The slinky

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Abstract

In this short note, we define a one-dimensional compact Hausdorff space called the slinky, which has strong computable type and properly contains a copy of itself. The previous example of such a space was infinite-dimensional.

1 Introduction

In [AH23], it is proved that if a space is minimal satisfying some Σ_2^0 invariant, then it has strong computable type. However the converse fails in general, and a counter-example is presented in [AH24]. That space is actually not minimal for *any* invariant, because it property contains a set that is homeomorphic to itself. This latter property is achieved by building the space so that it contains the Hilbert cube, which contains every compact Hausdorff space. Whether a finite-dimensional example exists was left open.

In this note, we give a positive answer by building a simple one-dimensional space which has strong computable type and properly contains a copy of itself. We call this space the *slinky*. It is depicted in Figure 1. An interactive model can be consulted at the address given in this footnote[1].

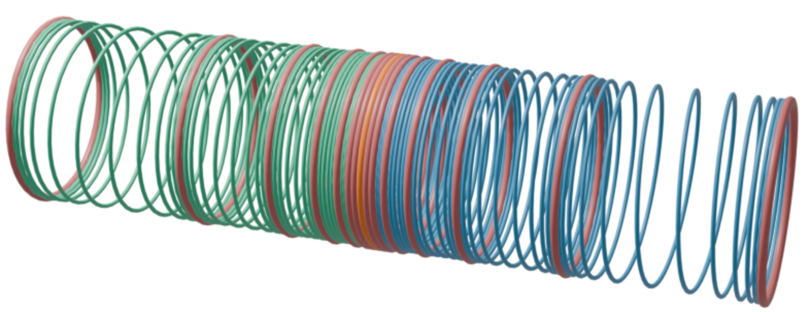

Figure 1: The slinky

[1] https://members.loria.fr/MHoyrup/slinky.html

2 The slinky

We first set a family of vertical circles:

$$\begin{aligned} C_n^+ &= \{(2^{-n}, y, z) : y^2 + z^2 = 1\} \\ C_n^- &= \{(-2^{-n}, y, z) : y^2 + z^2 = 1\} \\ C_\infty &= \{(0, y, z) : y^2 + z^2 = 1\} \end{aligned}$$

and then join consecutive circles by spirals accumulating along them:

$$\begin{aligned} x_n(t) &= 2^{-n-1}(1+t) \\ \theta(t) &= \tan(\pi t - \tfrac{\pi}{2}) \\ S_n^+ &= \{(x_n(t), \cos(\theta(t)), \sin(\theta(t))) : t \in (0,1)\} \\ S_n^- &= \{(-x_n(t), \cos(\theta(t)), \sin(\theta(t))) : t \in (0,1)\}. \end{aligned}$$

Note that as t ranges from 0 to 1, $x_n(t)$ goes from 2^{-n-1} to 2^{-n} and $\theta(t)$ goes from $-\infty$ to $+\infty$. Finally, the slinky is

$$X = C_\infty \cup \bigcup_{n\in\mathbb{N}} (S_n^+ \cup C_n^+ \cup S_n^- \cup C_n^-).$$

This space embeds in the plane, because it embeds in the cylinder which embeds in the plane.

Theorem 2.1. *The slinky has strong computable type.*

Proof. We show that it has computable type. The proof holds relative to any oracle, showing that it has strong computable type. We recall that Q is the Hilbert cube.

Let $K \subseteq Q$ be homeomorphic to the slinky and be semicomputable. We still denote by C_0^+ and C_0^- the embedded versions of the two extremal circles.

Claim 2.1. The sets C_0^+ and C_0^- are c.e. closed, i.e. the sets $\{i \in \mathbb{N} : B_i \cap C_0^\pm \neq \emptyset\}$ are c.e.

Proof. By symmetry, it is sufficient to prove the result for C_0^+.

Let W be a finite union of rational balls that contains all the circles except C_0^+, and is disjoint from C_0^+. Let $K' = K \setminus W$, and let $f : K' \to C_0^+$ be a retraction. Note that f is not null-homotopic. If a rational ball B does not intersect C^+, then $f|_{K'\setminus B}$ is not null-homotopic. However, if a rational ball B intersects C_0^+, then $f|_{K'\setminus B}$ is null-homotopic.

As a result, for a rational ball B, one has $K' \cap B \neq \emptyset \iff f|_{K'\setminus B}$ is null-homotopic. This condition is c.e. by Lemma 2.1, so C_0^+ is c.e. closed. □

Let B be a rational ball. We claim that B intersects K if and only if there exist two rational open sets U, V such that:

$$\begin{aligned}\overline{U} \cap \overline{V} &= \emptyset,\\ K \setminus B &\subseteq U \cup V,\\ U \cap C_0^- &\neq \emptyset,\\ V \cap C_0^+ &\neq \emptyset.\end{aligned}$$

Indeed, if U, V exist, then $K \setminus B$ is disconnected; as K is connected, B must intersect K. Conversely, if B intersects K, then B contains a cut-point x of K (the cut-points are the points belonging to the $S_n^{\pm}$'s, which are dense). The set $K \setminus \{x\}$ has two connected components, one containing C_0^- and the other containing C_0^+. These components are open in K, so there exist disjoint open sets $U_0, V_0 \subseteq Q$ such that $K \setminus \{x\} \subseteq U_0 \cup V_0$, $C_0^- \subseteq U_0$ and $C_0^+ \subseteq V_0$. Therefore, $K \setminus B \subseteq K \setminus \{x\} \subseteq U_0 \cup V_0$. As $K \setminus B$ is compact, one can replace U_0 and V_0 by finite unions of rational balls U and V, whose closures are disjoint, and satisfying the expected inclusions.

The existence of such U, V is c.e., so K is c.e. closed. □

Lemma 2.1. *Let $K \subseteq Q$ and $f : K \to S_1$ be continuous. The set*

$$\{i \in \mathbb{N} : f|_{K\setminus B_i} \textit{ is null-homotopic}\}$$

is c.e.

Proof. In [AH23] we show that to each $X \subseteq Q$ can be associated a partial numbering $\nu_{[X;S_1]}$ of the homotopy classes of continuous functions from X to S_1. When X is semicomputable, the relation

$$\{(m, n) \in \mathbb{N}^2 : \nu_{[X;S_1]}(m) = \nu_{[X;S_1]}(n)\}$$

is c.e., uniformly in X. Moreover, if $X' \subseteq X$ and n is an index of the homotopy class of $f : X \to S_1$, then n is also an index of the homotopy class of $f|_{X'} : X' \to S_1$.

Let $n \in \mathbb{N}$ be an index of the homotopy class of f in $[K; S_1]$. Then n is also an index of $f|_{K\setminus B_i}$ in $[K \setminus B_i; S_1]$ for any i. Let m be an index of the homotopy class of some constant function $f_0 : Q \to S_1$. Similarly, m is also an index of the the restriction of f_0 to $K \setminus B_i$ for any i. The function $f|_{K\cap B_i}$ is null-homotopic if and only if $\nu_{[K\setminus B_i;S_1]}(n) = \nu_{[K\setminus B_i;S_1]}(m)$, which is a c.e. relation, uniformly in i. □

Finally, the proper subset of X obtained by removing one of the extremal sections, say $S_0^+ \cup C_0^+$, is homeomorphic to X, so X indeed properly contains a copy of itself.

References

[AH23] Djamel Eddine Amir and Mathieu Hoyrup. Strong computable type. *Computability*, 12(3):227–269, 2023.

[AH24] Djamel Eddine Amir and Mathieu Hoyrup. The surjection property and computable type. *Topology and its Applications*, 355:109020, 2024.