An Application of Martin-Löf Randomness to Effective Probability Theory

Mathieu Hoyrup and Cristóbal Rojas

Abstract

In this paper we provide a framework for computable analysis of measure, probability and integration theories. We work on computable metric spaces with computable Borel probability measures. We introduce and study the framework of *layerwise computability* which lies on Martin-Löf randomness and the existence of a universal randomness test. We then prove characterizations of effective measure and integration notions in terms of layerwise computability. On the one hand it gives a simple way of handling effective measure theory, on the other hand it provides powerful tools to study Martin-Löf randomness.

Keywords. Algorithmic randomness, universal test, computable analysis, effective probability theory, Lebesgue integration, layerwise computability.

Contents

1 Introduction

Computable analysis is mainly focused on topological spaces. This can be observed in the main two frameworks of computable analysis: domain theory reduces every infinite computation to a convergence process in the Scott topology (see [EH98] e.g.) while in the theory of representations (see [Wei00] e.g.), the criterion for a representation to be acceptable (the technical term is *admissible*) is that it is equivalent to the representation induced by the topology of the space (the *standard representation*).

Computable analysis for measurable spaces and probability spaces has been much less investigated. An effective presentation of measurable spaces is proposed in [WD06]. Computability on L^p-spaces has been studied in [ZZ99], [Kun04], both for euclidean spaces with the Lebesgue measure. Computability of measurable sets has been studied, on the real line with the Lebesgue measure in [S̃an68] and on second countable locally compact Hausdorff spaces with a computable σ-finite measure in [Eda07]. In the latter a computability framework for bounded integrable functions is also introduced, when the measure is finite. A general computable framework for integration is still lacking: nothing is developed for non locally compact spaces, or for unbounded functions on general spaces with general measures.

On the other hand, another effective approach to probability theory has already been deeply investigated, namely algorithmic randomness, as introduced by Martin-Löf in [ML66]. This theory was originally developed on the Cantor space, i.e. the space of infinite binary sequences, endowed with a computable probability measure. Since then, the theory has been mainly studied on the Cantor space from the point of view of recursion theory, focused on the interaction between randomness and reducibility degrees. The theory has been recently extended to more general spaces in [HW03, Gác05, HR09b].

In this paper, we propose a general unified framework for the computable analysis of measure and integration theory, and establish intimate relations with algorithmic randomness. We first consider two natural ways (more or less already present in the literature) of giving effective versions of the notions of *measurable set*, *measurable map* and *integrable function*.

Then we develop a third approach which we call *layerwise computability* and that, in a sense, follows the idea that probability theory could be grounded on the algorithmic theory of randomness. This new approach is based on the existence of a universal randomness test.

This fundamental result proved by Martin-Löf in his seminal paper is a peculiarity of the effective approach of mathematics, having no counterpart in the classical world. Making a systematic use of this has quite unexpected strong consequences: (i) it gives *topological* characterizations of effective measurability notions; (ii) measure-theoretic notions, usually defined almost everywhere, become set-theoretic when restricting to effective objects; (iii) the practice of these notions is rather light: most of the basic manipulations on computability notions on topological spaces can be straightforwardly transposed to effective measurability notions, by the simple insertion of the term “layerwise”. This language trick may look suspicious, but in a sense this paper provides the background for this to make sense and being practiced.

In this way, Martin-Löf randomness and the existence of a universal test find an application in computable analysis. In [HR09a] we show how this framework in turn provides powerful tools to the study of algorithmic randomness, and a general way of deriving results in the spirit of [Dav01, Dav04].

In Sect. 2 we recall the background on computable probability spaces and define the notion of *layering of the space*, which will be the cornerstone of our approach. In Sect. 3 we present two approaches to make measure-theoretical notions on computable probability space effective. Some definitions are direct adaptations of preceding works, some others are new (in particular the notions of effectively measurable maps and effectively integrable functions). In Sect. 4 we present our main contribution, namely *layerwise computability*, and state several characterizations. Being rather long, the proofs are gathered in the appendix.

2 Preliminaries

2.0.1 Computable metric space.

Let us first recall some basic results established in [Gác05, HR09b]. We work on the well-studied computable metric spaces (see [EH98], [YMT99], [Wei00], [Hem02], [BP03]).

Definition 2.0.1. A ***computable metric space*** is a triple $(X, d, \mathcal{S})$ where:

1. (X, d) is a separable metric space,
2. $\mathcal{S} = \{s_i : i \in \mathbb{N}\}$ is a countable dense subset of X with a fixed numbering,
3. $d(s_i, s_j)$ are uniformly computable real numbers.

$\mathcal{S}$ is called the set of ***ideal points***. If $x \in X$ and $r > 0$, the metric ball $B(x, r)$ is defined as $\{y \in X : d(x, y) < r\}$. The set $\mathcal{B} := \{B(s, q) : s \in \mathcal{S}, q \in \mathbb{Q}, q > 0\}$ of ***ideal balls***, which is a basis of the topology, has a canonical numbering $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$. An ***effective open set*** is an open set U such that there is a r.e. set $E \subseteq \mathbb{N}$ with $U = \bigcup_{i\in E} B_i$. If $B_i = B(s, r)$ we denote by $\overline{B}_i$ the closed ball $\overline{B}(s, r) = \{x \in X : d(x, s) \leq r\}$. The complement of $\overline{B}_i$ is effectively open, uniformly in i. If X' is another computable metric space, a function $f : X \to X'$ is ***computable*** if the sets $f^{-1}(B'_i)$ are effectively open, uniformly in i. Let

$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. A function $f : X \to \overline{\mathbb{R}}$ is ***lower (resp. upper) semi-computable*** if $f^{-1}(q_i, +\infty]$ (resp. $f^{-1}[-\infty, q_i)$ is effectively opn, uniformly in i (where $q_0, q_1, \ldots$ is a fixed effective enumeration of the set of rational numbers $\mathbb{Q}$). We remind the reader that there is an effective enumeration $(f_i)_{i\in\mathbb{N}}$ of all the lower semi-computable functions $f : X \to [0, +\infty]$.

2.0.2 Computable probability space.

In [WD06] is studied an effective version of measurable spaces. Here, we restrict our attention to metric spaces endowed with the Borel σ-field (the σ-field generated by the open sets).

Let $(X, d, \mathcal{S})$ be a computable metric space. We first recall what it means for a Borel probability measure over X to be computable.

Theorem 2.0.1 (from [Gác05]). *The set $\mathcal{M}(X)$ of Borel probability measures over X can be made a computable metric space, with the Prokhorov distance and the finite rational convex combinations of Dirac measures as ideal measures.*

The induced topology is the weak topology, characterized by the weak convergence: μ_n ***weakly converge*** to μ if and only if:

$$\int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu \quad \text{for all continuous bounded } f : X \to \mathbb{R}.$$

Theorem 2.0.2 (from [Eda96, Sch07, HR09b]). *Let μ be a Borel probability measure. The following statements are equivalent:*

1. *μ is computable,*
2. *$\mu(B_{i_1} \cup \ldots \cup B_{i_n})$ are lower semi-computable, uniformly in $i_1, \ldots, i_n$,*
3. *$\int f_i \, \mathrm{d}\mu$ are uniformly lower semi-computable (f_i are the lower semi-computable functions).*

Proposition 2.0.1. *Let μ be a computable Borel probability measure. If $f : X \to [0, +\infty)$ is upper semi-computable and bounded by M then $\int f \, \mathrm{d}\mu$ is upper semi-computable (uniformly in a description of f and M).*

Following [HR09b] we introduce:

Definition 2.0.2 (from [HR09b]). A ***computable probability space*** is a pair (X, μ) where X is a computable metric space and μ is a computable Borel probability measure on X.

From now and beyond, we will always work on computable probability spaces.

The measures of ideal balls are generally only lower semi-computable. One can prove that the radii of the balls can be adjusted so that the measure of their boundaries are null (i.e. so that the balls become sets of μ-continuity). A ball $B(s, r)$ is said to be ***μ-almost decidable ball*** if r is a computable positive real number and $\mu(\{x : d(s, x) = r\}) = 0$.

Theorem 2.0.3 (from [HR09b])**.** *Let (X, μ) be a computable probability space. There is a basis $\mathcal{B}^\mu = \{B_1^\mu, B_2^\mu, \dots\}$ of uniformly μ-almost decidable balls which is effectively equivalent to the basis $\mathcal{B}$ of ideal balls. The measures of their finite unions are then uniformly computable.*

Effective equivalence between $\mathcal{B}$ and $\mathcal{B}^\mu$ means that every B_i^μ is an effective union of elements of $\mathcal{B}$, uniformly in i, and every B_i is an effective union of elements of $\mathcal{B}^\mu$, uniformly in i.

2.0.3 Algorithmic randomness.

Here, (X, μ) is a computable probability space. Martin-Löf randomness was first defined in [ML66] on the space of infinite symbolic sequences. Generalizations to abstract spaces have been investigated in [ZL70, HW03, Gác05, HR09b]. We follow the latter two approaches, developed on computable metric spaces.

Definition 2.0.3. A ***Martin-Löf test (ML-test)*** is a sequence of uniformly effective open sets U_n such that $\mu(U_n) < 2^{-n}$.

A point x ***passes*** a ML-test U if $x \notin \bigcap_n U_n$. A point is ***Martin-Löf random (ML-random)*** if it passes all ML-tests. We denote the set of ML-random points by ML_μ.

If a set $A \subseteq X$ can be enclosed in a ML-test (U_n), i.e. $A \subseteq \bigcap_n U_n$ then we say that A is an ***effective null set***.

The following fundamental result, proved by Martin-Löf on the Cantor space with a computable probability measure, can be extended to any computable probability space using Thm. 2.0.3 (almost decidable balls behave in some way as the cylinders in the Cantor space, as their measures are computable).

Theorem 2.0.4 (adapted from [ML66])**.** *Every computable probability space (X, μ) admits a universal Martin-Löf test, i.e. a ML-test U such that for all $x \in X$, x is ML-random $\iff$ x passes the test U. Moreover, for each ML-test V there is a constant c (computable from a description of V) such that $V_{n+c} \subseteq U_n$ for all n.*

We will often use the following result, proved by Kurtz on the Cantor space, but easily generalizable to any computable probability spaces using once again Thm. 2.0.3.

Proposition 2.0.2 (adapted from [Kur81])**.** *ML_μ is contained in every effective open set having measure one.*

One can suppose w.l.o.g. that the universal test is decreasing: $U_{n+1} \subseteq U_n$.

Definition 2.0.4. Let (X, μ) be a computable probability space. Let $(U_n)_{n\in\mathbb{N}}$ be a universal ML-test. We call $K_n := X \setminus U_n$ the ***n^{th} layer*** of the space and the sequence $(K_n)_{n\in\mathbb{N}}$ the ***layering*** of the space.

The set ML_μ of ML-random points can be expressed as an increasing union: $\mathrm{ML}_\mu = \bigcup_n K_n$.

We now introduce effective versions of notions from measure and integration theory on computable probability spaces.

3 Effective versions of measurability notions

In this section $T : (X, \mu) \to Y$ will denote a measurable function between the computable probability space (X, μ) and the computable metric space Y. We will consider effective versions of the notions of measurable set, measurable map, and integrable function. There are two main natural ways to define these effective versions:

3.1 The approach up to null sets

This approach is by *equivalent classes*. As a concequence, the obtained definitions cannot distinguish between objects which coincide *up to a null set.*

3.1.1 Measurable sets.

This approach to computability of measurable sets was first proposed by Šanin [Šan68] on $\mathbb{R}$ with the Lebesgue measure, and generalized by Edalat [Eda07] to any second countable locally compact Hausdorff spaces with a computable regular σ-finite measure. We present the adaptation of this approach to computable probability spaces (which are not necessarily locally compact).

Let (X, μ) be a computable probability space and $\mathfrak{S}$ the set of Borel subsets of X. The function $d_\mu : \mathfrak{S}^2 \to [0, 1]$ defined by $d_\mu(A, B) = \mu(A\Delta B)$ for all Borel sets A, B is a pseudo-metric. Let $[\mathfrak{S}]_\mu$ be the quotient of $\mathfrak{S}$ by the equivalence relation $A \sim_\mu B \iff d_\mu(A, B) = 0$ and $\mathcal{A}_\mu$ be the set of finite unions of μ-almost decidable balls with a natural numbering $\mathcal{A}_\mu = \{A_1, A_2, \ldots\}$. We denote by $[A]_\mu$ the equivalence class of a Borel set A.

Proposition 3.1.1. *$([\mathfrak{S}]_\mu, d_\mu, \mathcal{A}_\mu)$ is a computable metric space.*

The following definition is then the straightforward adaptation of [Šan68, Eda07].

Definition 3.1.1. A Borel set A is called a ***μ-recursive set*** if its equivalence class $[A]_\mu$ is a computable point of the computable metric space $[\mathfrak{S}]_\mu$.

In other words, there is a total recursive function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\mu(A\Delta A_{\varphi(n)}) < 2^{-n}$ for all n. The measure of any μ-recursive is computable. Observe that an ideal ball need not be μ-recursive as its measure is in general only lower semi-computable. On the other hand, μ-almost decidable balls are always μ-recursive.

3.1.2 Measurable maps.

To the notion of μ-recursive set corresponds a natural effective version of μ-recursive map:

Definition 3.1.2. A measurable map $T : (X, \mu) \to Y$ is called a ***μ-recursive map*** if there exists a basis of balls $\hat{\mathcal{B}} = \{\hat{B}_1, \hat{B}_2, \ldots\}$ of Y, which is effectively equivalent to the basis of ideal balls $\mathcal{B}$, and such that $T^{-1}(\hat{B}_i)$ are uniformly μ-recursive sets.

3.1.3 Integrable functions.

Computability on L^p spaces has been studied in [ZZ99, Kun04] for euclidean spaces with the Lebesgue measure. The L^1 case can be easily generalized to any computable probability space, and a further generalization including σ-finite measures might be carried out without difficulties.

Let (X, μ) be a computable probability space. Let $\mathcal{F}$ be the set of measurable functions $f : X \to \overline{\mathbb{R}}$ which are integrable. Let $I_\mu : \mathcal{F} \times \mathcal{F} \to [0, +\infty)$ be defined by $I_\mu(f, g) = \int |f - g| \, \mathrm{d}\mu$. I_μ is a metric on the quotient space $L^1(X, \mu)$ with the relation $f \sim_\mu g \iff I_\mu(f, g) = 0$. There is a set $\mathcal{F}_0 = \{f_0, f_1, \ldots\}$ of uniformly computable effectively bounded functions ($|f_i| < M_i$ with M_i computable from i) which is dense in $L^1(X, \mu)$. $\mathcal{F}_0$ is called the set of *ideal functions*.

Proposition 3.1.2. *$(L^1(X, \mu), d_\mu, \mathcal{F}_0)$ is a computable metric space.*

This leads to a first effective notion of integrable function:

Definition 3.1.3. A function $f : X \to \overline{\mathbb{R}}$ is a ***μ-recursive integrable function*** if its equivalence class is a computable point of the space $L^1(X, \mu)$, i.e. f can be effectively approximated by ideal functions in the L^1 norm.

If $f : X \to \overline{\mathbb{R}}$ is integrable, then f is a μ-recursive integrable function if and only if so are $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

3.2 The approach up to *effective* null sets

On a metric space, every Borel probability measure is *regular*, i.e. for every Borel set A and every $\epsilon > 0$ there is a closed set F and an open set U such that $F \subseteq A \subseteq U$ and $\mu(U \setminus F) < \epsilon$ (see [Bil68]). An *effective μ-null set* is then a null set for which this is effective. It can be generalized to sets with positive measure, through the notion of *effectively μ-measurable* set. We will see how to define *effectively μ-measurable* maps and *effectively μ-integrable* functions using the same idea.

3.2.1 Measurable sets.

Edalat [Eda07] already used regularity of measures to define μ-computable sets, a notion that is stronger than μ-recursivity. Let us consider the adaptation of this notion to computable probability spaces (for coherence reasons, we use the expression "effective μ-measurability" instead of "μ-computability").

Definition 3.2.1. A Borel set A is ***effectively μ-measurable*** if there are uniformly effective open sets U_i, V_i such that $X \setminus V_i \subseteq A \subseteq U_i$ and $\mu(U_i \cap V_i) < 2^{-i}$.

example: The whole space X is effectively μ-measurable. More generally, an effective open set is effectively μ-measurable if and only if its measure is computable. The Smith-Volterra-Cantor set, which is an effective compact subset of $[0, 1]$ whose Lebesgue measure is $1/2$, is effectively λ-measurable (λ denotes the Lebesgue measure). $\square$

3.2.2 Measurable maps.

To the notion of effectively μ-measurable set corresponds a natural effective version of measurable map:

Definition 3.2.2. A measurable map $T : (X, \mu) \to Y$ is ***effectively μ-measurable*** if there exists a basis of balls $\hat{\mathcal{B}} = \{\hat{B}_1, \hat{B}_2, \dots\}$ of Y, which is effectively equivalent to the basis of ideal balls $\mathcal{B}$, and such that $T^{-1}(\hat{B}_i)$ are uniformly effectively μ-measurable sets.

3.2.3 Integrable functions.

In [Eda07] a notion of *μ-computable integrable function* is proposed: such a function can be effectively approximated from above and below by simple functions. This notion is developed on any second countable locally compact Hausdorff spaces endowed with a computable finite Borel measure. In this approach only bounded functions can be handled, as they are dominated by simple functions, which are bounded by definition. We overcome this problem, providing at the same time a framework for metric spaces that are not locally compact, as function spaces.

The following definition is a natural extension of the counterpart of Def. 3.2.1 for the characteristic function $\mathbf{1}_A$ of an effectively μ-measurable set A.

Definition 3.2.3. A function $f : X \to [0, +\infty]$ is ***effectively μ-integrable*** if there are uniformly lower semi-computable functions $g_n : X \to [0, +\infty]$ and upper semi-computable functions $h_n : X \to [0, +\infty)$ such that:

1. $h_n \le f \le g_n$,
2. $\int (g_n - h_n) \, \mathrm{d}\mu < 2^{-n}$,
3. h_n is bounded by some M_n which is computable from n.

Remark 3.2.1. Let us define the hypographs (see [BZW99] for a study of these sets)

$$\begin{aligned} \mathrm{hypo}(f) &:= \{(x, y) \in X \times [0, +\infty] : y < f(x)\}, \\ \overline{\mathrm{hypo}}(f) &:= \{(x, y) \in X \times [0, +\infty] : y \le f(x)\}. \end{aligned}$$

Let $A = \overline{\mathrm{hypo}}(f)$: one has $\int f \, \mathrm{d}\mu = (\mu \times \lambda)(A)$. Let $F_n := \overline{\mathrm{hypo}}(h_n)$ and $U_n := \mathrm{hypo}(g_n + 2^{-n})$. In the computable metric space $X \times [0, +\infty]$, U_n as well as the complement of F_n are effectively open, $F_n \subseteq A \subseteq U_n$ and $(\mu \times \lambda)(U_n \setminus F_n) < 2^{-n+1}$.

Hence if effectively measurability of sets was defined for σ-finite measures, the set A would be effectively $(\mu \times \lambda)$-measurable.

Observe that a set A is effectively μ-measurable if and only if its characteristic function $\mathbf{1}_A$ is effectively μ-integrable.

4 The algorithmic randomness approach: Layerwise computability

4.1 Layerwise computability

We remind the reader that every computable probability space comes with a canonical layering $(K_n)_{n\in\mathbb{N}}$ (see Def. 2.0.4).

Definition 4.1.1. A set A is ***layerwise semi-decidable*** if it is semi-decidable on every K_n, uniformly in n. In other words, there are uniformly effective open sets U_n such that $A \cap K_n = U_n \cap K_n$ for all n.

In the language of representations, a set A is layerwise semi-decidable if there is a machine which takes n and a Cauchy representation of $x \in K_n$ as inputs, and eventually halts if and only if $x \in A$ (if $x \notin K_n$, nothing is assumed about the behavior the machine).

Definition 4.1.2. A set A is ***layerwise decidable*** if it is decidable on every K_n, uniformly in n. In other words, both A and its complement are layerwise semi-decidable.

In the language of representations, a set A is layerwise decidable if there is a machine which takes n and a Cauchy representation of $x \in K_n$ as inputs, halts and outputs 1 if $x \in A$, 0 if $x \notin A$.

Definition 4.1.3. A function $T : (X, \mu) \to Y$ is ***layerwise computable*** if it is computable on every K_n, uniformly in n. In other words, there are uniformly effective open sets $U_{n,i}$ such that $T^{-1}(B_i) \cap K_n = U_{n,i} \cap K_n$ for all n, i.

Here, $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ is the basis of ideal balls of Y. Using the language of representations, T is layerwise computable if there is a machine which takes n and a Cauchy representation of $x \in K_n$ as inputs and outputs a Cauchy representation of $T(x)$. We could also say that the sets $T^{-1}(B_i)$ are uniformly layerwise semi-decidable.

Actually, every computability notion on computable metric spaces has in principle its layerwise version. For instance one can define *layerwise lower semi-computable* functions $f : X \to \overline{\mathbb{R}}$.

Let us state some basic properties of layerwise computable maps, when considering the push-forward measure ν defined by $\nu(A) = \mu(T^{-1}(A))$.

Proposition 4.1.1. *Let $T : (X, \mu) \to Y$ be a layerwise computable map.*

- *The push-forward measure $\nu := \mu \circ T^{-1} \in \mathcal{M}(Y)$ is computable.*
- *T preserves ML-randomness, i.e. $T(ML_\mu) \subseteq ML_\nu$. Moreover, there is a constant c (computable from a description of T) such that $T(K_n) \subseteq K'_{n+c}$ for all n, where (K'_n) is the canonical layering of (Y, ν).*
- *If $f : (Y, \nu) \to Z$ is layerwise computable then so is $f \circ T$.*

- *If $A \subseteq Y$ is layerwise decidable (resp. semi-decidable) then so is $T^{-1}(A)$.*

The first point implies that in the particular case when $Y = \mathbb{R}$, a layerwise computable function is then a computable random variable as defined in [Mül99]: its distribution ν over $\mathbb{R}$ is computable. Observe that when ν is the push-forward of μ, layerwise computability notions interact as the corresponding plain computability ones; however, without assumption on ν the last three points may not hold.

As shown by the following proposition, if layerwise computable objects differ at one ML-random point then they essentially differ, i.e. on a set of positive measure.

Proposition 4.1.2. *Let $A, B \subseteq X$ be layerwise decidable sets and $T_1, T_2 : (X, \mu) \to Y$ layerwise computable functions.*

- *If $A = B \mod 0$ then $A \cap ML_\mu = B \cap ML_\mu$.*
- *If $T_1 = T_2$ almost everywhere then $T_1 = T_2$ on ML_μ.*

We can even strengthen this result, obtaining a layerwise version of Prop. 3.1.4.5 in [Hoy08].

Proposition 4.1.3. *Let A, B be layerwise semi-decidable sets. If $A \subseteq B \mod 0$ then $A \cap ML_\mu \subseteq B \cap ML_\mu$. More generally, if $f, f' : X \to [0, +\infty]$ are layerwise lower semi-computable functions such that $f \leq f'$ almost everywhere then $f \leq f'$ on ML_μ.*

4.2 Characterizations of effective measure-theoretic notions

4.2.1 Measurable sets.

The notion of effective μ-measurable set is strongly related to Martin-Löf approach to randomness. Indeed, if A is a Borel set such that $\mu(A) = 0$ then A is effectively μ-measurable if and only if it is an effective μ-null set. If A is effectively μ-measurable, coming with C_n, U_n, then $\bigcup_n C_n$ and $\bigcap_n U_n$ are two particular representative of $[A]_\mu$ which coincide with A on ML_μ. We can even go further, as the following result proves.

Theorem 4.2.1. *Let A be a Borel set. We have:*

1. *A is μ-recursive $\iff$ A is equivalent to an effectively μ-measurable set.*
2. *A is effectively μ-measurable $\iff$ A is layerwise decidable.*

The equivalences are uniform. Let A be a μ-recursive set: it is equivalent to a layerwise decidable set B. By Prop. 4.1.2 the set $A^* := B \cap \mathrm{ML}_\mu$ is well-defined and constitutes a canonical representative of the equivalence class of A under $\sim_\mu$. If A is already layerwise decidable then $A^* = A \cap \mathrm{ML}_\mu$. The operator $*$ is idempotent, it commutes with finite unions, finite intersections and complements. For instance, if A, B are μ-recursive then $A^* \cup B^*$ is a layerwise decidable set which is equivalent to $A \cup B$, so it coincides with $(A \cup B)^*$ by the preceding lemma.

Proposition 4.2.1. *If A be a layerwise semi-decidable then*

- *$\mu(A)$ is lower semi-computable,*
- *$\mu(A)$ is computable if and only if A is layerwise decidable.*

4.2.2 Measurable maps.

We obtain a version of Thm. 4.2.1 of measurable maps.

Theorem 4.2.2. *Let $T : (X, \mu) \to Y$ be a measurable map. We have:*

1. *T is μ-recursive $\iff$ T coincides almost everywhere with an effectively μ-measurable map.*
2. *T is effectively μ-measurable $\iff$ T is layerwise computable.*

The equivalences are uniform. Observe that if almost all implications directly derive from Thm.4.2.1, the first one is not so easy as we have to carry out the explicit construction of an effectively μ-measurable function from the equivalence class of T.

Let T be μ-recursive: there is a layerwise computable function T' which is equivalent to T. Let T^* be the restriction of T' to ML_μ. By Prop. 4.1.2 T^* is uniquely defined.

4.2.3 Integrable functions.

We know from Thm. 4.2.1 that A is effectively μ-measurable if and only if A is layerwise decidable, which is equivalent to the layerwise computability of $\mathbf{1}_A$. As a result, $\mathbf{1}_A$ is effectively μ-integrable if and only if $\mathbf{1}_A$ is layerwise computable. The picture is not so simple for unbounded integrable functions: although $\int f \,\mathrm{d}\mu$ is always computable when f is effectively μ-integrable, it is only *lower semi-computable* when f is layerwise computable.

Proposition 4.2.2. *Let $f : X \to [0, +\infty]$.*

- *If f is layerwise lower semi-computable then $\int f \,\mathrm{d}\mu$ is lower semi-computable (uniformly in a description of f).*
- *If f is bounded and layerwise computable then $\int f \,\mathrm{d}\mu$ is computable (uniformly in a description of f and a bound on f).*

Hence, we have to add the computability of $\int f \,\mathrm{d}\mu$ to get a characterization.

Theorem 4.2.3. *Let $f : X \to [0, +\infty]$ be a μ-integrable function. We have:*

1. *f is a μ-recursive integrable function $\iff$ f is equivalent to an effectively μ-integrable function.*
2. *f is effectively μ-integrable $\iff$ f is layerwise computable and $\int f \,\mathrm{d}\mu$ is computable.*

The equivalences are uniform, but a description of $\int f\,\mathrm{d}\mu$ as a computable real number must be provided.

If f is effectively μ-integrable then $f = \sup_n h_n = \inf_n g_n$ on ML_μ. Observe that the equivalence relation induced by the L^1 norm coincides with the equivalence relation "being equal μ-almost everywhere". Hence by Prop. 4.1.2, to the equivalence class of any μ-recursive integrable function f corresponds a unique layerwise computable function f^* defined on ML_μ.

We now get a rather surprising result, which is a weak version of Prop. 4.2.1 for integrable functions.

Proposition 4.2.3. *Let $f : X \to [0, +\infty]$ be a layerwise lower semi-computable function. If $\int f\,\mathrm{d}\mu$ is computable then f is layerwise computable.*

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A Proofs from Section 4.1

Proof of Proposition 4.1.1

- Let V be an effective open subset of Y. $T^{-1}(V)$ is a layerwise semi-decidable set, so there are effective open sets U_n such that $T^{-1}(V) \cap K_n = U_n \cap K_n$. Hence$\nu(V) = \mu(T^{-1}(V)) = \sup_n(\mu(U_n) - 2^{-n})$ is lower semi-computable. Everything is uniform in V.

- Let U (resp. U') be the universal test in (X, μ) (resp. (Y, ν)). $T^{-1}(U'_n)$ are uniformly layerwise semi-decidable sets and $\mu(T^{-1}(U'_n)) = \nu(U'_n) < 2^{-n}$. By Lem. B.0.1 there is a constant c such that $T^{-1}(U'_{n+c}) \subseteq U_n$. In other words, $K_n \subseteq T^{-1}(K'_{n+c})$.

- There is c such that $T(K_n) \subseteq K'_{n+c}$. As T is computable on K_n and f is computable on K'_{n+c}, uniformly in n, $f \circ T$ is computable on K_n, uniformly in n.

- The same kind of argument can be used.

Proof of Proposition 4.1.2.

We show that $A \cap K_n \subseteq B \cap K_n$ for all n. There are effective open sets U_n, V_n such that $A \cap K_n = (X \setminus V_n) \cap K_n$ and $B \cap K_n = U_n \cap K_n$. Hence $(A \cap K_n) \setminus (B \cap K_n) = K_n \setminus (U_n \cup V_n)$ is the complement of an effective open set of measure one, so it contains no ML-random point by Prop. 2.0.2, hence it is empty as $K_n \subseteq \mathrm{ML}_\mu$.

The second point is a corollary. We know from Prop. 4.1.1 that the common push-forward measure ν is computable. Let $\mathcal{B}^\nu = \{B_1^\nu, B_2^\nu, \ldots\}$ be a basis of ν-almost decidable balls provided by Thm. 2.0.3. For each i, the ball B_i^ν is layerwise decidable so the sets $T_1^{-1}(B_i^\nu)$ and $T_2^{-1}(B_i^\nu)$ are layerwise decidable and equivalent, so they coincide on ML_μ by the first point. In other words, the restrictions of T_1 and T_2 to ML_μ coincide.

Proof of Proposition 4.1.3

We first prove it for effective open sets U, V such that $U \subseteq V \mod 0$. Let $F \subseteq U$ be a finite union of closed balls. $F \setminus V$ is the complement of an effective open set and $\mu(F \setminus V) = 0$, so it contains no ML-random point (Prop. 2.0.2) hence $F \cap \mathrm{ML}_\mu \subseteq V \cap \mathrm{ML}_\mu$. As this is true for every finite union of closed balls $F \subseteq U$, $U \cap \mathrm{ML}_\mu \subseteq V \cap \mathrm{ML}$.

Now, let A, B be layerwise semi-decidable sets such that $A \subseteq B \mod 0$. Let $n \in \mathbb{N}$: there are effective open sets U, V such that $A \cap K_n = U \cap K_n$ and $B \cap K_n = V \cap K_n$. Let $U' = U \cup U_n$ and $V' = V \cup U_n$ (where $U_n = X \setminus K_n$): $U' \subseteq V' \mod 0$ so $A \cap K_n = U' \cap K_n \subseteq V' \cap K_n = B \cap K_n$. And this is true for all n.

Finally, if f, f' are layerwise lower semi-computable and $f \leq f'$ almost everywhere, then for $q \in \mathbb{Q}$, $A = f'^{-1}(q, +\infty]$ and $B = f^{-1}(q, +\infty]$ are layerwise semi-decidable sets satisfying $A \subseteq B \mod 0$, so $A \cap \mathrm{ML}_\mu \subseteq B \cap \mathrm{ML}_\mu$. Hence if $x \in \mathrm{ML}_\mu$, $f'(x) > q \implies f(x) > q$ for all $q \in \mathbb{Q}$. In other words, $f(x) \leq f'(x)$.

B Proofs from Section 4.2

Proof of Theorem 4.2.1.

1. Let A_i be an effective sequence of finite unions of μ-almost decidable open balls such that $\mu(A\Delta A_i) < 2^{-i}$. A_i can be expressed as an effective union of μ-almost decidable balls whose closure are contained in A_i. Let $\overline{A}'_i$ be a finite union of the corresponding closed balls such that $\mu(A'_i) > \mu(A_i) - 2^{-i}$. One has $\overline{A}'_i \subseteq A_i$ and the complements of $\overline{A}'_i$ are uniformly effective open sets. The class $[A]_\mu$ has two canonical representatives that are effectively μ-measurable: $\liminf \overline{A}'_i$ and $\limsup A_i$ and the sets $F_n = \bigcap_{i>n} \overline{A}'_i$ and $U_n = \bigcup_{i>n} A_i$ witness their effective μ-measurability. Conversely, let A be effectively μ-measurable, coming with F_n, U_n. Expressing U_n as an effective union of μ-almost decidable balls, one can effectively extract a finite union A_n such that $\mu(A_n) > \mu(F_n)$. Then $A\Delta A_n \subseteq U_n \setminus F_n$ so $\mu(A\Delta A_n) < 2^{-n}$.

2. If A is effectively μ-measurable, coming with F_n, U_n then $U_n \setminus F_n$ is a ML-test, so there is c such that $(U_{n+c} \setminus F_{n+c}) \cap K_n = \emptyset$ for all n. One easily gets $A \cap K_n = U_{n+c} \cap K_n = F_{n+c} \cap K_n$. Conversely, if A is layerwise decidable, then the complements of $F_n := A \cap K_n$ and $F'_n := (X \setminus A) \cap K_n$ are uniformly effective open sets. The sets F_n and $U_n = X \setminus F'_n$ make A effectively μ-measurable.

Proof of Proposition 4.2.1

Let U_n be effective open sets such that $A \cap K_n = U_n \cap K_n$. First, $\mu(A) = \sup_n(\mu(U_n) - 2^{-n})$ is lower semi-computable. We suppose now that $\mu(A)$ is computable and show that A is effectively μ-measurable. Let $V_n = U_n \cup (X \setminus K_n)$. $A \subseteq V_n$ and $\mu(V_n) \leq \mu(A) + 2^{-n}$. Expressing U_n as an effective union of μ-almost decidable balls, one can effectively extract a finite union A_n such that $\mu(A_n) > \mu(A) - 2^{-n}$ (indeed, $\mu(U_n) > \mu(A) - 2^{-n}$). Now, $F_n := A_n \cap K_n \subseteq U_n \cap K_n \subseteq A$. So F_n, V_n make A effectively μ-measurable, hence layerwise decidable.

Proof of Theorem 4.2.2

1. Let T be a μ-recursive map. We construct an effecively μ-measurable function T^* such that $T = T*$ almost everywhere. We will need some lemmas:

Lemma B.0.1. *Let A_n be uniformly layerwise semi-decidable sets such that $\mu(A_n) < 2^{-n}$. There exists c such that $K_n \cap A_{n+c} = \emptyset$ for all n.*

Proof. There are uniformly effective open sets U_p^n such that $K_p \cap A_n = K_p \cap U_p^n$. Let $V_n = U_{n+1}^{n+1}$. As $\mu(X \setminus K_{n+1}) < 2^{-n-1}$, $\mu(V_n) \leq 2^{-n}$ so $(V_n)_{n\in\mathbb{N}}$ is a ML-test, hence there is c such that $K_n \cap V_{n+c} = \emptyset$ for all n. We conclude observing that $K_n \cap A_{n+c+1} = K_n \cap U_n^{n+c+1} \subseteq K_n \cap V_{n+c}$. □

Lemma B.0.2. *Let A_i be uniformly μ-recursive sets. The set $\bigcup_i A_i$ is μ-recursive if and only if its measure is computable. In this case, $(\bigcup_i A_i)^* = \bigcup_i A_i^*$.*

Proof. This is a corollary of Prop. 4.2.1. Indeed, A_i^* being uniformly layerwise decidable, $\bigcup_i A_i^*$ is layerwise semi-decidable. If its measure is computable then it is layerwise decidable by Prop. 4.2.1. Hence $A := \bigcup_i A_i$, which is equivalent to it, satisfies $A^* = (\bigcup_i A_i^*)^* = (\bigcup_i A_i^*) \cap \mathrm{ML}_\mu = \bigcup_i A_i^*$. $\square$

Construction of T^*.

Let $T : (X, \mu) \to Y$ be a μ-recursive map, coming with a basis $\hat{\mathcal{B}} = \{\hat{B}_1, \hat{B}_2, \dots\}$ of the topology on Y such that the sets $A_i := T^{-1}(\hat{B}_i)$ are uniformly μ-recursive. The sets A_i^* are then well defined. Let $x \in \mathrm{ML}$: we define $S_x := \bigcap_{i:x\in A_i^*} \hat{B}_i$.

Claim. S_x contains at most one point.

Proof. Let $\epsilon > 0$ and $E = \{i : \hat{B}_i \text{ has radius } < \epsilon\}$. As $Y = \bigcup_{i\in E} \hat{B}_i$, $\mathrm{ML}_\mu = X^* = (T^{-1}(Y))^* = (\bigcup_{i\in E} A_i)^* = \bigcup_{i\in E} A_i^*$ by Lem. B.0.2). Hence there is $i \in E$ such that $x \in A_i^*$, so $S_x \subseteq \hat{B}_i$ whose diameter is less than 2ϵ. As this is true for every $\epsilon > 0$, it follows that $\mathrm{diam}(S_x) = 0$. $\square$

Lemma B.0.3. *S_x is not empty.*

Proof. As we have just seen, for each i there is a ball $\hat{B}_{n_i}$ such that $x \in (T^{-1}(\hat{B}_{n_i}))^*$ and $\hat{B}_{n_i}$ has a radius $< 2^{-i}$. Let s_i be the center of the ball $\hat{B}_{n_i}$. The sequence $(s_i)_{i\in\mathbb{N}}$ is a Cauchy sequence. Indeed, let $i < j$: $x \in A_{n_i}^* \cap A_{n_j}^* = (T^{-1}(\hat{B}_{n_i} \cap \hat{B}_{n_j}))^*$, so $\hat{B}_{n_i} \cap \hat{B}_{n_j} \neq \emptyset$ and hence $d(s_i, s_j) < 2^{-i+1}$. Let y be the limit of s_i, which exists by completeness of Y. We now state and prove two claims which will enable us to conclude.

Claim. For every k, if $x \in A_k^*$ then $y \in \mathrm{cl}(\hat{B}_k)$.

of the claim. Indeed, for all i, $\hat{B}_k \cap \hat{B}_{n_i} \neq \emptyset$ as $x \in A_k^* \cap A_{n_i}^*$. So y is also the limit of a sequence of points belonging to $\hat{B}_k$. $\square$

Claim. For every k, if $x \in A_k^*$ then $y \in \hat{B}_k$.

of the claim. There is a r.e. set E such that $\hat{B}_k = \bigcup_{i\in E} \hat{B}_i$ and $\mathrm{cl}(\hat{B}_i) \subseteq \hat{B}_k$ for all $i \in E$. As $\hat{B}_k, \hat{B}_i$ are all effectively μ-measurable, $A_k^* = \bigcup_{i:\mathrm{cl}(B_i)\subseteq B_k} A_i^*$ by Lem. B.0.2 so there is $i \in E$ such that $x \in A_i^*$ hence $y \in \mathrm{cl}(\hat{B}_i) \subseteq \hat{B}_k$ by the preceding claim. $\square$

Now we conclude the proof of lemma B.0.3: $y \in \bigcap_{k:x\in A_k^*} \hat{B}_k = S_x$. $\square$

Let us define $T^* : \mathrm{ML}_\mu \to Y$ by $\{T^*(x)\} := S_x$ for all $x \in \mathrm{ML}_\mu$. Of course, the function T^* can be seen as a function of X by extending it in an arbitrary measurable way.

Claim. For every k, if $T^*(x) \in \hat{B}_k$ then $x \in A_k^*$.

Proof. If $T^*(x) \in \hat{B}_k$ then there is i such that $B(T^*(x), 2^{-i+1}) \subseteq \hat{B}_k$. Hence $\hat{B}_{n_i} \subseteq \hat{B}_k$ so $x \in A_{n_i}^* \subseteq A_k^*$. $\square$

Finally, for $x \in \mathrm{ML}$ we have $T^*(x) \in \hat{B}_k \iff x \in A_k^*$, so $T^{*-1}(\hat{B}_k) = A_k^* = (T^{-1}(\hat{B}_k))^*$ hence we get the following property:

$$T^{*-1}(\hat{B}_i) = (T^{-1}(\hat{B}_i))^* \qquad \text{for all } i. \tag{1}$$

From this it directly follows that T^* is effectively μ-measurable and that it coincides with T almost everywhere. So we have proved the first implication of point 1.

Conversely, if there is a basis $\hat{\mathcal{B}} = \{\hat{B}_1, \hat{B}_2, \dots\}$ such that the sets $T^{-1}(\hat{B}_i)$ are uniformly effectively μ-measurable, then these sets are uniformly μ-recursive by Thm. 4.2.1 (point 1) so T is μ-recursive. Moreover, by the same theorem (point 2.) the sets $T^{-1}(\hat{B}_i)$ are uniformly layerwise decidable so there are uniformly effective open sets $U_{n,i}$ such that $K_n \cap T^{-1}(\hat{B}_i) = K_n \cap U_{n,i}$. As $\hat{B}$ is effectively equivalent to $\mathcal{B}$, T is layerwise computable.

Suppose now that T is layerwise computable. By Prop. 4.1.1 the push-forward measure $\nu = \mu \circ T^{-1}$ is computable: let $\hat{B}$ be a basis of ν-almost decidable balls provided by Thm. 2.0.3. As T is layerwise computable, the sets $T^{-1}(\hat{B}_i)$ are uniformly layerwise decidable, hence effectively μ-measurable, so T is effectively μ-measurable.

Proof of Proposition 4.2.2

- For $n, \delta > 0$, let $A_{\delta,n} = f^{-1}(\delta n, +\infty]$. One has $\delta \sum_{n>0} \mathbf{1}_{A_{\delta,n}} < f \leq \delta(1 + \sum_{n>0} \mathbf{1}_{A_{\delta,n}})$, so $\int f \, \mathrm{d}\mu = \sup_\delta \sum_{n>0} \mu(A_{\delta,n})$. As $A_{\delta,n}$ is a layerwise semi-decidable, uniformly in δ, n, $\int f \, \mathrm{d}\mu$ is lower semi-computable.

- Let a be a bound on f. Then $\int f \, \mathrm{d}\mu = \inf_n (\int f \mathbf{1}_{F_n} \, \mathrm{d}\mu + a 2^{-n})$ is upper semi-computable by Prop. 2.0.1. Applying the same argument to $a - f$ gives that $\int f \, \mathrm{d}\mu = a - \int (a - f) \, \mathrm{d}\mu$ is lower semi-computable.

Proof of Theorem 4.2.3

The proof goes this way:

(a) f is a computable point of $L^1(X, \mu) \Rightarrow f$ is equivalent to an effectively μ-integrable function,

(b) f is effectively μ-integrable $\Rightarrow \int f \, \mathrm{d}\mu$ is computable and f is layerwise computable,

(c) $\int f \, \mathrm{d}\mu$ is computable and f is layerwise computable $\Rightarrow f$ is a computable point of $L^1(X, \mu)$

This will imply point 1. and one implication of point 2. To derive the other implication, let us make a preliminary observation.

Lemma B.0.4. *If f is effectively μ-integrable and $f' = f$ on ML_μ then f' is also effectively μ-integrable.*

Proof. Let h_n, g_n be associated to f. The problem is that $h_n \leq f' \leq g_n$ may not be satisfied outside ML_μ. To correct this we construct g'_n (resp. h'_n) which coincides with g_n (resp. h_n) on ML_μ and such that $g'_n = +\infty$ and $h'_n = 0$ outside ML_μ. We put $g'_n := g_n + 2^{-n} t_\mu$ where $t_\mu = \sum_n \mathbf{1}_{U_n}$ (U_n being the universal ML-test) and $h'_n := h_n \mathbf{1}_{K_{i_n}}$ where i_n is a computable sequence such that $M_n 2^{-i_n} < 2^{-n}$ where M_n is a bound on h_n. □

Hence if f is layerwise computable and $\int f \,\mathrm{d}\mu$ is computable then f is equivalent to an effectively μ-integrable function f_2 (by (c) and (a)) which is in turn layerwise computable (by (b)). Using Prop. 4.1.2, $f_2 = f$ on ML_μ. By the preceding lemma, f is then effectively μ-integrable, so the other implication of point 2. is proved.

We now prove (a), (b) and (c).

(a) Let f be a computable point of $L^1(X, \mu)$ and f_n a computable sequence of ideal functions such that $\int |f - f_n| \,\mathrm{d}\mu < 2^{-n}$. The class of f in L^1 has two effectively μ-integrable representatives $\liminf f_i$ and $\limsup f_i$. The functions $g_n := \sup_{i>n} f_i$ and $h_n := \inf_{i>n} f_i$ witness their effective μ-integrability.

(b) Let f be effectively μ-integrable, coming with h_n, g_n. First, $\int f \,\mathrm{d}\mu = \sup_n \left(\int g_n \,\mathrm{d}\mu - 2^{-n}\right) = \inf_n \left(\int h_n \,\mathrm{d}\mu + 2^{-n}\right)$ is both lower and upper semi-computable.

Let $U_n = \{x : \exists p, g_{n+2p}(x) - h_{n+2p}(x) > 2^{-p}\}$. By Tchebychev inequality, $\mu\{x : g_{n+2p}(x) - h_{n+2p}(x) > 2^{-p}\} \leq 2^p \int (g_{n+2p} - h_{n+2p}) \,\mathrm{d}\mu \leq 2^{-n-p}$ so $\mu(U_n) \leq \sum_p 2^{-n-p} \leq 2^{-n}$. U_n is then a ML-test so there is c such that $K_n \cap U_{n+c} = \emptyset$ for all n. Hence on K_n, $g_{n+c+2p} - 2^{-p} \leq h_{n+c+2p} \leq f \leq g_{n+c+2p} \leq h_{n+c+2p} + 2^{-p}$ for all p, so $f = \sup_p (g_{n+c+2p} - 2^{-p}) = \inf_p (h_{n+c+2p} + 2^{-p})$ which is both lower and upper semi-computable, uniformly in n.

(c) Let f be a layerwise computable function such that $\int f \,\mathrm{d}\mu$ is computable. We first use the following (easy) equality

$$\int |f - g| \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu - 2 \int \min(f, g) \,\mathrm{d}\mu \tag{2}$$

which holds for nonnegative integrable real functions f, g. Then we use Prop. 4.2.2: if g is a layerwise computable bounded function then so is $\min(f, g)$, hence $\int \min(f, g) \,\mathrm{d}\mu$ is computable from g and a bound on g. From this is follows that if $g = f_i \in \mathcal{F}_0$ is an ideal function then $\int |f - f_i| \,\mathrm{d}\mu$ is computable, uniformly in i. In other words, the distances of f to ideal points of $L^1(X, \mu)$ are uniformly computable so f is a computable point of $L^1(X, \mu)$.

Proof of Proposition 4.2.3

Let f be a layerwise lower semi-computable such that $\int f \,\mathrm{d}\mu$ is computable. Using equality (2) in the Proof of Theorem 4.2.3 and Prop. 4.2.2, $\int |f - f_i| \,\mathrm{d}\mu$ is upper semi-computable for $f_i \in \mathcal{F}_0$, uniformly in i. It follows that f is a computable point of $L^1(X, \mu)$, as for each n one can effectively find f_i such that $\int |f - f_i| \,\mathrm{d}\mu < 2^{-n}$. By Thm. 4.2.3 f is then equivalent to a layerwise computable function f'. We now apply Prop. 4.1.3 to f and f': they are layerwise lower semi-computable and almost everywhere so they coincide on ML_μ, so f is layerwise computable.