Presentations of topological spaces

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–

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Introduction

Computable analysis:

- How to represent mathematical objects,
- Difficulty of computing them,
- Difficulty of extracting information.

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- How to represent mathematical objects,
- Difficulty of computing them,
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In this talk, the objects are **topological spaces**.

We mostly focus on **compact metrizable spaces**, represented in **three** ways.

Introduction: presentations

Let $(B_i)_{i\in\mathbb{N}}$ be the rational boxes in $[0,1]^n$ $(n = 1, 2, \ldots, \infty)$.

A compact metrizable space X can be represented as a compact set $K \subseteq [0,1]^n$ homeomorphic to X, with:

1. Positive information

An enumeration of $\{i \in \mathbb{N} : B_i \cap K \neq \emptyset\}$.

2. Negative information

An enumeration of $\{i \in \mathbb{N} : \overline{B_i} \cap K = \emptyset\}$.

3. Both

Introduction

General questions

How difficult is it to recognize the topological shape?
Can shape encode information?

Computable type

Encoding information in a space

Countably-based spaces

Computable type

Knowing the shape can be used to derive positive information from negative information.

Computable type

Let $(B_i)_{i\in\mathbb{N}}$ be the rational boxes in $[0,1]^n$.

Definition

A compact set $K \subseteq [0,1]^n$ is

- **Semicomputable** if $\{i \in \mathbb{N} : K \cap \overline{B}_i = \emptyset\}$ is c.e.,
- **Computable** if moreover $\{i \in \mathbb{N} : K \cap B_i \neq \emptyset\}$ is c.e.

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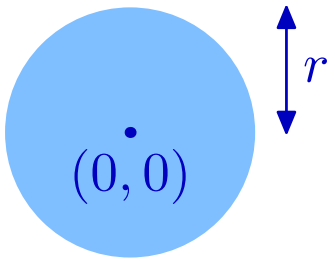
Example

A disk $D \subseteq \mathbb{R}^2$ that is semicomputable but not computable.



$$r = 1 - \sum_{n:M_n \text{ halts}} 2^{-n-1}.$$

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Question

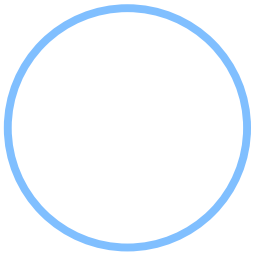
What are the possible shapes of sets that are semicomputable but not computable? What about the circle?

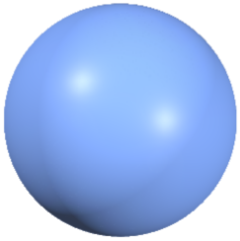
Computable type

Theorem (J. Miller, 2002)

If $K \subseteq \mathbb{R}^n$ *is homeomorphic to a sphere* $\mathbb{S}_m$*, then*

$$K \text{ is semicomputable} \iff K \text{ is computable.}$$





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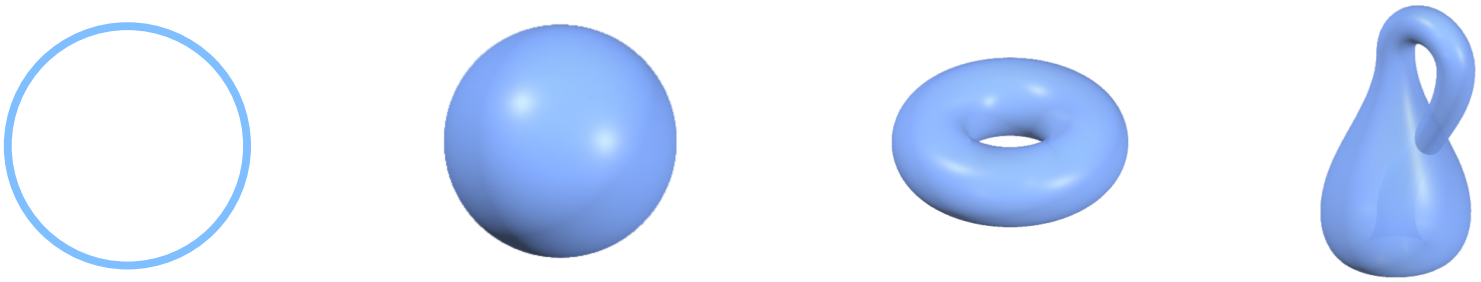

Theorem (Iljazović, 2013)

If $K \subseteq [0,1]^n$ *is homeomorphic to a closed manifold, then*

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Definition (Iljazović)

A space has **computable type** if for every $K \subseteq [0,1]^n$ that is homeomorphic to X,

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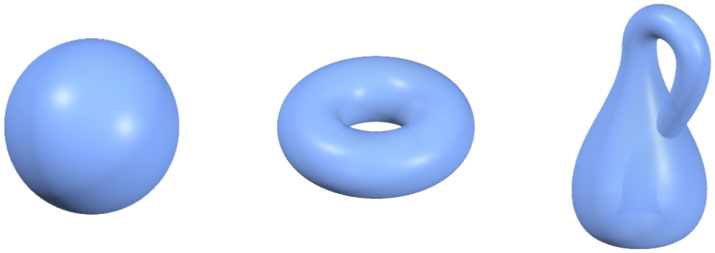

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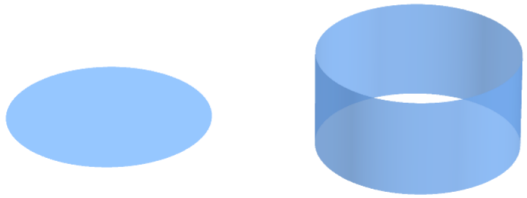

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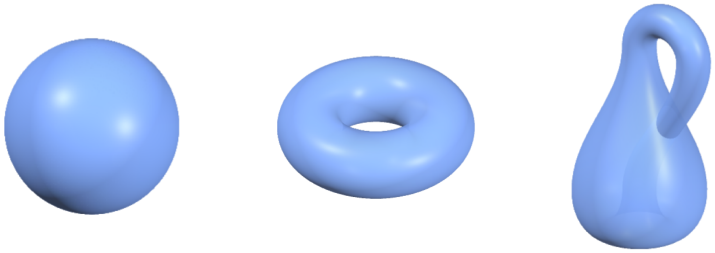

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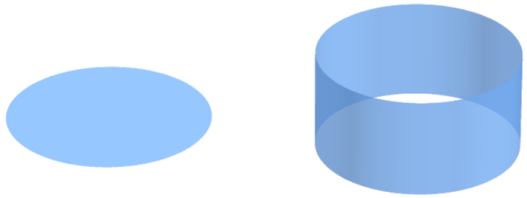

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What about other simple spaces? (simple=triangulable)

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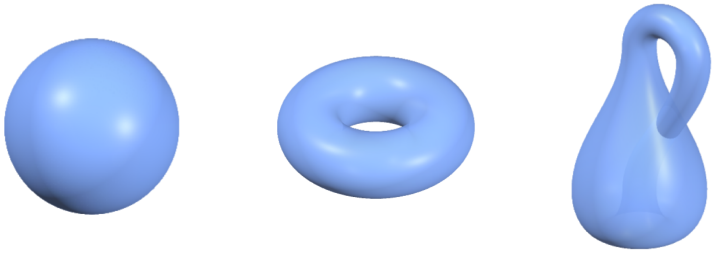

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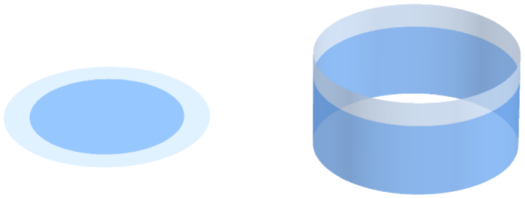

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A simple space does not have computable type $\iff$ it can be "slightly squeezed".

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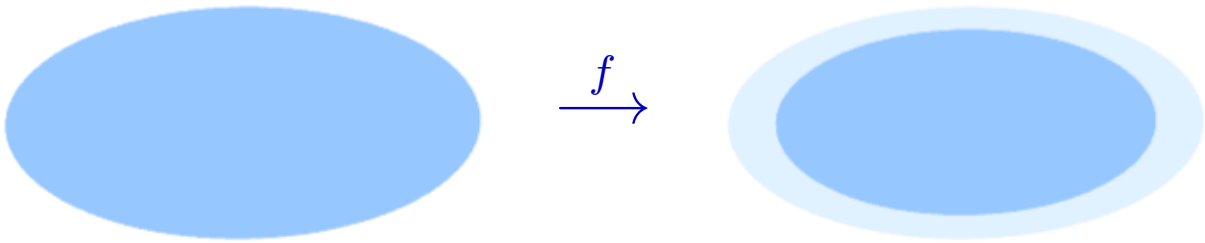
Theorem (Amir, H. 2022)

A simple space X does not have computable type $\iff$ $\forall \epsilon > 0$, there is a non-surjective continuous function $f : X \to X$ such that $d(f(x), x) < \epsilon$ for all $x \in X$.



Computable type

Example

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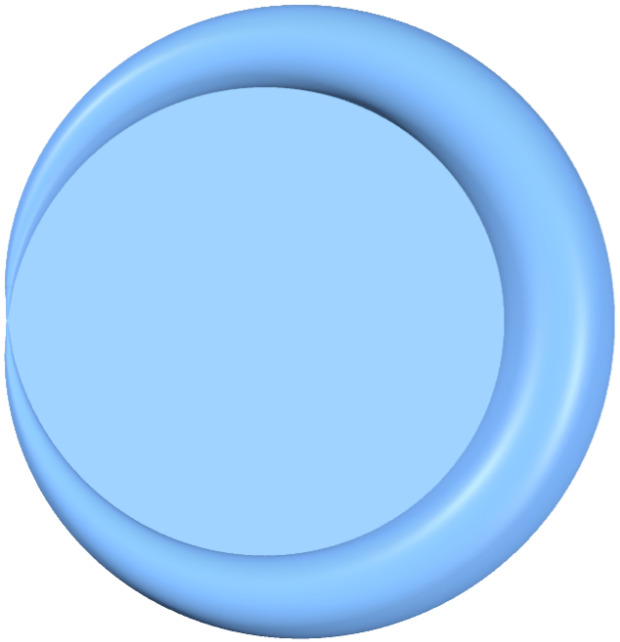

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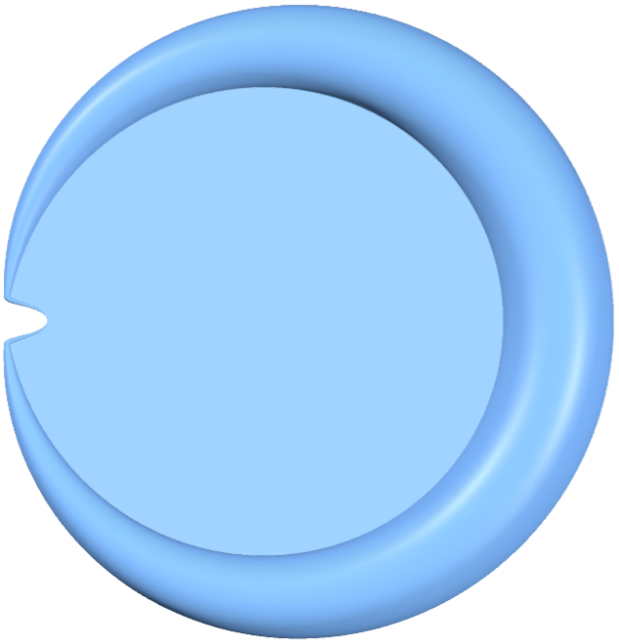

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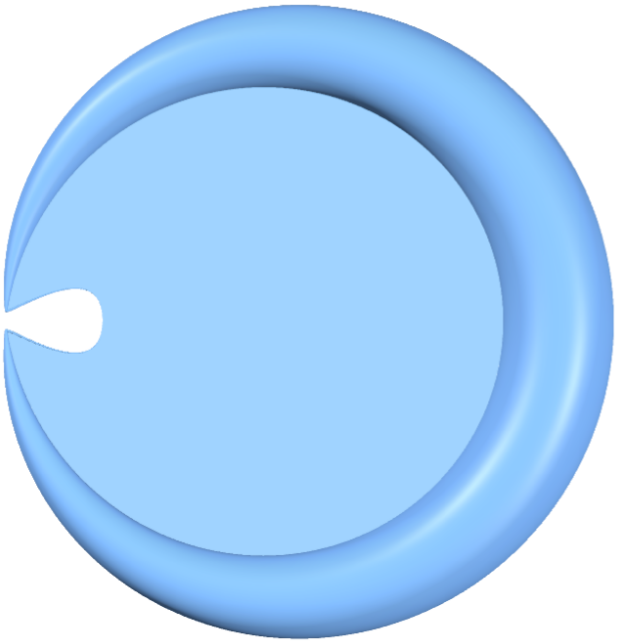

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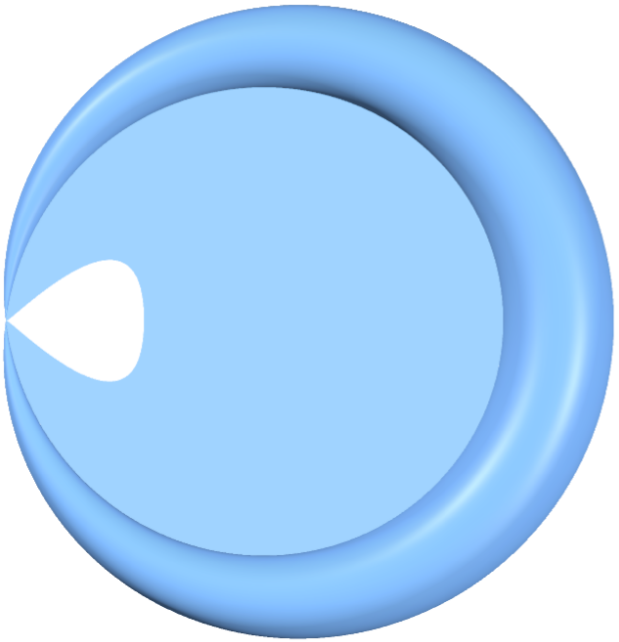

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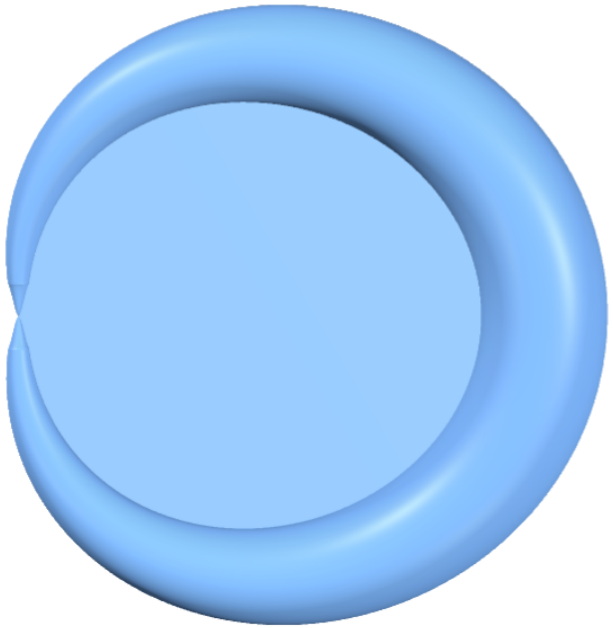
A semicomputable, non-computable realization:



- Let $(n_0, n_1, n_2, \ldots)$ be a computable enumeration of the halting set H,
- Let the kth hole have size $\frac{1}{2^{n_k}}$,
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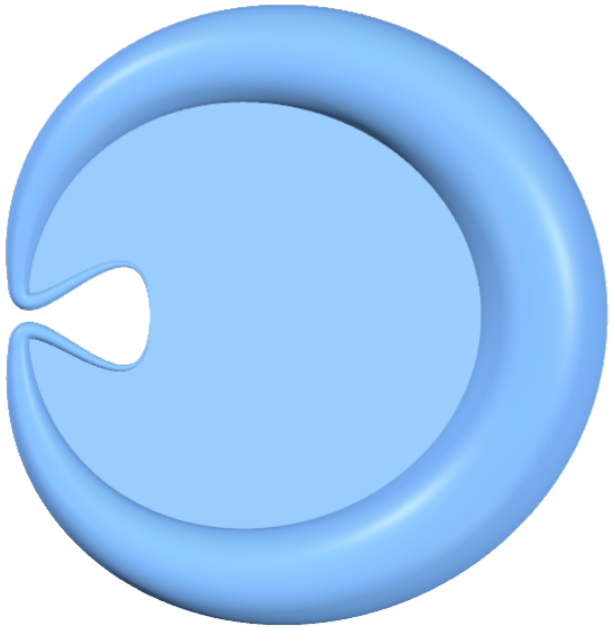

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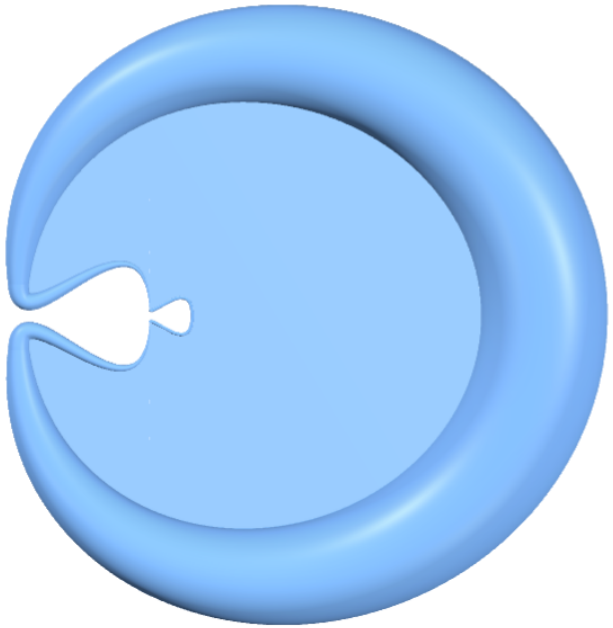

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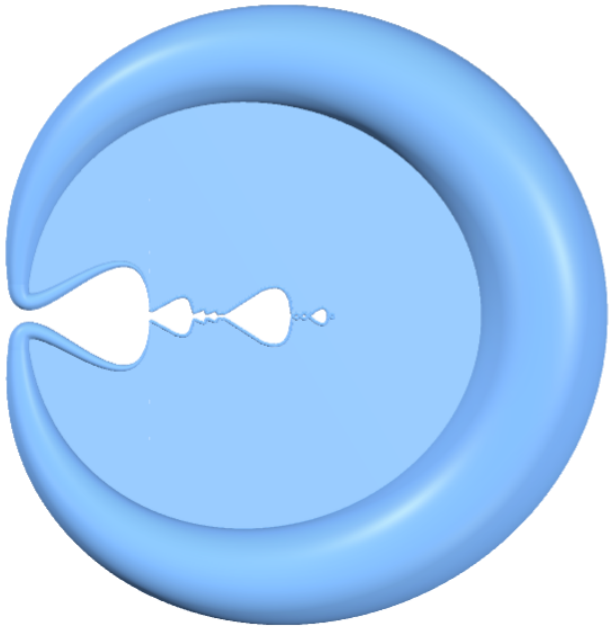

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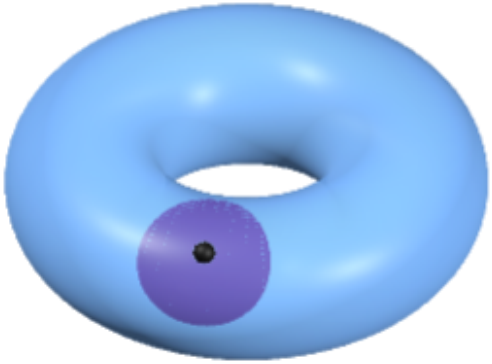

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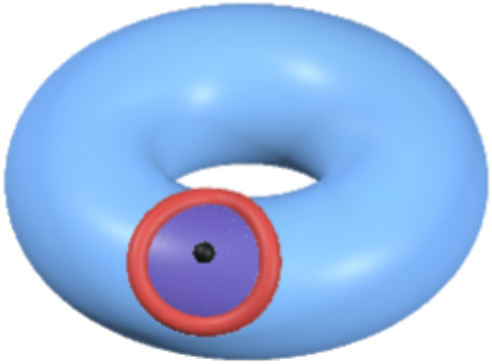

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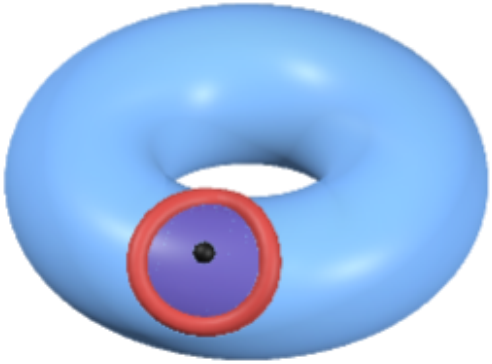
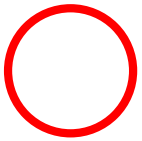


Links: 

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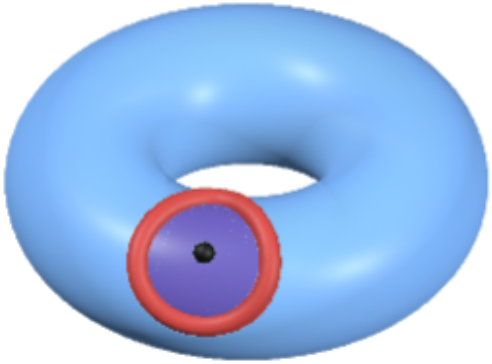
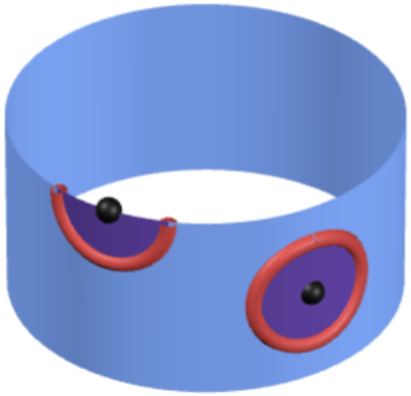
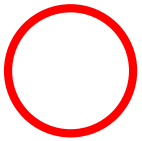
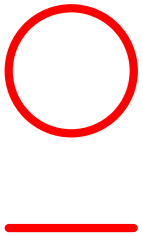




Links: 



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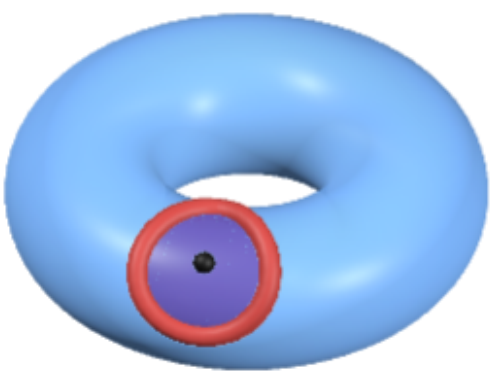

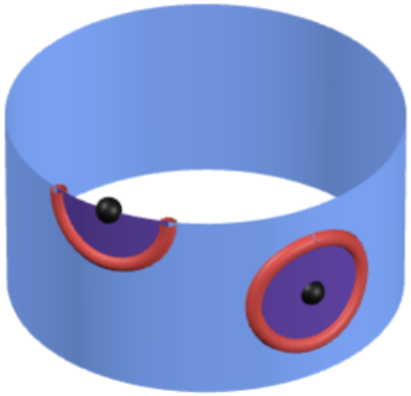

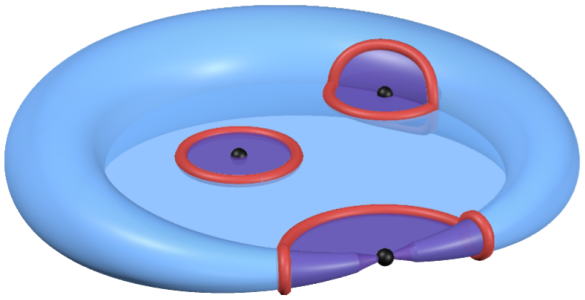

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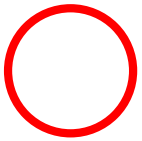

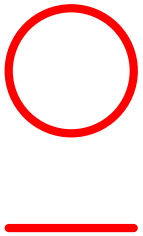

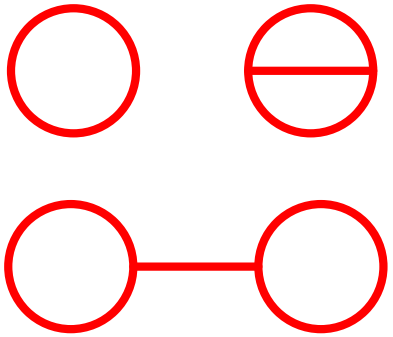

Computable type: links

In a two-dimensional simple space, the links are finite graphs.

Theorem (Amir, H. 2022)

Such a space has computable type $\iff$ *each link is a union of cycles.*

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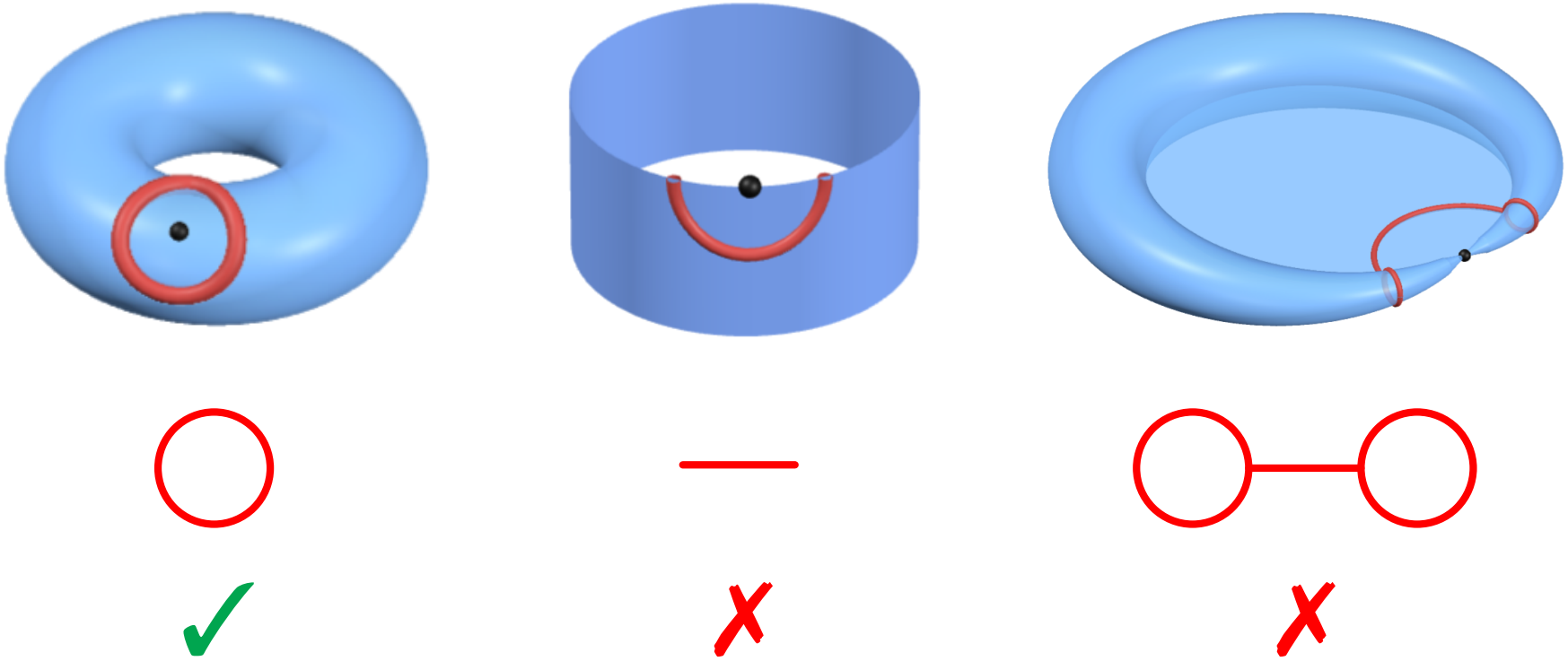

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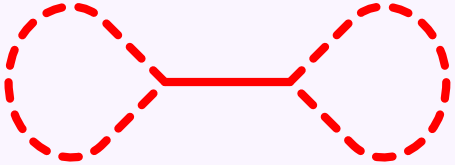
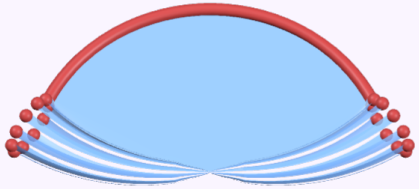
Proof idea of $\Longrightarrow$ **.**

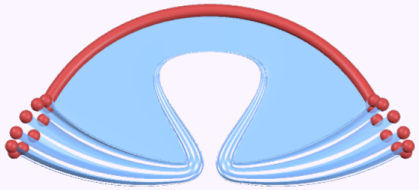
- If a link is not a union of cycles:



- Then locally, the space looks like:



- So it can be distorted this way:

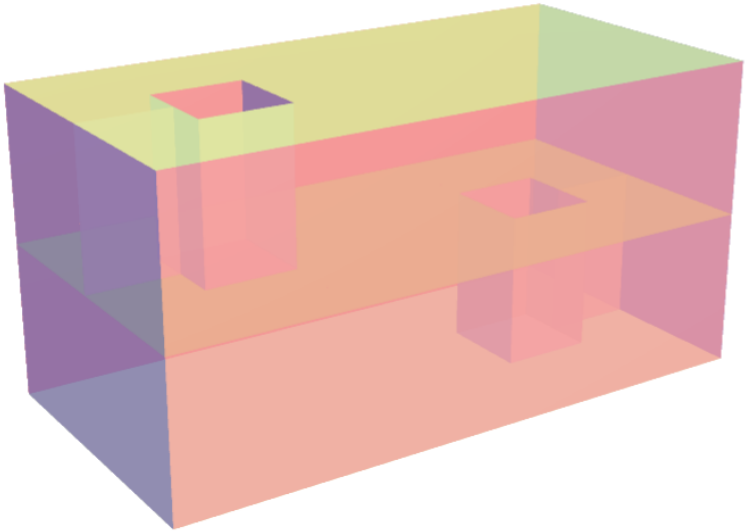


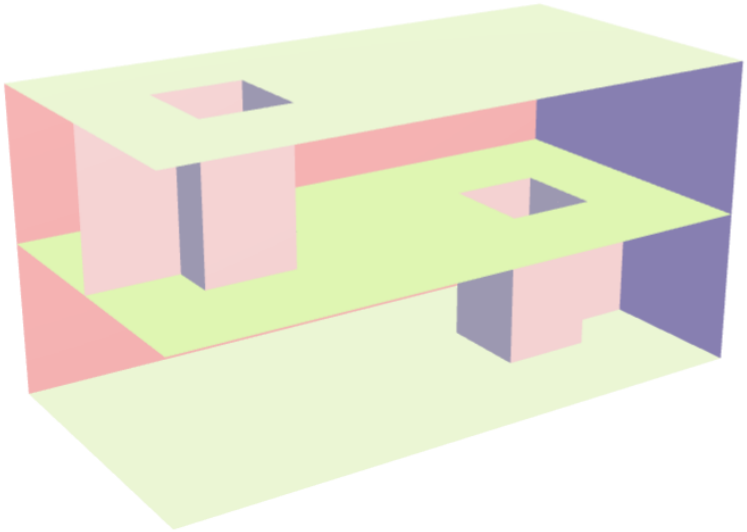
- We can encode the halting set.

$\square$

Computable type: links

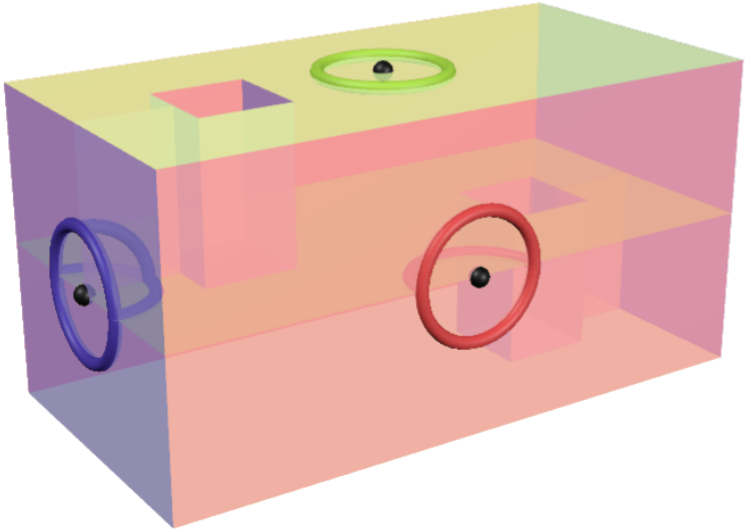
Example: Bing's house

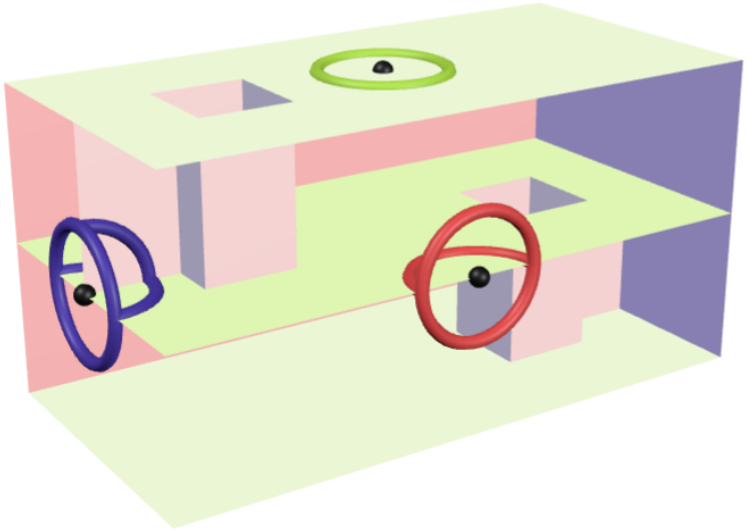




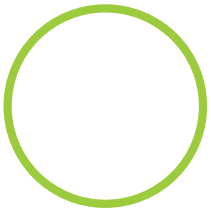
Computable type: links

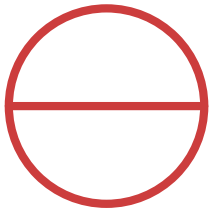
Example: Bing's house

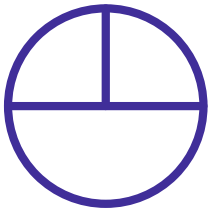




Bing's house has computable type, because its links are unions of cycles:







Computable type: links

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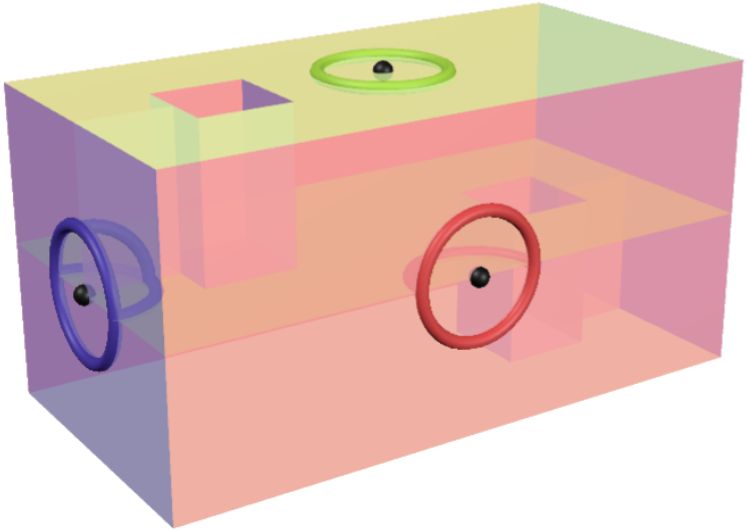

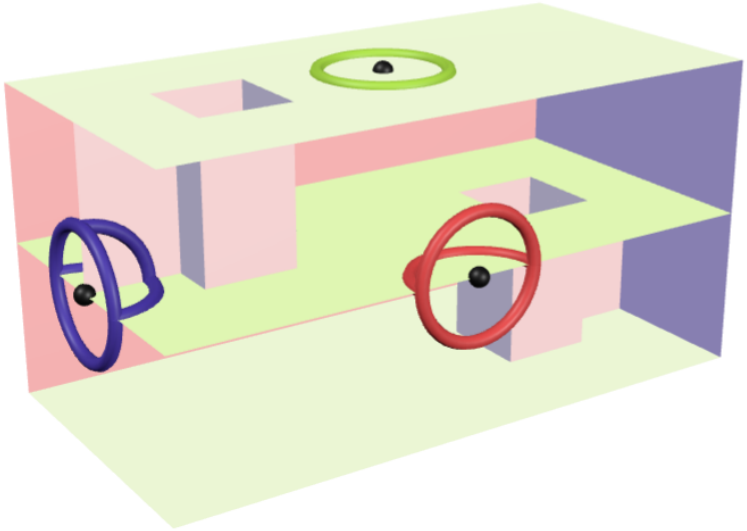

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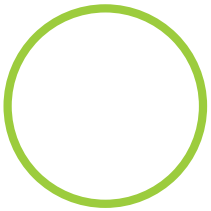
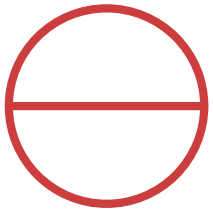
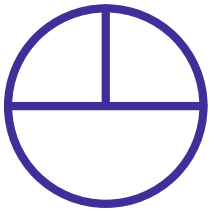






Bing's house is contractible, but cannot be "slightly squeezed".

Computable type: links

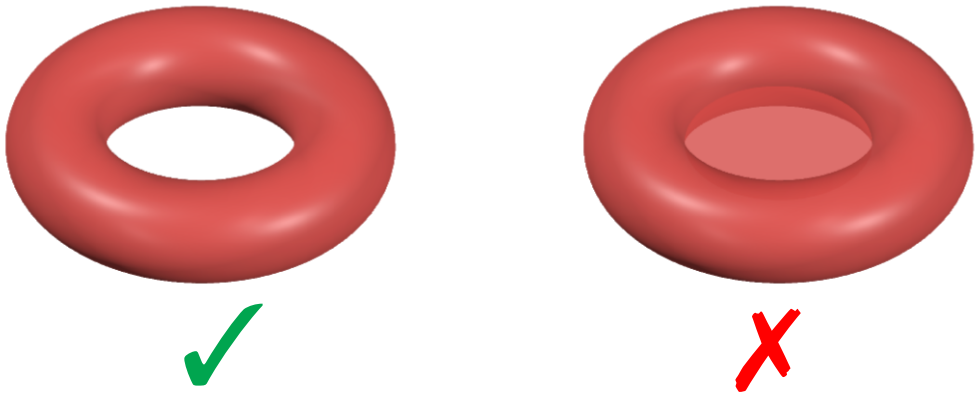
Higher dimensions

In a space of dimension n, the links have dimension $n-1$.

Theorem (Amir, H. 2024)

A purely n-dimensional simple space has computable type $\iff$ its links are unions of cycles[1].

Examples of links of dimension 2:



[1] homology cycles

Computable type

Encoding information in a space

Countably-based spaces

Encoding information in a space

Can shape encode information?

Encoding information in a space

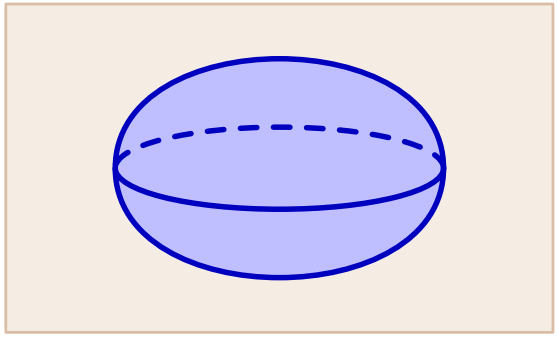
- Alice wants to send a message $M = 0100101110\ldots$ to Bob,

Alice $\xrightarrow{M}$ Bob

Encoding information in a space

- Alice wants to send a message $M = 0100101110\ldots$ to Bob,
- Alice encodes M in a topological space X_M and sends X_M,

Alice



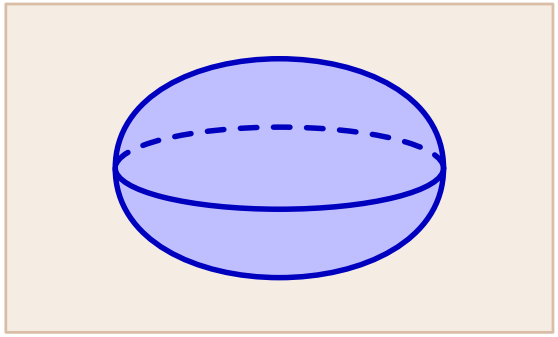
$\longrightarrow$

Bob

Encoding information in a space

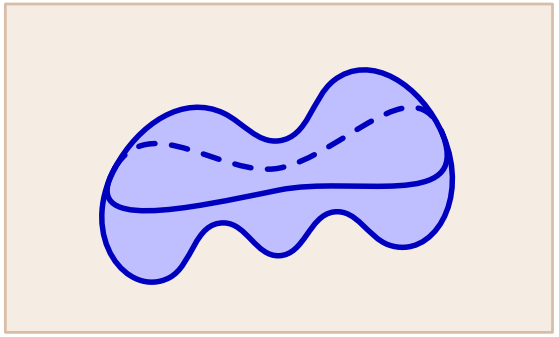
- Alice wants to send a message $M = 0100101110\ldots$ to Bob,
- Alice encodes M in a topological space X_M and sends X_M,
- Bob receives a continuous deformation of X_M.

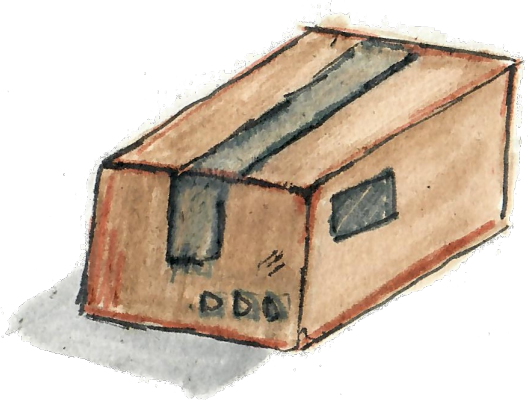
Alice



$\longrightarrow$

Bob





Presentations

We only consider compact metrizable spaces.

Definition

A **presentation** of a space X is:

- A compact set $K \subseteq [0,1]^n$ homeomorphic to X,
- Positive + negative information.

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Alice sends a presentation of X_M, Bob receives another presentation of X_M.

Sending messages: naive attempt

Alice sends this message:

Sending messages: naive attempt

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0|0 0|0||| 0 ...

Bob may receive this message:

0|0|0|0|0| ...

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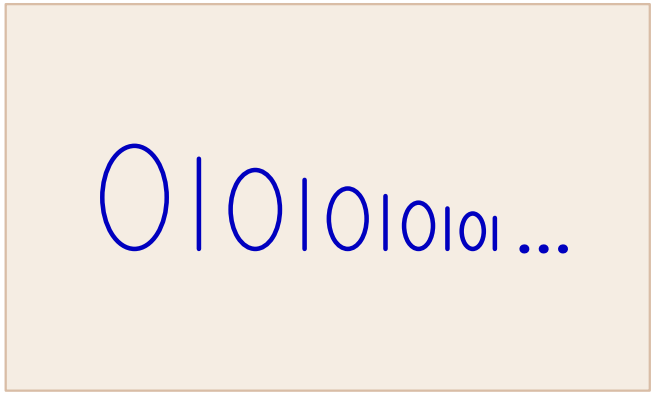
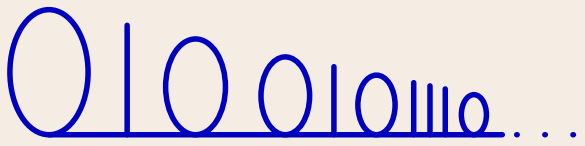


Problem

All messages with infinitely many 0's and 1's are homeomorphic.

Sending messages

Alice sends this message:



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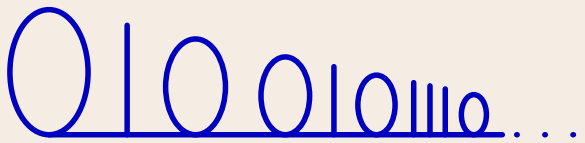

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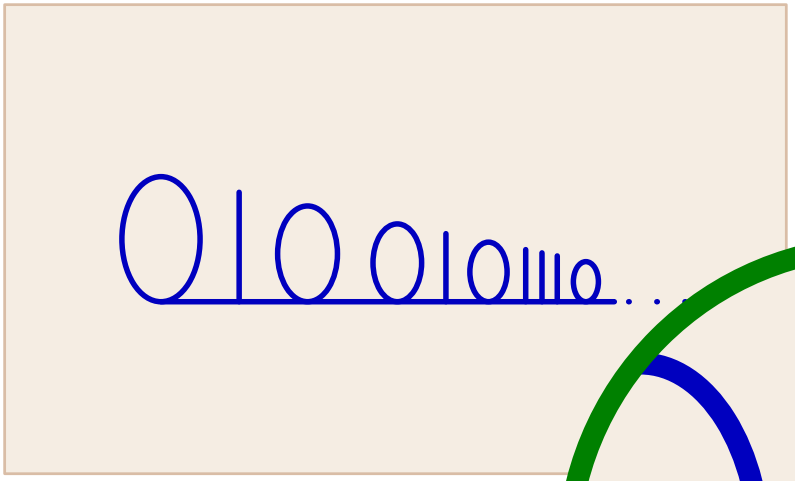

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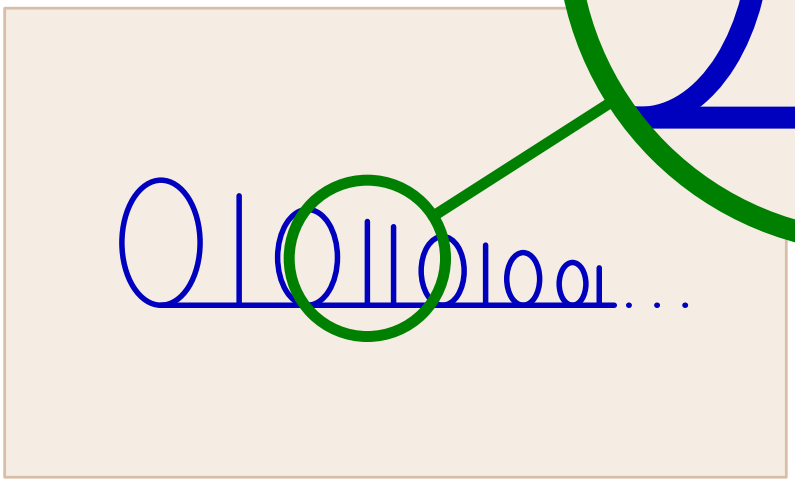

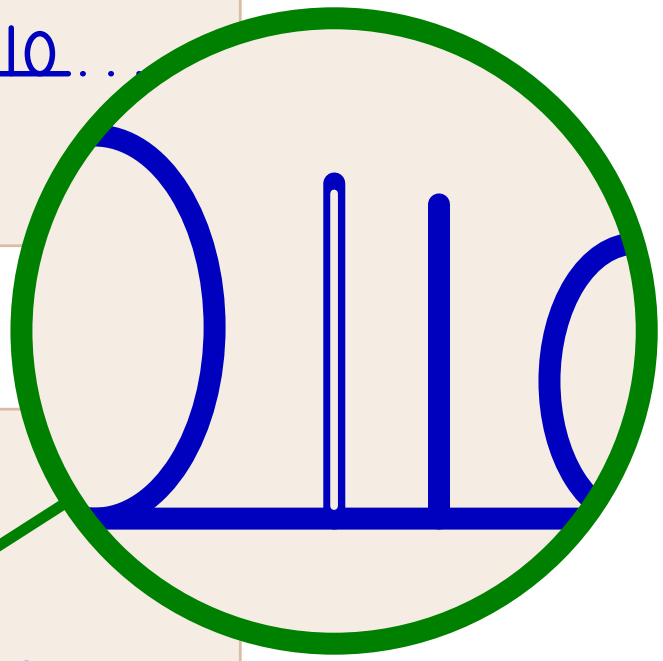

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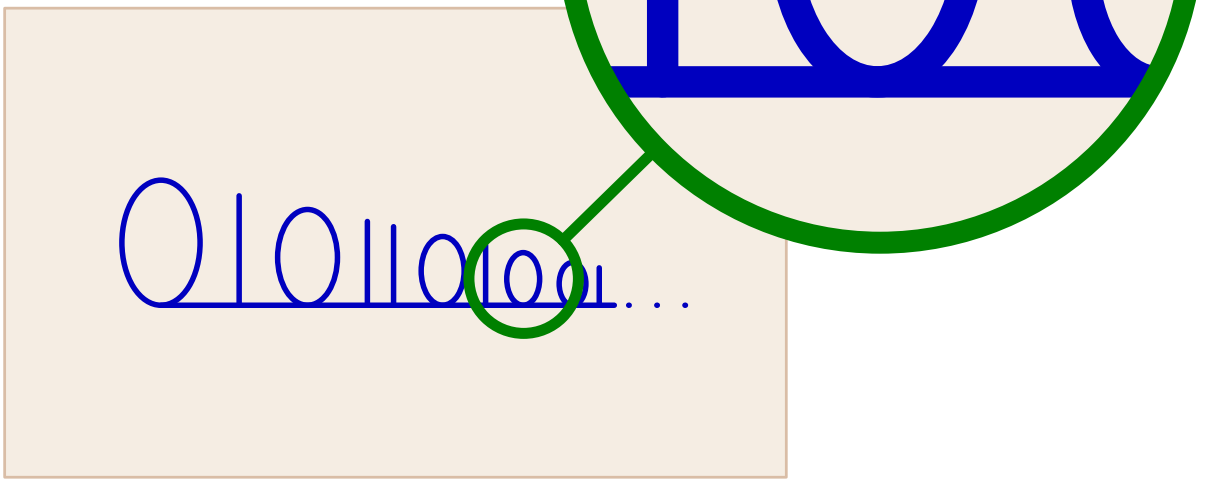

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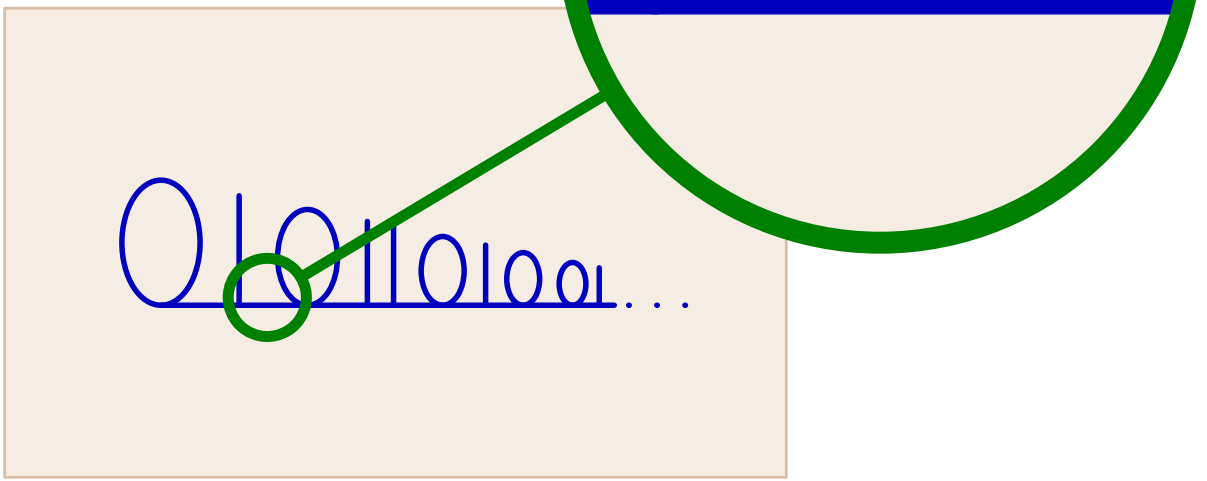


Binary sequences vs sets

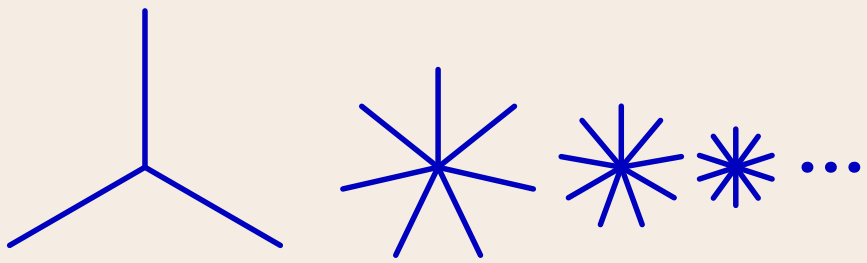
Now the message is a set $A \subseteq \mathbb{N}$.

$$\text{Computing a binary sequence } x_0 x_1 x_2 \ldots$$
$$\Longleftrightarrow$$
$$\text{Enumerating the set } \{2n : x_n = 0\} \cup \{2n+1 : x_n = 1\}.$$

Stars

Message: $\{3, 7, 9, 10, \ldots\}$.

Alice sends:



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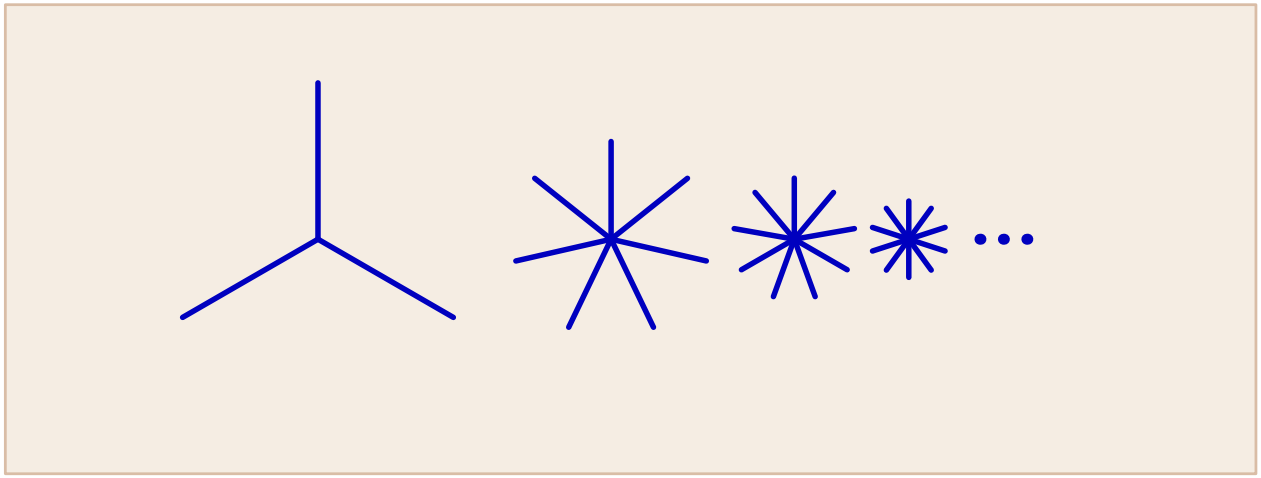

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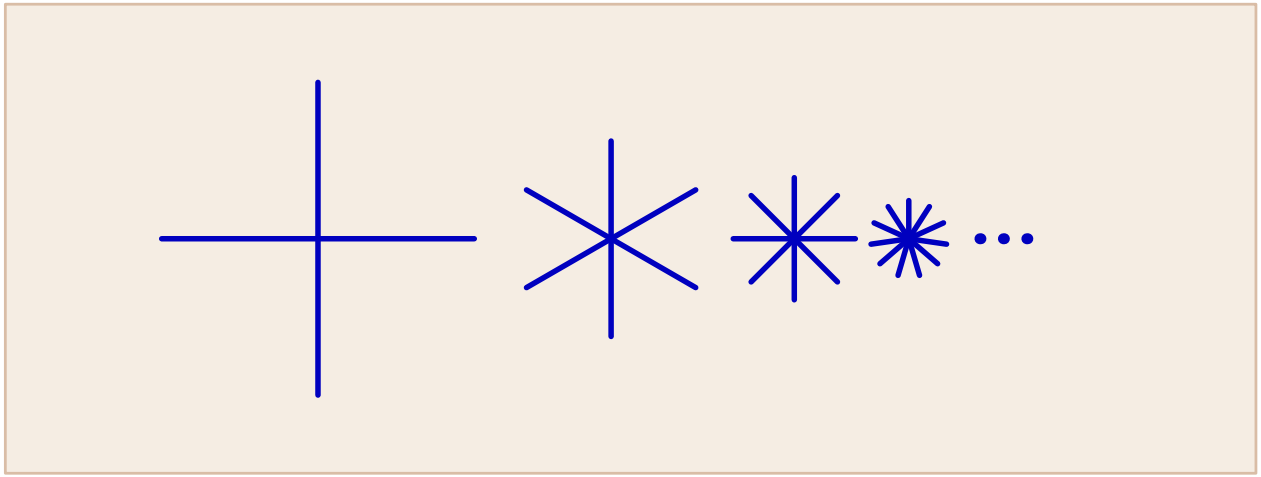

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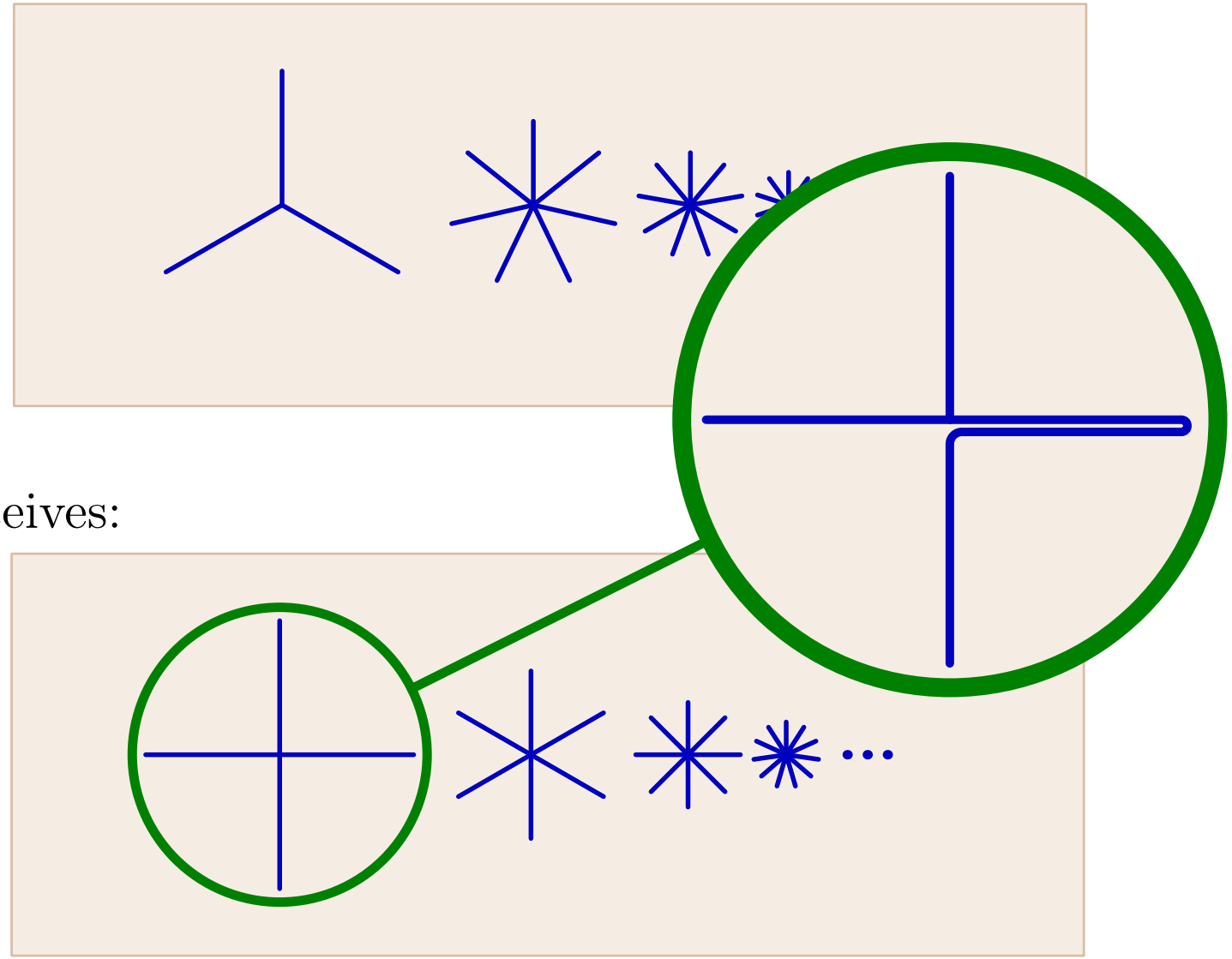

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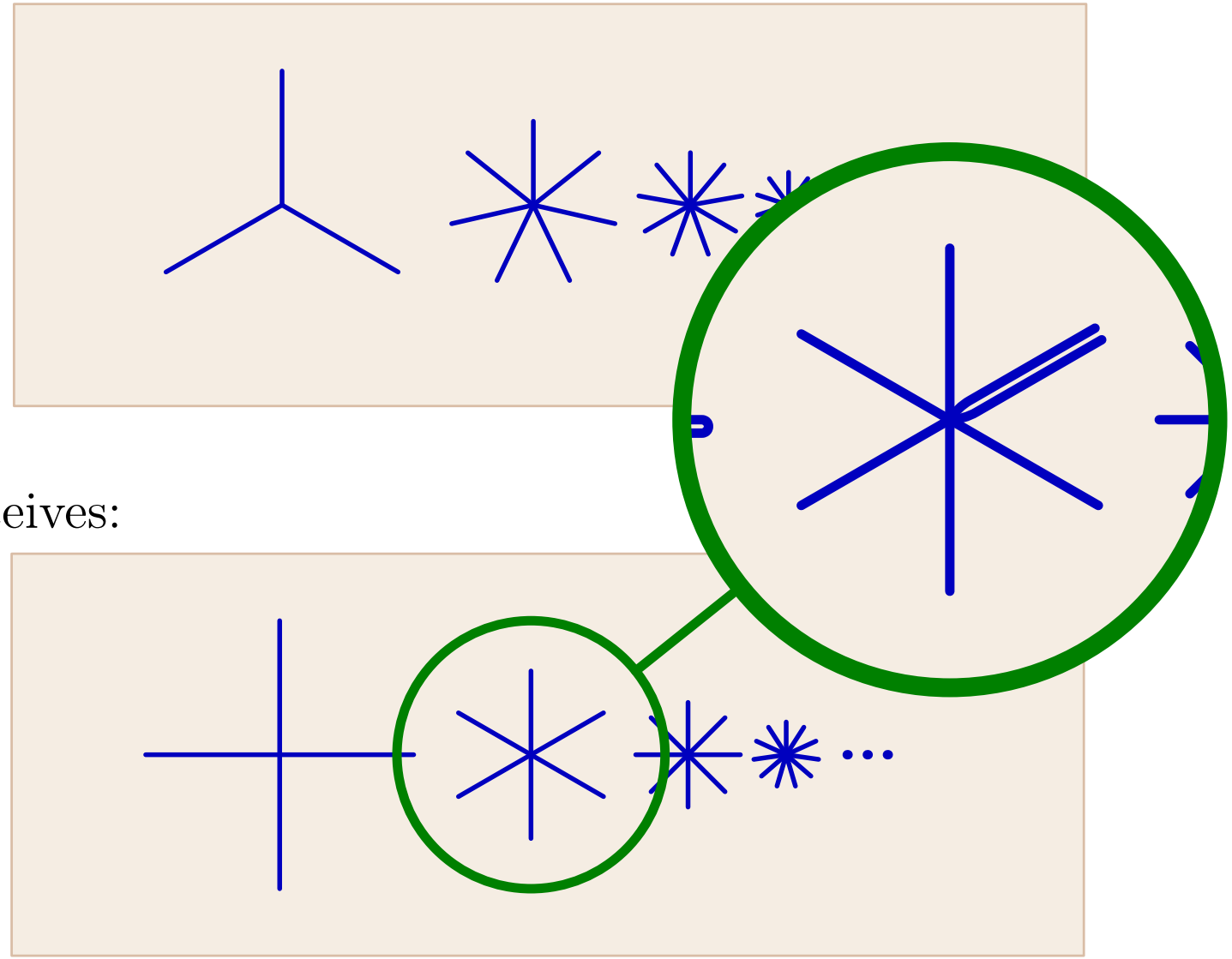

Bob receives:

Stars

Decyphering

Bob can only know A in a Σ^0_3 fashion: from the received message he can compute a set $B \subseteq \mathbb{N}^4$ such that

$$n \in A \iff \exists i, \forall j, \exists k, (n, i, j, k) \in B.$$

This is optimal.

Stars

Precise result

Let X_A be the disjoint union of:

- One star S_{2k} for each $k \in A$,
- Infinitely many stars S_{2k+1} for each $k \in \mathbb{N}$.

Theorem (Melnikov, Harrison-Trainor, Ng, 2020)

An oracle Z computes a presentation of $X_A \iff A$ is Σ^0_3 relative to Z.

Stars

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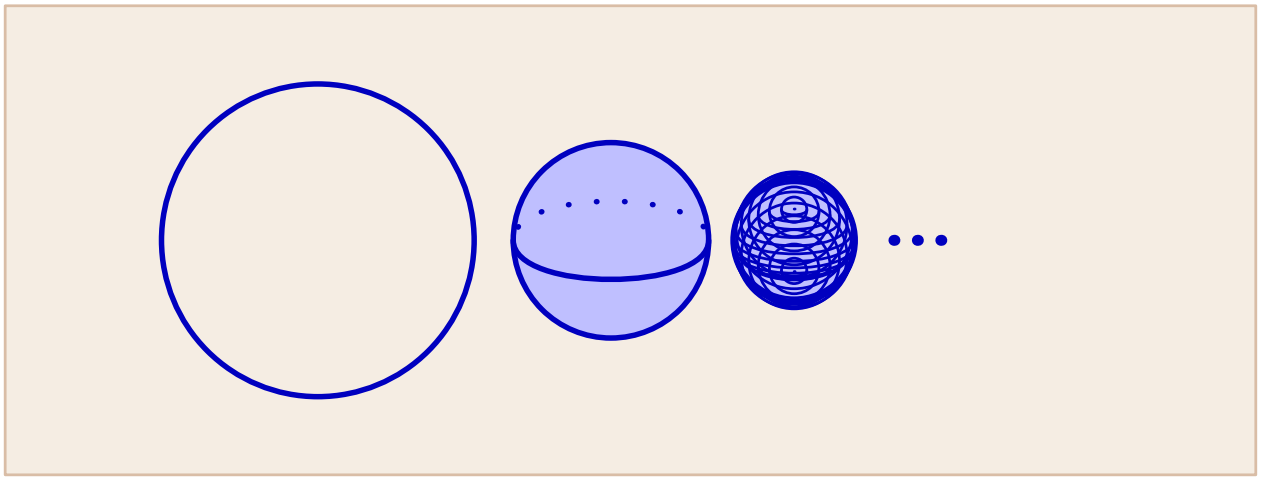
An oracle Z computes a presentation of $X_A \iff A$ is Σ^0_3 relative to Z.

Question

Can we encode A into a space Y_A in a Σ^0_2 way? in a Σ^0_1 way?

Spheres

Let $Y_A = \bigsqcup_{k \in \mathcal{A}} \mathbb{S}_{2k} \sqcup \bigsqcup_{k \notin A} \mathbb{S}_{2k+1}$.



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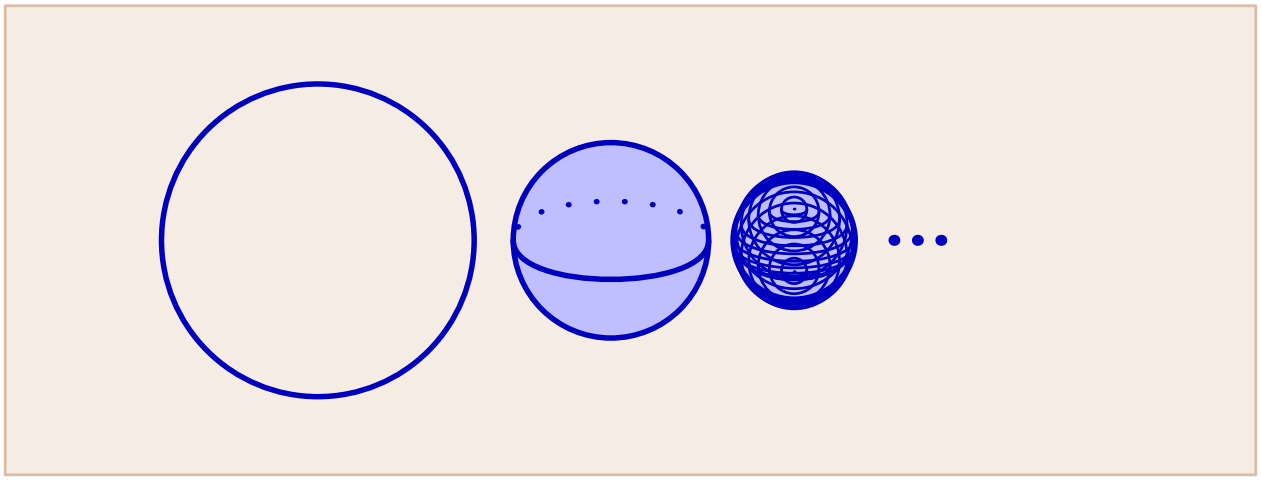

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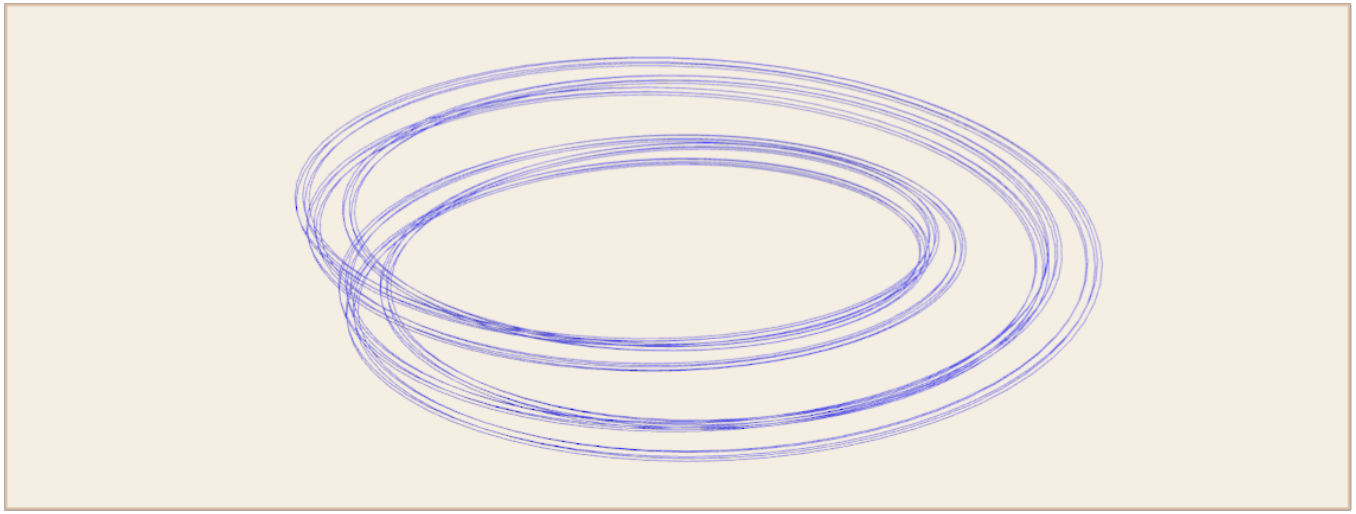
Question

What about Σ^0_1?

Solenoid



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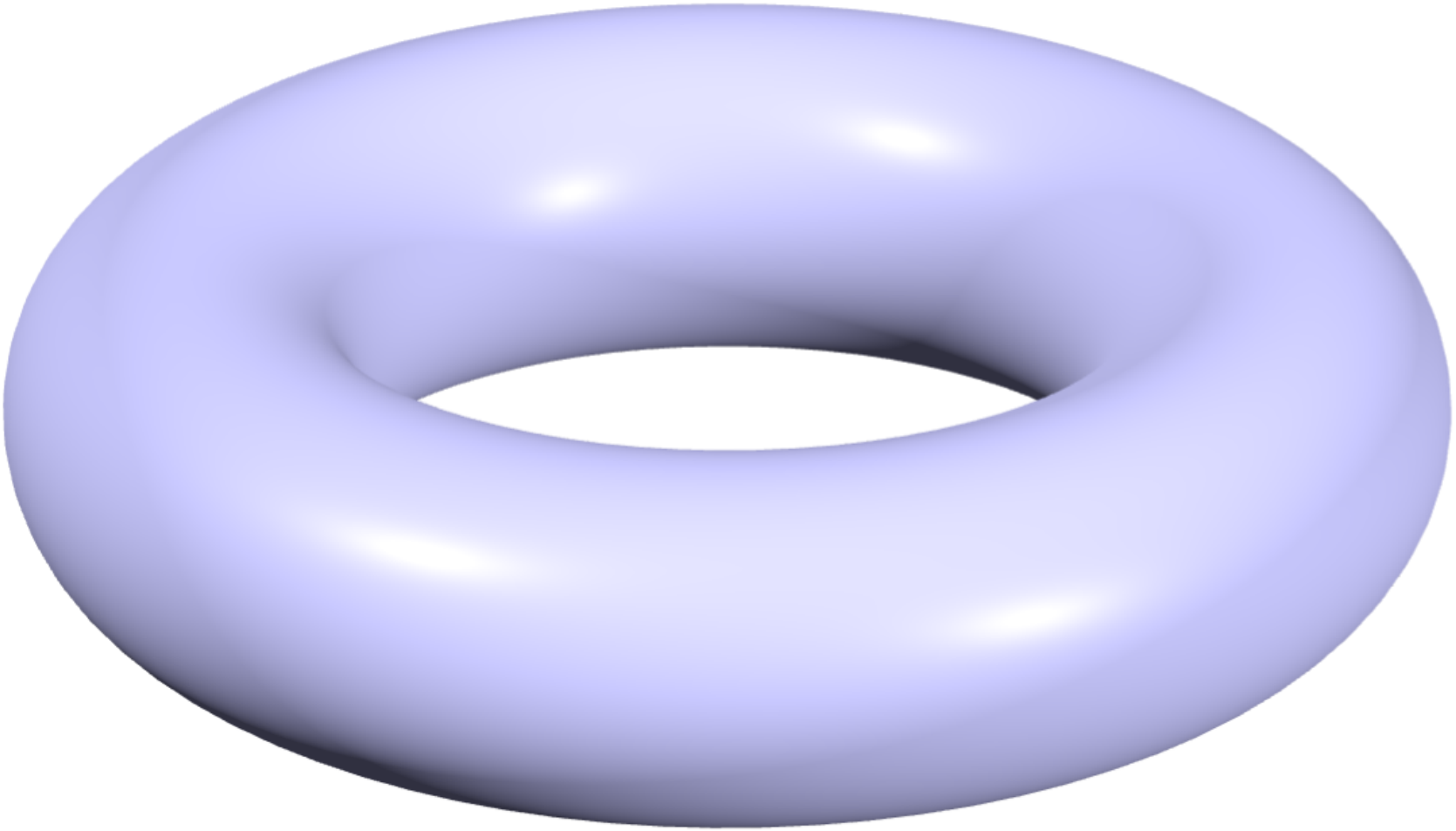
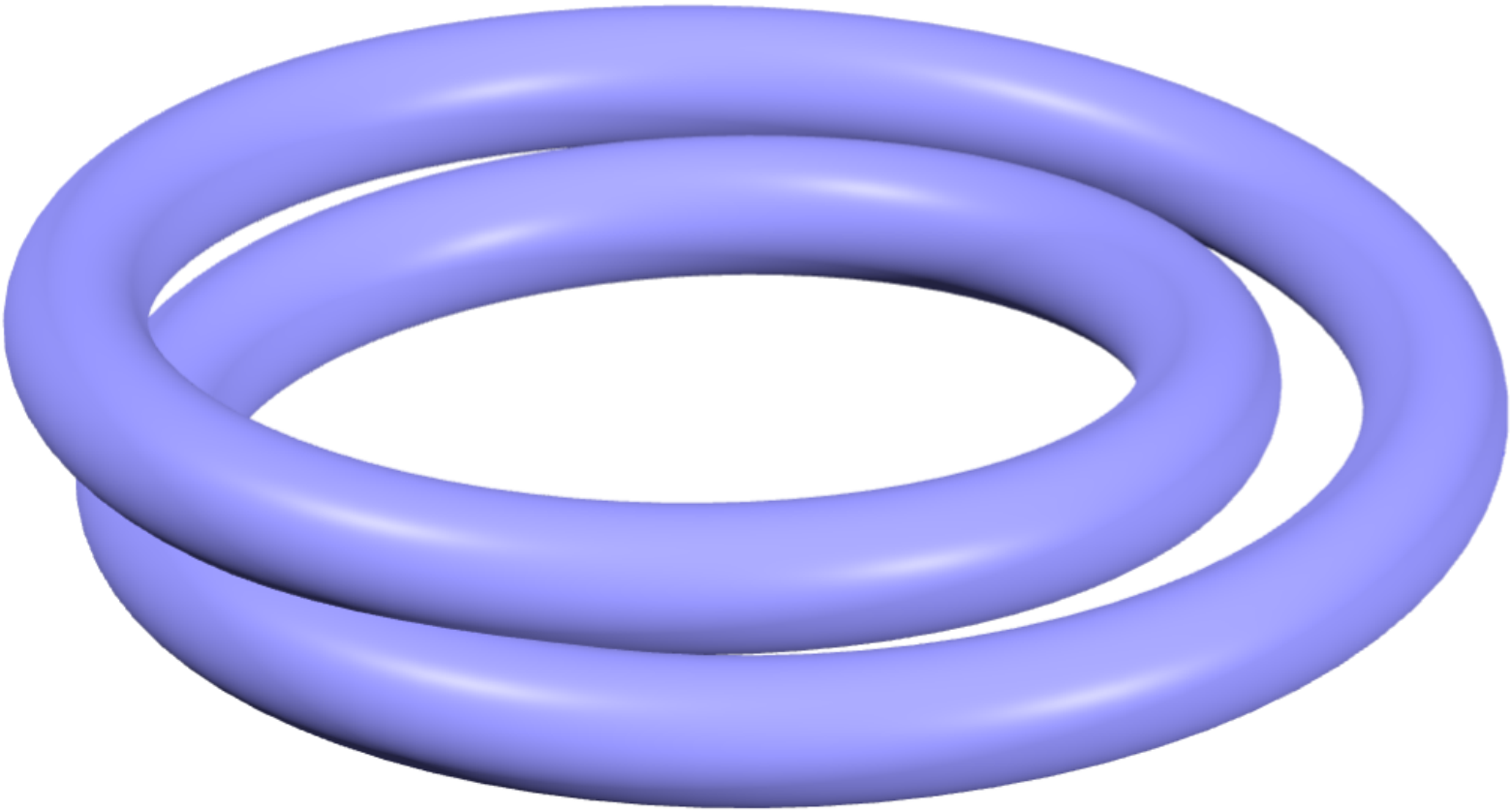


$$A = \{2, 3, 5, \ldots\}$$

Solenoid



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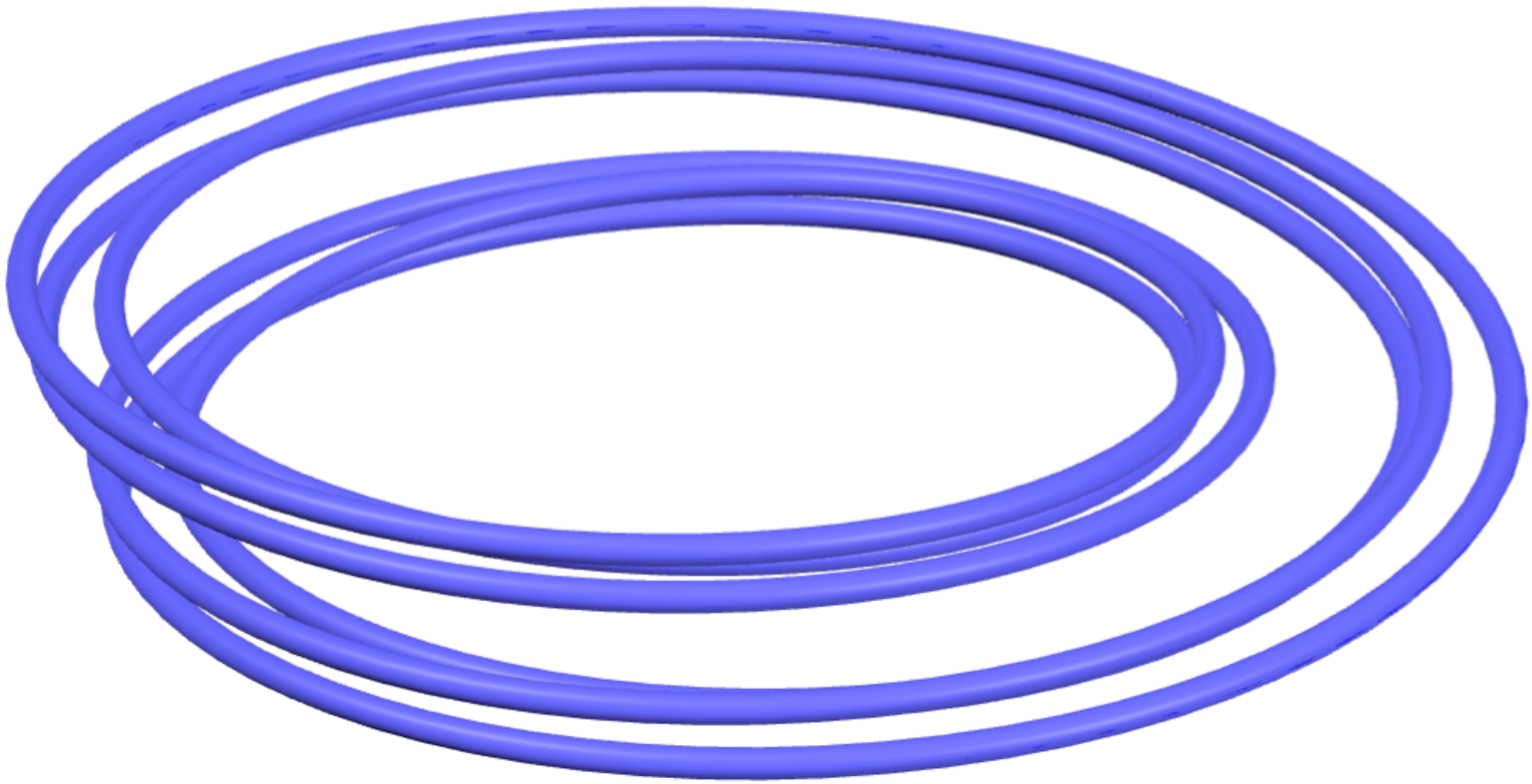

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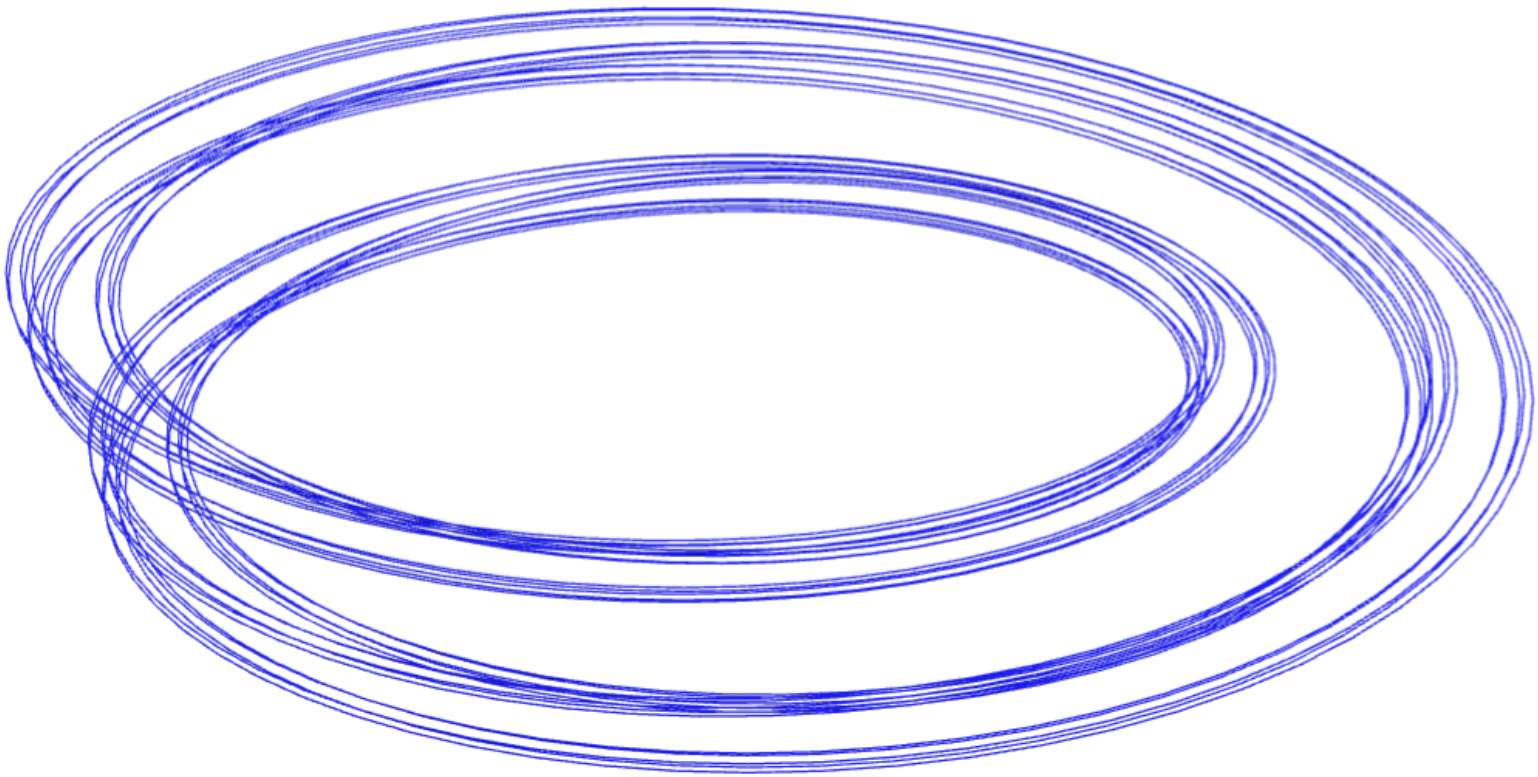

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Solenoid

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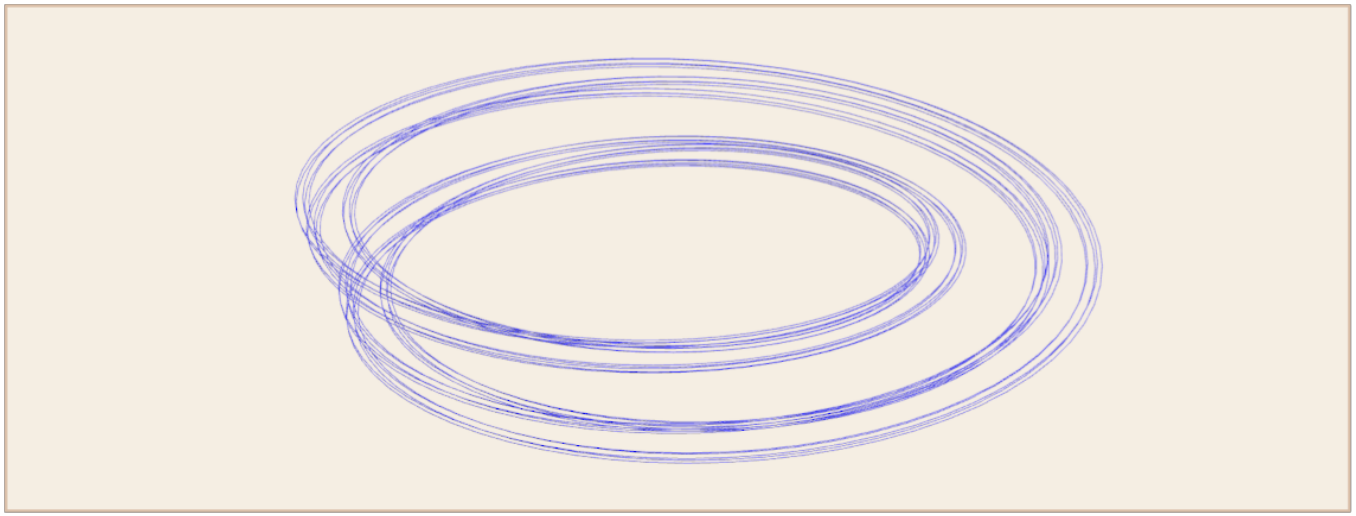
Alice sends $\text{Solenoid}(2, 3, 5, \ldots)$:



Bob can enumerate A from any presentation of that space.

Solenoid

Let $p_0, p_1, \ldots$ be the prime numbers. For $A = \{a_0, a_1, \ldots\} \subseteq \mathbb{N}$, let

$$X_A = \mathrm{Solenoid}(p_{a_0}, p_{a_1}, \ldots).$$

Theorem (Lupini, Melnikov, Nies 2023)

An oracle Z computes a presentation of $X_A \iff A$ is Σ^0_1 relative to Z.

Proof.

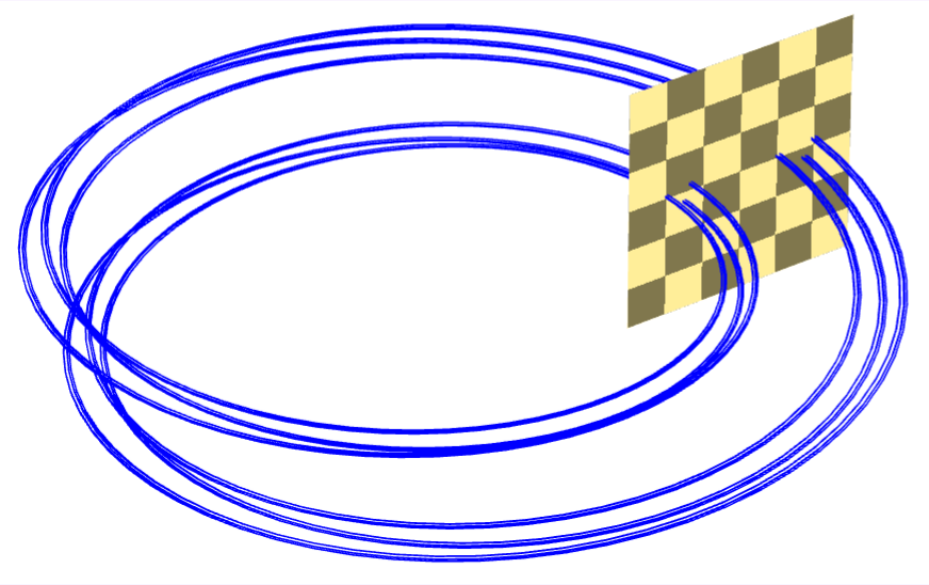
Use cohomology... $\square$

Solenoid: cohomology

To each space X is associated a group $\check{H}^1(X)$, the **Čech cohomology** group.

Elements of $\check{H}^1(X)$:

Ways of counting turns. Operators assigning a number to each path.



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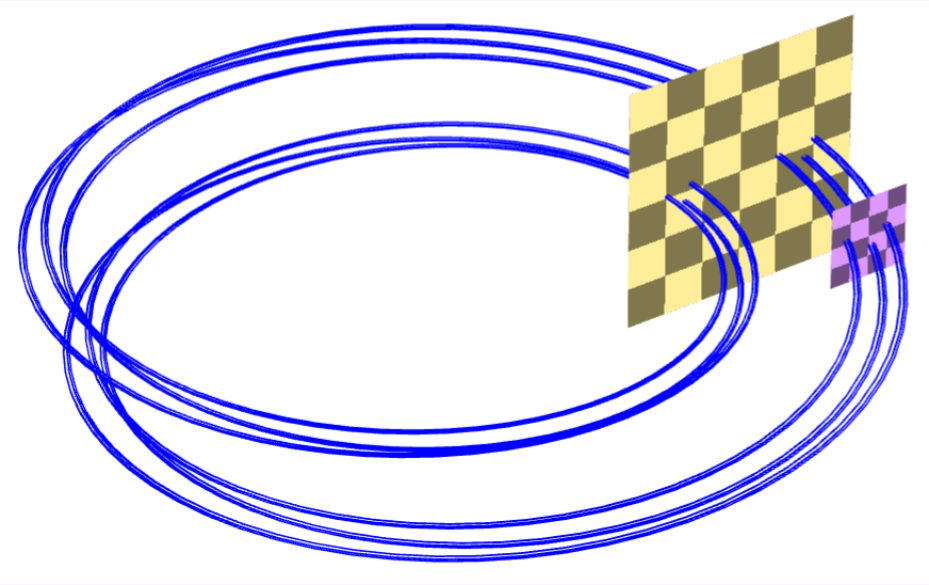

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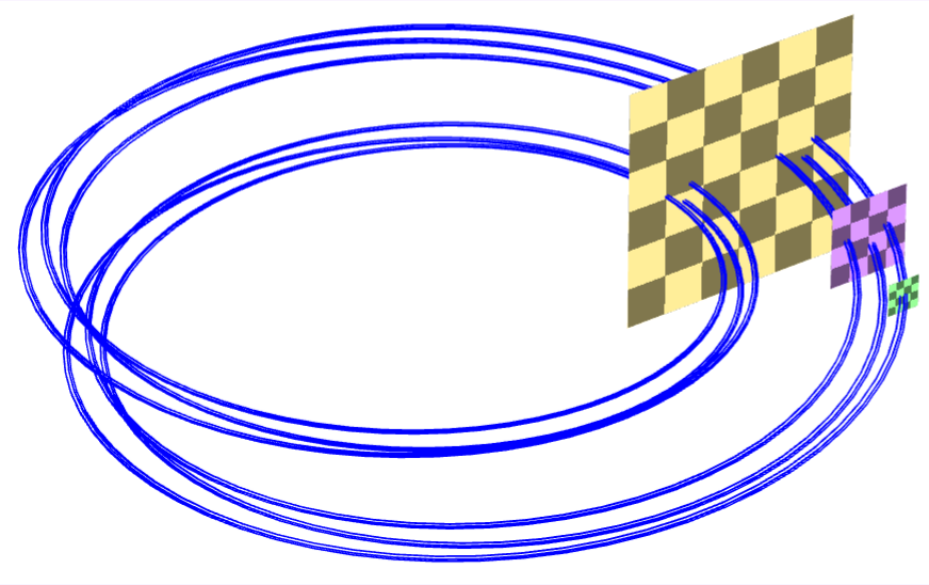

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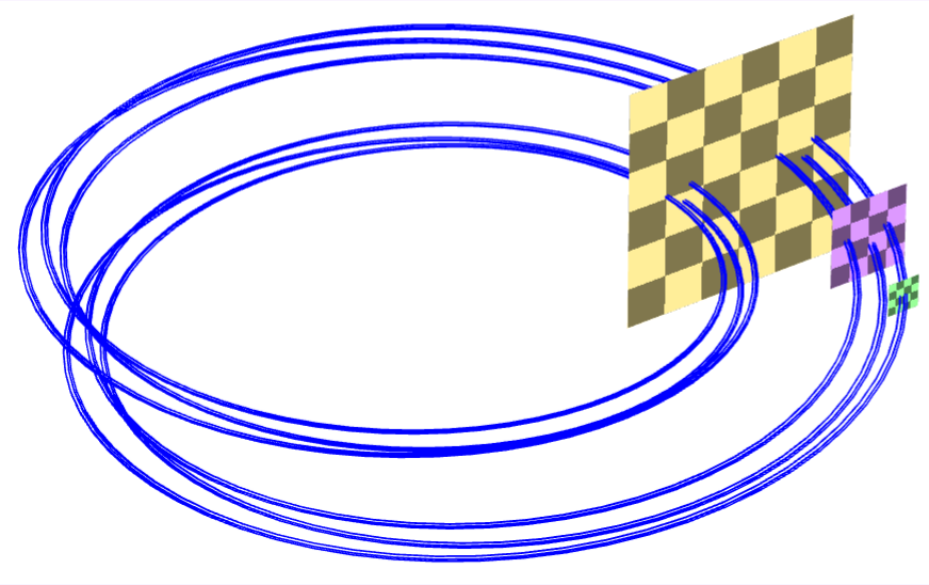
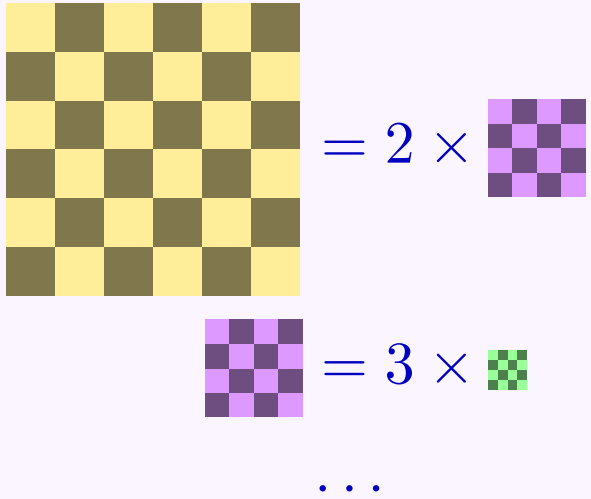
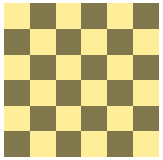
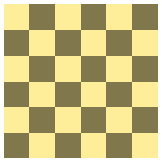


Group relations:

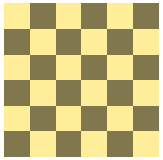


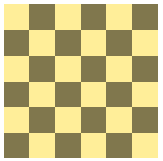
Solenoid: cohomology

$n \in A \iff$  is divisible by p_n

$\iff \exists x \in \check{H}^1(X_A)$ such that  $= p_n \times x$.

Solenoid: cohomology

$n \in A \iff$  is divisible by p_n

$\iff \exists x \in \check{H}^1(X_A)$ such that  $= p_n \times x$.

Theorem (Lupini, Melnikov, Nies 2023)

A presentation of $\check{H}^1(X)$ can be computed given X.

Corollary

A can be enumerated given X_A.

Alice goes pointillist

Summary

Information can be encoded in spaces presented using positive + negative information.

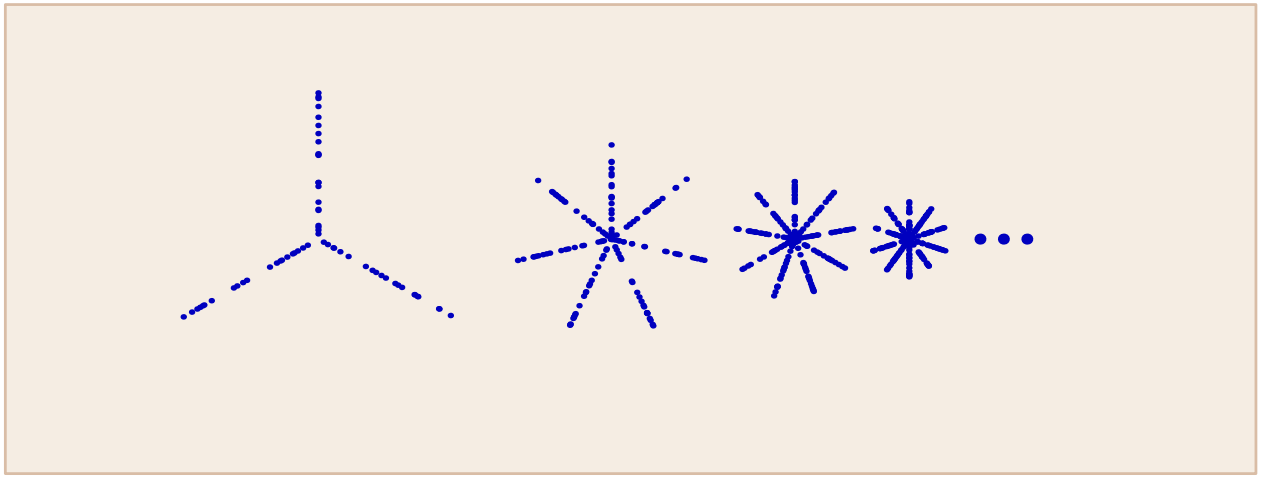
Alice goes pointillist

Summary

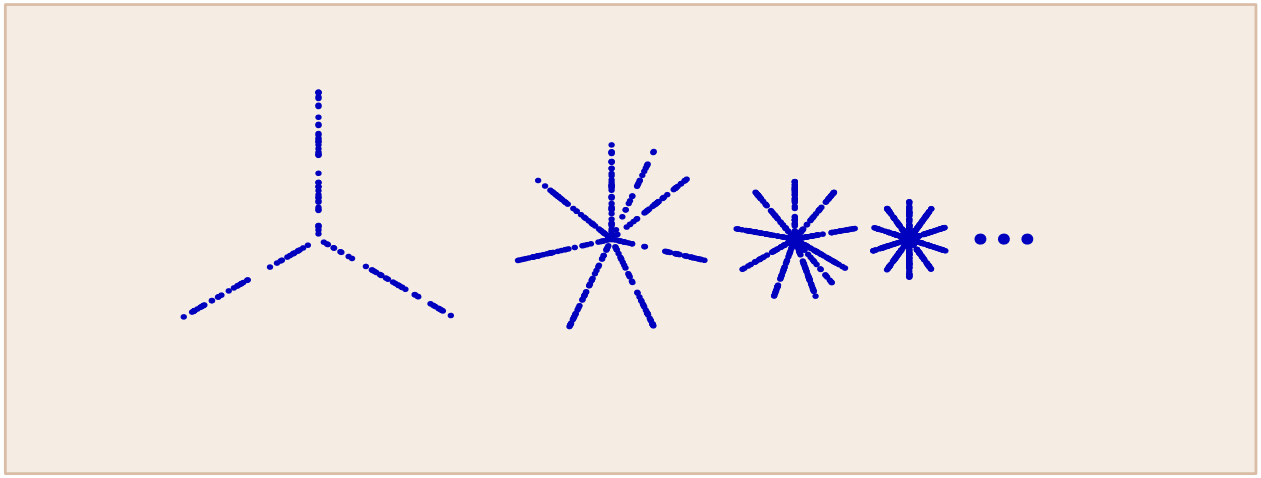
Information can be encoded in spaces presented using positive + negative information.

What about using positive information only?

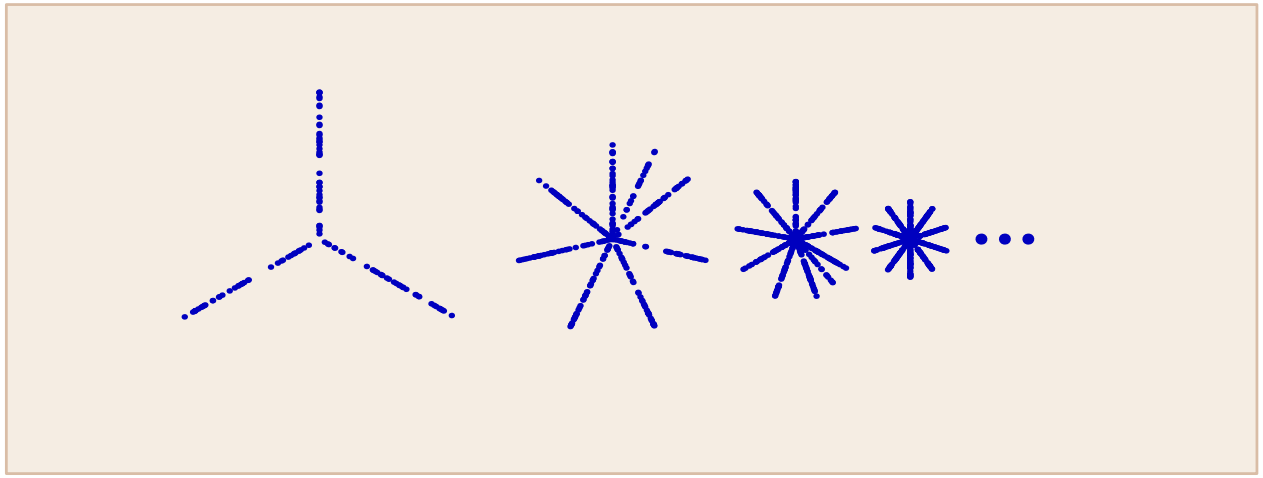
Alice goes pointillist



Alice goes pointillist



Alice goes pointillist

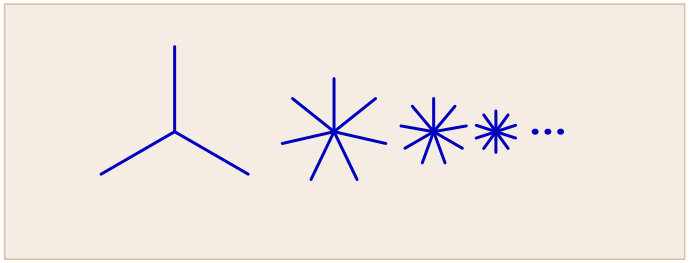


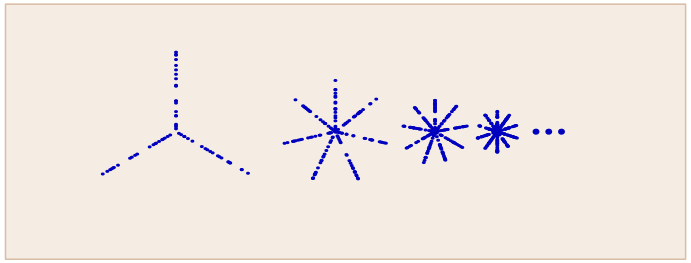
Polish presentation of X:

- A compact set $K \subseteq [0,1]^n$ homeomorphic to X,
- Enumeration of $\{i \in \mathbb{N} : B_i \cap K \neq \emptyset\}$ ($\simeq$ a dense sequence in K).

Alice goes pointillist

Σ^0_3

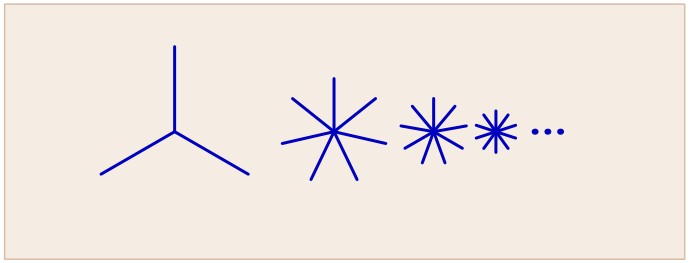


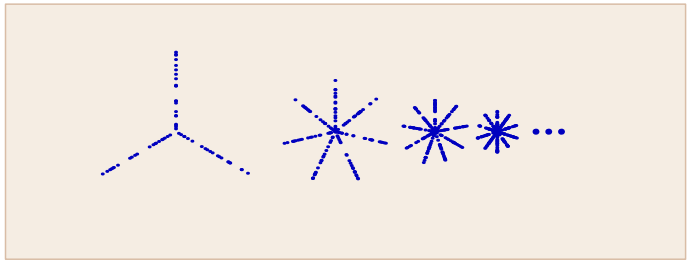


Σ^0_4

Alice goes pointillist

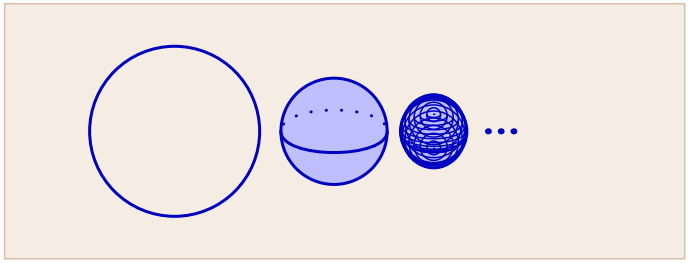
Σ^0_3

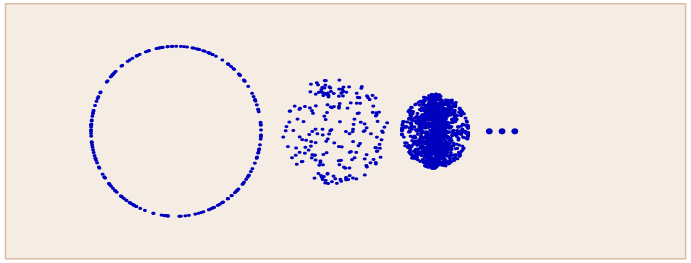




Σ^0_4

Σ^0_2

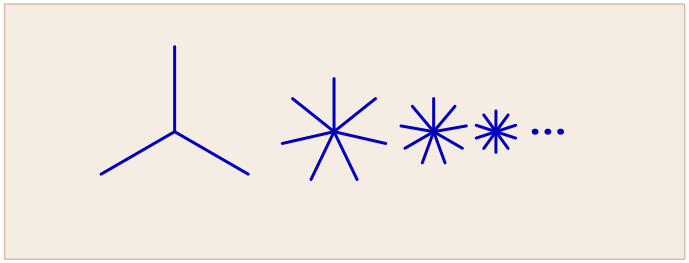


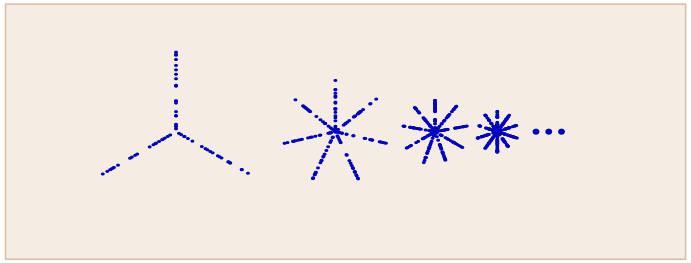


Σ^0_3

Alice goes pointillist

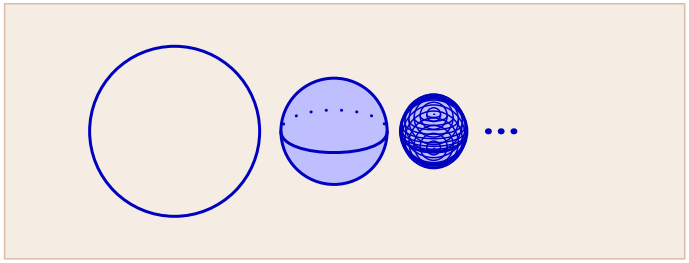
Σ^0_3

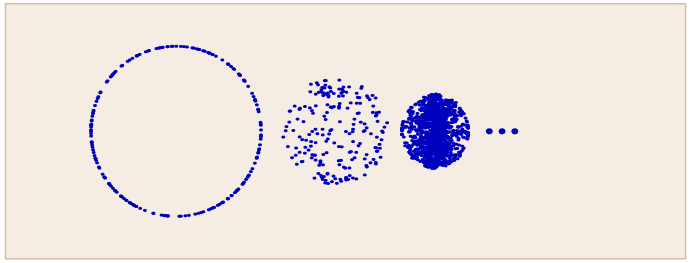




Σ^0_4

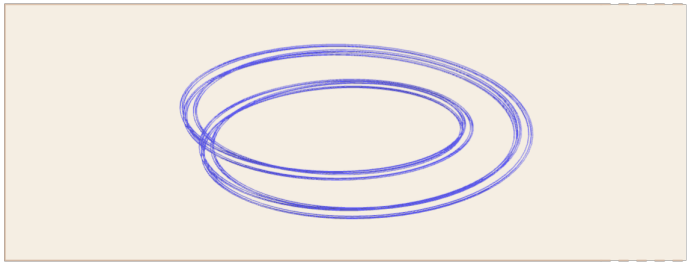
Σ^0_2

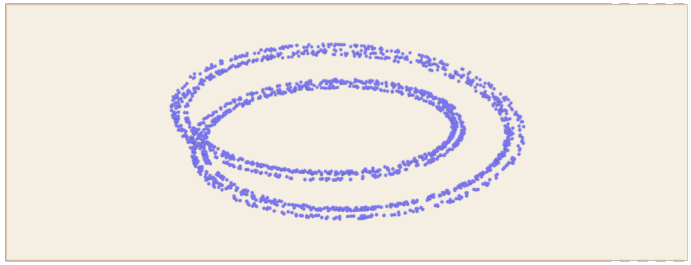




Σ^0_3

Σ^0_1

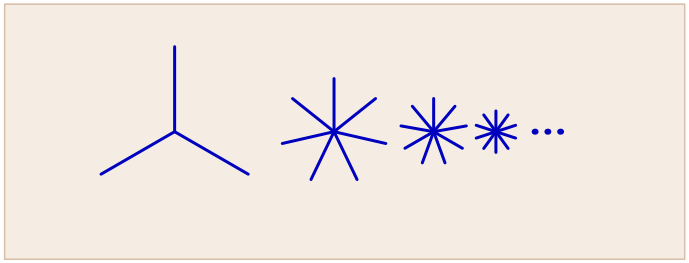




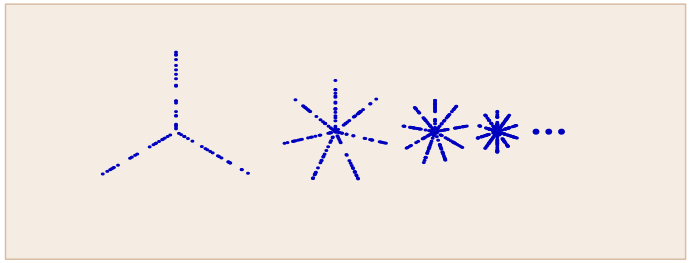
Σ^0_2

Alice goes pointillist

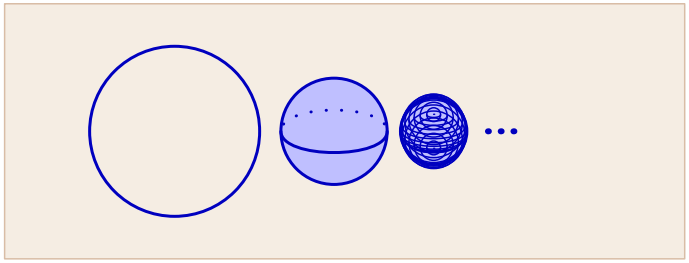
Σ^0_3



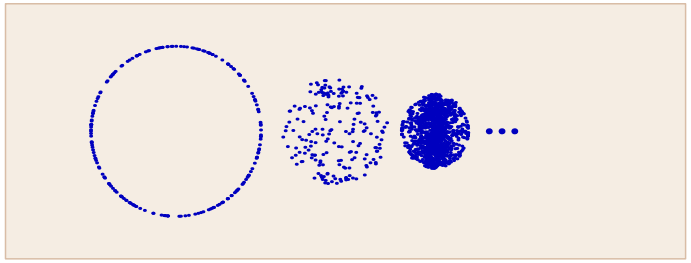
Σ^0_4



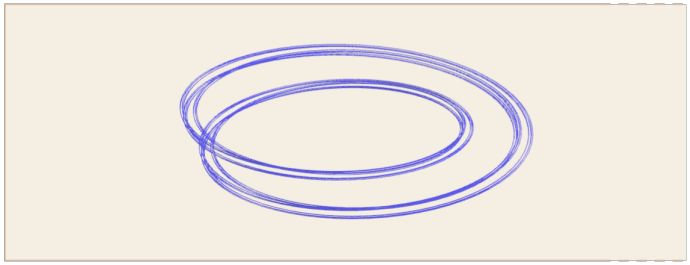
Σ^0_2



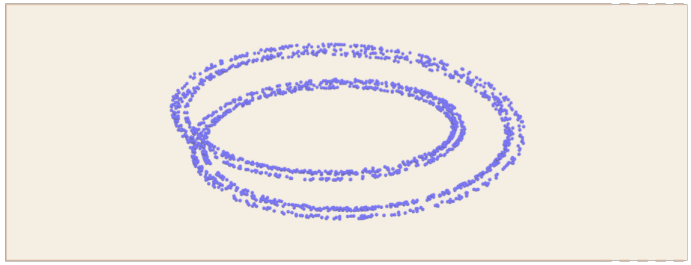
Σ^0_3

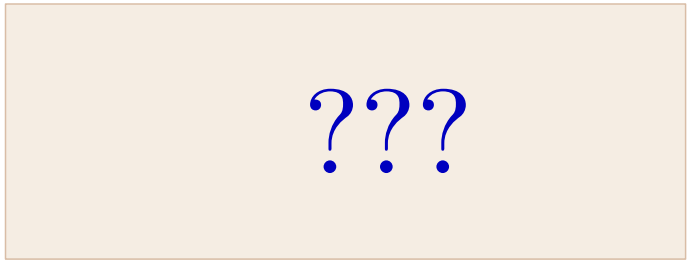


Σ^0_1



Σ^0_2





Σ^0_1?

Alice goes pointillist

Polish presentations cannot encode information in a Σ^0_1 way.

Theorem (H., Kihara, Selivanov, 2020)

Let $A \subseteq \mathbb{N}$ not Σ^0_1. There is no space X such that A is Σ^0_1 relative to any Polish presentation of X.

Computable type

Encoding information in a space

Countably-based spaces

Countably-based spaces

- Compact metrizable spaces embed in $[0,1]^{\mathbb{N}}$,
- Countably-based T_0-spaces embed in $\{0,1\}^{\mathbb{N}} = P(\mathbb{N})$ endowed with the **Scott topology**, whose basis is:

$$B_{\langle i_0,\ldots,i_n\rangle} = \{A \subseteq \mathbb{N} : \{i_0,\ldots,i_n\} \subseteq A\}.$$

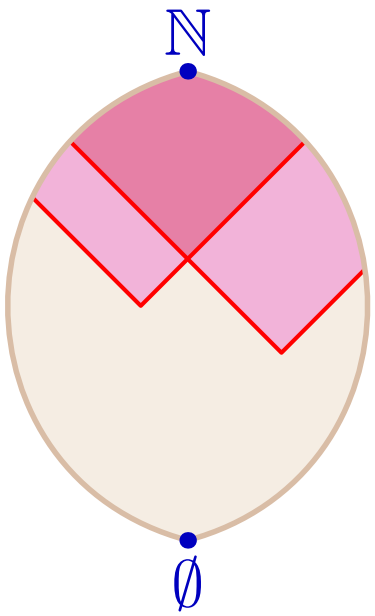


Figure: The Scott domain

Countably-based spaces

Let X be a countably-based T_0-space.

Definition

A **presentation** of X is given by:

- A set $Y \subseteq \{0,1\}^{\mathbb{N}}$ homeomorphic to X,
- Positive + negative information: characteristic function of

$$\{i \in \mathbb{N} : B_i \cap Y \neq \emptyset\}.$$

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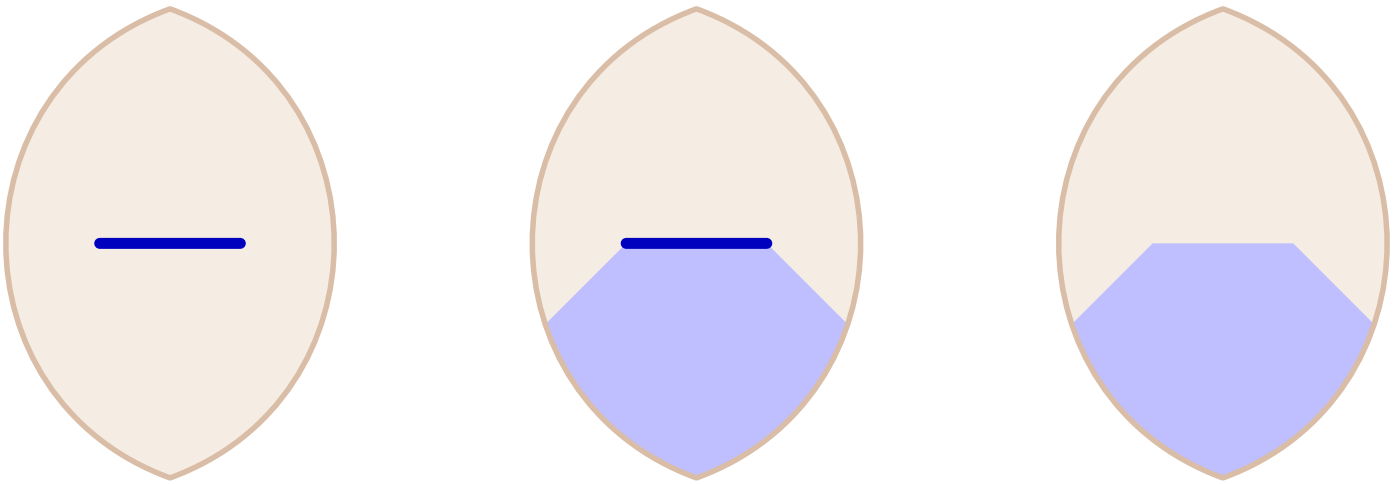
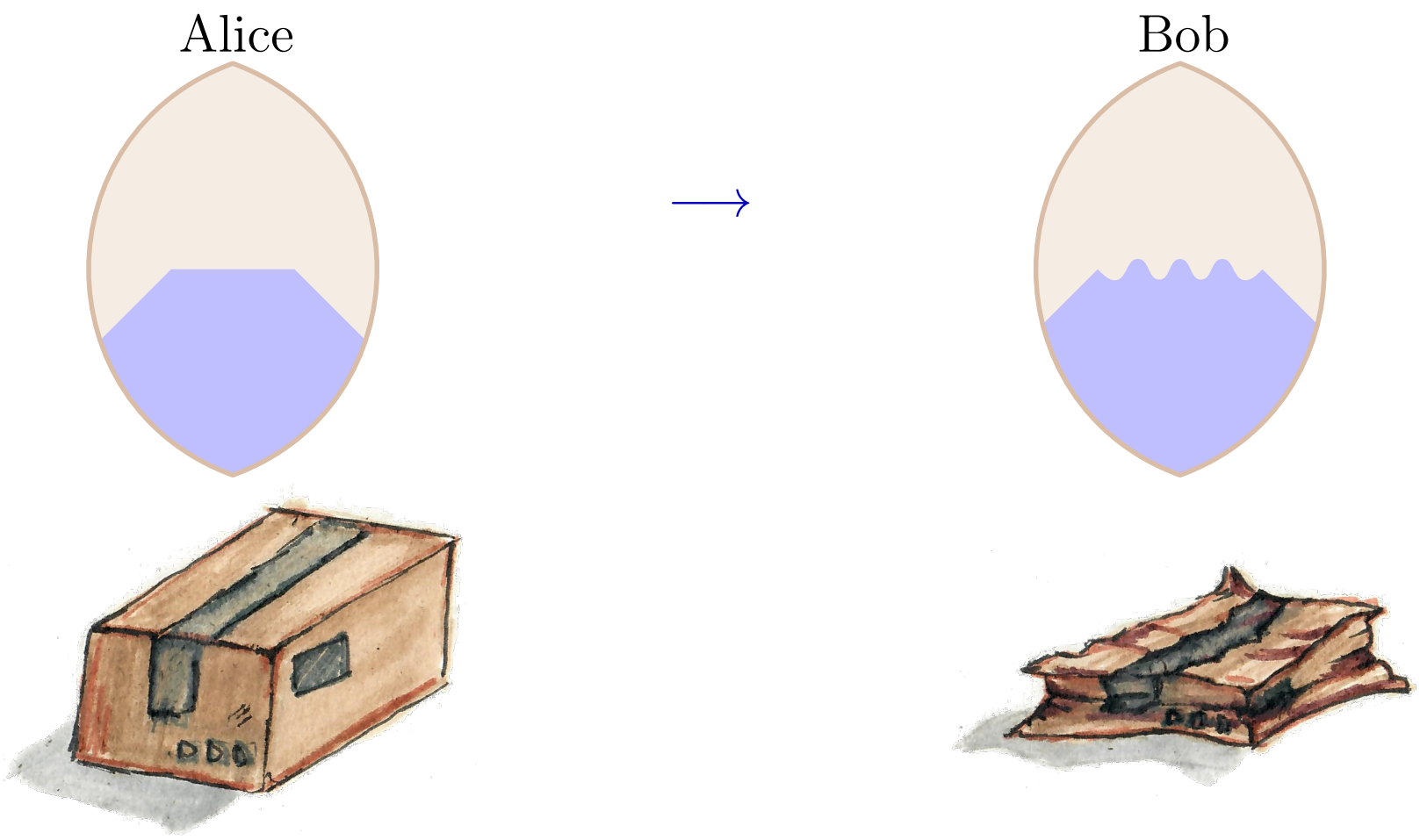


(a) Y (b) Y and its closure (c) Presentation

Countably-based space

Can we encode information in a countably-based space presented this way?



Countable-based spaces

Theorem (H., Melnikov, Ng 2025)

Every countably-based space has a computable presentation.

Countable-based spaces

Theorem (H., Melnikov, Ng 2025)

Every countably-based space has a computable presentation.

Theorem (H., Melnikov, Ng 2025)

All the perfect compact metrizable spaces share a computable presentation.

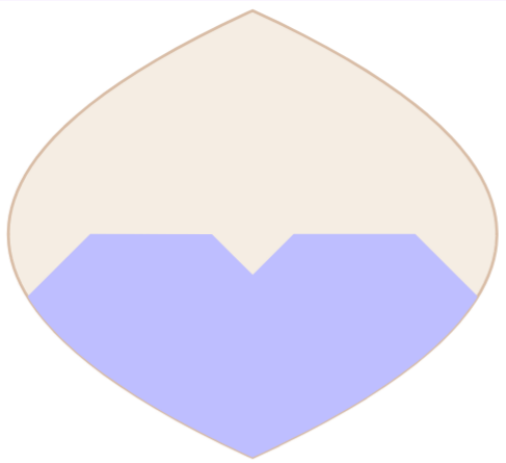
Proof.

They all "look like" the Cantor space. □

Countably-based spaces

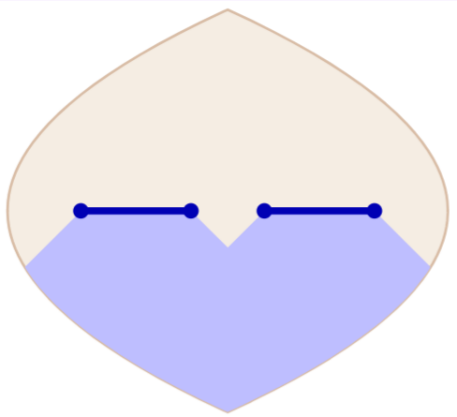
Actual space?

Presentation

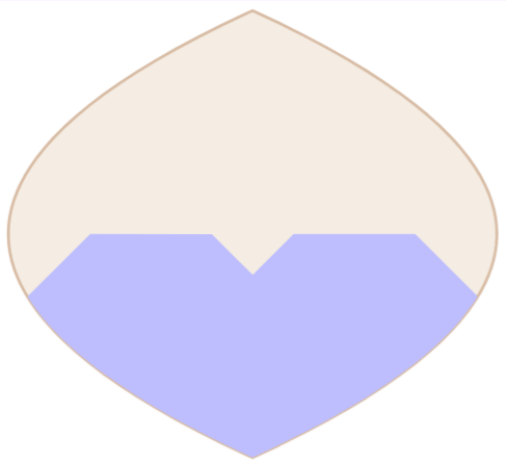


Countably-based spaces

Actual space? two segments?

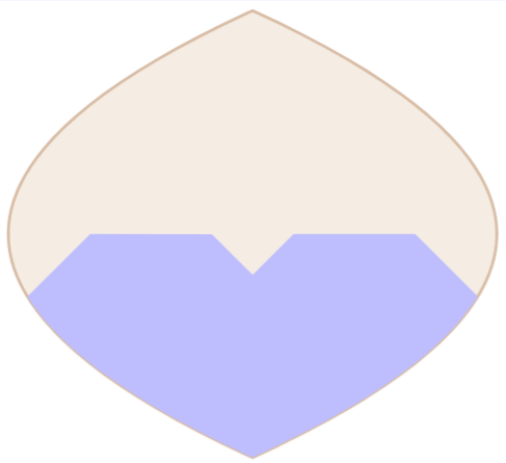


Presentation

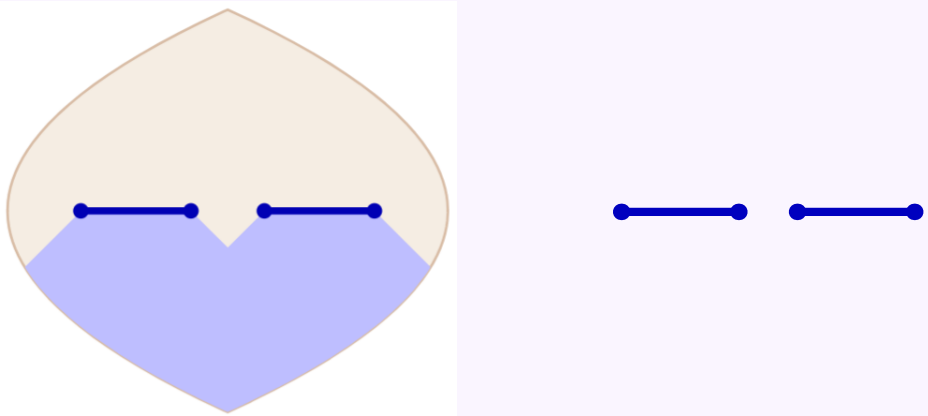


Countably-based spaces

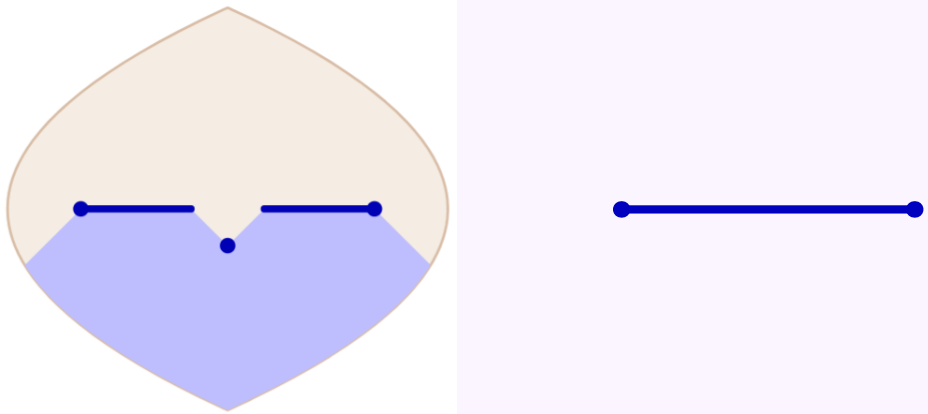
Presentation



Actual space? two segments?



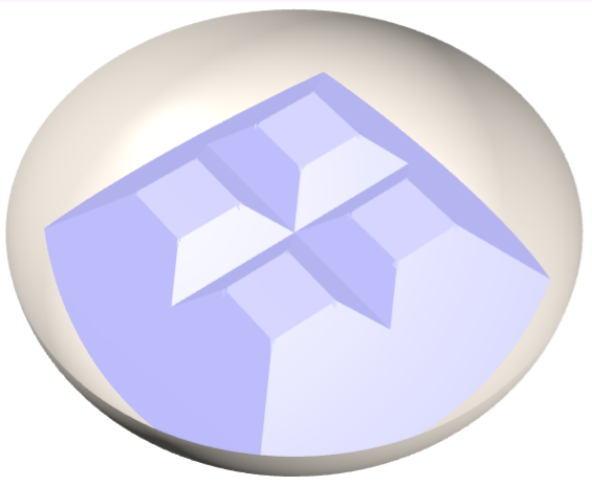
or one segment?



Countably-based spaces

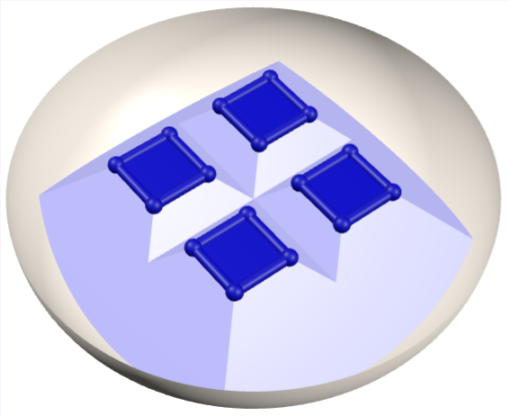
Actual space?

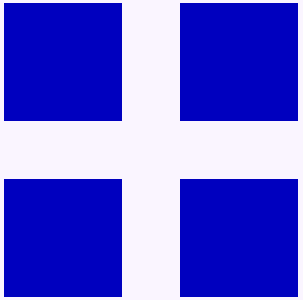
Presentation



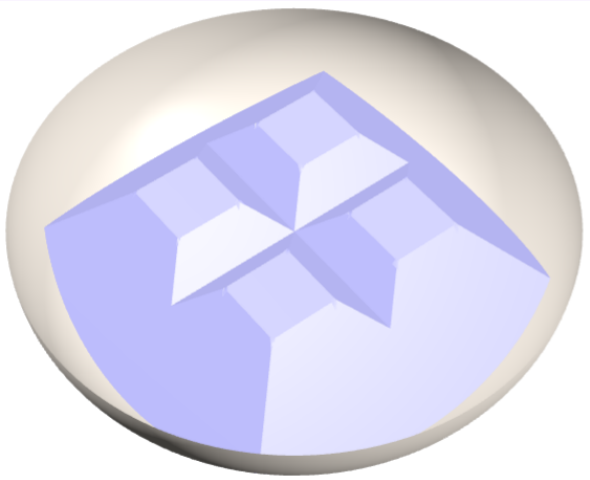
Countably-based spaces

Actual space? four squares?



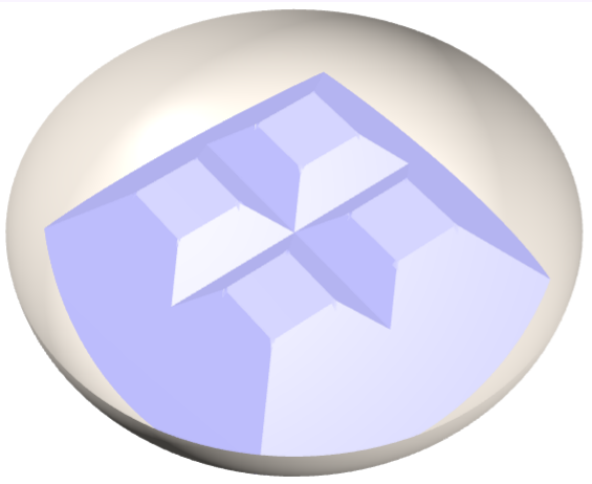


Presentation

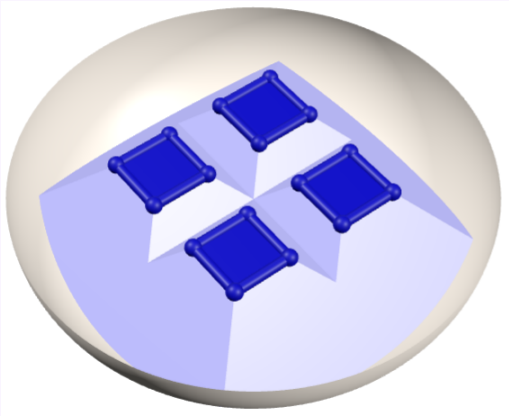


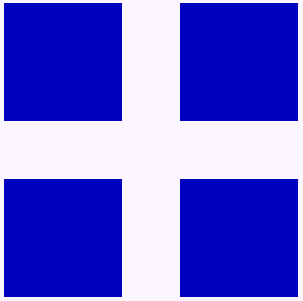
Countably-based spaces

Presentation

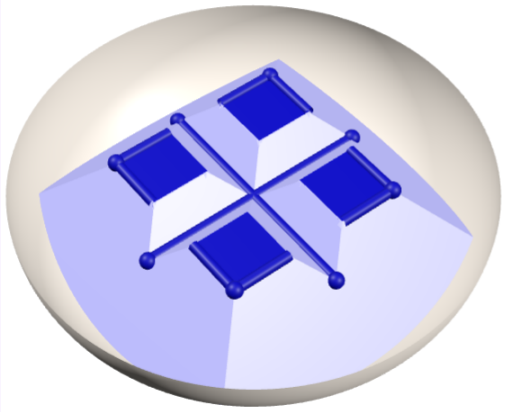


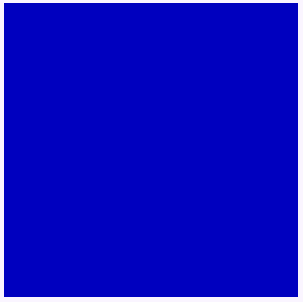
Actual space? four squares?





or just one?

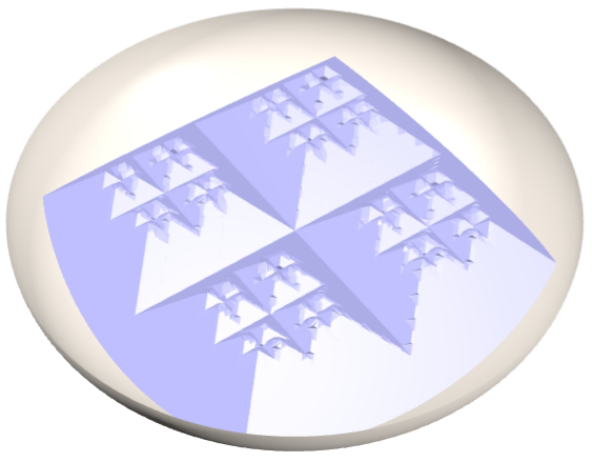




Countably-based spaces

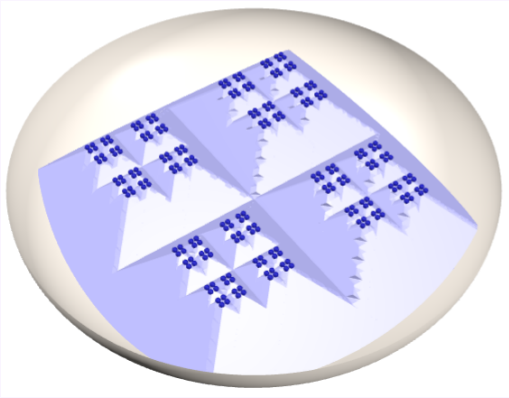
Actual space?

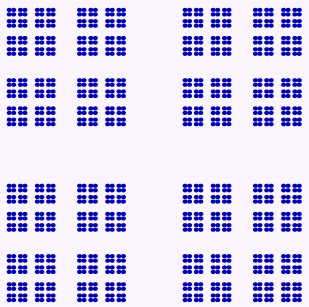
Presentation



Countably-based spaces

Actual space? Cantor space?





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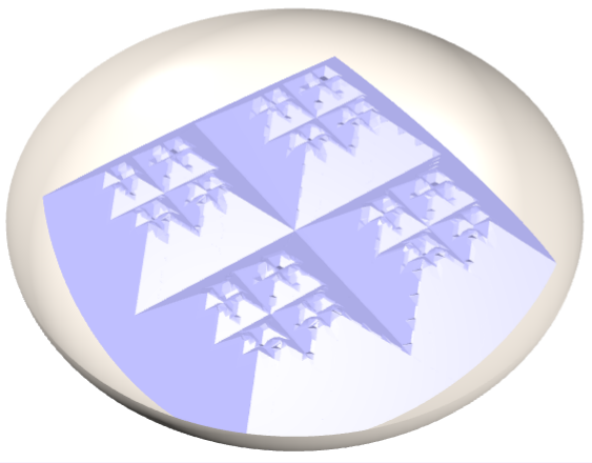
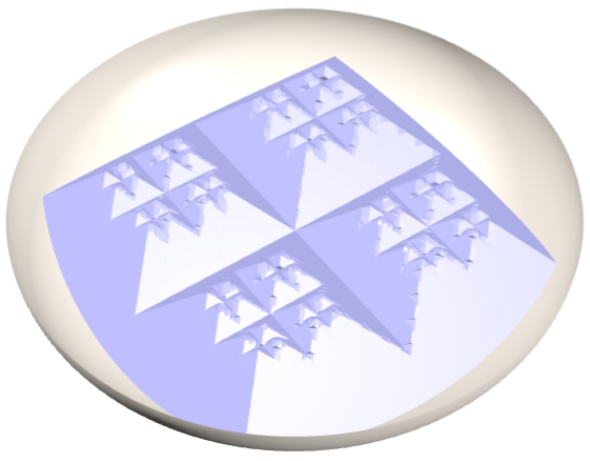
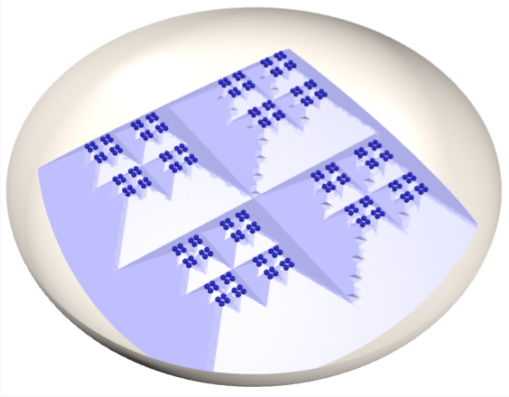


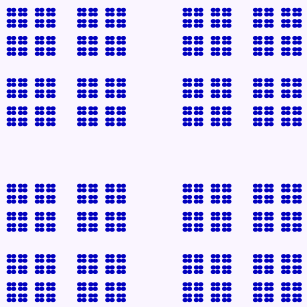
Countably-based spaces

Presentation

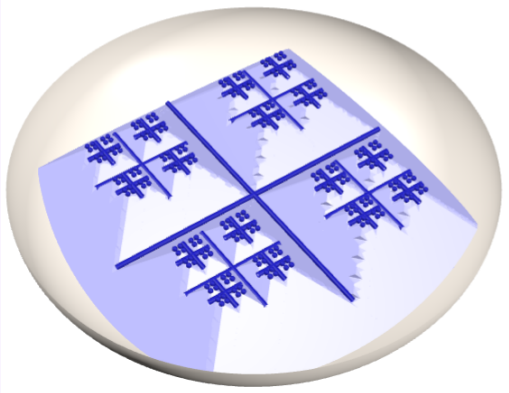


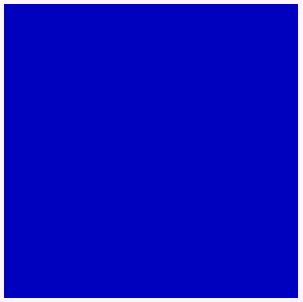
Actual space? Cantor space?





or the square?





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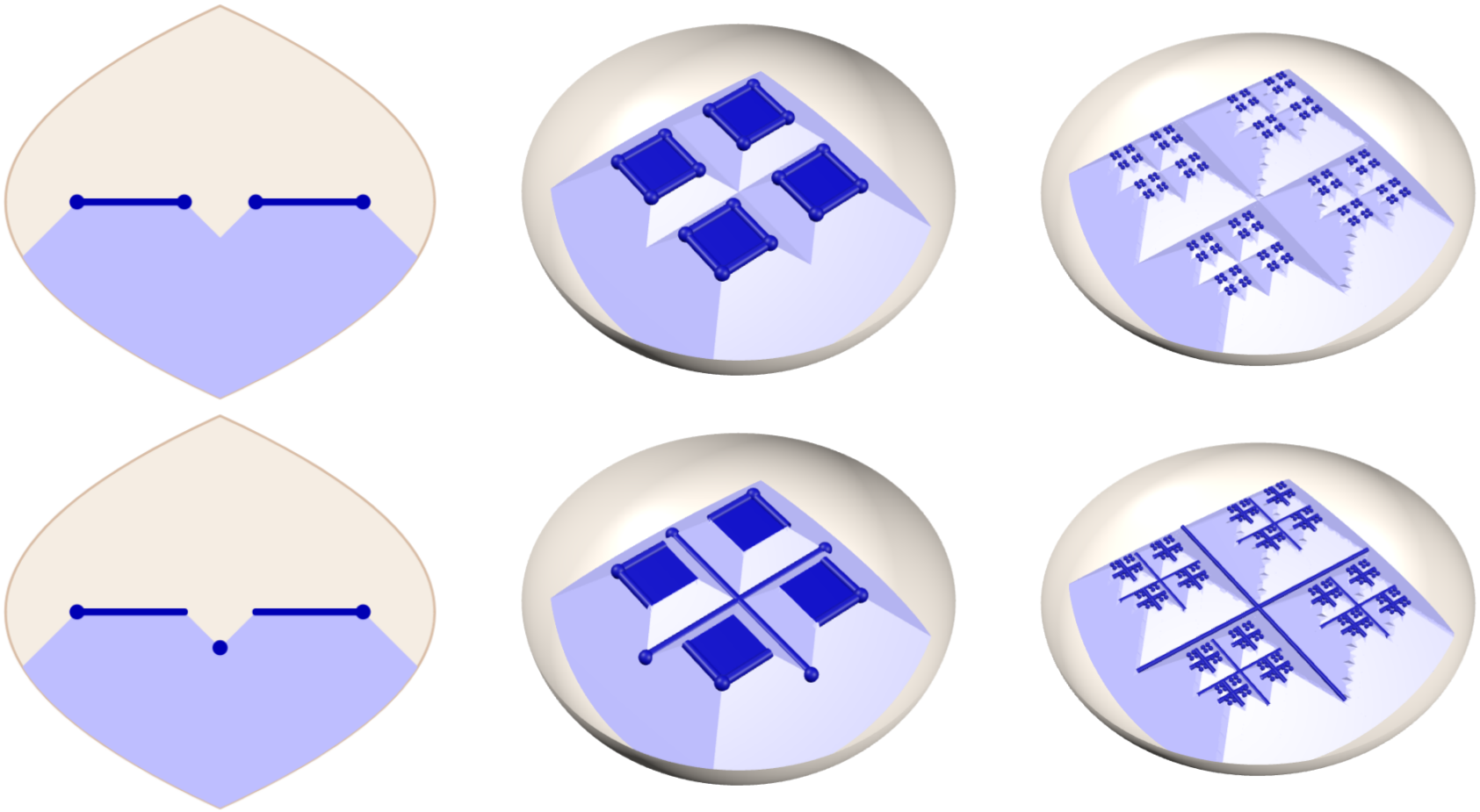

The lower spaces are obtained from the upper spaces by **identifying a few points**.

Countably-based spaces

Every compact metrizable space X without isolated points can be obtained from the Cantor space by identifying a few points.

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- There exists f such that for p in a dense set,

$$[p]_{\sim} = \{p\}.$$

Countably-based spaces

Theorem (H., Melnikov, Ng 2025)

All the perfect compact metrizable spaces share a computable presentation.

Proof idea.

- Let X be compact metrizable space without isolated points,
- X can be obtained from the Cantor space by identifying points,
- The canonical presentation of the Cantor space is also a presentation of X.

$\square$

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Question

What information can be encoded in such spaces?

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Thank you!