Local generation of languages

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Loria
Laboratoire lorrain de recherche en informatique et ses applications

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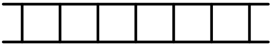
informatiques mathématiques

Motivating problem



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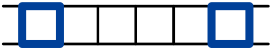

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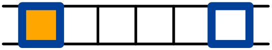

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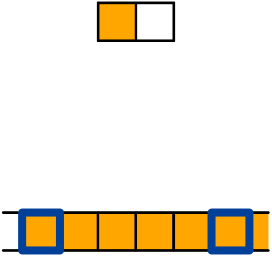

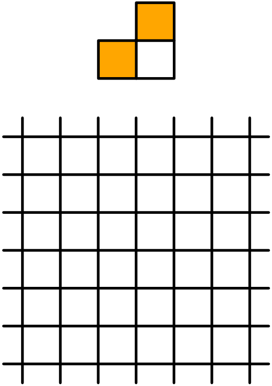

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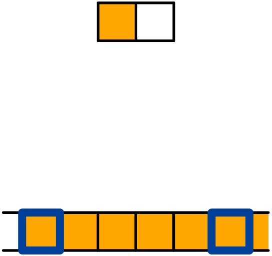

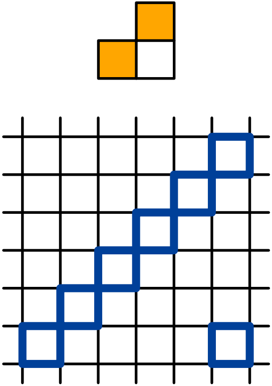

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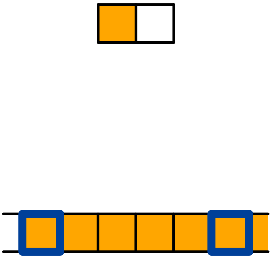

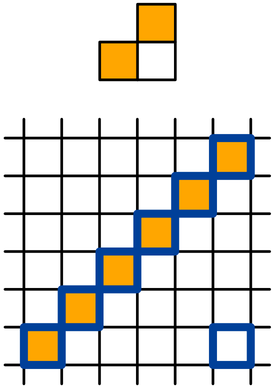

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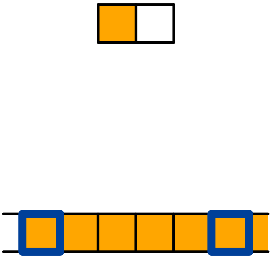

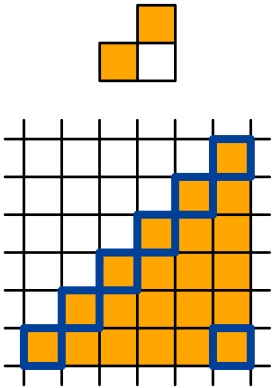

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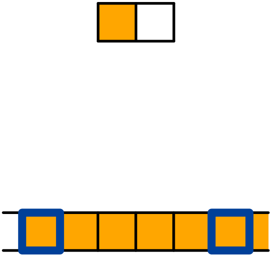

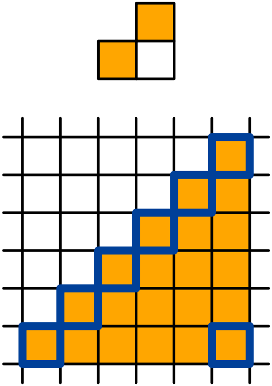

How should cells coordinate?

Coordination

- We fix a language $L \subseteq \{0,1\}^n$,
- Each position $i < n$ has to choose a value in $\{0,1\}$, so that they collectively produce any element of L,
- How should cells coordinate their choices?

Coordination: example

Let

$$L = \{000, 001, 100, 111\}.$$

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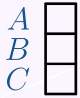
Generation procedure



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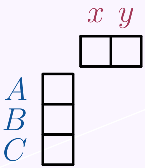

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$$B = \min(x, y)$$
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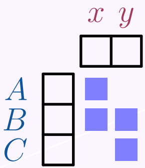

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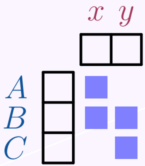

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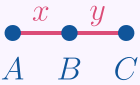
Coordination structure



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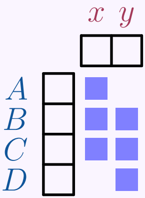

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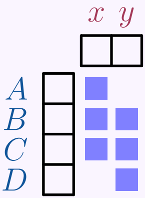

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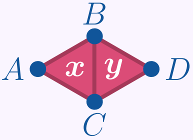

Language generation

Let $f : A^m \to B^n$.

Definition

The **communication complex** K_f of f is a simplicial complex:

- Vertex set $V = [0, n-1]$,
- A set $S \subseteq V$ belongs to K_f if the output cells in S depend on a common input.

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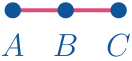
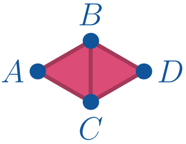


$f_1(x, y) = (x, \min(x, y), y)$



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A simplicial complex K over $[0, n-1]$ **generates** $L \subseteq B^n$ if there is a function $f : A^m \to B^n$ such that:

- $\mathrm{im}(f) = L$,
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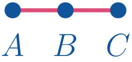

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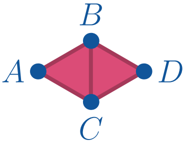

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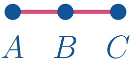
General goal

Given a language L, what are the (minimal) complexes generating L?



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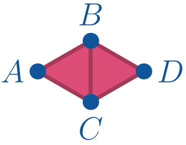

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Language generation: simple examples

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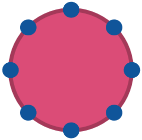

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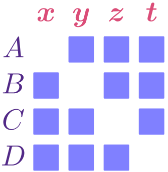
Assume otherwise: there is a function with this dependency relation:



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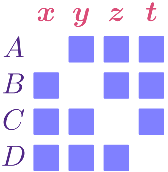

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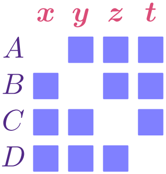

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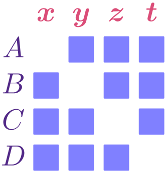

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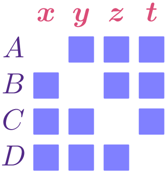

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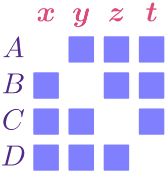

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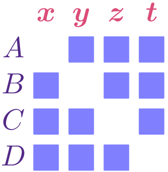

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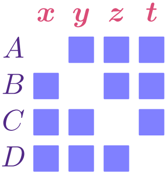

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	x	y	z	t
A		■	■	■
B	■		■	■
C	■	■		■
D	■	■	■	

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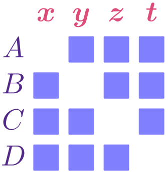

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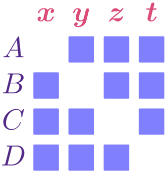

But D can take value 0: contradiction! $\square$

Simple languages

One occurrence of 1

Monotonic sequences

At most one occurrence of 1

Let $L_{\leq 1} = \{w \in \{0,1\}^n : |w|_1 \leq 1\}$.

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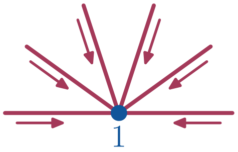

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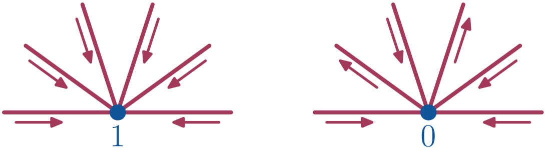

Answer

The only minimal complex generating $L_{\leq 1}$ is the **complete graph**.

At least one occurrence of 1

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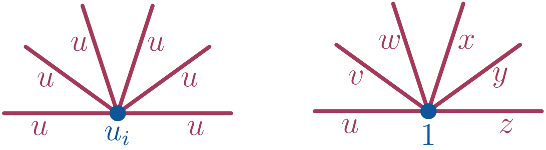
- Local rule over any tree:



where $u, v, \ldots \in L_{\geq 1}$

At least one occurrence of 1

Let $L_{\geq 1} = \{w \in \{0,1\}^n : |w|_1 \geq 1\}$.

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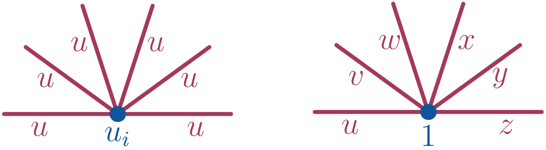

where $u, v, \ldots \in L_{\geq 1}$

Answer

The minimal complexes generating $L_{\geq 1}$ are the **trees**.

At least k occurrences of 1

Let $L_{\geq k} = \{w \in \{0,1\}^n : |w|_1 \geq k\}$.

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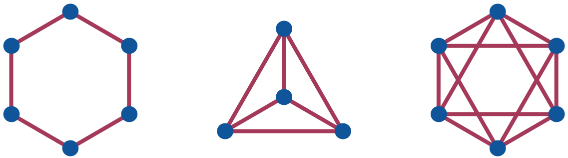
Answer

The minimal complexes generating L are the **minimal k-connected graphs**.

k-connected $=$ still connected when removing $k-1$ vertices.



(a) 2-connected (b) 3-connected (c) 4-connected

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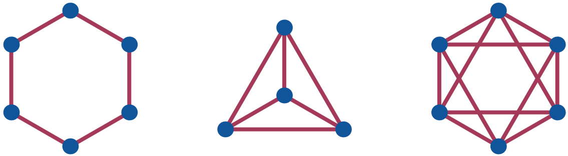

(a) 2-connected (b) 3-connected (c) 4-connected

- Erasing $k-1$ symbols, we get $L_{\geq 1}$ (of length $n-(k-1)$). It is irreducible, so the remaining complex is connected.
- Local rule over a k-connected graph: the same as for $k=1$.

Upwards closed languages

Endow $\{0,1\}^n$ with the componentwise order, and let L be upwards closed.

Answer

The minimal complexes generating L are the **minimal L-connected graphs**.

L-connected $=$ still connected when removing the vertices $\{i : w_i = 1\}$ for any maximal string $w \notin L$.

Simple languages

One occurrence of 1

Monotonic sequences

One occurrence of 1

Let $L = \{w \in \{0,1\}^n : |w|_1 = 1\}$.

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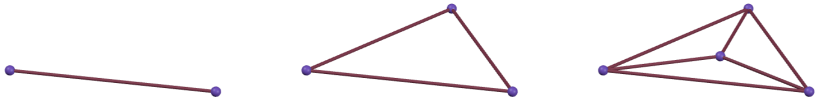
$\mathrm{Cone}(K_{n-1})$:

- Start from the complete graph K_{n-1} over $n-1$ vertices,



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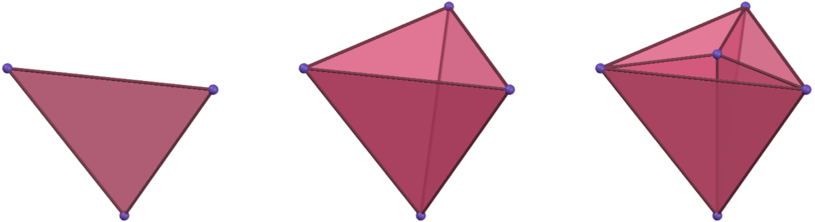
Let $L = \{w \in \{0,1\}^n : |w|_1 = 1\}$.

Theorem

The complex $\mathrm{Cone}(K_{n-1})$ *is minimal generating* L*.*

$\mathrm{Cone}(K_{n-1})$:

- Start from the complete graph K_{n-1} over $n-1$ vertices,
- Add a vertex v and a triangle joining v to each edge of K_{n-1}.



One occurrence of 1

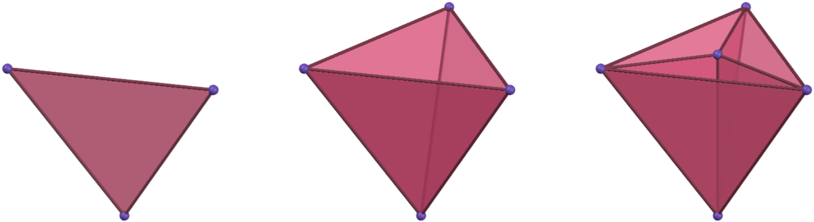
Let $L = \{w \in \{0,1\}^n : |w|_1 = 1\}$.

Theorem

The complex $\mathrm{Cone}(K_{n-1})$ *is minimal generating* L.

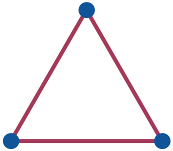
Procedure:

- The $n-1$ vertices produce at most one occurrence of 1,
- v sees everything and chooses its value accordingly.



One occurrence of 1: minimality

Assume that the hollow triangle generates $L_3 = \{100, 010, 001\}$:



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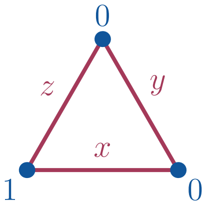

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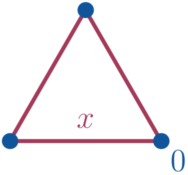

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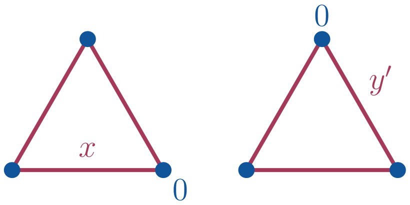

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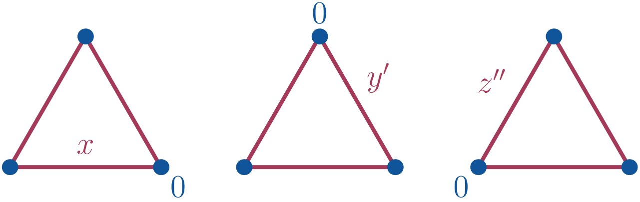

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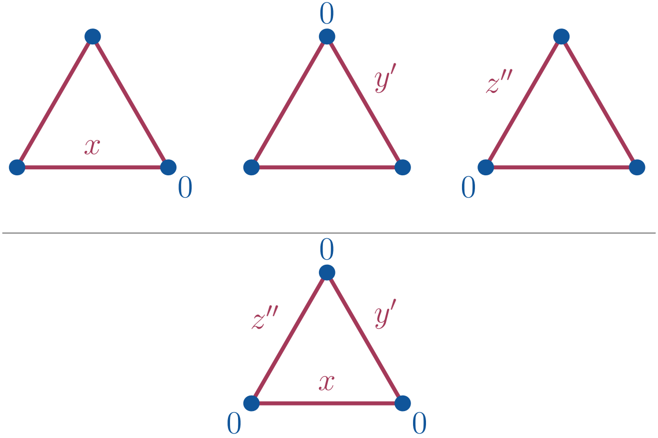

Contradiction: $000 \notin L_3$.

One occurrence of 1: minimality

Proposition

Let $n \geq 3$. Every pair $\{i, j\}$ belongs to a triangle.

One occurrence of 1: minimality

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Consider the surjective function $g : \mathsf{U}_n \to \mathsf{U}_3$:

$$g(x) = abc, \text{ where } \begin{cases} a = x_i, \\ b = x_j, \\ c = \max\{x_k : k \notin \{i, j\}\}. \end{cases}$$

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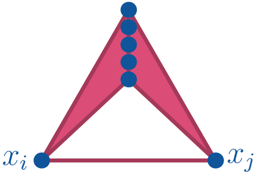
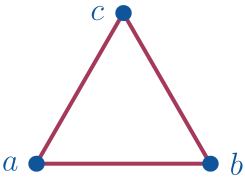
If L_n is generated by:



then L_3 is generated by:



One occurrence of 1: minimality

Proposition

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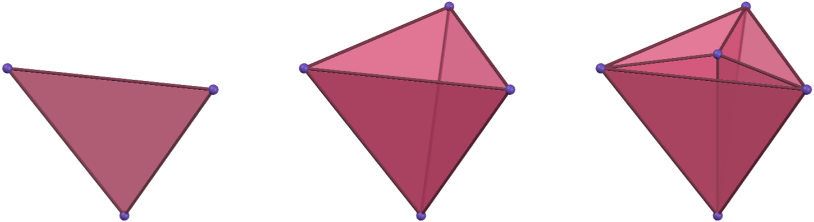
Minimality of $\mathrm{Cone}(K_{n-1})$:



Each edge in K_{n-1} belongs to a single triangle, which therefore cannot be removed.

One occurrence of 1

For $n = 3, 4, 5$, $\text{Cone}(K_{n-1})$ are the only minimal complexes generating L.

Open question

What about $n \geq 6$?

Simple languages

One occurrence of 1

Monotonic sequences

Monotonic sequences

Let

$$\begin{aligned}\mathsf{Mon}_n = \{&000\ldots000,\\ &000\ldots001,\\ &000\ldots011,\\ &\quad\ldots\\ &011\ldots111,\\ &111\ldots111,\\ &111\ldots110,\\ &111\ldots100,\\ &\quad\ldots\\ &100\ldots000\}\end{aligned}$$

Symmetries:

Monotonic sequences

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Symmetries:

- Reflection: $x_0x_1\ldots x_{n-1} \to x_{n-1}\ldots x_1x_0$,

Monotonic sequences

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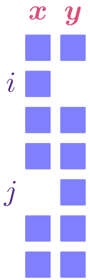
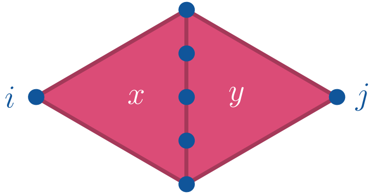
Symmetries:

- Reflection: $x_0x_1\ldots x_{n-1} \to x_{n-1}\ldots x_1x_0$,
- Rotation: $x_0x_1\ldots x_{n-1} \to x_1\ldots x_{n-1}\overline{x_0}$.

Monotonic sequences

- Two positions $i \neq j$ can be filled independently,
- So there is a generating function on this complex:





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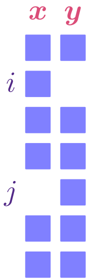

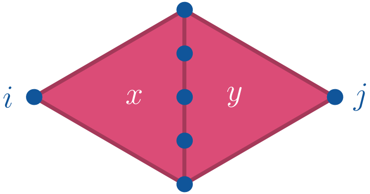

Proposition

It is minimal.

Monotonic sequences

For $n = 3, 4$, those are the only minimal complexes.
For $n = 5$, a new one appears.

Monotonic sequences

There is a correction function $f : \{0,1\}^5 \to \{0,1\}^5$.
It uses the local correction function $\mathrm{maj} : \{0,1\}^3 \to \{0,1\}$.

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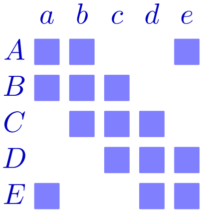
First attempt



$$A = \mathrm{maj}(\overline{e}, a, b)$$
$$B = \mathrm{maj}(a, b, c)$$
$$C = \mathrm{maj}(b, c, d)$$
$$D = \mathrm{maj}(c, d, e)$$
$$E = \mathrm{maj}(d, e, \overline{a})$$

Monotonic sequences

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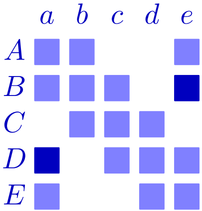
Second attempt



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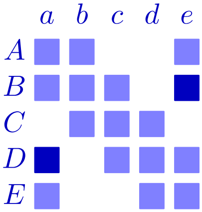

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This complex is minimal generating Mon_5.

Monotonic sequences

This complex is minimal generating Mon_7:

	a	b	d	f	g
A	■	■			■
B	■	■	■		■
C	■	■	■	■	
D		■	■	■	
E		■	■	■	■
F	■		■	■	■
G	■			■	■

$$A = \mathrm{maj}(\overline{g}, a, b)$$
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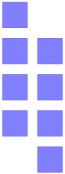
$$F = \mathrm{maj}(d, f, G)$$
$$G = \mathrm{maj}(f, g, \overline{a})$$

Monotonic sequences

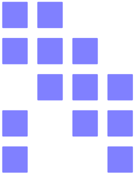
This complex is minimal generating Mon_8:

	a	b	c	d	e	f	g	h
A	■	■	■	■			■	■
B	■	■	■	■			■	■
C	■	■	■	■	■	■		
D	■	■	■	■	■	■		
E			■	■	■	■	■	■
F			■	■	■	■	■	■
G	■	■			■	■	■	■
H	■	■			■	■	■	■

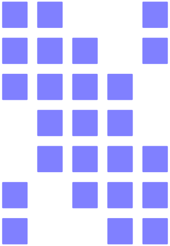
Monotonic sequences: a pattern?



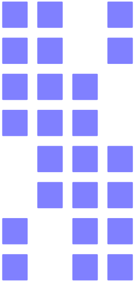
$n \geq 2$



$n = 5$

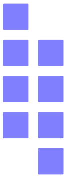


$n = 7$

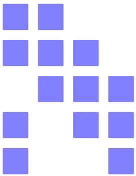


$n = 8$

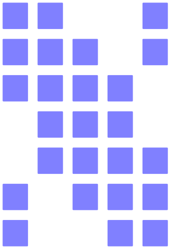
Monotonic sequences: a pattern?



$n \geq 2$



$n = 5$



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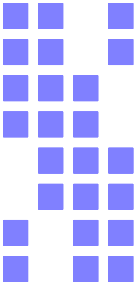


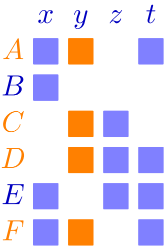
$n = 8$

Theorem

Let K be a minimal complex generating Mon_n. The maximal simplices of K are intervals.

Monotonic sequences: intervals

Assume that some maximal simplex is not an interval:



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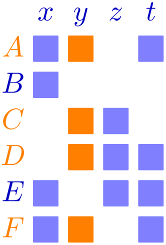
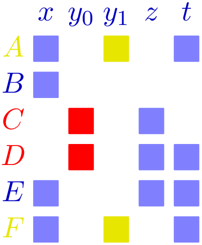


A proper subcomplex generates Mon_n:



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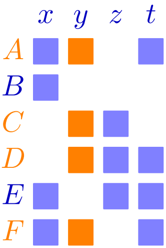

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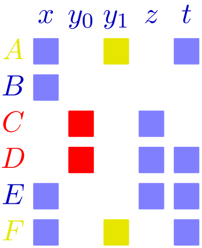
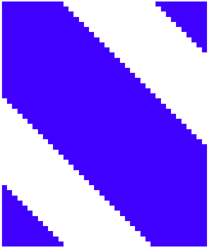


Key observation

If $w|_{[\![B,E]\!]}$ and $w|_{[\![E,B]\!]}$ are monotonic, then so is w.

Monotonic sequences: intervals

A minimal complex generating Mon_n always looks like:



Monotonic sequences: intervals

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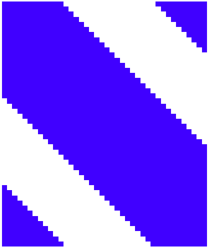


Can we say more?

Monotonic sequences: interval length

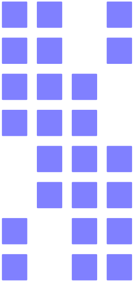
Let $\mu(n)$ be the minimal k such that Mon_n is generated by a complex whose intervals all have lengths $\leq k$.

Theorem

$$\left\lfloor \frac{3n+1}{4} \right\rfloor \leq \mu(n) \leq \left\lceil \frac{3n}{4} \right\rceil.$$

Monotonic sequences: interval length

Upper bound: $\mu(n) \leq \lceil \frac{3n}{4} \rceil$.



$n = 8$

Monotonic sequences: interval length

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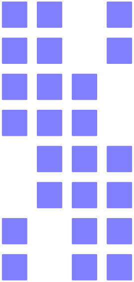
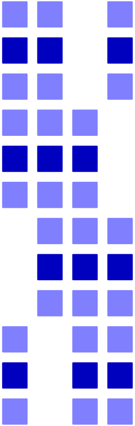


$n = 8$



$n = 12$

Etc.

Monotonic sequences: interval length

The dark side of the proof

Lower bound: $\left\lfloor \frac{3n+1}{4} \right\rfloor \leq \mu(n)$.

Lemma 0.1. *Assume that f generates Mon_n and K_f contains no k-interval. Let $b = 2(n-k)$. For $2k-n-1 \leq j \leq k-1$, there exists γ such that $f_0(\gamma) = 0$ and*

$$\mathrm{dom}(\gamma) = \mathcal{W}_f(\llbracket 0, j \rrbracket) \cup \mathcal{W}_f(\llbracket j+1, b+j+1 \rrbracket).$$

Proof. To lighten the notations, we drop the subscript f in $\mathcal{W}_f$. We prove the result by induction on j. Note that by symmetry, the statement implies more generally that for any $a \in I_n$, there exists γ such that $f_a(\gamma) = 0$ and $\mathrm{dom}(\gamma) = \mathcal{W}(\llbracket a, a+j \rrbracket) \cup \mathcal{W}(\llbracket a+j+1, a+b+j+1 \rrbracket)$. This more general statement will be used as induction hypothesis.

For $j = 2k-n-1$, the result is a direct application of Theorem **??**, giving $\mathrm{dom}(\gamma) = \mathcal{W}(\llbracket 0, j \rrbracket) \cup \mathcal{W}(\llbracket j+1, n \rrbracket)$, and indeed $b+j+1 = n$ (and the version for any a holds by symmetry).

We assume the result for j satisfying $2k-n-1 \leq j < k-1$ and prove it for $j+1$. Let $a = n-k$ and

$$\mathbb{I}_0 = \llbracket 0, j \rrbracket \qquad \mathbb{I}_1 = \llbracket j+1, b+j+1 \rrbracket$$
$$\mathbb{J}_0 = \llbracket a, a+j \rrbracket \qquad \mathbb{J}_1 = \llbracket a+j+1, a+b+j+1 \rrbracket.$$

From the induction hypothesis, there exist partial inputs α and β such that:

- $f_0(\alpha) = 0$ and $\mathrm{dom}(\alpha) = \mathcal{W}(\mathbb{I}_0) \cup \mathcal{W}(\mathbb{I}_1)$,
- $f_a(\beta) = 0$ on $\mathrm{dom}(\beta) = \mathcal{W}(\mathbb{J}_0) \cup \mathcal{W}(\mathbb{J}_1)$.

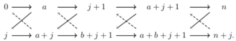
Claim 1. The domains of α and β are disjoint.

The intersection of these domains is

$$\mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta) = \mathcal{W}(\mathbb{I}_0 \cup \mathbb{J}_0) \cup \mathcal{W}(\mathbb{J}_0 \cup \mathbb{I}_1) \cup \mathcal{W}(\mathbb{I}_1 \cup \mathbb{J}_1) \cup \mathcal{W}(\mathbb{J}_1 \cup \mathbb{I}_0).$$

We prove that it is empty by showing that each union $\mathbb{I}_u \cup \mathbb{J}_v$ is an interval of length at least k.

We check that $\mathbb{I}_0 \angle \mathbb{J}_0 \angle \mathbb{I}_1 \angle \mathbb{J}_1 \angle \mathbb{I}_0$ (illustrated in Figure **??**). Indeed, each pair of consecutive intervals can be expressed as $\llbracket x, z \rrbracket \angle \llbracket y, t \rrbracket$ where $x \leq y \leq z \leq t < x+n$ (Lemma **??**), as summarized by the following diagram (see paragraph after Lemma **??**):



These inequalities all follow from:

$$k < n,$$
$$2k-n-1 \leq j < k-1,$$
$$2n+1 \leq 3k.$$

Therefore, the consecutive unions of intervals are intervals:

- $\mathbb{I}_0 \cup \mathbb{J}_0 = \llbracket 0, a+j \rrbracket$ has size $a+j+1 \in [k, n]$,
- $\mathbb{J}_0 \cup \mathbb{I}_1 = \llbracket a, b+j+1 \rrbracket$ has size $b+j+2-a \in [k, n]$,
- $\mathbb{I}_1 \cup \mathbb{J}_1 = \llbracket j+1, a+b+j+1 \rrbracket$ has size $a+b+1 \in [k, n]$,
- $\mathbb{J}_1 \cup \mathbb{I}_0 = \llbracket a+j+1, n+j \rrbracket$ has size $n-a = k \in [k, n]$.

As a result, K does not contain any $\mathbb{I}_u \cup \mathbb{J}_v$, which means that $\mathcal{W}(\mathbb{I}_u \cup \mathbb{J}_v)$ is empty. Therefore, α and β have disjoint domains and the claim is proved.

This claim implies that α and β are compatible, so we can apply Lemma **??**. Let $c = n-1 \in \llbracket a, 0 \rrbracket$. One has $f_c(\gamma) = 0$ where

$$\gamma = \alpha|_{\mathcal{W}(\llbracket c,0 \rrbracket)} \cup \beta|_{\mathcal{W}(\llbracket a,c \rrbracket)}.$$

The interval $\llbracket a, c \rrbracket$ has size $c+1-a = n-a = k$ so $\mathcal{W}(\llbracket a, c \rrbracket)$ is empty. Therefore, $\gamma = \alpha|_{\mathcal{W}(\llbracket c,0 \rrbracket)}$ has domain

$$\begin{aligned} \mathrm{dom}(\gamma) &= \mathrm{dom}(\alpha) \cap \mathcal{W}(\llbracket c, 0 \rrbracket) \\ &= \big(\mathcal{W}(\llbracket 0, j \rrbracket) \cup \mathcal{W}(\llbracket j+1, b+j+1 \rrbracket)\big) \cap \mathcal{W}(\llbracket c, 0 \rrbracket) \\ &= \mathcal{W}(\llbracket n-1, j \rrbracket) \cup \mathcal{W}(\llbracket j+1, b+j+1 \rrbracket). \end{aligned}$$

By symmetry, we can apply the circular permutation $x \mapsto x+1 \bmod n$, implying that there exists a partial input δ such that $f_0(\delta) = 0$ and $\mathrm{dom}(\delta) = \mathcal{W}(\llbracket 0, j+1 \rrbracket) \cup \mathcal{W}(\llbracket j+2, b+j+2 \rrbracket)$ which proves the induction step. □

Proof. Let

$$\mathbb{I} = \llbracket k, 2n-k \rrbracket.$$

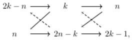
We apply Lemma 0.1 to $j = k-1$, giving γ such that $f_0(\gamma) = 0$ and $\mathrm{dom}(\gamma) = \mathcal{W}(\llbracket 0, k-1 \rrbracket) \cup \mathcal{W}(\llbracket k, 2n-k \rrbracket)$. As $\llbracket 0, k-1 \rrbracket$ has size k, $\mathcal{W}(\llbracket 0, k-1 \rrbracket)$ is empty so $\mathrm{dom}(\gamma) = \mathcal{W}(\mathbb{I})$.

Let

$$\mathbb{J}_0 = \llbracket 2k-n, 0 \rrbracket$$
$$\mathbb{J}_1 = \llbracket 0, 2k-n-1 \rrbracket = \llbracket 0, 2k-1 \rrbracket$$

By Theorem **??**, there exists α such that $f_0(\alpha) = 1$ and $\mathrm{dom}(\alpha) = \mathcal{W}(\llbracket 2k-n, n \rrbracket) \cup \mathcal{W}(\llbracket 0, 2k-n-1 \rrbracket) = \mathcal{W}(\mathbb{J}_0) \cup \mathcal{W}(\mathbb{J}_1)$.

We show that α and γ have disjoint domains. One has $\mathbb{J}_0 \angle \mathbb{I} \angle \mathbb{J}_1$, because the endpoints of these intervals satisfy the inequalities summarized by the following diagram,



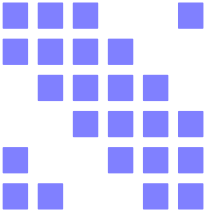
which all follow from $k \leq n$ and $3k \geq 2n+1$. Therefore,

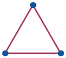
- $\mathbb{J}_0 \cup \mathbb{I} = \llbracket 2k-n, 2n-k \rrbracket$ has size $3(n-k)+1 \in [k, n]$,
- $\mathbb{I} \cup \mathbb{J}_1 = \llbracket k, 2k-1 \rrbracket$ has size k.

As a result, α and γ have disjoint domains so they are compatible, but they give opposite values to 0, which is a contradiction. □

Some open questions

- What are the complexes generating $\{x \in \{0,1\}^n : |x|_1 = 1\}$?
- Does this complex generate Mon_6?

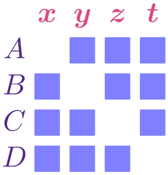


- What are the languages $L \subseteq A^3$ generated by  ?

Reminder

- Let $L = \{0^k 1^{n-k} : 0 \leq k \leq n\}$ (non-decreasing sequences). The only complex is the full complex.

Assume otherwise: there is a function following this scheme:



$$
\begin{aligned}
& f(x_1, y_1, z_1, t_1) = 1111 \\
\Longrightarrow\ & A(x_1, y_1, z_1, t_1) = 1 \\
\Longrightarrow\ & A(\_, y_1, z_1, t_1) = 1 \\
\Longrightarrow\ & B(\_, y_1, z_1, t_1) = 1 \\
\Longrightarrow\ & B(\_, \_, z_1, t_1) = 1 \\
\Longrightarrow\ & C(\_, \_, z_1, t_1) = 1 \\
\Longrightarrow\ & C(\_, \_, \_, t_1) = 1 \\
\Longrightarrow\ & D(\_, \_, \_, t_1) = 1 \\
\Longrightarrow\ & D(\_, \_, \_, \_) = 1
\end{aligned}
$$

But D can take value 0: contradiction!

Inference system

Language axioms. For every $w \in L$,

$$\frac{}{f(x_w) \succeq w}$$

Language rules. If L respects the rule $q \to r$:

$$\frac{f(p) \succeq q}{f(p) \succeq r}$$

Restriction rule.

$$\frac{f(p) \succeq q}{f(p|_{\mathcal{W}(q)}) \succeq q}$$

Join rule. If p_1, p_2 are compatible, and q_1, q_2 are compatible:

$$\frac{f(p_1) \succeq q_1 \qquad f(p_2) \succeq q_2}{f(p_1 \cup p_2) \succeq q_1 \cup q_2}$$

Conflict rule. If p_1, p_2 are compatible, and q_1, q_2 are incompatible:

$$\frac{f(p_1) \succeq q_1 \qquad f(p_2) \succeq q_2}{\bot}$$