity of the property of being homeomorphic to a fixed finite 2-complex, that is, a two-dimensional space that can be finitely triangulated.

The first result gives a lower bound.

Proposition 4.1. *If K is a Peano continuum admitting an open subset homeomorphic to the plane $\mathbb{R}^2$, then $\mathcal{H}(K)$ is Π^0_3-hard.*

Proof. We regard the closed half-disk $\mathbb{D}_2$ as the set

$$\{(r, \theta) \mid r \in [0, 1],\ \theta \in [0, \pi]\}$$

in polar coordinates.

We denote by S a continuum homeomorphic to

$$\left\{ (x, \sin(1/x)) \mid x \in (0,1] \right\} \cup \left(\{0\} \times [-1,1]\right) \cup \left([-1,0] \times \{0\}\right),$$

the endpoints of S are $(1, \sin(1))$ and $(-1, 0)$.

Finally, we denote by I a closed segment.

It follows from [AH25] that there exists a continuous map

$$f : 2^{\mathbb{N}^2} \longrightarrow \mathcal{K}(\mathbb{D}_2)$$

satisfying the following properties:

- the image of f is contained in
$$\mathcal{H}(I) \cup \mathcal{H}(S);$$
- the set
$$f^{-1}\big(\mathcal{H}(I)\big)$$
is a Π^0_3-hard subset of $2^{\mathbb{N}^2}$;
- For every element $u \in 2^{\mathbb{N}^2}$, the compactum $f(u)$ has endpoints $(0,0)$ and $(1, \pi/2)$, and satisfies
$$f(u) \subset \big\{(r,\theta) \mid r \in [0,1],\ \theta \in [0, \pi/2]\big\}.$$

We now define a continuous map
$$\Phi : \mathcal{K}(\mathbb{D}_2) \longrightarrow \mathcal{K}(\mathbb{B}_3),$$
where $\mathbb{B}_3$ denotes the solid ball obtained by rotating (2π) the half-disk $\mathbb{D}_2$, which we parametrize using spherical coordinates. More precisely, for every compact set $K \subset \mathbb{D}_2$, we set

$$\Phi(K) := \left\{ (r, \theta, \varphi) \;\middle|\; (r,\theta) \in K,\ \varphi \in [0, 2\pi) \right\}.$$

Observe that $\Phi(I)$ is homeomorphic to a disk, whereas $\Phi(S)$ is a continuum that is not locally connected and has the same boundary as a disk.

We thus obtain that the composition $\Phi \circ f$ is a continuous map

$$\Phi \circ f : 2^{\mathbb{N}^2} \longrightarrow \mathcal{K}(\mathbb{B}_3)$$

satisfying the following properties:

- its image is contained in
$$\mathcal{H}(\mathbb{B}_3) \cup \mathcal{H}(\Phi(S));$$
- the set
$$(\Phi \circ f)^{-1}\big(\mathcal{H}(\mathbb{B}_2)\big) = f^{-1}\big(\mathcal{H}(I)\big)$$
is a Π^0_3-hard subset of $2^{\mathbb{N}^2}$;

- For any two distinct elements $u, u' \in 2^{\mathbb{N}^2}$, the boundaries of $\Phi(f(u))$ and $\Phi(f(u'))$ are equal as subsets of $\mathbb{B}_2$. In particular,

$$\partial\Phi(f(u)) = \partial\Phi(f(u')) \approx \mathbb{S}^1.$$

It is important to observe that, throughout this construction, the boundary of the disk is never altered. Consequently, if K is a Peano continuum admitting an open subset homeomorphic to the plane, then K is Π^0_3-hard to separate from a continuum obtained by removing a disk and replacing it with $\Phi(S)$. □

4.1 Closed surfaces

A *closed surface* is a compact two-dimensional manifold without boundary. In the wake of the work of Ganea [Gan59] and of Mardešić and Segal [MS63] in the early 1960s, Patkowska [Pat77] established in the 1970s the following result.

Theorem 4.1. *Let M be a connected, closed surface. Every 2-dimensional Peano continuum that is M-like is in fact homeomorphic to M.*

In conjunction with Theorem 3.2 from [CDM05], it has the following consequence.

Corollary 4.1. *Let M be a connected, closed surface. The property of being homeomorphic to M is Π^0_3-complete. It is $D_2(\Sigma^0_2)$-complete among the Peano continua.*

Proof. Since $\mathcal{H}(M) = \mathcal{P} \cap \mathcal{D}_2 \cap \mathcal{L}_M$, it follows that $\mathcal{H}(M)$ is Π^0_3 and $D_2(\Sigma^0_2)$ among the Peano continua.

By Proposition 4.1, $\mathcal{H}(M)$ is also Π^0_3-hard. Let I be the line segment. One has $I \preceq M \preceq M \vee I$, so $\mathcal{H}(M)$ is neither Π^0_2 nor Σ^0_2 among the Peano continua, therefore it is $D_2(\Sigma^0_2)$-hard among them by Theorem 2.1. □

The classification of connected surfaces implies that there exists a countable family of pairwise non-homeomorphic connected surfaces, which we may denote by $(M_n)_{n\in\mathbb{N}}$ (see Theorem 6.3 in Section 6.4 of [GX13]). Moreover, among the connected closed surfaces, each one is uniquely determined by its homotopy equivalence class. This understanding naturally leads to the following proposition.

Proposition 4.2. *The property of being homeomorphic to some closed, connected surface is Π^0_3-complete. It is $D_2(\Sigma^0_2)$-complete among the Peano continua.*

Proof. Let $\mathcal{S} \subseteq \mathcal{K}(Q)$ be the set of compact sets that are homeomorphic to a closed connected surface. One has

$$\mathcal{S} = \mathcal{P} \cap \mathcal{D}_2 \cap \bigcup_{n\in\mathbb{N}} \mathcal{L}_{M_n},$$

which by itself only implies that $\mathcal{S}$ has complexity $D_2(\Sigma^0_3)$, and Σ^0_3 relative to $\mathcal{P}$. We can improve these complexities as follows.

For each n, let A_n be the set of continua that are like $\{M_p : p \geq n\}$, which has complexity Π^0_2 by Theorem 3.2, and observe that $A_{n+1} \subseteq A_n$. We show that

$$\mathcal{S} = \mathcal{P} \cap \mathcal{D}_2 \cap A_0 \setminus \bigcap_{n\in\mathbb{N}} A_n,$$

which has complexity Π^0_3 and $D_2(\Sigma^0_2)$ relative to $\mathcal{P}$.

For every n, $M_n \in \mathcal{P} \cap D_2 \cap A_0$. Ganea [Gan59, Theorem 4.1] showed in particular that there exists an $\epsilon > 0$ such that for every closed surface N, every surjective ϵ-map $f : M_n \to N$ is a homotopy equivalence, hence a homeomorphism. Therefore, M_n cannot be like $\{M_p : p \geq n+1\}$, i.e. $M_n \notin A_{n+1}$ hence $M_n \notin \bigcap_{m\in\mathbb{N}} A_m$. Conversely, let $X \in \mathcal{P} \cap \mathcal{D}_2 \cap A_0 \setminus \bigcap_n A_n$. Let n be maximal such that $X \in A_n$. As $X \notin A_{n+1}$, there exists $\epsilon_0 > 0$ such that there is no surjective ϵ_0-map from X to any M_p, $p \geq n+1$. As $X \in A_n$, for any positive $\epsilon < \epsilon_0$ there exists a surjective ϵ-map from X to some A_p, $p \geq n$, and p can only be n. Therefore, X is M_n-like. As X is a Peano continuum of dimension 2, X is homeomorphic to M_n by Theorem 4.1.

The hardness follows from the same argument as in Corollary 4.1. □

4.2 Cyclic Peano continuum

The article of Mardešić and Segal [MS63] prompted us to pay particular attention to the notion of a cyclic space. This notion appears to go back at least to the early works of Whyburn on Peano continua, carried out in the mid-1920s. Whyburn defined a Peano continuum C to be cyclic if, for any two points of C, there exists a simple closed curve (that is, a “topological circle”) in C containing them. In one of his early papers, he proved that a Peano continuum is cyclic if and only if it contains no cut point, i.e., no point whose removal disconnects the space (see Theorem 1 of [Why27]). This result sheds light on the current definition of a cyclic space.

Definition 4.1. A topological space X is said to be *cyclic* if, for every point $x \in X$, the space $X \setminus \{x\}$ is connected.

Proposition 4.3. *The property of being a cyclic Peano continuum (denoted by $\mathcal{CP}$) is Π^0_3-complete.*

Proof. Let $\mathcal{B}$ be a countable basis of open sets of the Hilbert cube. We define :

$$\mathcal{E} := \{(B_1, B_2) \in \mathcal{B}^2 \mid \overline{B_1} \cap \overline{B_2} = \emptyset\}$$

$$\mathcal{M}(B_1, B_2) := \left\{ \begin{matrix} C \in \mathcal{C} \mid \exists C_1, C_2 \in \mathcal{C} \text{ such that } C_1, C_2 \subset C, C_1 \cap C_2 \subset B_1 \cup B_2, \\ C_i \cap B_k \neq \emptyset \text{ for } i,k \in \{1,2\} \end{matrix} \right\}.$$

We shall begin by proving that

$$\mathcal{CP} = \mathcal{P} \cap \left(\bigcap_{(B_1,B_2)\in\mathcal{E}} ([B_1] \cap [B_2])^c \cup \mathcal{M}(B_1, B_2) \right) =: \mathcal{F}.$$

Let C be a cyclic Peano continuum, and let $(B_1, B_2) \in \mathcal{E}$ such that $C \cap B_1 \neq \emptyset$ and $C \cap B_2 \neq \emptyset$. Choose two points x_1 and x_2, respectively in $C \cap B_1$ and $C \cap B_2$. Since C is a cyclic Peano continuum, there exists a closed arc containing x_1 and x_2. We can then deduce that there exist two arcs, C_1 and C_2, with endpoints x_1 and x_2 and intersecting only at these points. Consequently, $C \in \mathcal{M}(B_1, B_2)$. Thus, we obtain that $\mathcal{CP} \subset \mathcal{F}$.

Let C be an element of $\mathcal{F}$ and x a point of C. Consider two distinct points x_1 and x_2 in $C \setminus x$. There then exist two disjoint connected neighborhoods, V_{x_1} and V_{x_2}, of x_1 and x_2 respectively, contained in $C \setminus x$. Choose two basis open sets, $B_1 \subseteq V_{x_1}$ and $B_2 \subseteq V_{x_2}$, containing x_1 and x_2, respectively. Since $C \in \mathcal{F}$, there exist two continua C_1 and C_2 contained in C such that their intersection is included in the union of B_1 and B_2, and each intersects both B_1 and B_2. At least one of C_1 or C_2 does not contain x, say C_1. We then have that $V_{x_1} \cup V_{x_2} \cup C_1$ is connected. It follows that $C \setminus x$ is connected, that is, C is a cyclic Peano continuum.

We shall now show that $\mathcal{F}$ has complexity Π^0_3. To this end, we introduce the following notations:

$$\mathcal{R}(B_1, B_2) := \mathcal{C} \times (\mathcal{C} \cap [B_1] \cap [B_2])^2$$

$$\mathcal{L}(B_1, B_2) := \mathcal{C} \times \{(C_1, C_2) \in \mathcal{C}^2 \mid C_1 \cap C_2 \subset B_1 \cup B_2\}$$

$$\mathcal{N} := \{(C, C_1, C_2) \in \mathcal{C}^3 \mid C_1 \cap C^c \neq \emptyset \text{ or } C_2 \cap C^c \neq \emptyset\}$$

where $(B_1, B_2) \in \mathcal{E}$.

We can observe that $\mathcal{R}(B_1, B_2)$, $\mathcal{L}(B_1, B_2)$, and $\mathcal{N}$ are open subsets of $\mathcal{C}^3$. By combining these observations, we deduce that the intersection

$$\mathcal{M}'(B_1, B_2) := \mathcal{R}(B_1, B_2) \cap \mathcal{L}(B_1, B_2) \cap \mathcal{N}^c$$

is a set of class Σ^0_2. Since the projection π_1 from $\mathcal{C}^3$ to $\mathcal{C}$ is a continuous map from a compact space into a Hausdorff space, we obtain that

$$\pi_1(\mathcal{M}'(B_1, B_2)) = \mathcal{M}(B_1, B_2)$$

is of complexity Σ^0_2.

Since $\mathcal{P}$ has complexity Π^0_3, we can conclude that $\mathcal{F}$ is also Π^0_3.

Since $\mathcal{CP}$ is a class of Peano continua, closed under homeomorphism, and contains a continuum admitting an open free arc (for instance S^1), it follows from Theorem 7.3 of [CDM05] that $\mathcal{CP}$ is Π^0_3-hard.

□

4.3 The disk

The article [MS63] by Mardešić and Segal also provides a characterization of the disk.

Theorem 4.2 (Mardešić–Segal [MS63])**.** *A topological space X is homeomorphic to $\mathbb{D}_2$ if and only if X is a cyclic Peano continuum that is $\mathbb{D}_2$-like*

Combining Theorem 3.2 from [CDM05] with Proposition 4.3, we obtain the following corollary.

Corollary 4.2. *Being homeomorphic to the disk is Π^0_3-complete.*

Proof. Since $\mathcal{H}(\mathbb{D}_2) = \mathcal{CP} \cap \mathcal{L}_{\mathbb{D}_2}$, it follows that $\mathcal{H}(\mathbb{D}_2)$ is Π^0_3. Moreover, by Proposition 4.1, $\mathcal{H}(\mathbb{D}_2)$ is also Π^0_3-hard. □

The following lemma revisits a classical result from descriptive set theory, reformulating it in a form adapted to the proof of the proposition that follows. We first introduce the following set.

$$H = \left\{ M \in 2^{\mathbb{N}^2} : \text{at most one row of } M \text{ contains infinitely many 1's} \right\}.$$

Lemma 4.1. *Let $A \subseteq 2^{\mathbb{N}}$ be a Σ^0_3 set. Then there exists a continuous map $g : 2^{\mathbb{N}} \to H$ such that, for every $u \in 2^{\mathbb{N}}$,*

$$u \in A \quad \Longleftrightarrow \quad \textit{for some } n \in \mathbb{N}, \textit{ the } n\textit{-th row of } g(u) \textit{ contains infinitely many } 1\textit{'s.}$$

Proof. According to Proposition 22.16 of Kechris [Kec95], there exists a set

$$R \subseteq 2^{\mathbb{N}} \times \mathbb{N}$$

of complexity Π^0_2 such that the family $\big(R(\,\cdot\,, n)\big)_{n \in \mathbb{N}}$ forms a partition of A, where $R(\,\cdot\,, n)$ denotes the following subset of A:

$$\{\, u \in 2^{\mathbb{N}} \mid (u, n) \in R \,\}.$$

Since the problem

$$N_2 := \{\, u \in 2^{\mathbb{N}} \mid u \text{ contains infinitely many 1's} \,\}$$

is Π^0_2-complete, it follows that there exists a continuous map

$$r : 2^{\mathbb{N}} \times \mathbb{N} \longrightarrow 2^{\mathbb{N}}$$

such that, for all $u \in 2^{\mathbb{N}}$ and all $n \in \mathbb{N}$,

$$(u, n) \in R \quad \Longleftrightarrow \quad r(u, n) \text{ contains infinitely many 1's.}$$

By currying r, we obtain a continuous map

$$g : 2^{\mathbb{N}} \longrightarrow (2^{\mathbb{N}})^{\mathbb{N}} \cong 2^{\mathbb{N}^2},$$

which assigns to each $u \in 2^{\mathbb{N}}$ the matrix

$$g(u) := \big(r(u, n)\big)_{n \in \mathbb{N}}$$

□

Proposition 4.4. *The disk is Π^0_3-hard to separate from two adjacent disks sharing a boundary point.*

Proof. The proof consists in constructing a continuous map

$$h : 2^{\mathbb{N}^2} \longrightarrow \mathcal{K}(I^2)$$

such that, for every u, the compactum $h(u)$ is either homeomorphic to a disk or to two adjacent disks sharing a boundary point, and such that the preimage

$$h^{-1}\big(\mathcal{H}(\mathbb{D}_2)\big)$$

is Π^0_3-hard.

We begin by constructing a continuous map

$$f : 2^{\mathbb{N}^2} \longrightarrow \mathcal{K}(I^2).$$

For each matrix $u \in 2^{\mathbb{N}^2}$, we define a compact set $f(u) \subseteq I^2$ as the union of the compact sets associated with each row u^n of u.

We begin by partitioning I^2 into vertical strips, within which we shall construct the compact sets associated with each row u^n. Let

$$a_n := 1 - 2^{-n} \quad (n \geq 0),$$

and

$$d_n := a_{n+1} - a_n = 2^{-n-1}.$$

Thus, the vertical strip is given by

$$R_n = [a_n, a_{n+1}] \times [a_n, 1].$$

The compact set associated with the row u^n will be a subset of R_n. We now introduce parameters that will allow us to locate points within R_n. For each n, we define two real sequences:

- horizontally:
$$b_i^n := a_n + \frac{d_n}{2} a_i \qquad (i \in \mathbb{N}),$$
- vertically:
$$c_i^n := a_n + (1 - a_n) a_i \qquad (i \in \mathbb{N}).$$

The sequence (b_i^n) increases from a_n toward $a_n + \frac{d_n}{2}$, whereas the sequence (c_i^n) increases from a_n toward 1.

We now define the points that will be used to encode the boundary of the compact set associated with the line u^n. We first define a sequence of ordinates $(y_i^n)_{i \in \mathbb{N}}$ in the strip R_n by:

- initial value:
$$y_0^n := a_n;$$
- recurrence:
$$y_{i+1}^n := u_i^n c_{i+1}^n + (1 - u_i^n) y_i^n.$$

We then set

$$p_i^n := (b_i^n, y_i^n) \qquad (i \in \mathbb{N}).$$

Thus, if $u_i^n = 1$, we move vertically up to the level c_{i+1}^n; if $u_i^n = 0$, we keep the same height as the previous point. The segments joining p_i^n to p_{i+1}^n then define a polygonal path in R_n.

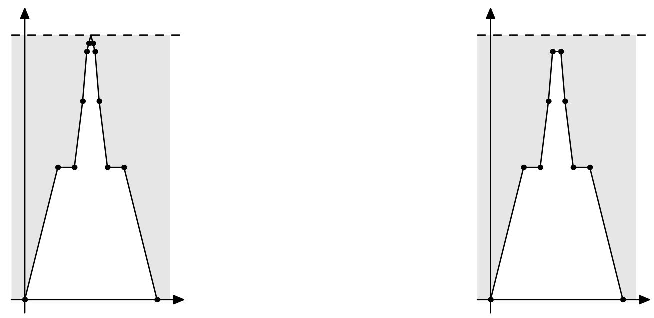
For each n, we define a symmetric copy (q_i^n) of the points (p_i^n) by reflection with respect to the vertical axis

$$x = a_n + \frac{d_n}{2}.$$

The union of the polygonal path (p_i^n) with its symmetric copy (q_i^n) forms an "arch" in the strip R_n.

We then set

$$K_n(u) := \{(x, y) \in R_n \mid (x, y) \text{ lies above the polygonal path}\}.$$



(a) When u^0 has infinitely many 1's (b) When $u^0 = 10110^\omega$

Figure 1: The arc corresponding to the row u^0

Finally, we define

$$f(u) := \bigcup_{n\in\mathbb{N}} K_n(u).$$

Observe that this arch reaches the top of the strip R_n if and only if the sequence u^n contains infinitely many 1's. Otherwise, if u^n contains only finitely many 1's, the arch stops at the level c_i^n, where i is the last index such that $u_i^n = 1$.

Suppose that the matrices u and u' coincide on the initial $N \times N$ square, that is,

$$u(n,i) = u'(n,i) \quad \text{for all } n < N \text{ and } i < N.$$

Then the Hausdorff distance between $f(u)$ and $f(u')$, with respect to the Euclidean metric on I^2, is bounded above by a quantity of order 2^{-N}.

The restriction $f_{\restriction H}$ of f to H defines a continuous map such that, for every $u \in H$, the compactum $f(u)$ is either homeomorphic to a disk or to two adjacent disks sharing a boundary point. Moreover, we have

$$f_{\restriction H}^{-1}\big(\mathcal{H}(\mathbb{D}_2)\big) = \Big\{u \in 2^{\mathbb{N}^2} \;\Big|\; \text{each row of } u \text{ contains only finitely many 1's}\Big\} \subset H.$$

Since the set

$$S_3 = \Big\{u \in 2^{\mathbb{N}^2} \;\Big|\; \text{there exists a row of } u \text{ containing infinitely many 1's}\Big\}$$

is Σ_3^0-complete, Lemma 4.1 ensures the existence of a continuous map

$$g : 2^{\mathbb{N}} \longrightarrow H$$

such that, for every $u \in 2^{\mathbb{N}}$, u has at least one row containing infinitely many 1's if and only if $g(u)$ has a unique row containing infinitely many 1's.

We conclude by setting $h := g \circ f$. □

The two previous results immediately lead to the following.

Corollary 4.3. *Being homeomorphism to the disk is Π_3^0-complete among the finitely triangulable continua.*

5 Higher complexity levels

There exist continua for which the homeomorphism problem has arbitrarily high Borel complexity. However, as there are countably many finitely triangulable continua, their complexity levels are bounded, i.e. there exists a countable ordinal α such that $\mathcal{H}(X) \in \Pi^0_\alpha$ for every finitely triangulable continuum. It is therefore natural to ask for the optimal value of α. We show that it is at least 4, by identifying two spaces that are Π^0_4-hard: the product of the triod with $[0,1]$, and the 3-dimensional ball.

In our proof, the reductions of a Π^0_4-complete subset of the Cantor space to X produce either a copy of $\mathcal{H}(X)$, or a set which is not an ANR. This kind of construction cannot be found for higher complexity levels, because Dobrowolski and Rubin [DR94] proved that the set of ANRs is Π^0_4-complete. Therefore, if a finite simplicial complex is Γ-hard for a complexity level Γ which is not contained in Π^0_4, then there is a reduction that only produces ANRs.

5.1 The cylinder of the triod

Let X be the product of the triod with $[0,1]$, shown in Figure 2.

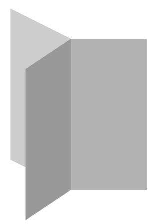


Figure 2: The product of the triod with $[0,1]$

Theorem 5.1. *The set $\mathcal{H}(X)$ is Π^0_4-hard.*

Proof. We reduce the following Π^0_4-complete problem P to $\mathcal{H}(X)$. P is a subset of the Cantor space. We interpret an element of the Cantor space as a sequence of matrices $M := (M^n)_{n\in\mathbb{N}}$ with binary coefficients $(M^n_{i,j})_{i,j\in\mathbb{N}}$. We let $M \in P$ if and only if for every $n \in \mathbb{N}$, almost every row of M^n contains a 1, i.e. there exists i_0 such that for all $i \geq i_0$, there exists j such that $M^n_{i,j} = 1$. By standard arguments [Kec95, Exercise 23.3], P is a Π^0_4-complete set.

Let M be such a sequence of matrices. Our first goal is to define a set $V \subseteq [0,2] \times [0,3]$ which is homeomorphic to the disk if and only if $M \in P$.

We define

$$E_n = \{i \in \mathbb{N} : \forall j \in \mathbb{N}, M^n_{i,j} = 0\},$$
$$V_n = \left[0, \tfrac{1}{2}\right] \cup \bigcup_{i\in E_n} \left[\tfrac{1}{2} + \tfrac{1}{2i+4}, \tfrac{1}{2} + \tfrac{1}{2i+3}\right],$$

illustrated in Figure 3.

Observe that V_n is a union of intervals, which is finite if and only if E_n is. In particular, if E_n is infinite, then V_n is not locally connected.

(a) When E_n is infinite (b) When E_n is finite

Figure 3: The set $V_n \times [0, \epsilon]$

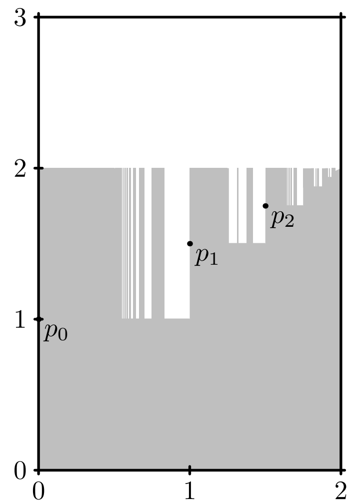


Figure 4: The set V

We then define a closed set $V \subseteq [0,2] \times [0,3]$ obtained by taking scaling down each $V_n \times [0,1]$ and combining them as in Figure 4.

Precisely, let

$$p_n = (a_n, b_n) := (2 - 2^{-n+1}, 2 - 2^{-n})$$
$$V = \bigcup_n (p_n + 2^{-n}(V_n \times [0,1])) \cup ([a_n, a_n + 2^{-n}] \times [0, b_n]).$$

Claim 1. V is homeomorphic to the disk if and only if M satisfies property P.

Proof of the claim. If $M \in P$, then each E_n is finite so it is visually clear that V is indeed a topological disk. If $M \notin P$, then some E_n is infinite so V_n is not locally connected, as well as V. □

In the Hausdorff metric, V does not depend continuously on the input M because neither do the sets E_n. However, it is possible to define a function $f : [0,2] \times [0,3] \to [0,1]$ that continuously depend on M, and whose zero set is V.

First, the complement of V_n in $[0,1)$ is U_n, defined as follows:

$$U_{n,i,j} = \left(\tfrac{1}{2} + \tfrac{1}{2i+3+2M^n_{i,j}}, \tfrac{1}{2} + \tfrac{1}{2i+2}\right),$$
$$U_n = \bigcup_{i,j} U_{n,i,j}.$$

Claim 2. U_n is the complement of V_n in $[0,1)$.

Proof. We first show that U_n and V_n are disjoint. Assume for a contradiction that for some $i,j \in \mathbb{N}$ and $k \in E_n$, one has

$$\left(\tfrac{1}{2i+3+2M^n_{i,j}}, \tfrac{1}{2i+2}\right) \cap \left[\tfrac{1}{2k+4}, \tfrac{1}{2k+3}\right] \neq \emptyset.$$

(for simplicity, we have remove the term $\frac{1}{2}$ everywhere).

It implies that $\frac{1}{2i+3+2M^n_{i,j}} < \frac{1}{2k+3}$ so $i+M^n_{i,j} > k$, and that $\frac{1}{2i+2} > \frac{1}{2k+4}$ so $i \leq k$. Therefore $i = k$ and $M^n_{i,j} = 1$, contradicting $k \in E_n$.

We next show that $U_n \cup V_n$ contains $(0,1)$. Let $x \in (0,1)$. If $x \leq \frac{1}{2}$, then $x \in V_n$. Now assume that $x > \frac{1}{2}$ and let $y = x - \frac{1}{2}$. Let $i \in \mathbb{N}$ be such that $\frac{1}{2i+4} \leq y < \frac{1}{2i+2}$. There are three cases:

- If $y > \frac{1}{2i+3}$, then for any j, $y \in (\frac{1}{2i+3+2M^n_{i,j}}, \frac{1}{2i+2})$ so $x \in U_n$,
- If $y \leq \frac{1}{2i+3}$ and there exists j such that $M^n_{i,j} = 1$, then $y \in (\frac{1}{2i+3+2M^n_{i,j}}, \frac{1}{2i+2})$ so $x \in U_n$,
- Otherwise, $i \in E_n$ and $y \in [\frac{1}{2i+4}, \frac{1}{2i+3}]$ so $x \in V_n$. □

Therefore, the complement of V in $[0,2] \times [0,3]$ is

$$U := ([0,2] \times (2,3]) \cup \bigcup_n (p_n + 2^{-n} U_n).$$

For n,i,j, let $f_{n,i,j}$ be the distance function to the complement of $p_n + 2^{-n} U_{n,i,j}$. Let $f_0(x,y) = \max(y-2,0)$. We define a continuous function $f_M : [0,2] \times [0,3] \to [0,+\infty)$ by

$$f_M = f_0 + \sum_{n,i,j} 2^{-n-i-j} f_{n,i,j}.$$

The zero set of f_M is the complement of U, i.e. V, and f_M depends continuously on M: there exists a constant c such that if $M^n_{i,j} = N^n_{i,j}$ for all $n,i,j \leq k$, then $d(f_M, f_N) \leq c.2^{-k}$.

Finally, let $X_M \subseteq [0,3]^2 \times [0,+\infty)$ be defined as the union of the graph of f_M with a flat rectangle, i.e.

$$X_M = \{(x, f_M(x)) : x \in [0,2] \times [0,3]\} \cup \{(x,0) : x \in [0,2] \times [0,3]\}.$$

The set X_M, illustrated in Figure 5 and viewable at [Hoy], is made of two disks attached along their common subset V. If $M \in P$, then V is a disk and X_M is homeomorphic to X. If $M \notin P$, then V is not locally connected and X_M is not homeomorphic to X because it is not locally 1-connected: let n be such that E_n is infinite, and let $x = (2 - 3.2^{-n-1}, 2)$ and observe that as V is not locally connected at x, X_M contains arbitrarily small loops around x that cannot be contracted in the 2^{-n}-neighborhood of x. □

More generally, the same construction shows that if X is a finite simplicial complex made of triangles, and such that some edge belongs to at least three triangles, then $\mathcal{H}(X)$ is Π^0_4-hard.

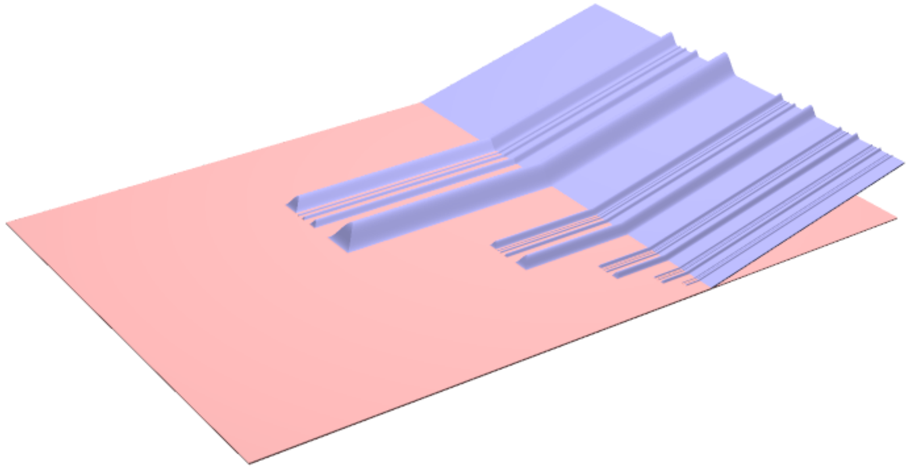


Figure 5: An illustration of the set X_M

5.2 The 3-dimensional ball

From the previous result, we can derive a lower bound on the complexity of the 3-dimensional ball.

Corollary 5.1. *Let $\mathbb{B}_3$ be the 3-dimensional ball. The set $\mathcal{H}(\mathbb{B}_3)$ is Π^0_4-hard.*

Proof. The idea is to thicken the set X_M in the proof of Theorem 5.1. More precisely, let

$$Y_M = \{(x, f(x) + y) : x \in [0,2] \times [0,3], y \in [0,1]\} \cup \{(x, -y) : x \in [0,2] \times [0,3], y \in [0,1]\}.$$

When $M \in P$, Y_M is homeomorphic to $\mathbb{B}_3$ and when $M \notin P$, X_M is not locally 1-connected so X_M is not homeomorphic to $\mathbb{B}_3$. □

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