# Products do not preserve computable type

Djamel Eddine Amir, Mathieu Hoyrup

Inria, Loria Nancy (France)

Computability in Europe 2023

# Computability of compact sets

A compact set $K \subseteq \mathbb{R}^n$ is:

- **Computable** if the set of rational balls intersecting $K$ is decidable,
- **Semicomputable** if the set of rational balls that are disjoint from $K$ is recursively enumerable (r.e.).

# Computability of compact sets

A compact set $K \subseteq \mathbb{R}^n$ is:

- **Computable** if the set of rational balls intersecting $K$ is decidable,
- **Semicomputable** if the set of rational balls that are disjoint from $K$ is recursively enumerable (r.e.).

**Example**

There is a semicomputable disk in $\mathbb{R}^2$ which is not computable. Center $(0, 0)$, radius $1 - \sum_{n \in \text{halting set}} 2^{-n}$.

# Computability of compact sets

A compact set $K \subseteq \mathbb{R}^n$ is:

- **Computable** if the set of rational balls intersecting $K$ is decidable,
- **Semicomputable** if the set of rational balls that are disjoint from $K$ is recursively enumerable (r.e.).

**Example**

There is a semicomputable disk in $\mathbb{R}^2$ which is not computable. Center $(0, 0)$, radius $1 - \sum_{n \in \text{halting set}} 2^{-n}$.

**Question**

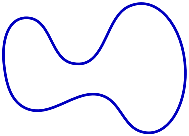
Is there a semicomputable *circle* which is not computable?

# Spheres

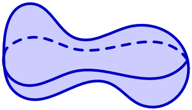
**Theorem ([Miller 2002])**

*If $X \subseteq \mathbb{R}^m$ is homeomorphic to the $n$-dimensional sphere $\mathbb{S}_n$, then*

$$X \text{ is semicomputable} \iff X \text{ is computable.}$$



1-sphere



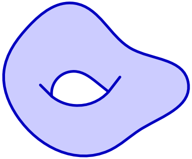
2-sphere

# Manifolds

**Theorem ([Iljazović 2013])**

*If $X \subseteq \mathbb{R}^m$ is a closed manifold, then*

$$X \text{ is semicomputable} \iff X \text{ is computable}.$$



Torus

# Computable type

**Definition**

A compact space $X$ has **computable type** if for every set $K \subseteq \mathbb{R}^m$ that is homeomorphic to $X$,

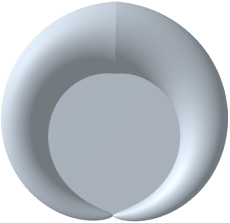
$$K \text{ is semicomputable} \iff K \text{ is computable.}$$

# Computable type

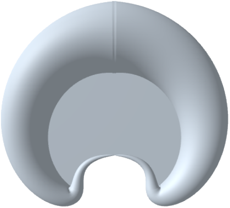
Let $X$ be a finite simplicial complex.

**Theorem (Amir, H, 2022)**

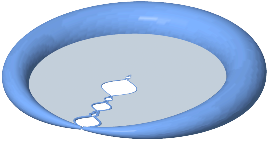
*$X$ has computable type $\iff$ $\exists\epsilon > 0$ such that no $\epsilon$-deformations of $X$ converge to a proper subset of $X$.*



(a) $X$



(b) $\epsilon$-deformations of $X$



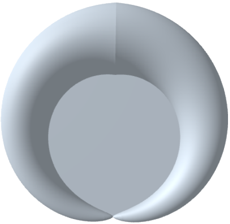
(c) Semicomputable, non-computable copy

# Computable type

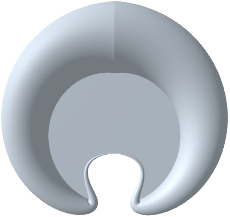
Let $X$ be a finite simplicial complex.

**Theorem (Amir, H, 2022)**

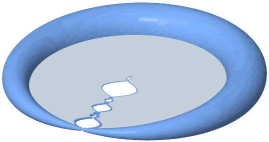
*$X$ has computable type $\iff$ $\exists\epsilon > 0$ such that no $\epsilon$-deformations of $X$ converge to a proper subset of $X$.*



(a) $X$



(b) $\epsilon$-deformations of $X$



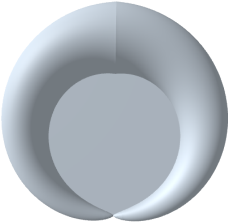
(c) Semicomputable, non-computable copy

# Computable type

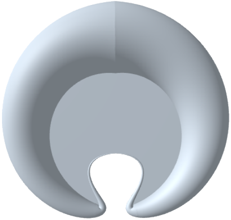
Let $X$ be a finite simplicial complex.

**Theorem (Amir, H, 2022)**

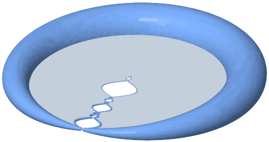
*$X$ has computable type $\iff \exists \epsilon > 0$ such that no $\epsilon$-deformations of $X$ converge to a proper subset of $X$.*



(a) $X$



(b) $\epsilon$-deformations of $X$



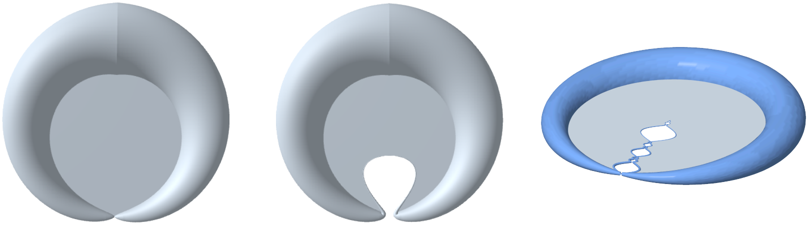
(c) Semicomputable, non-computable copy

# Computable type

Let $X$ be a finite simplicial complex.

**Theorem (Amir, H, 2022)**

*$X$ has computable type $\iff \exists\epsilon > 0$ such that no $\epsilon$-deformations of $X$ converge to a proper subset of $X$.*



(a) $X$

(b) $\epsilon$-deformations of $X$

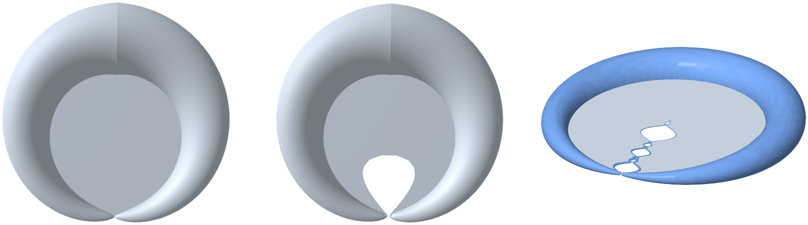
(c) Semicomputable, non-computable copy

# Computable type

Let $X$ be a finite simplicial complex.

**Theorem (Amir, H, 2022)**

*$X$ has computable type $\iff$ $\exists \epsilon > 0$ such that no $\epsilon$-deformations of $X$ converge to a proper subset of $X$.*



(a) $X$

(b) $\epsilon$-deformations of $X$

(c) Semicomputable, non-computable copy

# Computable type

**Question [Čelar, Iljazović 2021]**

If $X$ and $Y$ both have computabe type, does $X \times Y$ have computable type?

# Computable type

**Question [Čelar, Iljazović 2021]**

If $X$ and $Y$ both have computabe type, does $X \times Y$ have computable type?

**Answer [Amir, H. 2023]**

No. There exists $X$ that has computable type, but $X \times \mathbb{S}_1$ does not.

# Roadmap

1. For a family of spaces, we reduce computable type to **homotopy** properties of certain functions,
2. For a smaller family of spaces, these functions are between **spheres**,
3. We then apply results about homotopy groups of spheres.

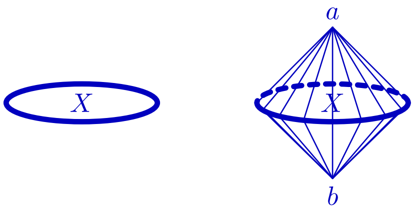
Suspension

A family of spaces

# Suspension

The **suspension** of a space $X$ is the space $\Sigma X$ obtained as follows:

- Add two points $a, b$ to $X$,
- For each $x \in X$, add a segment from $x$ to $a$, and a segment from $x$ to $b$.



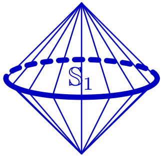

(a) $X$ (b) $\Sigma X$

# Suspension

The suspension of a sphere is a sphere:

$$\Sigma\mathbb{S}_n = \mathbb{S}_{n+1}.$$

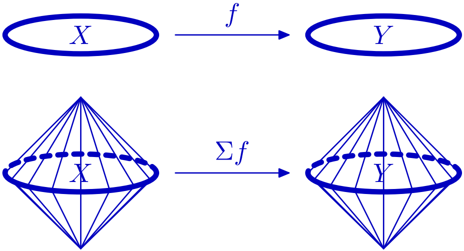

(a) $\mathbb{S}_1$




(b) $\Sigma\mathbb{S}_1 \cong \mathbb{S}_2$

# Suspension

The **suspension** of a function $f : X \to Y$ is $\Sigma f : \Sigma X \to \Sigma Y$.



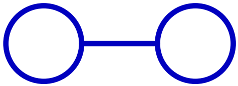
# Suspension

When $X$ is nice[1], we obtain a further characterization of the $X$'s such that $\Sigma X$ has computable type.

[1] simplicial complex

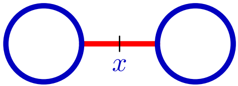
# Suspension

- Let $X$ be a space,



# Suspension

- Let $X$ be a space,
- Let $x \in X$ have a neighborhood $U \cong \mathbb{R}^n$,

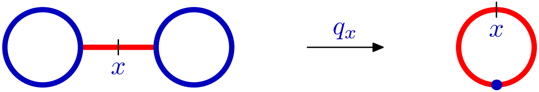


# Suspension

- Let $X$ be a space,
- Let $x \in X$ have a neighborhood $U \cong \mathbb{R}^n$,
- The quotient space $X/U^c$ is $\mathbb{S}_n$,

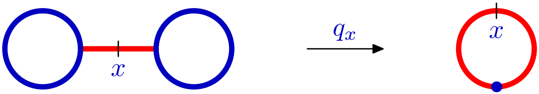
# Suspension

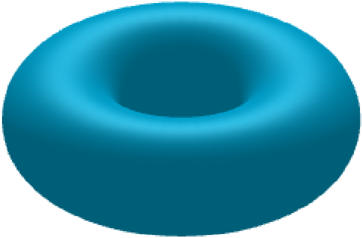
- Let $X$ be a space,
- Let $x \in X$ have a neighborhood $U \cong \mathbb{R}^n$,
- The quotient space $X/U^c$ is $\mathbb{S}_n$,
- Let $q_x : X \to \mathbb{S}_n$ be the quotient map.



# Suspension

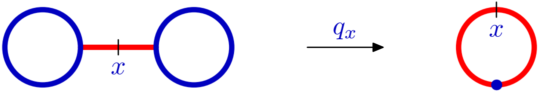
- Let $X$ be a space,
- Let $x \in X$ have a neighborhood $U \cong \mathbb{R}^n$,
- The quotient space $X/U^c$ is $\mathbb{S}_n$,
- Let $q_x : X \to \mathbb{S}_n$ be the quotient map.

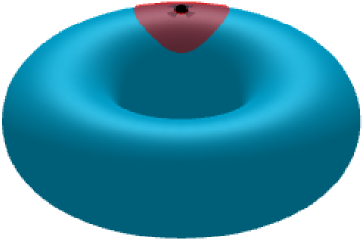


# Suspension

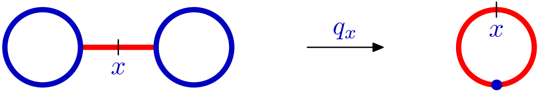
- Let $X$ be a space,
- Let $x \in X$ have a neighborhood $U \cong \mathbb{R}^n$,
- The quotient space $X/U^c$ is $\mathbb{S}_n$,
- Let $q_x : X \to \mathbb{S}_n$ be the quotient map.

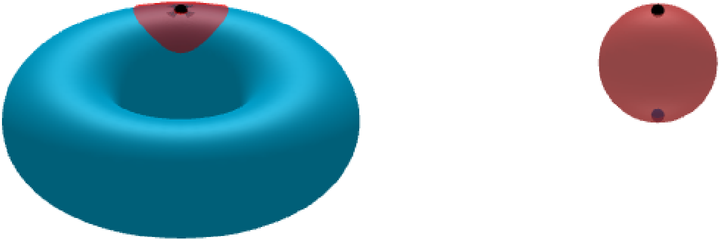


# Suspension

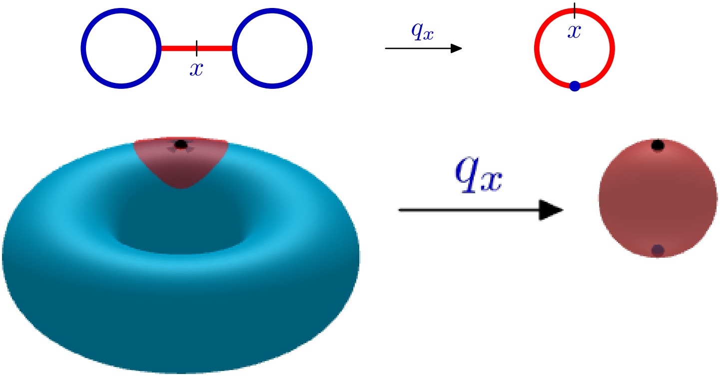
- Let $X$ be a space,
- Let $x \in X$ have a neighborhood $U \cong \mathbb{R}^n$,
- The quotient space $X/U^c$ is $\mathbb{S}_n$,
- Let $q_x : X \to \mathbb{S}_n$ be the quotient map.



# Suspension

- Let $X$ be a space,
- Let $x \in X$ have a neighborhood $U \cong \mathbb{R}^n$,
- The quotient space $X/U^c$ is $\mathbb{S}_n$,
- Let $q_x : X \to \mathbb{S}_n$ be the quotient map.



# Suspension

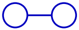
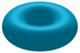
Let $X$ be a simplicial complex.

**Theorem**

*$\Sigma X$ has computable type $\iff$ no quotient map $q_x : X \to \mathbb{S}_n$ is homotopic to a constant.*

(homotopic to a constant: $\exists h_t : X \to \mathbb{S}_n$ with $h_0 = q_x$ and $h_1$ constant)

**Examples**

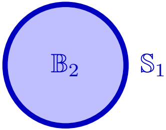
- The suspension of  does **not** have computable type.
- The suspension of  has computable type.

Suspension

A family of spaces

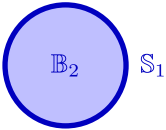
# A family of spaces

- The boundary of the ball $\mathbb{B}_{n+1}$ is $\mathbb{S}_n$.

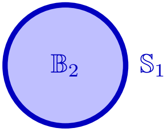


# A family of spaces

- The boundary of the ball $\mathbb{B}_{n+1}$ is $\mathbb{S}_n$.
- Let $f : \mathbb{S}_n \to \mathbb{S}_p$. We attach $\mathbb{B}_{n+1}$ to $\mathbb{B}_{p+1}$ along their boundaries using $f$: each $x \in \mathbb{S}_n$ is glued to $f(x) \in \mathbb{S}_p$.



# A family of spaces




- The boundary of the ball $\mathbb{B}_{n+1}$ is $\mathbb{S}_n$.
- Let $f : \mathbb{S}_n \to \mathbb{S}_p$. We attach $\mathbb{B}_{n+1}$ to $\mathbb{B}_{p+1}$ along their boundaries using $f$: each $x \in \mathbb{S}_n$ is glued to $f(x) \in \mathbb{S}_p$.
- We obtain the space $X_f = \mathbb{B}_{p+1} \cup_f \mathbb{B}_{n+1}$

  *(click on the picture below to launch animation)*

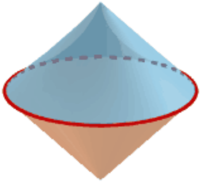


Figure: $X_f$ where $f : \mathbb{S}_1 \to \mathbb{S}_1$ is the doubling map

# A family of spaces

**Theorem**

$\Sigma X_f$ *has computable type* $\iff$ $\Sigma f$ *is not homotopic to a constant.*

$\Sigma X_f \times \mathbb{S}_1$ *has computable type* $\iff$ $\Sigma^2 f$ *is not homotopic to a constant.*

# A family of spaces

**Theorem**

*$\Sigma X_f$ has computable type $\iff$ $\Sigma f$ is not homotopic to a constant.*
*$\Sigma X_f \times \mathbb{S}_1$ has computable type $\iff$ $\Sigma^2 f$ is not homotopic to a constant.*

From the literature on homotopy groups of spheres (Freudenthal, Whitehead, Toda), there exists $f : \mathbb{S}_7 \to \mathbb{S}_3$ such that:

- $\Sigma f : \mathbb{S}_8 \to \mathbb{S}_4$ is not homotopic to a constant,
- $\Sigma^2 f : \mathbb{S}_9 \to \mathbb{S}_5$ is homotopic to a constant.

# A family of spaces

**Theorem**

*$\Sigma X_f$ has computable type $\iff$ $\Sigma f$ is not homotopic to a constant.*
*$\Sigma X_f \times \mathbb{S}_1$ has computable type $\iff$ $\Sigma^2 f$ is not homotopic to a constant.*

From the literature on homotopy groups of spheres (Freudenthal, Whitehead, Toda), there exists $f : \mathbb{S}_7 \to \mathbb{S}_3$ such that:

- $\Sigma f : \mathbb{S}_8 \to \mathbb{S}_4$ is not homotopic to a constant,
- $\Sigma^2 f : \mathbb{S}_9 \to \mathbb{S}_5$ is homotopic to a constant.

**Corollary**

*$\Sigma X_f$ and $\mathbb{S}_1$ have computable type, but $\Sigma X_f \times \mathbb{S}_1$ does not.*

# Summary

- $\Sigma X$ has computable type $\iff$ no quotient map $q_x : X \to \mathbb{S}_n$ is homotopic to a constant,

# Summary

- $\Sigma X$ has computable type $\iff$ no quotient map $q_x : X \to \mathbb{S}_n$ is homotopic to a constant,
- For $f : \mathbb{S}_n \to \mathbb{S}_p$,

# Summary

- $\Sigma X$ has computable type $\iff$ no quotient map $q_x : X \to \mathbb{S}_n$ is homotopic to a constant,
- For $f : \mathbb{S}_n \to \mathbb{S}_p$,
  - $\Sigma X_f$ has computable type $\iff$ $\Sigma f$ is not homotopic to a constant,

# Summary

- $\Sigma X$ has computable type $\iff$ no quotient map $q_x : X \to \mathbb{S}_n$ is homotopic to a constant,
- For $f : \mathbb{S}_n \to \mathbb{S}_p$,
  - $\Sigma X_f$ has computable type $\iff$ $\Sigma f$ is not homotopic to a constant,
  - $\Sigma X_f \times \mathbb{S}_1$ has computable type $\iff$ $\Sigma^2 f$ is not homotopic to a constant,

# Summary

- $\Sigma X$ has computable type $\iff$ no quotient map $q_x : X \to \mathbb{S}_n$ is homotopic to a constant,
- For $f : \mathbb{S}_n \to \mathbb{S}_p$,
  - $\Sigma X_f$ has computable type $\iff$ $\Sigma f$ is not homotopic to a constant,
  - $\Sigma X_f \times \mathbb{S}_1$ has computable type $\iff$ $\Sigma^2 f$ is not homotopic to a constant,
- There exists $f : \mathbb{S}_7 \to \mathbb{S}_3$ such that $\Sigma f$ is not homotopic to a constant, but $\Sigma^2 f$ is.

# A question

The counter-example $\Sigma X_f$ has dimension $9$. Can it be lowered?

We know that if $X, Y$ are simplicial complexes of dimensions $\leq 4$, then $X, Y$ have computable type $\iff$ $X \times Y$ has computable type. This is because for dimension $\leq 4$, computable type can be characterized using homology, which behaves well w.r.t. products.