Computable presentations of topological spaces

Mathieu Hoyrup, Alexander Melnikov, Keng Meng Ng

Inria Nancy

Presentations

Let (X, τ) be a countably-based topological space.

Definition

A **precomputable topological presentation** of (X, τ) is an indexed basis $(B_i)_{i \in \mathbb{N}}$ together with a c.e. set $E \subseteq \mathbb{N}$ such that

$$B_i \cap B_j = \bigcup_{(i,j,k) \in E} B_k.$$

Definition

A **computable topological presentation** of (X, τ) is a precomputable presentation $(B_i)_{i \in \mathbb{N}}$ such that moreover the set

$$\{i \in \mathbb{N} : B_i \neq \emptyset\}$$

is c.e.

Presentations

Computable presentations appear in several works:

- Grubba, Schröder, Weihrauch 2007,
- Korovina, Kudinov 2008,

in combination with other properties: computable regularity, domain-theoretic properties, etc.

It is closely related to **computable overtness**, used in many other works.

Presentations

Every countably-based space X has a **precomputable** presentation:

- $(\mathcal{P}(\omega), \tau_{\text{Scott}})$ has a (pre)computable presentation $(B_i)_{i \in \mathbb{N}}$,
- X embeds in $\mathcal{P}(\omega)$,
- The induced presentation $(B_i \cap X)_{i \in \mathbb{N}}$ is precomputable.

Note: the induced presentation is not **computable** in general.

Computable presentations

Theorem (Melnikov, Ng, 2023)

There is a Polish space which has a computable topological presentation, but no arithmetical Polish presentation.

Theorem (Bazhenov, Melnikov, Ng, 2023)

Every $0'$-computable Polish space has a computable topological presentation.

How far can we extend the latter results? $0''$? beyond?

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How far can we extend the latter results? $0''$? beyond?

Theorem (H., Melnikov, Ng, 2023)

Actually, every countably-based space has a computable topological presentation.

Computable presentations

Before proving the result, let us illustrate how computable presentations give little information about the space.

What properties of the space can be detected from a computable presentation?

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What properties of the space can be detected from a computable presentation?

Dense subspace

A computable topological presentation of (X, τ) is also a topological presentation of any dense subspace $Y \subseteq X$.
Therefore, most properties cannot be detected:

- Connectedness: $[0, 1/2) \cup (1/2, 1]$ is dense in $[0, 1]$,
- Dimension: $\mathbb{Q}$ is dense in $\mathbb{R}$.

Compact Polish spaces

Let us assume that (X, τ) is compact Polish. The dense subset trick does not apply anymore.

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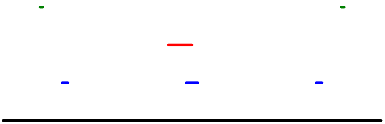

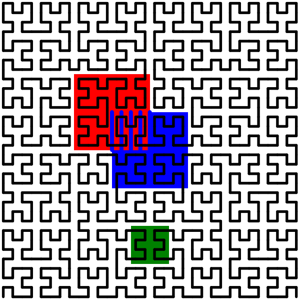

Compact Polish spaces

Some invariants can be detected:

- Whether X has an isolated point:

$$\exists i, \underbrace{\forall j, k, [(B_i \cap B_j \neq \emptyset \text{ and } B_k \cap B_i \neq \emptyset) \implies B_j \cap B_k \neq \emptyset]}_{B_i \text{ is a singleton}}.$$

- Whether the isolated points are dense:

$$\forall l, \exists i, B_i \cap B_l \neq \emptyset \text{ and } B_i \text{ is a singleton}.$$

Compact Polish spaces

Theorem

Every compact Polish space has a computable topological presentation.
Moreover,

- *All the perfect compact Polish spaces share a common comp. top. pres.*
- *All the compact Polish spaces with an infinite dense set of isolated points share a common comp. top. pres.*

Compact Polish spaces

Definition

A function $f : X \to Y$ is **almost injective** if the set

$$\{x \in X : f^{-1}(f(x)) = \{x\}\}$$

is dense.

Lemma

Let $f : X \to Y$ be continuous, almost injective, surjective.
Let $(B_i)_{i\in\mathbb{N}}$ be a computable topological presentation of X, which is closed under finite unions.
Define

$$C_i = \{y : f^{-1}(y) \subseteq B_i\} = Y \setminus f(X \setminus B_i).$$

Then $(C_i)_{i\in\mathbb{N}}$ is a computable topological presentation of Y, formally equivalent to $(B_i)_{i\in\mathbb{N}}$.

Compact Polish spaces

Lemma (Binary expansion)

Every perfect compact Polish space is the continuous image of an almost injective function $f : 2^\omega \to X$.

Let $(B_i)_{i\in\mathbb{N}}$ be the family of clopen subsets of 2^ω.

Corollary

The family $(B_i)_{i\in\mathbb{N}}$ *is a computable presentation of any perfect compact Polish space.*

Compact Polish spaces

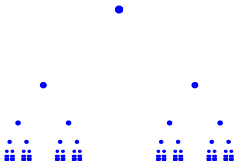

Figure: The space $2^{\leq\omega}$

Lemma

Every compact Polish space whose isolated are dense is the continuous image of an almost injective function $f : 2^{\leq\omega} \to X$.

Let $(B_i)_{i\in\mathbb{N}}$ be the family of clopen subsets of $2^{\leq\omega}$.

Corollary

The family $(B_i)_{i\in\mathbb{N}}$ is a computable presentation of any compact Polish space whose isolated points are dense.

From compact Polish to metrizable

Corollary

Every separable metrizable space has a computable presentation.

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Proof.

Compactification:

- Embed X in the Hilbert cube $[0,1]^\omega$,

From compact Polish to metrizable

Corollary

Every separable metrizable space has a computable presentation.

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Compactification:

- Embed X in the Hilbert cube $[0,1]^\omega$,
- $\mathrm{cl}(X)$ is compact Polish, so it has a computable presentation,

From compact Polish to metrizable

Corollary

Every separable metrizable space has a computable presentation.

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Compactification:

- Embed X in the Hilbert cube $[0, 1]^\omega$,
- $\mathrm{cl}(X)$ is compact Polish, so it has a computable presentation,
- X is dense in $\mathrm{cl}(X)$. □

From metrizable to countably-based

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Every countably-based space has a computable presentation.

Proof.

- Embed X in $\mathcal{P}(\omega)$,
- X is dense in $\mathrm{cl}(X)$,
- The space $M = \max(\mathrm{cl}(X))$ is zero-dimensional and countably-based, hence metrizable, so M has a computable presentation,

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Proof.

- Embed X in $\mathcal{P}(\omega)$,
- X is dense in $\mathrm{cl}(X)$,
- The space $M = \max(\mathrm{cl}(X))$ is zero-dimensional and countably-based, hence metrizable, so M has a computable presentation,
- Transfer the computable presentation of M to X.

□

Conclusion

The notion of computable presentation is actually not restrictive.
However, in combination with other computability properties it is restrictive.

[1]Strong $\equiv$ the set $\{i : B_i \neq \emptyset\}$ is computable

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- Computably topological + computably regular $\iff$ computable metric space

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Assuming **computable compactness**,

- Strong[1] computably topological $\iff$ Computably Polish
- Computably topological $\overset{?}{\iff}$ Right-c.e. Polish

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