On the information carried by programs about the objects they compute*

Mathieu Hoyrup and Cristóbal Rojas

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Abstract

In computability theory and computable analysis, finite programs can compute infinite objects. Presenting a computable object via any program for it, provides at least as much information as presenting the object itself, written on an infinite tape. What additional information do programs provide? We characterize this additional information to be any upper bound on the Kolmogorov complexity of the object. Hence we identify the exact relationship between Markov-computability and Type-2-computability. We then use this relationship to obtain several results characterizing the computational and topological structure of Markov-semidecidable sets.

1 Introduction

We assume that the reader is familiar with Turing machines and basic computability theory over the natural numbers. To define computability over infinite objects, one still uses Turing Machines but has to set up a way for them to access such objects. In any case, the input of the machine is a finite or infinite sequence of symbols written on the input tape and one has to choose a suitable way to describe infinite objects by such symbolic sequences. We now briefly describe the two main approaches that have been developed.

The first one was introduced and studied by Turing [Tur36], Grzegorczyk [Grz57], Lacombe [Lac55] and later Kreitz and Weihrauch [Wei00] and is nowadays known as Type-Two computability. In this model, the description itself is completely written on the input tape of the machine. At any time, the machine can read a finite portion of this description. We will call this the ***Type-2-model***. The second approach, promoted by the Russian school led by Markov [Mar54, Kus06], gives an alternative. In this model one restricts the action of the machine to operate on computable (infinite) objects only, in the sense that they have computable descriptions. Instead of having access to description themselves as in the Type-2-model, the machine here has access to a *program* computing a description. We will call this the ***Markov-model***. These two approaches provide a priori different computability notions, and their comparison has been an important subject of study [Ric53, MS55, Sha56, KLS57, Cei62, Fri58, PE60, Her96, Spr01].

It is clear that the Markov-model is at least as powerful as the Type-2-model, so the question is: does it allow to compute strictly more than the Type-2-model? The answer depends on the objects

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that we consider, and the algorithmic tasks we want to perform on them. The computational power of these models can therefore be classified according to these parameters. Table 1 summarizes the most celebrated results in this direction. The computable objects considered are the partial computable functions and the total computable functions. The algorithmic tasks considered are decidability and semidecidability of properties about these objects.

Table 1: Some celebrated results comparing Markov-computability to Type-2-computability.

Objects	Decidability	Semidecidability
Partial computable functions	Markov $\equiv$ Type-2 *Rice*	Markov $\equiv$ Type-2 *Rice−Shapiro*
Total computable functions	Markov $\equiv$ Type-2 *Kreisel et al/Ceitin*	Markov $>$ Type-2 *Friedberg*

Kreisel-Lacombe-Shœnfield/Ceitin's Theorem [KLS57, Cei62] for instance, states that over total computable functions, Markov-decidability is equivalent to Type-2-decidability[1]. This means that the machine trying to decide a property, when provided with a program p for a function f, cannot do better than just running p to evaluate f. The machine gains no additional information about f from p. We note that Ceitin's version of this result shows that over the real line, Markov-computable functions and Type-2-computable functions coincide.

On the other hand, Friedberg [Fri58] exhibited properties about total computable functions that are Markov-semidecidable but not Type-2-semidecidable. So that for semidecidability, a program p for a function f *does* give some additional information that can be exploited by the machine. The main question we raise in this paper is the following:

Can we characterize the additional useful information contained in a program computing an object, as compared to having the object itself ?

To get some intuition, consider the following fundamental difference between the two models. In the Type-2-model, at any given time only a finite portion of the description of x is provided, which corresponds to a finite approximation of x. Clearly, this approximation is also good for infinitely many other objects – all the ones that are "close enough" to x. In particular, *x is never completely specified.* In the Markov-model on the other hand, the program provided to the machine completely specifies x from the beginning of the calculation! This increases the predictive power of M, which might therefore be able to perform stronger calculations. The point is to understand in which situations this fact can be exploited. A trivial example is obtained when one considers the relativized setting: *every* function is Markov-computable relative to an appropriate (powerful enough) oracle. Whereas whatever oracle A we consider, Type-2-computable functions relative to A must always be continuous.

This observation takes us to another interesting point that separates the Markov-model from the Type-2-model, namely their topological structure. It is well known that Type-2-computability and topology are closely related: e.g. the Type-2-computable functions are exactly the effectively continuous ones, and the Type-2-semidecidable properties exactly correspond to the effectively open sets. The connection between Markov-computability and topology, on the other hand, appears to be

[1] In its original form, this theorem is stated for functionals.

much less clear. In particular, Friedberg's construction provides a Markov-semidecidable set which is not open (for the standard topology restricted to computable elements).

An obvious solution to relate Markov-computability to topology is to consider precisely the topology generated by all the Markov-semidecidable sets – the so called Ershov's topology. The question then becomes:

How do Markov-semidecidable sets look like? can we characterize Ershov's open sets ?

In the present paper we make use of Kolmogorov Complexity to provide a fairly complete answer to these and other questions in different settings. Our main result is a characterization of the additional information provided by a program, when the class of objects considered are the computable points of an effective topological space. It can be informally stated as follows (see Section 3):

Theorem A *Over effective topological spaces, a program computing x provides as much information as a description of x itself together with* ***any upper bound on the Kolmogorov complexity of x.***

Here, the Kolmogorov complexity $K(x)$ of a computable infinite object x is to be understood as the size of the shortest program computing a description of x. Obviously, any program for x trivially provides, in addition to a description, an upper bound on its Kolmogorov complexity. Theorem A says that this bound is all the exploitable additional information it provides.

Thus, we have a third model to deal with computable infinite objects. In this model, input x is presented to the machine as a pair (d, k), where d is a description for x and k a bound on the Kolmogorov complexity of x. We shall call this the ***K-model.*** In these terms, a particular case of Theorem A can be stated as follows: if $\mathcal{X}$, $\mathcal{Y}$ are *effective topological spaces* (not necessarily metric) and X_c, Y_c are the corresponding set of computable points, then a function $f : X_c \to Y_c$ is Markov-computable if and only if it is K-computable.

A simple observation shows that one can not in general compute a program for x from a K-description of x, meaning that the two notions are not fully equivalent. Despite this fact, Theorem A is valid in great generality: it holds for decidability, semidecidability and also higher in the hierarchy. In proving this we make a fundamental use of the Recursion Theorem. Interestingly, although the Recursion Theorem does not relativize (a well known fact), Theorem A relative to the halting set also holds in many cases, but for entirely different reasons.

The K-model also sheds light into the structure of the open sets of Ershov's topology, providing a nice characterization in terms of Kolmogorov complexity, at least in the particular case of the extended natural numbers $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Theorem B *On the extended natural numbers, the Ershov topology is generated by the sets $\{n\}$ and $\{x \in \overline{\mathbb{N}} : K(x) < h(x)\}$ for some computable order h.*

With the same techniques, we are able to prove several other related results that are interesting on their own. For example, we show that there is no effective enumeration of the Markov-semidecidable sets of $\overline{\mathbb{N}}$ and that there is a Markov-semidecidable subset of $\{0,1\}^{\mathbb{N}}$ that is not Σ^0_2.

Finally, in the search of the limitations of our techniques, we turn our attention to more general spaces and analyze functions with values on topological spaces that have an admissible representation but are not countably-based. In particular, when this is the space of open subsets of Baire space $\mathcal{O}(\mathbb{B})$, we show that Markov-computability can be strictly stronger than K-computability:

Theorem C. *For functions from the partial computable functions with values on $\mathcal{O}(\mathbb{B})$ one has that:*

$$\textit{Markov-computability} \quad > \quad \textit{K-computability} \quad > \quad \textit{Type-2-computability.}$$

One of the main question that remains open is whether the first strict inequality in Theorem C holds if we replace the *partial* computable functions by the *total* ones. The situation is summarized in table 2.

Table 2: Some results comparing Markov-computability, K-computability and Type-2-computability. $\mathbb{S} = \{\bot, \top\}$ is the Sierpiński space whose topology is generated by $\{\top\}$.

Space $\mathcal{X}$	Semidecidable	$\emptyset'$-Semidecidable	F: $\mathcal{X} \to \mathcal{O}(\mathbb{B})$
$\mathbb{S}$	Markov $\equiv$ K $\equiv$ Type-2	Markov $>$ K $\equiv$ Type-2	Markov $>$ K $\equiv$ Type-2
Partial functions	Markov $\equiv$ K $\equiv$ Type-2	Markov $>$ K $\equiv$ Type-2	Markov $>$ K $>$ Type-2
Total functions	Markov $\equiv$ K $>$ Type-2	Markov $\equiv$ K $>$ Type-2	Markov ? K $>$ Type-2

The paper is organized as follows. We start by providing the basic notions and definitions in Section 2. In Section 3 we introduce the K-model and present our main results. Section 4 contains several results that shed light on the structure of Markov-semidecidable sets and in Section 5 we present the announced negative results. Finally, Section 6 contains a list of related problems for possible future work.

2 Background

2.1 Notations and basic definitions.

We assume the reader is familiar with computability theory. Let $\{\varphi_e\}_{e\in\mathbb{N}}$ be an effective enumeration of the set of computable partial functions. We denote by $P_c(\mathbb{N})$ the collection of c.e. subsets of $\mathbb{N}$ and $W_e = \text{dom}(\varphi_e)$ the induced effective enumerations of its elements. If $A \in P_c(\mathbb{N})$, an ***index*** of A is a number e such that $W_e = A$. If A is a c.e. set, implicitly given by an index, $A[s]$ is the finite subset of A enumerated by stage s, so that $A[s] \subseteq A[s+1]$ and $A = \bigcup_s A[s]$. We use the notation $A[\text{at } s] = A[s] \setminus A[s-1]$ if $s \geq 1$ and $A[\text{at } 0] = A[0]$. If F is a finite subset of $\mathbb{N}$ then $[F]$ is the collection of supersets of F. $\mathbb{B} = \mathbb{N}^{\mathbb{N}}$ will denote Baire space.

2.2 Effective topological spaces.

An ***effective topological space*** is a tuple $(\mathcal{X}, \tau, \mathcal{B})$ where $(\mathcal{X}, \tau)$ is a non-empty topological space, $\mathcal{B} = \{\mathcal{B}_i\}_{i\in\mathbb{N}}$ is numbered basis such that there exists a computable function $f : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ satisfying $\mathcal{B}_i \cap \mathcal{B}_j = \bigcup_{k\in W_{f(i,j)}} \mathcal{B}_k$.

Given an effective topological space $\mathcal{X}$, the standard representation is defined as a surjective map $\rho : \text{dom}(\rho) \subseteq \mathbb{B} \to \mathcal{X}$ satisfying $\rho(f) = x$ whenever $\{f(n) : n \in \mathbb{N}\} = \{i : x \in \mathcal{B}_i\}$. We will call any $f \in \rho^{-1}(x)$, a ***Type-2-name*** of x. An element x is ***computable*** if it has a computable Type-2-name. We denote by $\boldsymbol{X_c}$ the set of computable points. The countable set X_c has a canonical numbering ν defined by $\nu(i) = x$ if φ_i is a name of x. We will call such an e a ***Markov-name*** of x. To facilitate the reading of the paper, we will use the font A, N, U when working on the space X_c, and the fonts $\mathcal{A}, \mathcal{N}, \mathcal{U}$ when working on $\mathcal{X}$.

2.2.1 Type-2-computability and Markov-computability

Let $(\mathcal{X}, \tau, \mathcal{B})$ and $(\mathcal{Y}, \tau', \mathcal{B}')$ be effective topological spaces. In what follows R stands for both *Type-2* and *Markov*. A set $A \subseteq X_c$ is ***R-semidecidable*** if there is a Turing machine M which, when provided with an R-name of x, halts if and only if $x \in A$. A function $F : X_c \to Y_c$ is ***R-computable*** if there is a Turing machine M which, when provided with an R-name of x, writes an R-name for $f(x)$ on its one-way output tape. It is not hard to see that a function $f : X_c \to Y_c$ is R-computable if and only if the sets $f^{-1}(\mathcal{B}'_i)$ are uniformly R-semidecidable.

Remark 2.1. It is worth noting that for a function $f : X_c \to Y_c$, being Markov-computable is equivalent to having a Machine M which, provided with a Markov-name of x, outputs a *Type-2-name* of $f(x)$. Indeed, combining the program for x with the program for M gives a program for $f(x)$. We also note that a function $f : X_c \to Y_c$ which is Type-2-computable does not necessarily extends to a Type-2-computable function $\overline{f} : \mathcal{X} \to \mathcal{Y}$.

A numbering η of X_c is ***admissible*** if it is equivalent to the canonical numbering ν in the sense that there exists partial computable functions f and g such that $\nu = \eta \circ f$ on $\mathrm{dom}(\nu)$ and $\eta = \nu \circ g$ on $\mathrm{dom}(\eta)$. The Markov-computability notions do not depend on the choice of the admissible numbering. We will often use the admissible numbering η of X_c defined by $\eta(e) = x$ whenever $W_e = \{i \in \mathbb{N} : x \in \mathcal{B}_i\}$.

Type-2-computability and topology are closely related. A set $\mathcal{U} \subseteq \mathcal{X}$ is an ***effective open set*** if there exists $e \in \mathbb{N}$ such that $\mathcal{U} = \bigcup_{i \in W_e} \mathcal{B}_i$. If $A = \mathcal{U} \cap X_c$, we will then say that A is effectively open ***in*** $\boldsymbol{X_c}$. The connection is established by the following result (see [Wei00]).

Theorem 2.1. *A set $A \subseteq X_c$ is Type-2-semidecidable if and only if it is effectively open in X_c. Therefore, a function $f : X_c \to Y_c$ is Type-2-computable if and only if it is effectively continuous, i.e. the sets $f^{-1}(\mathcal{B}'_i)$ are uniformly effectively open in X_c.*

As mentioned in the introduction, in order to have an analogous result for Markov-computability, we have to use Ershov's topology on X_c, which may be different from the topology of $\mathcal{X}$ restricted to X_c.

Example 2.1. Let $\mathbb{B} = \mathbb{N}^{\mathbb{N}}$ be the Baire space. For each finite sequence u, let $[u]$ be the set of infinite extensions of u, called a cylinder. We endow $\mathbb{B}$ with the topology generated by the cylinders, which is an effective topology. The standard numbering φ_e of partial computable functions, restricted to the indices of total functions is an admissible numbering of $\mathbb{B}_c$.

Example 2.2. Let $\mathcal{P}(\mathbb{N})$ be space of subsets of $\mathbb{N}$. For each finite set $F \subseteq \mathbb{N}$, let $[F]$ be the set of supersets of F. We endow $\mathcal{P}(\mathbb{N})$ with the Scott topology, generated by the sets $[F]$, which is an effective topology. The standard numbering $W_e = \mathrm{dom}(\varphi_e)$ of c.e. sets is admissible numbering of $P(\mathbb{N})_c$

3 Main results

In this section, $(\mathcal{X}, \tau, \mathcal{B}, \nu_{\mathcal{B}})$ is always an effective topological space and X_c is the subset of computable elements. We start by explaining the main idea behind our results. Let $x \in X_c$ be a fixed element. From a machine Type-2-semideciding a set A containing x, one can compute a neighborhood $\mathcal{N}$ of x such that for every element $y \in X_c$ the following implication holds:

$$y \in \mathcal{N} \implies y \in A. \tag{1}$$

Now assume that A has the weaker property of being Markov-semidecidable, and still contains x. From a machine Markov-semideciding A one cannot in general compute such a neighborhood, which may not exist as shown by Friedberg's example. However, from the Markov-name of any other element $y \in X_c$ one can still compute a neighborhood $\mathcal{N}_y$ of x such that implication (1) holds. Further, as a finite intersection of neighborhoods is still a neighborhood, one can compute a neighborhood $\mathcal{N}$ satisfying implication (1) for all y in a given finite set. Using this argument we can show that the problem $x \in A$ can be Type-2-semidecided *as soon as we know, in addition, a finite list of programs containing at least one for x*. This additional information is equivalent to having any upper bound on the Kolmogorov complexity of x, which leads us to the notion of K-computability that we now introduce.

3.1 K-computability

Definition 3.1. The ***Kolmogorov complexity K(x)*** of a computable element $x \in X_c$ is the length of a shortest program computing a Type-2-name of x.

In this paper, whether we use prefix-free, monotone or plain machines will not make any difference so we do not need to specify the definition any further.

Definition 3.2. A ***K-name*** of a computable element $x \in X_c$ consists of a pair (k, f) where $k \geq K(x)$ and f is a Type-2-name of x.

Remark 3.1. Note that k is only an upper bound on the Kolmogorov complexity of x and not necessarily of f, which may even be non computable. Note also that knowing any such k is effectively equivalent to knowing any upper bound on a Markov-name of x. This is what we will rather use in our proofs.

The ***K-computability*** notions are defined in the same way as in the previous section. We will denote by $\boldsymbol{X_c(k)}$ the set of computable elements whose Kolmogorov complexity is at most k. Note that $X_c = \bigcup_k X_c(k)$ and that K-computability is the same as Type-2-computability on $X_c(k)$, uniformly in k. In particular, a set $A \subseteq X_c$ is K-semidecidable iff there exists uniformly effective open sets $\mathcal{U}_k$ such that $A \cap X_c(k) = \mathcal{U}_k \cap X_c(k)$.

Thus, for each notion of computability we have so far three versions, depending on the way the objects are represented.

It is clear that one can compute K-names from Markov-names. An important first observation is the fact that the converse does not necessarily holds. In other words, the representations underlying Markov-computability and K-computability are not equivalent.

Proposition 3.1. *In general, it is not possible to compute Markov-names from K-names.*

Proof. Let $f : \mathbb{N} \to \mathbb{N}$ be a computable function such that $x_{f(i)} = 0^\omega$ if $\varphi_i(i)$ does not halt, $x_{f(i)} = 0^t 1^\omega$ if $\varphi_i(i)$ halts in time t.

Assume that there is an oracle Turing machine M that converts each K-name of a sequence into a program computing the sequence (a Markov-name). There is a computable function $k : \mathbb{N} \to \mathbb{N}$ such that $k(i)$ is an upper bound on the complexity of $x_{f(i)}$ and of 0^ω. Given i, run M on $0^{k(i)}10^\omega$, let u_i be its use when it halts. M halts and outputs a program computing the sequence 0^ω. On $0^{k(i)}1x_{f(i)}$ it must halt and output a program computing $x_{f(i)}$, so if $x_{f(i)}$ contains a 1 then it must occur early in the sequence, namely before $u_i - k(i) - 1$. As a result, if $\varphi_i(i)$ halts then it must halt in time $u_i - k(i) - 1$. It enables one to decide the Halting problem, which is a contradiction. □

One can show that on Cantor space, Markov-names are limit-computable (can be *learned*) from K-names: given x and $k \geq K(x)$, one can compute a sequence of natural numbers converging to an index of x (this problem was investigated in the context of inductive inference [FW79]). A c.e. set, however, cannot be learned in this way. Actually one can prove a stronger statement.

Proposition 3.2. *There is no Turing functional Φ that, given an index e and a Type-2-name of a set W which is either $\mathbb{N}$ or W_e, computes a sequence of numbers converging to an index of W.*

Proof. Assume that such a Φ exists. Using the Recursion theorem, we define an index a in the following way. At the same time we enumerate a c.e. set W_a and we build an oracle f.

We define $W_a = \bigcup_i F_i$ where F_i is a computable sequence of finite sets and $F_i \subseteq F_{i+1}$. At the same time we define an oracle t as the limit of a computable sequence of finite strings t_i such that t_{i+1} extends t_i. The finite string t_i contains exactly the elements of F_i.

The strings t_i are such that $\Phi^{t_i}(a)$ outputs a (finite or infinite) sequence of indices taking at least i values. Hence $\Phi^t(a)$ outputs an infinite sequence of indices taking infinitely many values.

We start with $F_0 = \emptyset$ and t_0 is the empty string. Assume F_i and t_i have been defined. Look for a finite extension u of t_i such that $\Phi^u(a)$ eventually outputs an index i such that W_i contains some number that is not in u (such a u must be found: on a representation of $\mathbb{N}$ starting with t_i, Φ must eventually output a program enumerating $\mathbb{N}$, let u be the finite part of the oracle that is read when such a program is output). Let F_{i+1} be the set of elements enumerated in u. Run $\Phi^{u0^\omega}(a)$ and look for a number $j \neq i$ output later than i (such a j must be found, otherwise the oracle is a representation of W_a so it must eventually stabilize on a program j enumerating the elements of u, and j must be different from i). Let t_{i+1} be the part of the oracle that is used in the computation of j. □

The rest of this section is devoted to show that, despite the facts above, the notions of Markov-computability and K-computability are indeed equivalent to a large extent.

3.2 Equivalence between Markov-computability and K-computability

We will use the Recursion Theorem. See [Rog87].

Theorem 3.1 (Recursion Theorem)**.** *For every computable total function f, there exists e such that $\varphi_e = \varphi_{f(e)}$. Moreover, p can be computed from an index of f.*

The following Lemma contains the main technical arguments.

Lemma 3.1. *Let A be a c.e. subset of $\mathbb{N}$. There exist uniformly effective Scott open sets $\mathcal{U}_k \subseteq \mathcal{P}(\mathbb{N})$, such that for every c.e. set E the following hold:*

i) if all the indices of E belong to A then $E \in \mathcal{U}_k$ for every k,

ii) if no index of E belongs to A then $E \notin \mathcal{U}_k$ for every $k \geq K(E)$.

The argument is uniform: the open sets $\mathcal{U}_k$ are effective, uniformly in a c.e. index of A.

Proof. Using the Recursion theorem, there is a computable function $e(a, b)$ such that for all $a, b \in \mathbb{N}$,

$$W_{e(a,b)} = \begin{cases} W_a & \text{if } e(a,b) \notin A, \\ W_a[t] \cup W_b & \text{if } e(a,b) \in A[\text{at } t]. \end{cases}$$

Let $k \in \mathbb{N}$. We define an effective open set $\mathcal{U}_k$. Compute b_k such every element whose complexity is less than k has an index less than b_k. If a is such that for all $b \leq b_k$, $e(a,b) \in A$ then let t be minimal such that $e(a,b) \in A[t]$ for all $b \leq b_k$, enumerate $[W_a[t]]$ into $\mathcal{U}_k$.

We now check the two announced conditions. *i)* Let $E \subseteq \mathbb{N}$ be a c.e. set. Assume that every index of E belongs to A and let a be an index of E. For all b, $e(a,b) \in A$ (otherwise $e(a,b)$ is an index of $W_a = E$ but $e(a,b) \notin A$, contradiction), so $\mathcal{U}_k$ contains $[W_a[t]]$ for some t, which contains E. *ii)* Assume that $K(E) \leq k$, that no index of E belongs to A and that $E \in \mathcal{U}_k$. Let $b \leq b_k$ be an index of E. As $E \in \mathcal{U}_k$, E belongs to some $[W_a[t]]$ enumerated into $\mathcal{U}_k$ (here a is not the same as above and is not assumed to be an index of E). As $e(a,b) \in A$, $W_{e(a,b)} = W_a[t'] \cup W_b$ for some $t' \leq t$. As $W_a[t'] \subseteq W_a[t] \subseteq W_b$, $e(a,b)$ is an index of E that belongs to A, contradicting the assumption. □

We know state the main explicit versions of Theorems A and B.

Theorem 3.2. *Let $\mathcal{X}$ be an effective topological space. A set $A \subseteq X_c$ is Markov-semidecidable iff it is K-semidecidable. The equivalence is uniform.*

Proof. Every effective topological space is Type-2-computably homeomorphic to a subspace of $\mathcal{P}(\mathbb{N})$: to $x \in \mathcal{X}$, associate $\{i \in \mathbb{N} : x \in \mathcal{B}_i\}$ where $\mathcal{B}_i$ is enumeration of the basis of $\mathcal{X}$. Hence we can assume that $\mathcal{X}$ is a subspace of $\mathcal{P}(\mathbb{N})$. Let $I \subseteq \mathbb{N}$ be a c.e. set such that for all $e \in \mathbb{N}$ for which $W_e \in X_c$, it holds $W_e \in A \iff e \in I$. Each c.e. set $E \in X_c$ either has all its indices in I or has no index in I, so the effective open sets $\mathcal{U}_k$ provided by Lemma 3.1 coincide with A on the set of elements of X_c whose complexity is at most k. Now, a machine K-semideciding A works as follows: given a Type-2-name of $E \in X_c$ and $k \geq K(E)$, it tests whether $E \in \mathcal{U}_k$ and halts in this case only. □

Corollary 3.1. *Let $\mathcal{X}, \mathcal{Y}$ be effective topological spaces. A function $f : X_c \to Y_c$ is Markov-computable iff f is K-computable. The equivalence is uniform.*

Proof. Let B_i be the numbered basis of $\mathcal{Y}$. f is Markov-computable iff the sets $f^{-1}(\mathcal{B}_i)$ are uniformly Markov-semidecidable iff these sets are K-semidecidable (Theorem 3.2) iff f is K-computable. □

We now show that the argument in the proof of Lemma 3.1 can be extended from semidecidability to weaker classes of properties, showing that for most algorithmic tasks, the additional information given by programs is indeed just an upper bound on the Kolmogorov complexity.

Hierarchies. Let $\mathcal{X}$ be an effective topological space. We consider the finite levels of the effective Borel hierarchy, defined as follows. The class Σ^0_1 consists of the effective open sets. The class Σ^0_{n+1} consists of the effective unions of differences of Σ^0_n-sets. The classes Π^0_n consists of complements of Σ^0_n-sets. The class Δ^0_n is the intersection of Σ^0_n and Π^0_n. Inside the class Δ^0_2 we consider the finite levels of the effective difference hierarchy. For $n \in \mathbb{N}$, the class $\mathcal{D}_n$ consists of the differences of n effective open sets, i.e. the sets $(\mathcal{U}_0 \setminus \mathcal{U}_1) \cup \ldots (\mathcal{U}_{n-2} \setminus \mathcal{U}_{n-1})$ if n is even and the sets $(\mathcal{U}_0 \setminus \mathcal{U}_1) \cup \ldots \mathcal{U}_{n-1}$ if n is odd. In the case $\mathcal{X} = \mathbb{N}$ with the discrete topology, the effective Borel hierarchy is exactly the arithmetical hierarchy, the class $\mathcal{D}_n$ of effective difference hierarchy is exactly the class of n-c.e. sets.

Theorem 3.3. *A set $A \subseteq X_c$ is Markov-n-c.e. iff it is K-n-c.e. More precisely, the set of indices of elements of A is n-c.e. on the set of indices of X_c iff there exist uniformly effective open sets $\mathcal{U}^1_k, \ldots, \mathcal{U}^n_k$ such that $A \cap X_c(k) = D_n(\mathcal{U}^1_k, \ldots, \mathcal{U}^n_k) \cap X_c(k)$.*

Proof. Again we can assume w.l.o.g. that $\mathcal{X}$ is a subspace of $\mathcal{P}(\mathbb{N})$. Let $A_0 \supseteq A_1 \supseteq \ldots \supseteq A_{n-1}$ such that if $W_e \in X_c$ then $W_e \in A \iff e \in A_0 \setminus A_1 \cup A_2 \setminus A_3 \ldots$. We denote a tuple $(a_0, \ldots, a_n) \in \mathbb{N}^{n+1}$ by $\overline{a}$. There is a computable function $e(\overline{a})$ such that

$$W_{e(\overline{a})} = \begin{cases} W_{a_0} & \text{if } e(\overline{a}) \notin A_0, \\ W_{a_0}[t_0] \cup W_{a_1} & \text{if } e(\overline{a}) \in A_0[\text{at } t_0] \setminus A_1, \\ W_{a_0}[t_0] \cup W_{a_1}[t_1] \cup W_{a_2} & \text{if } e(\overline{a}) \in A_0[\text{at } t_0] \cap A_1[\text{at } t_1] \setminus A_2, \\ \ldots & \\ W_{a_0}[t_0] \cup W_{a_1}[t_1] \cup \ldots \cup W_{a_{n-1}} & \text{if } e(\overline{a}) \in A_0[\text{at } t_0] \cap \ldots \cap A_{n-2}[\text{at } t_{n-2}] \setminus A_{n-1}, \\ W_{a_0}[t_0] \cup W_{a_1}[t_1] \cup \ldots \cup W_{a_n} & \text{if } e(\overline{a}) \in A_0[\text{at } t_0] \cap \ldots \cap A_{n-2}[\text{at } t_{n-2}] \cap A_{n-1}[\text{at } t_{n-1}]. \end{cases}$$

Given k, let b_k be un upper bound on the indices of elements whose Kolmogorov complexity is at most k. If a_0 is such that for all $a_1, \ldots, a_n \leq b_k$, $e(a_0, \ldots, a_n) \in A_0$ then let t_0 be minimal such that all these numbers belong to $A_0[t_0]$, enumerate $[W_{a_0}[t_0]]$ in U_0. By induction, let $1 \leq i < n$ and assume $a_0, \ldots, a_{i-1}$ have been accepted with $t_0, \ldots, t_{i-1}$. If a_i is such that for all $a_{i+1}, \ldots, a_n \leq b_k$, $e(a_0, \ldots, a_n) \in A_i$ then let t_i be minimal such that all these numbers belong to $A_i[t_i]$ and enumerate $[W_{a_0}[t_0] \cup \ldots \cup W_{a_i}[t_i]]$ in U_i.

In the same way, one checks that if $W_e \in X_c(k)$ then $W_e \in A \iff W_e \in U_0 \setminus U_1 \cup U_2 \setminus U_3 \cup \ldots$. □

It is known from [Sel84] that there exists a Markov-2-c.e. subset of $\mathcal{P}(\mathbb{N})$ that is not even Π^0_2. Hence Markov-2-c.e. sets are not the differences of Markov-semidecidable sets.

In the following theorem, we need to assume an additional property on the space $\mathcal{X}$. Namely, that the domain of the standard representation on $\mathcal{X}$ is a Π^0_2 set. This is case for example for the so called *quasi-Polish spaces* (see [dB13]).

Theorem 3.4. *A set $A \subseteq X_c$ is Markov-Σ^0_2 iff it is K-Σ^0_2. More precisely, the set of indices of elements of A is Σ^0_2 on the set of indices of X_c iff there exist uniformly effective open sets $\mathcal{U}^n_k, \mathcal{V}^n_k$ such that $A \cap X_c(k) = \bigcup_n (\mathcal{U}^n_k \setminus \mathcal{V}^n_k) \cap X_c(k)$.*

Remark 3.2. In case $\mathcal{X}$ is a Polish space, the sets $\mathcal{V}_k$ are not needed and therefore the last part of the statement reads $A \cap X_c(k) = \bigcup_n \mathcal{U}^n_k \cap X_c(k)$.

Proof of Theorem 3.4. We show that if A is Markov-Π^0_2 then A is K-Π^0_2, which is equivalent to the statement by replacing A with its complement. We use the numbering $x_e = x$ if W_e is the set of indices of basic neighborhoods of x. Let $P = \bigcap_n P_n \subseteq \mathbb{N}$ be Π^0_2 (P_n are uniformly c.e.) such that if e is an index of $x \in X_c$ then $x \in A \iff e \in P$. The assumption about the space implies that the set of indices of elements of X_c is a Π^0_2-set $Q = \bigcap_n Q_n \subseteq \mathbb{N}$, where Q_n are uniformly c.e. sets.

Given i, let $C_i = \{x_i\}$ if i is an index and $i \in P \cap Q$, $C_i = \emptyset$ otherwise. C_i is Π^0_2, uniformly in i. Indeed, for each n, define the uniformly effective open sets

$$(\mathcal{U}_{2n}, \mathcal{V}_{2n}) = \begin{cases} (B_n, \emptyset) & \text{if } n \notin W_i, \\ (X, B_n) & \text{if } n \in W_i, \end{cases}$$

and

$$(\mathcal{U}_{2n+1}, \mathcal{V}_{2n+1}) = \begin{cases} (X, \emptyset) & \text{if } i \notin P_n \cap Q_n, \\ (X, X) & \text{if } i \in P_n \cap Q_n. \end{cases}$$

One has $C_i = \bigcap_n (\mathcal{U}_n \setminus \mathcal{V}_n)^c$.

Given $k \in \mathbb{N}$, compute b_k such that every element of Kolmogorov complexity at most k has an index $\leq b_k$. The set $\bigcup_{i \leq b_k} C_i$ is Π^0_2, uniformly in k. This set is exactly $A \cap X_c(k)$. □

4 Structure of Markov-semidecidable sets

Here we provide several results that shed light on the computational and topological structure of Markov-semidecidable sets. Our first result shows that Markov-semidecidable sets share some of the nice properties of Type-2-semidecidable sets.

Proposition 4.1. *Assume that $\mathcal{X}$ contains a dense computable sequence. Given a Markov-semidecidable set A, it is semi-decidable whether A is non-empty. If A is non-empty, one can compute a sequence of points $\{x_i\} \subseteq A$ which is dense in A.*

Proof. Using the Recursion theorem, there is a computable function $e(a)$ such that $x_{e(a)} = x_a$ if $e(a) \notin A$, or $x_{e(a)}$ is some point from the dense sequence in some neighborhood of x_a if $e(a) \in A$ at time t. A is non-empty iff there is a such that $e(a) \in A$. When A is non-empty, one can compute an element in A: look for a such that $e(a) \in A$, $x_{e(a)}$ is such a point. To get a computable dense sequence, apply this argument in parallel to the intersection of A with each basic open set $\mathcal{B}_i$. □

The following result provides an upper bound on the effective Borel complexity of Markov-semidecidable sets.

Proposition 4.2. *Let $A \subseteq X_c$ be Markov-semidecidable. There exist uniformly effective open sets $\mathcal{U}_k \subseteq \mathcal{X}$ such that $A = \bigcap_k \mathcal{U}_k \cap X_c$.*

Proof. Let $\mathcal{U}_k$ be the effective open sets from the proof of Theorem 3.2 and define $\mathcal{S} = \bigcap_k \mathcal{U}_k$. We already know that $A \subseteq \mathcal{U}_k$ for all k. If $x \in X_c \cap \mathcal{S}$ then let $k \geq K(x)$. Since $x \in X_c(k) \cap \mathcal{U}_k = X_c(k) \cap A$, we conclude that $x \in A$. □

The result above is actually tight, as the following theorem shows. For a finite string u, let us define the monotone complexity $Km(u)$ of u as the length of a shortest program computing a (finite or infinite) binary sequence extending u. The program writes its output on a one-way output tape and may never halt. Again the precise definition of $Km(u)$ (Levin or Schnorr monotone or process complexity) does not make any difference for our purposes. The only important property is that for a computable sequence x, $Km(x\restriction_n) \leq Km(x)$ for all n. For the seek of completeness, let us recall original Friedberg's example. We present it in a way that is more convenient for our purposes.

Theorem 4.1 (Friedberg)**.** *On the Cantor space, the set*

$$\mathcal{A} = \{0^\omega\} \cup \bigcup_{n : Km(0^n 1) < \log(n) - 1} [0^n 1].$$

is Markov-semidecidable but not open. Hence the Ershov topology is strictly stronger than the Cantor topology.

Proof. We show that $\mathcal{A}$ is K-semidecidable. Given an infinite binary sequence x (a Type-2-description) and a bound k on $K(x)$, we only need to read the first $e = 2^{k+2}$ bits of x. If we see only zeros, we accept. Otherwise one gets $0^n 1 \ldots$ for some $n < e$, then test whether $Km(0^n 1) < \log(n) - 1$. □

Remark 4.1. Friedberg's example happens to be Σ^0_2. It is an effective open set appended with a limit point. We strengthen Friedberg's example by constructing a Markov-semidecidable set which is far from being open.

Theorem 4.2. *There is a Markov-semidecidable subset of $\{0,1\}^{\mathbb{N}}_c$ that is not Σ^0_2. It is a non-empty closed subset of $\{0,1\}^{\mathbb{N}}_c$ with empty interior, defined by*

$$A = \{x \in \{0,1\}^{\mathbb{N}}_c : \forall n, Km(x\restriction_n) < n/2 + c\} \quad \text{for some sufficiently large } c \in \mathbb{N}.$$

Proof. We choose c such that for some computable sequence x, $K(x) \leq c$, hence A is non-empty as it contains x. We first show that A is K-semidecidable. First, the function $u \mapsto Km(u)$ is right-c.e. Now, given x and some $k \geq K(x)$, $x \in A$ iff for all $n \leq 2(k-c)$, $Km(x\restriction_n) < n/2 + c$, as for all $n > 2(k-c)$, $Km(x\restriction_n) \leq K(x) \leq k < n/2 + c$.

Here we denote $\{0,1\}^{\mathbb{N}}$ by $\mathcal{X}$ and the set of computable sequences by X_c. A is a subset of X_c. We show that there is no Σ^0_2-subset of $\mathcal{X}$ whose intersection with X_c is A. Let $\overline{A}$ be the closure of A in $\mathcal{X}$ (it might not be $\{x \in \mathcal{X} : \forall n, Km(x\restriction_n) < n/2 + c\}$). Here is the argument:

1. A has empty interior in X_c, i.e. there is no cylinder $[u]$ such that $[u] \cap X_c \subseteq A$. Indeed, given a finite string u and a sufficiently large k, for most words v of length k, $Km(uv) \geq |uv|/2 + c$ so $[uv]$ is disjoint from A.

2. If $\mathcal{P} \subseteq \mathcal{X}$ is a Π^0_1-set and $\mathcal{P} \cap X_c \subseteq A$ then $\mathcal{P}$ is nowhere dense in A. Indeed, if there exists a cylinder $[u]$ such that $\emptyset \neq A \cap [u] \subseteq \mathcal{P}$ then $A \cap [u] = \mathcal{P} \cap [u] \cap X_c$ is both Markov-semidecidable and Markov-co-semidecidable hence by Kreisel-Lacombe-Shœnfield/Ceitin theorem it is clopen on X_c, so A has non-empty interior in X_c, contradicting the first point.

3. By Proposition 4.1, $\overline{A}$ is a c.e. closed subset of $\mathcal{X}$ (it contains a dense computable sequence) hence a Π^0_2-set. Let $\mathcal{S} \subseteq \mathcal{X}$ be the Π^0_2-set given by Proposition 4.2, satisfying $A = \mathcal{S} \cap X_c$. Let $\mathcal{S}' = \overline{A} \cap \mathcal{S}$. $\mathcal{S}'$ is a Π^0_2-set which contains a dense computable sequence, and A is exactly the set of computable elements of $\mathcal{S}'$. From this it follows that computable Baire theorem holds on $\mathcal{S}'$: if the sets $\mathcal{P}_i$ are uniformly Π^0_1-sets that have empty interior in $\mathcal{S}'$ then one can compute some x in $\mathcal{S}' \setminus \bigcup_i \mathcal{P}_i$.

4. Now, if $\mathcal{P}_i$ are uniformly Π^0_1-sets such that each $\mathcal{P}_i \cap X_c$ is contained in A then by the second point $\mathcal{P}_i$ has empty interior in A, and also in $\mathcal{S}' \subseteq A$, so by the third point one can compute some x in $\mathcal{S}' \setminus \bigcup_i \mathcal{P}_i$. As x is computable and belongs to $\mathcal{S}'$, x belongs to A so $\bigcup_i \mathcal{P}_i$ does not cover A. □

For the following results, we restrict our attention to the space $\mathcal{X} = \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ whose topology is generated by the singletons $\{n\}$ and the semi-lines $[n, \infty]$, for $n \in \mathbb{N}$. Note that every point in this space is computable, so that $\mathcal{X} = X_c$.

Friedberg's example translated to this space reads $\{x \in \overline{\mathbb{N}} : K(x) < \log(x) - 1\}$, which inspires the following definition.

Definition 4.1. We define the ***Friedberg sets*** of $\overline{\mathbb{N}}$ to be the ones of the form $\{x \in \overline{\mathbb{N}} : K(x) < h(x)\}$, where $h : \mathbb{N} \to \mathbb{N}$ is any computable order, namely, any non decreasing unbounded computable function.

Note that a computable order can always be extended to a computable function $h : \overline{\mathbb{N}} \to \overline{\mathbb{N}}$, with $h(\infty) = \infty$.

Friedberg sets are Markov-semidecidable just like the original set. The next two results show that, unlike Cantor space, the only Markov-semidecidable sets over $\overline{\mathbb{N}}$ which are not Type-2-semidecidable are essentially the Friedberg sets.

Proposition 4.3. *If $A \subseteq \overline{\mathbb{N}}$ is Markov-semidecidable and contains ∞ then there is a computable order h such that A contains a Friedberg set.*

Proof. Since A is K-semidecidable, for each k one can compute $p(k)$ such that $[p(k), \infty] \cap \{x : K(x) \leq k\} \subseteq A$. One can assume that $p(k)$ is increasing. Let $h(n) = \min\{i : p(i) > n\}$. If $n \notin A$ then $p(K(n)) > n$ (just take $k = K(n)$), so one has $h(n) \leq K(n)$. □

Remark 4.2. Observe that $K(x)$ here coincides with the usual notion of Kolmogorov complexity of natural numbers.

Proposition 4.3 provides a nice characterization of the Ershov's open sets.

Corollary 4.1. *The Ershov topology is generated by the singletons $\{n\}$ and the Friedberg sets.*

Remark 4.3. Note that the sets $[n, \infty]$ can be expressed as the Friedberg sets $\{x \in \overline{\mathbb{N}} : K(x) < h(x)\}$ where $h(x) = 0$ for $x < n$ and $h(x) = c(x + 1)$ for $x \geq n$, where c is such that $K(n) \leq c(n + 1)$ for all $n \in \mathbb{N}$.

Whether or not one can find such a characterization on other spaces such as the Cantor space is an interesting question.

We end this section by observing that, unlike Type-2-semidecidable sets, Markov-semidecidable sets cannot be effectively enumerated.

Proposition 4.4. *There is no effective enumeration of the Markov-semidecidable subsets of $\overline{\mathbb{N}}$.*

Proof. Let A_i be a sequence of uniformly Markov-semidecidable sets, coming with uniformly c.e. sets $E_i \subseteq \mathbb{N}$ such that $E_i \cap \mathrm{dom}(\nu) = \nu^{-1}(A_i)$. One can extract the sets that contain ∞ (let e_0 be some index of ∞, one can enumerate the numbers i such that E_i contains e_0), so we can assume w.l.o.g. that each A_i contains ∞. For each i one can compute an increasing computable function $f_i : \mathbb{N} \to \mathbb{N}$ whose range is contained in A_i. Now we build a Markov-semidecidable set A that contains ∞ and differs from each A_i. Let f be a computable order such that for each i and all sufficiently large k, $f_i(k) < f(k)$ (for instance, $f(k) = \max(f_0(k), \ldots, f_k(k)) + 1$). Here we use another version of Kolmogorov complexity: $C(x)$ is the minimal index of x. We now define $A = \{x \in \overline{\mathbb{N}} : f(C(x)) \leq x\}$. A is Markov-semidecidable by the usual argument.

We show that A differs from each A_i. Let $i \in \mathbb{N}$. As $C \circ f_i$ is one-to-one, there exist infinitely many $k \in \mathbb{N}$ such that $C(f_i(k)) \geq k$. Moreover if k is sufficiently large then $f_i(C(f_i(k))) < f(C(f_i(k)))$. Hence there exists k such that $f_i(k) \leq f_i(C(f_i(k))) < f(C(f_i(k))$. Let $x = f_i(k)$: $x \in A_i$ by construction of f_i and $x < f(C(x))$ so $x \notin A$. □

5 When Markov beats Kolmogorov

In this section we explore the limits of our methods. We first look at the relativized case, and show that there are simple cases that separate Markov-computability from K-computability. However, we also show that, interestingly, the equivalence persists if the space has a Polish structure.

5.1 Relativization

Let $\mathbb{S} = \{\bot, \top\}$ be the Sierpiński space with topology given by $\{\emptyset, \{\top\}, \{\bot, \top\}\}$. Note that as $\mathbb{S}$ is finite, K-computability is trivially equivalent to Type-2-computability simply because all the elements share a common upper bound on their Kolmogorov complexities, which therefore provides no interesting information. Relativizing w.r.t. the Halting set, we can then separate Markov-decidability from Type-2-decidability, and therefore from K-decidability.

Remark 5.1. The set $\{\bot\} \subseteq \mathbb{S}$ is Markov-decidable relative to the halting set but is not Type-2-decidable relative to any oracle.

Proof. It is not decidable relative to any oracle simply because it is not clopen. □

Similarly, using $\emptyset''$ we can separate, over $\mathcal{P}(\mathbb{N})$, K-semidecidability from Type-2-semidecidability (without oracle, the two notions coincide with Markov-decidability by Rice-Shapiro theorem).

Proposition 5.1. *The set $\{\mathbb{N}\} \subseteq \mathcal{P}(\mathbb{N})$ is K-semidecidable relative to $\emptyset''$ but is not Type-2-semidecidable relative to any oracle.*

Proof. Let $E \subseteq \mathbb{N}$ be a c.e. set and k an upper bound on its Kolmogorov complexity. From k we know that A has an index in some finite set F. Using $\emptyset''$ we, for each $e \in F$, decide whether $W_e = \mathbb{N}$ and compute, when $W_e \neq \mathbb{N}$, an element outside W_e. We then wait that each one of this finite set of elements appears in A, enumerated on the input tape, and accept A if it is the case. □

However, metric spaces behave differently. Although stated on Cantor space, the next result extends to any computable metric space [Wei00].

Proposition 5.2. *Let A be an oracle computing the Halting set. A subset of $\{0,1\}^{\mathbb{N}}_c$ is Markov-semidecidable relative to A if and only if it is K-semidecidable relative to A.*

Proof. Assume that $\mathcal{A}$ is Markov-semidecidable relative to the halting set, via some oracle Turing machine M. Assume we are given the halting set, $x \in \{0,1\}^{\mathbb{N}}_c$ and an upper bound k on its Kolmogorov complexity. Look for a partition of $\{0, \dots, k\}$ into three sets A, B, C such that:

- for every $a \in A$, x_a is incompatible with x (there exists n such that $x_a(n)\downarrow \neq x(n)$),
- for every $b \in B$, x_b is partial (there exists n such that $x_b(n)\uparrow$),
- for every $c \in C$, c is accepted by M with the halting set as oracle.

Once such a partition is found, accept x.

First, the tests can be effectively done relative to the halting set: incompatibility of x_a with x is semidecidable, partiality of x_b is semidecidable relative to the halting set. If such a partition is found then x must have an index in C so x must satisfy the property. Conversely if x satisfies the property then for every $i \leq k$, either i is an index of x or x_i is incompatible with x or x_i is partial, so a partition exists and will be eventually found. □

5.2 Functions to non-effective topological spaces

In this section we provide results that strictly separate our three notions: Markov-computability, K-computability and Type-2-computability. The idea of the constructions is to build uniform versions of the examples given in Remark 5.1 and Proposition 5.1. For this, one can imagine a function with two arguments, where the second argument $A \in \mathbb{B}$ is always provided to the machine by a Type-2-name and plays the role of the oracle. The only difficulty is then to make the computation work in a uniform way in the oracle parameter. In order to get a well defined function w.r.t. our models, we express it as a function of the first argument only, but with values on $\mathcal{O}(\mathbb{B})$, which is the set of open subsets of the Baire space endowed with the topology generated by the following sets: given a compact set $K \subseteq \mathbb{B}$, the class of open subsets of $\mathbb{B}$ containing K is open. This topology is not countably-based, and hence it is not an effective topology. However it does have an admissible representation [Sch02].

We now present the details of the simplest case, a uniform version of Remark 5.1. This result contrasts with Corollary 3.1.

Theorem 5.1. *There exists a Markov-computable function $F : \mathbb{S} \to \mathcal{O}(\mathbb{B})$ that is not K-computable.*

Proof. We use the admissible numbering $\nu_{\mathbb{S}}$ of $\mathbb{S}$ defined by $\nu_{\mathbb{S}}(e) = \top$ if $\varphi_e(e)\downarrow$, $\nu_{\mathbb{S}}(e) = \bot$ otherwise. We define two effective open sets $U_\bot, U_\top$ and define $F(\bot) = U_\bot$ and $F(\top) = U_\top$. First, let $U_\bot = \mathbb{B}$. Let $T : \mathbb{N} \to \mathbb{N}$ be defined as follows: $T(n)$ is the halting time of $\varphi_n(n)$ if it halts, $T(n) = 0$ otherwise. The open set $U_\top := \mathbb{B} \setminus \{T\}$ happens to be effective. First the function F is not Lacombe computable because it is not continuous: indeed, F is not monotonic as $\bot \leq \top$ but $U_\bot = \mathbb{B}$ is not contained in $U_\top \subsetneq \mathbb{B}$. As $\mathbb{S}$ is finite, F is not K-computable neither. However F is Markov-computable. Given an index e of $s \in \mathbb{S}$, enumerate $U_\top$ and enumerate the set of functions f such that $\varphi_e(e)$ does not halt in exactly $f(e)$ steps. The latter set of functions is effectively open, uniformly in e. If $\varphi_e(e)\uparrow$ then the whole space $\mathbb{B}$ is enumerated. If $\varphi_e(e)\downarrow$ then nothing more than $U_\top$ is enumerated. Intuitively, given e and f, from T one can decide whether $\varphi_e(e)$ halts, i.e. whether $\nu_{\mathbb{S}}(e) = \bot$. □

A similar construction, based on Proposition 5.1, yields a function $F : \mathcal{P}(\mathbb{N}) \to \mathcal{O}(\mathbb{B})$ which is K-computable but not Type-2-computable by replacing the function T from Theorem 5.1 by a function T' computing $\emptyset''$ and such that $\mathbb{B} \setminus \{T'\}$ is effectively open.

Combining all these results, and using that fact that Theorem 5.1 can clearly also be realized using $\mathcal{P}(\mathbb{N})$ in place of $\mathbb{S}$, we obtain our announced Theorem C.

Theorem 5.2. *For functions from $\mathcal{P}(\mathbb{N})$ with values on $\mathcal{O}(\mathbb{B})$ on has that:*

$$\text{Markov-computability} \;>\; \text{K-computability} \;>\; \text{Type-2-computability}.$$

While Type-2-computable functions are always Scott continuous (i.e. monotone and compact), one can show that K-computable functions are always monotone but not necessarily compact. Markov-computable functions may even not be monotone.

Let us now briefly discuss whether Theorem 5.2 holds for functions from the Cantor space to $\mathcal{O}(\mathbb{B})$. Friedberg's example of a Markov (hence K)-semidecidable set that is not Type-2-semidecidable directly implies the second inequality. However the idea behind the proof of the first inequality cannot be applied on Cantor space. Indeed, using Proposition 5.2 one can show that the analog of the function of Theorem 5.1 is actually K-computable.

Proposition 5.3. *The function $G : \{0,1\}^{\mathbb{N}} \to \mathcal{O}(\mathbb{B})$ mapping 0^ω to $\mathbb{B}$ and any other sequence to $\mathbb{B} \setminus \{T\}$ is K-computable.*

Proof. Given x, k and f, apply the algorithm given by Proposition 5.2 to semi-decide, if $f = T$, whether $x = 0^\omega$. In parallel, semidecide whether $f \neq T$. □

We leave the following question open: is there a Markov-computable function from the Cantor space to $\mathcal{O}(\mathbb{B})$ that is not K-computable? A simpler version of this question: is there an oracle that separates K-semidecidability from Markov-semidecidability on Cantor space? We note that, by Proposition 5.2, such an oracle should not compute $\emptyset'$.

6 Future work

We list a few problems for future work.

- Find a characterization of the Ershov topology on other spaces than $\overline{\mathbb{N}}$, like the Cantor space.
- Determine for which levels of the effective difference hierarchy the Markov-model and the K-model are equivalent. We know from Theorem 3.3 that the equivalence holds for the *finite* levels. What about the level ω?
- All our results hold when the space $\mathcal{X}$ is an effective topological space. However the three models also make sense on any represented space. It seems like an interesting research program to study the extent to which our results are valid in this case.
- Compare the effective Borel hierarchy induced by the Markov-semidecidable sets, the hierarchy induced by the arithmetical hierarchy on the indices and the effective Borel hierarchy induced by the standard topology.

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