

Borel complexity of continua

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- We study its interactions with Descriptive Set Theory.

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- Topological invariants are used to distinguish spaces: what are the “simplest” ones?
- Given a complexity level, what topological properties can be expressed?

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- If X is a continuum, then $\mathcal{H}(X) \subseteq C([0, 1]^{\mathbb{N}})$ is the set of compact sets homeomorphic to X ,
- The homeomorphism relation is not Borel, but each $\mathcal{H}(X)$ is Borel (Ryll-Nardewski 1965).

Likeness

Homeomorphism classes

First complexity levels

Higher complexity levels

Likeness

- Continua: a vast family,
- Finite simplicial complexes: easier to understand,
- So let's approximate continua by simplicial complexes.

Likeness

A notion of similarity between continua.

Definition

A continuous function $f : X \rightarrow Y$ is an ϵ -map if

$$d(x, y) > \epsilon \implies f(x) \neq f(y).$$

Intuitively, f is almost injective.

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Definition

We say that X is Y -like if for every $\epsilon > 0$, there exists a surjective ϵ -map $f : X \rightarrow Y$. We write it $X \preceq Y$.

Likeness

Examples

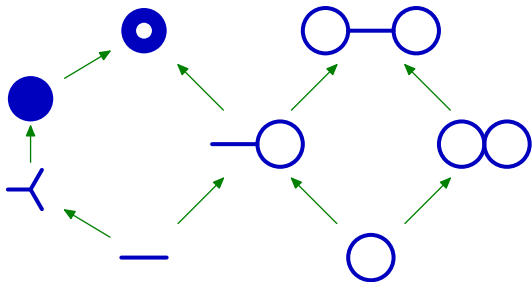


Figure: $X \rightarrow Y$ means: X is Y -like

Likeness

Inverse limits

Assume that Y is a finite simplicial complex.

Theorem (Mardešić, Segal 1963)

The following conditions are equivalent:

- X is Y -like,
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It means that

$$\dots Y \xrightarrow{f_2} Y \xrightarrow{f_1} Y \xrightarrow{f_0} Y$$

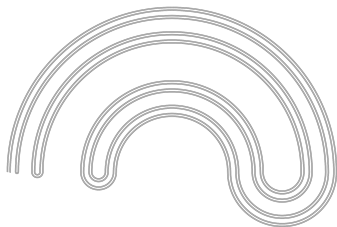
and

$$X \cong \{(y_i)_{i \in \mathbb{N}} : \forall i, f_i(y_{i+1}) = y_i\}.$$

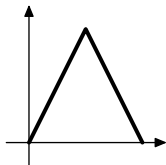
Likeness

Inverse limits

For instance, let B be the **buckethandle**:



B is $[0, 1]$ -like, and $B = \varprojlim ([0, 1], f)$ where f is the **tent map**:



Likeness

If X is Y -like, then:

- X inherits many properties from Y ,
- First step towards proving that $X \cong Y$,
- Or gives an example of some $X \not\cong Y$ that has many properties of Y .

Likeness

Preservation

If X is Y -like, then X inherits many properties of Y :

- Having dimension $\leq n$ (Alexandrov 1928),
- Having trivial Čech cohomology group (Ganea 1959),
- Indecomposability (Nadler 1971),
- Unicoherence (Nadler 1971),
- Etc.

Likeness

Complexity

Let Y be a finite simplicial complex and

$$\mathcal{L}_Y := \{X : X \text{ is } Y\text{-like}\}.$$

Theorem (Camerlo, Darji, Marcone 2005)

\mathcal{L}_Y is Π_2^0 -complete.

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Theorem (Gamard, H., Terrassin 2025)

\mathcal{L}_Y is the smallest Π_2^0 invariant property satisfied by Y .

In other words, if X is Y -like, then X inherits all the Π_2^0 invariant properties of Y .

Likeness

Complexity

Definition (Amir, H 2020)

$X \sqsubseteq Y$ if every Π_2^0 invariant property of Y is satisfied by X .

When Y is simple,

$$X \text{ is } Y\text{-like} \iff X \sqsubseteq Y$$

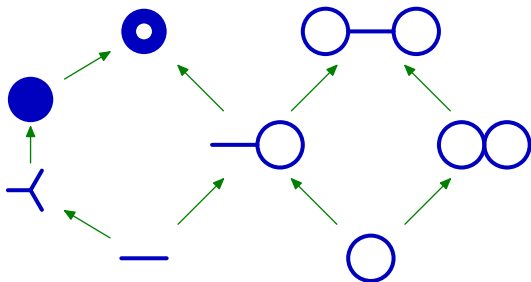


Figure: $X \rightarrow Y$ means: $X \sqsubseteq Y$ (from WDCM 2021)

Likeness

Preservation

Reminder

If X is Y -like, then X inherits many properties of Y :

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- Having trivial Čech cohomology group (Ganea 1959),
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Likeness

Preservation

Reformulation

The following invariant properties are Π_2^0 :

- Having dimension $\leq n$,
- Having trivial Čech cohomology group (Lupini, Melnikov, Nies 2023, Downey, Melnikov 2023),
- Indecomposability (Nadler 1971),
- Unicoherence (Camerlo, Darji, Marcone 2005),
- Etc.

Likeness

Homeomorphism classes

First complexity levels

Higher complexity levels

Homeomorphism classes

- Given a particular continuum X , how complex is

$$\mathcal{H}(X) = \{K \subseteq [0, 1]^{\mathbb{N}} : K \cong X\}?$$

- Given a complexity level $\Gamma \in \{\Pi_{\alpha}^0, \Sigma_{\alpha}^0\}$, for which X one has $\mathcal{H}(X) \in \Gamma$?

Likeness

Homeomorphism classes

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First level

The only Π_1^0 homeomorphism class is the singleton.

Proof.

All the classes $\mathcal{H}(X)$ are dense in $C([0, 1]^{\mathbb{N}})$ except this one. \square

Second level

There is at most one X such that $\mathcal{H}(X)$ is Π_2^0 -complete.

Proof.

If $X \neq \{0\}$, then $\mathcal{H}(X)$ is dense. If $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are Π_2^0 then they intersect by the Baire category theorem. \square

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Is there such an X ? Yes, the pseudo-arc.

Second level

Pseudo-arc

There are two continua that are homeomorphic to all of their subcontinua:

- The arc (=line segment),
- The pseudo-arc.



(a) The arc



(b) The pseudo-arc

Key references: Knaster 1922, Moise 1948, Bing 1951, Hoehn and Oversteegen 2019

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Theorem (Bing 1951, Camerlo, Darji, Marcone 2005)

$\mathcal{H}(A)$ is Π_2^0 -complete.

Second level

An unoccupied level

Theorem (Gamard, H., Terrassin 2025)

There is no continuum X such that $\mathcal{H}(X)$ is Σ_2^0 -complete.

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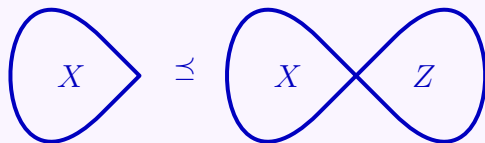
An unoccupied level

Theorem (Gamard, H., Terrassin 2025)

There is no continuum X such that $\mathcal{H}(X)$ is Σ_2^0 -complete.

Proof.

- If $X = \{x\}$, then $\mathcal{H}(X)$ is Π_1^0 -complete.
- If X is infinite, then X is Y -like for some $Y \not\cong X$ (Amir, H. 2025).



If $\mathcal{H}(X)$ is Σ_2^0 , then $Y \in \mathcal{H}(X)$, contradiction.

□

Likeness

Homeomorphism classes

First complexity levels

Higher complexity levels

Peano continua

For many simple Y 's, one has

$$X \cong Y \iff \begin{cases} X \text{ is Peano,} \\ X \text{ is } Y\text{-like,} \\ ??? \end{cases}$$

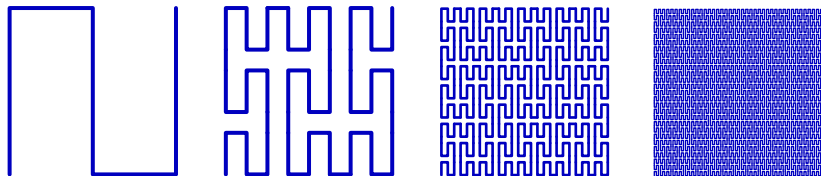
Peano continua

Definition

A continuum is **Peano** if it is locally connected.

Theorem (Hahn-Mazurkiewicz, 1910s)

X is Peano $\iff X$ has a space-filling curve.



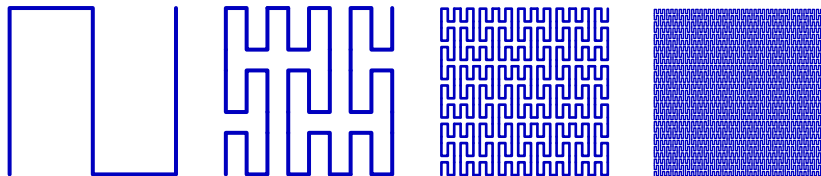
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Theorem (Kuratowski 1931, Mazurkiewicz 1931)

Being Peano is Π_3^0 -complete.

Graphs

Let G be a finite graph. There are finitely many G -like graphs $G_1, \dots, G_n \not\cong G$.

Theorem (Kato, Ye 2000)

One has

$$X \cong G \iff X \text{ is } \begin{cases} \text{Peano,} \\ G\text{-like,} \\ \text{not } G_1\text{-like, } \dots, G_n\text{-like.} \end{cases}$$

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This is optimal.

Theorem (Camerlo, Darji, Marcone 2005)

$\mathcal{H}(G)$ is Π_3^0 -complete, and $D_2(\Sigma_2^0)$ -complete among the locally connected continua.

The disk

Let \mathbb{D} be the disk.

Theorem (Mardešić, Segal 1963)

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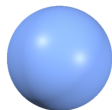
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Theorem (Gamard, H., Terrassin 2025)

$\mathcal{H}(\mathbb{D})$ is Π_3^0 -complete, even among the Peano continua.

Closed surfaces

Let S be a closed surface.

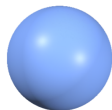


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Let X be Peano and S -like.

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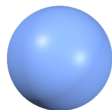
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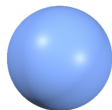
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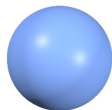
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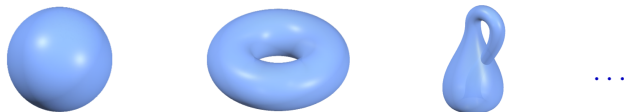
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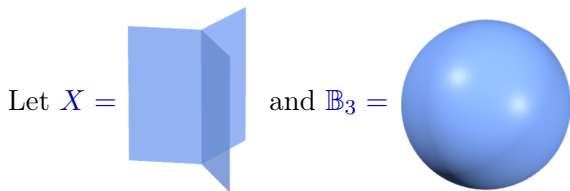
The latter is optimal.

Theorem (Gamard, H., Terrassin 2025)

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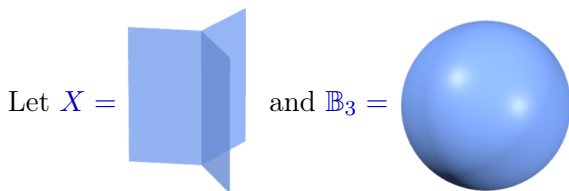
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Open question

Are they Π_4^0 ?

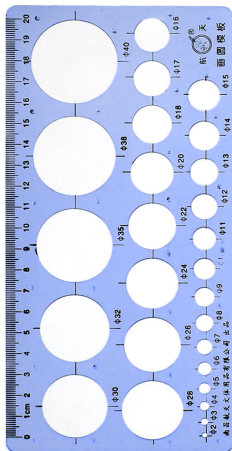
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We show that $\mathcal{H}(\mathbb{B}_3)$ is Σ_3^0 -hard, by reducing the Σ_3^0 -complete set $P \subseteq [0, 1]^{\mathbb{N}}$ of eventually positive sequences.

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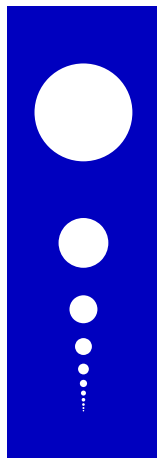
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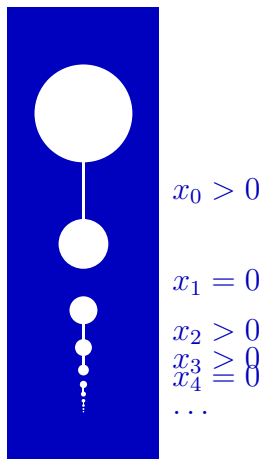
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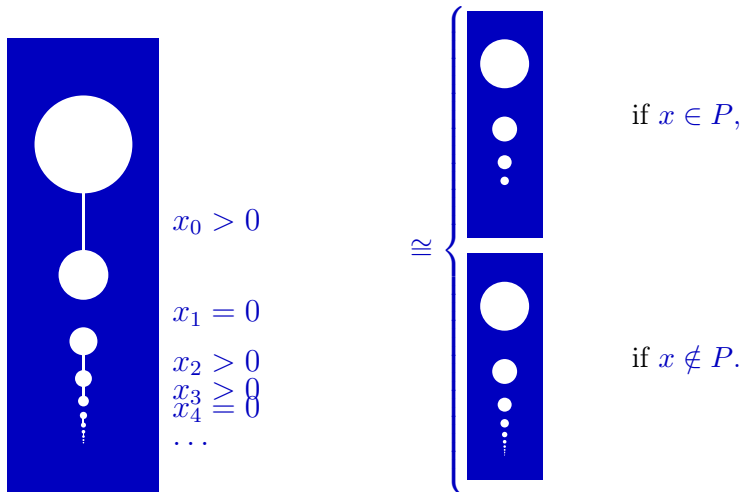
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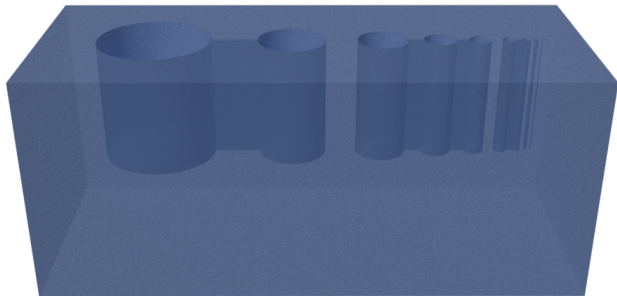
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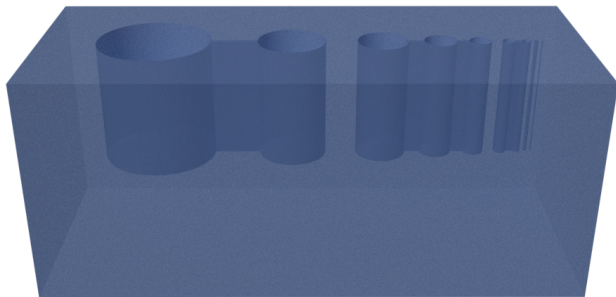
Finally, solidify:



Hence, $\mathcal{H}(\mathbb{B}_3)$ is Σ_3^0 -hard.

Higher complexity levels

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Hence, $\mathcal{H}(\mathbb{B}_3)$ is Σ_3^0 -hard.

It is Π_4^0 -hard by gluing infinitely many such building blocks.

Future directions

- Is $\mathcal{H}(\mathbb{B}_3)$ in Π_4^0 ?

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 - But being an ANR is Π_4^0 .
- Is there a continuum X such that $\mathcal{H}(X)$ is Σ_α^0 -complete?
- X is Y -like^(α) if every Π_α^0 property of Y is satisfied by X .
Is it interesting?