An introduction to computational geometry

ENSG Option Géologie Numérique

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présentation trés largement inspirée du travail de Xavier Goaoc

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Question 1: What is it all about?

It's about designing algorithmic solutions to geometric problems.

Ex: *path planning, geometric model manipulation, visibility computation...*

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In a mathematical sense.

Often based on geometric arguments.

Often requires new geometric insight.

⋆ Proven efficient.

In the sense of complexity theory.

Complexity of algorithms are analyzed in an adequate model of computation.

Understanding the complexity of the problems themselves is important.

⋆ Work in practice.

Algorithms that are simple enough to be implemented.

Implementations that handle degeneracies and finite precision arithmetic.

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Definition of the operations allowed in an algorithm and their cost.

Goal: estimate the ressources required by an algorithm

as a function of the input size.

Ex: execution time, memory space, number of I/O transfers, number of processors...

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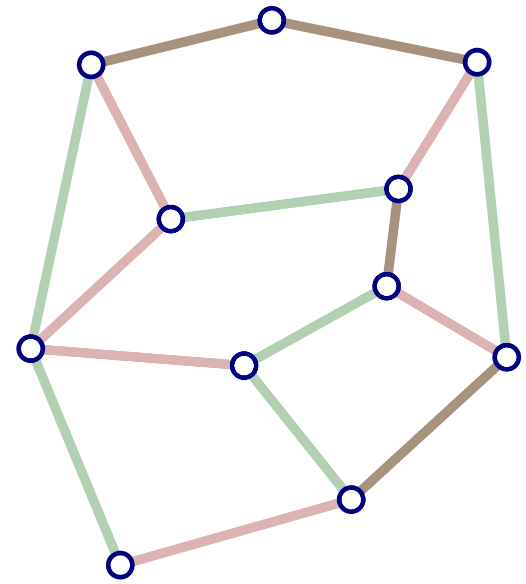
Classical model in CG: Real RAM model

Allows manipulation of real (as in $\mathbb{R}$) numbers.

Input size n → complexity $f(n) = \max_{\text{input } |X|=n} f(X)$

Care about asymptotic order of magnitude of f $(O(), \Omega(), \Theta())$.

Brute force does not scale well (or: why should we think?)



The ”Travelling salesman problem”.

Input: n cities and all inter-city distances.

Output: order on the cities that minimizes the distance travelled.

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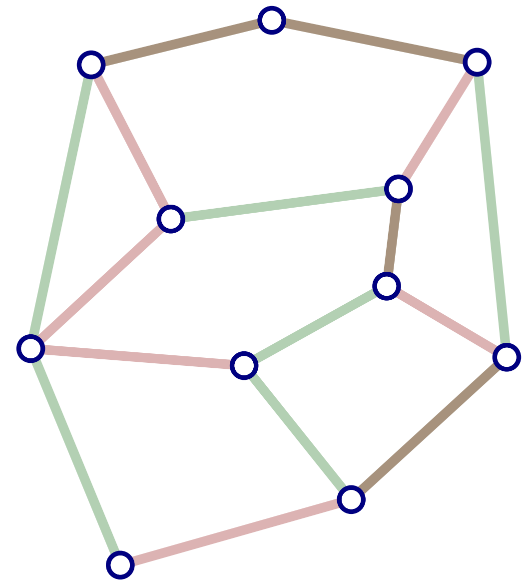

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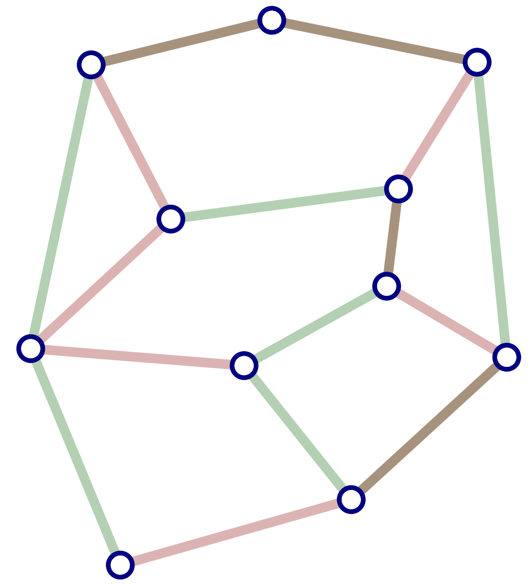

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Assume your computer can process 10^{15} orders per second.

Generate the order, add-up the distances, compare to the current best...

A very generous over-estimation.

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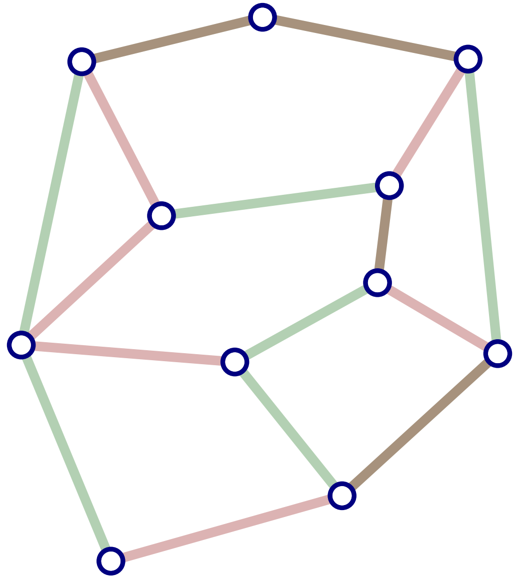

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Start the computation now. It will end...

in 30-60 min. for $n = 20$.

in two weeks for $n = 22$.

in twenty years for $n = 24$.

in four centuries for $n = 25$.

in the dark for $n = 30$.

Orders of magnitude

Sort by increasing asymptotic orders of magnitude:

$$n,\ 2^n,\ n^2,\ n!,\ \sqrt{n},\ \log n,\ \log^* n,\ 2^{n^2}$$

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$$\forall a, b, c, d, e \in \mathbb{R}_+$$

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Three classes of problems

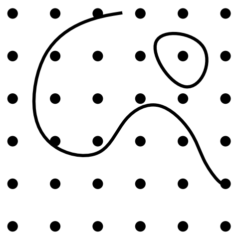
Undecidable: no algorithm will solve the problem. Ever.

NP-hard: conjectured unlikely that a polynomial-time algorithm exists.

Polynomial-time: solvable by an algorithm with complexity $O(n^c)$

for some constant c.

Hilbert's tenth problem



Input: a polynomial P in n variables with integer coefficients.

Output: yes if P has a integer solution, no otherwise.

Ex: $P(x_1, x_2, x_3) = x_1^2 + 3x_1x_2 - 2x_2^2 + 4x_3 + 3$

Tenth question in Hilbert's list of *Problèmes futurs des mathématiques*.

Raised in 1900. Algorithmic question before the age of computers.

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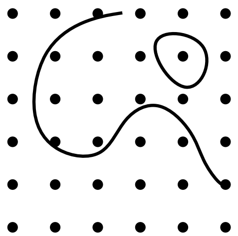

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UNDECIDABLE

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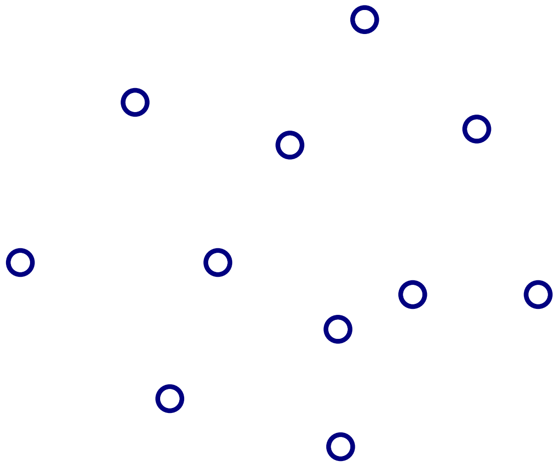
Given a set of points in the plane

find a triangulation of the convex hull

that minimizes the sum of edge lengths.



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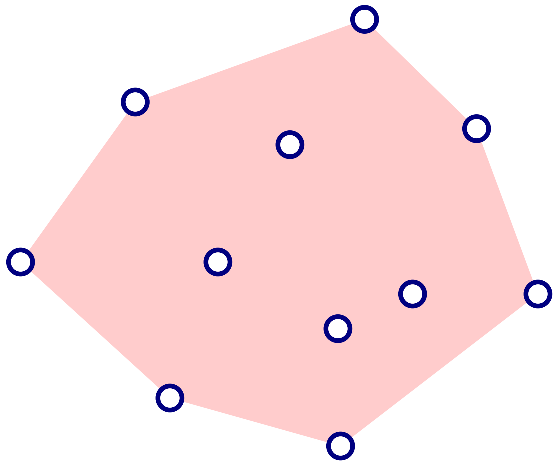

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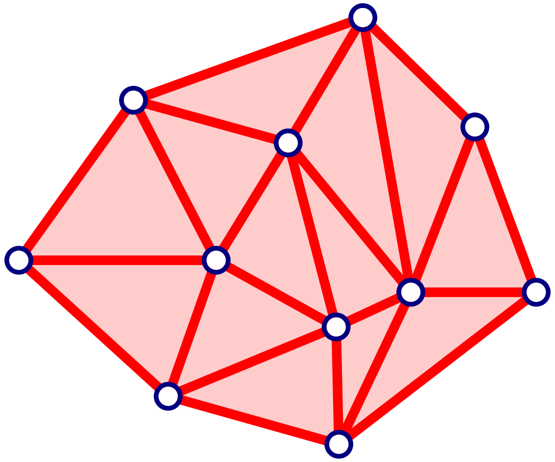

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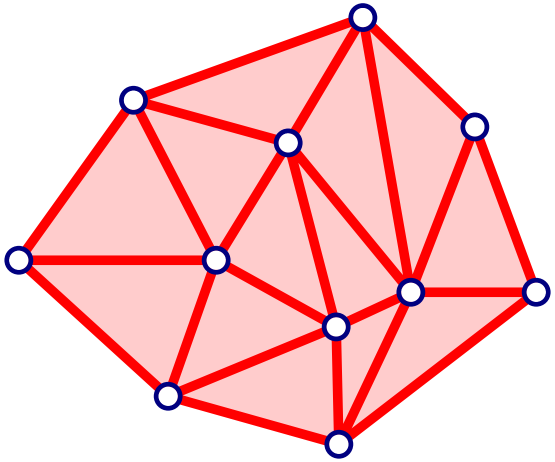

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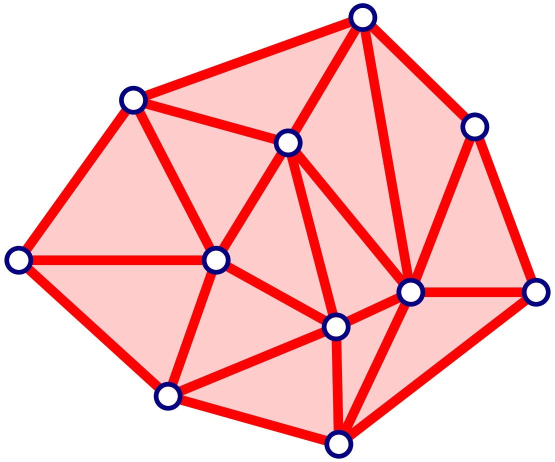

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NP-hard

Problems solvable in polynomial time

Algorithms for the same problem may have different complexities.

Ex: Merge sort has $\Theta(n \log n)$ complexity.

Bubble sort has $\Theta(n^2)$ complexity.

Quick sort has $\Theta(n^2)$ complexity but $O(n \log n)$ average-case complexity.

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This can have a drastic impact.

http://www.sorting-algorithms.com/

Wrap-up: what is it about?

Algorithmic solutions to geometric problems.

Proofs of correctness and complexity bounds.

Beware of undecidable or NP-hard problems.
Asymptotic complexity matters in practice.

(Attention to degeneracy and numerical issues.)

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How to compute the intersections among n segments in 2D?

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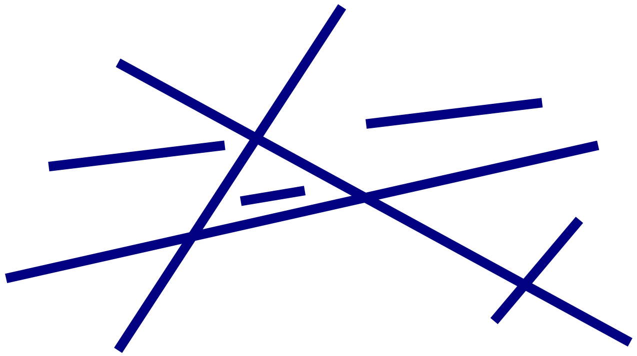

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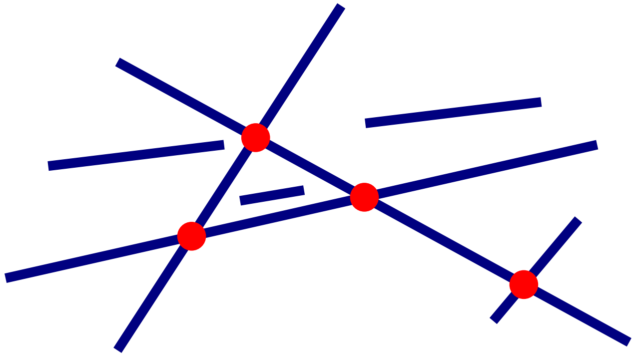

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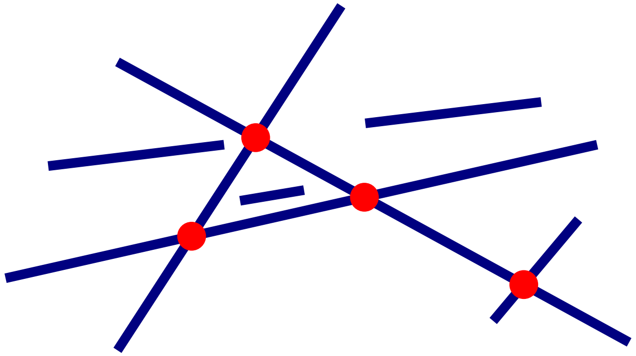
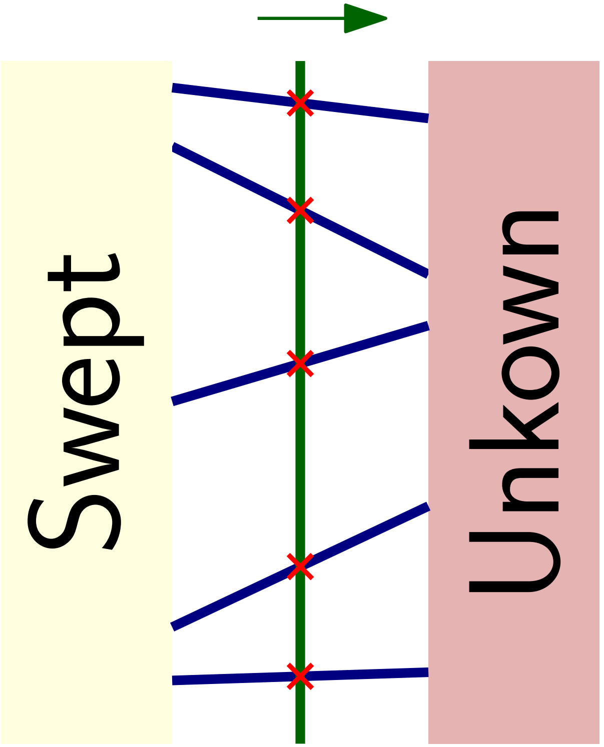


Any idea?

Use geometry to avoid testing unnecessary pairs.

Idea: sweep the plane using a line.



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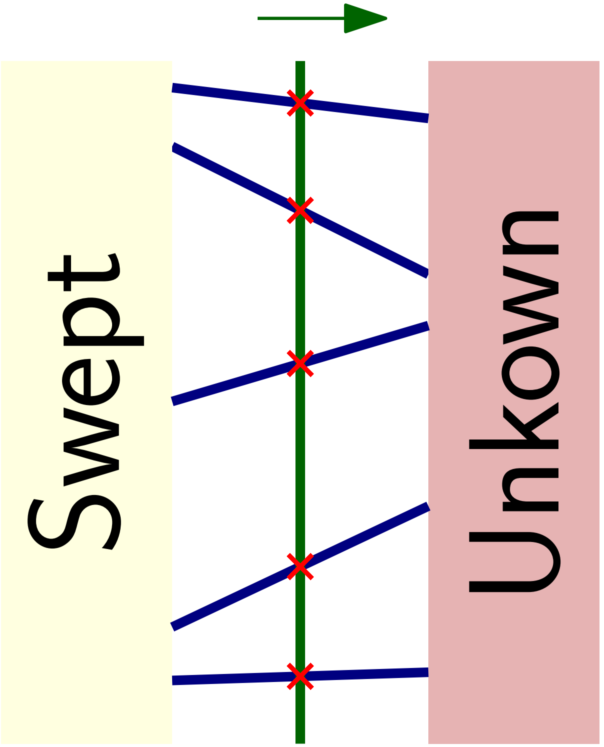

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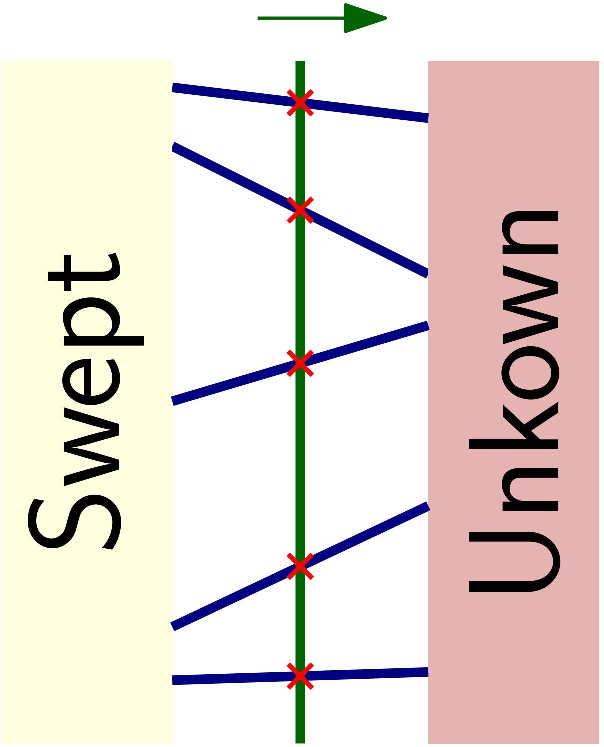

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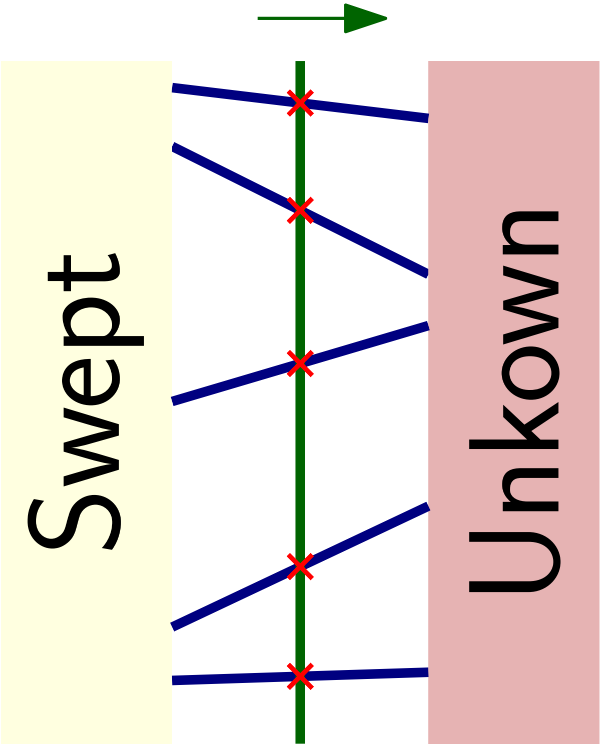

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Data structure? Predicates?

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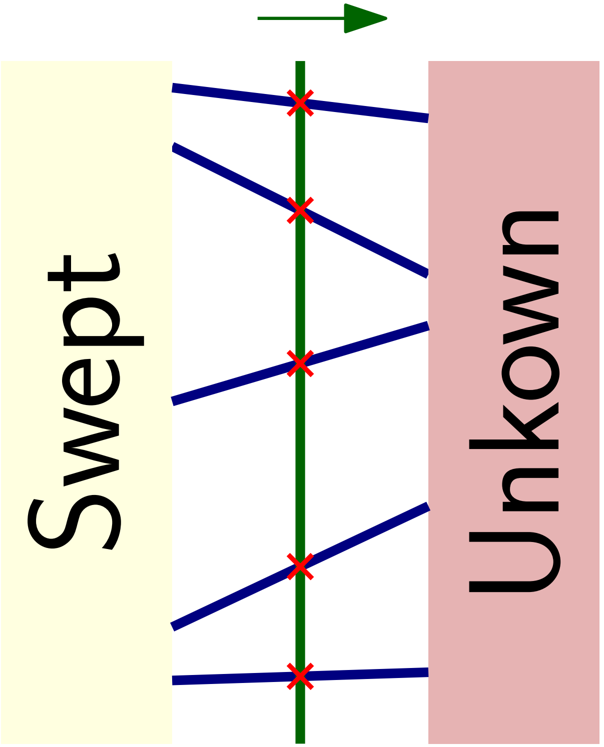

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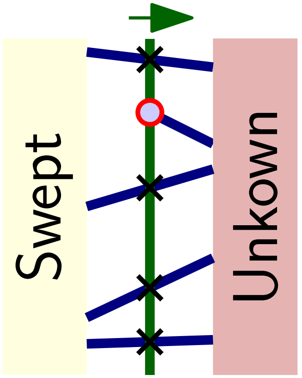

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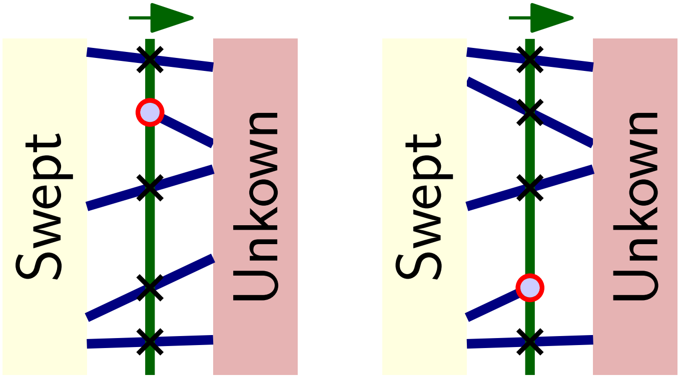

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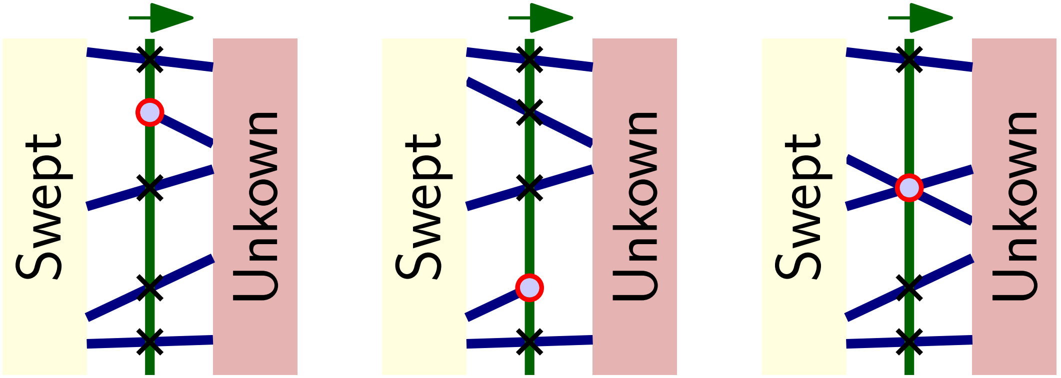

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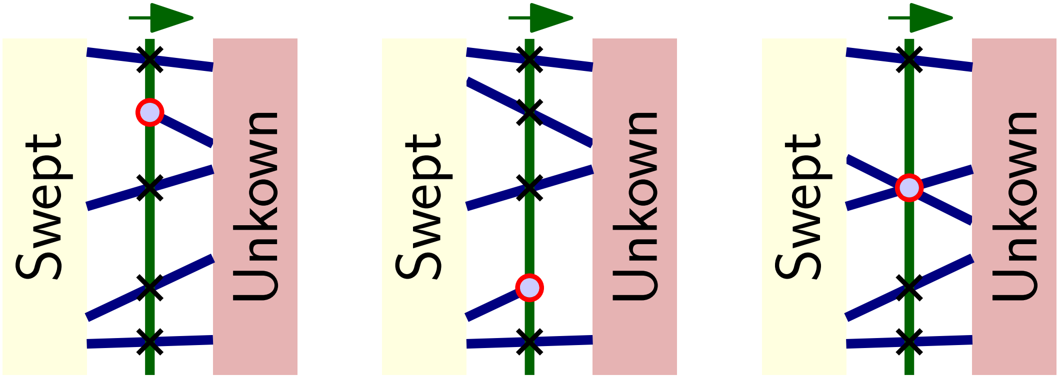
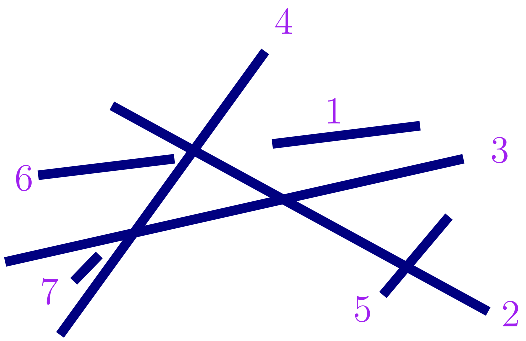


Data structures

Ordered list of segments intersected by the line.
Supports efficient insertion, deletion & exchange.

List of events sorted by x-coordinates.
Supports efficient insertion & deletion.

Algorithm



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While Events $\neq \emptyset$
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Events: sorted list of events.

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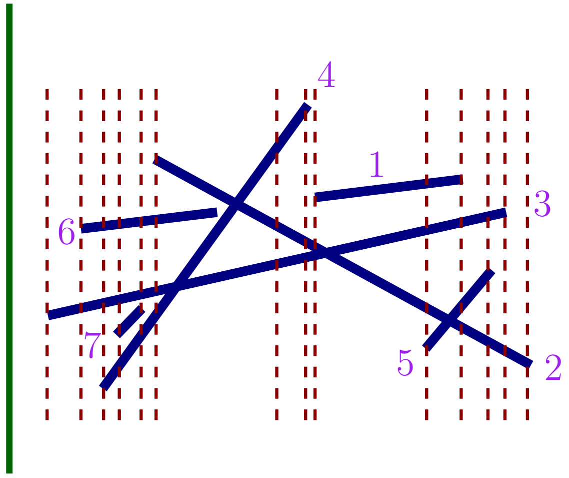

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Sweep$= \{\}$

Output$= \{\}$

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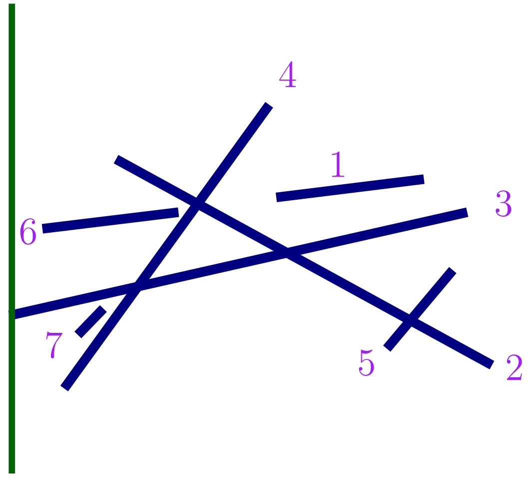

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Sweep$= \{3\}$

Output$= \{\}$

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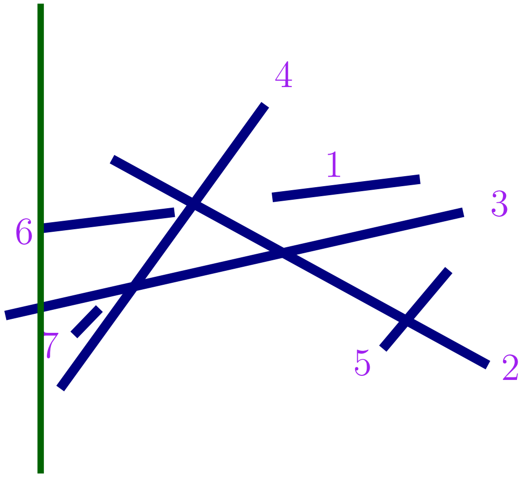

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Sweep$= \{6, 3\}$

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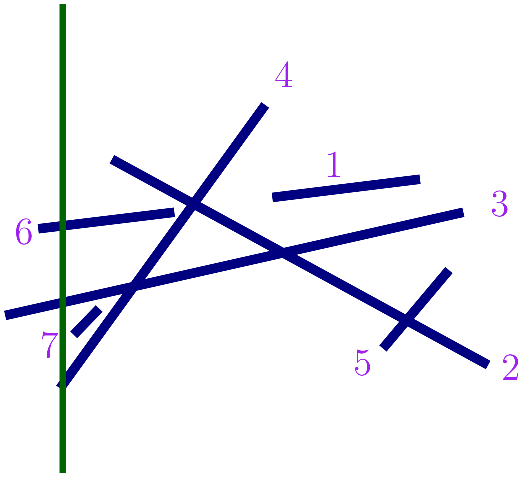
Algorithm



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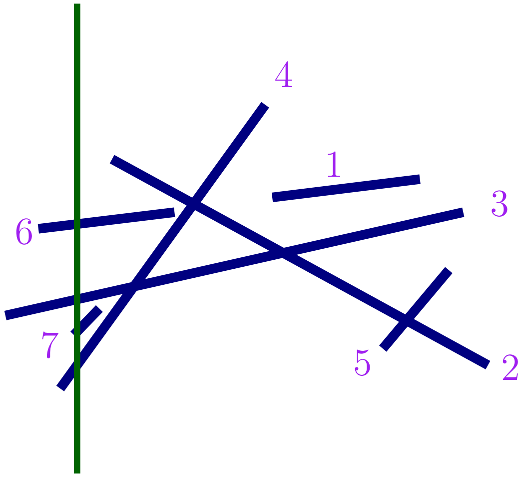
Sweep$= \{6, 3, 4\}$

Output$= \{(3, 4)\}$

Algorithm



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Sweep= $\{6, 3, 7, 4\}$

Output= $\{(3, 4)\}$

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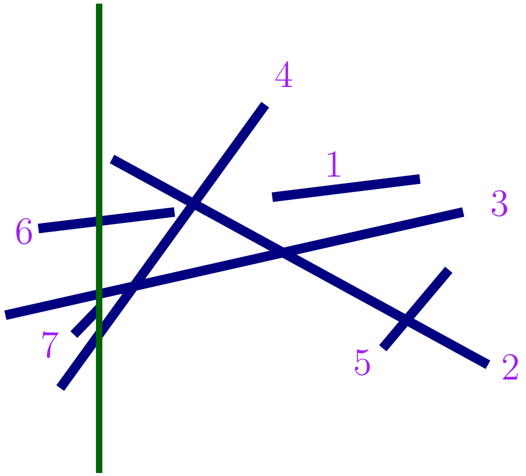

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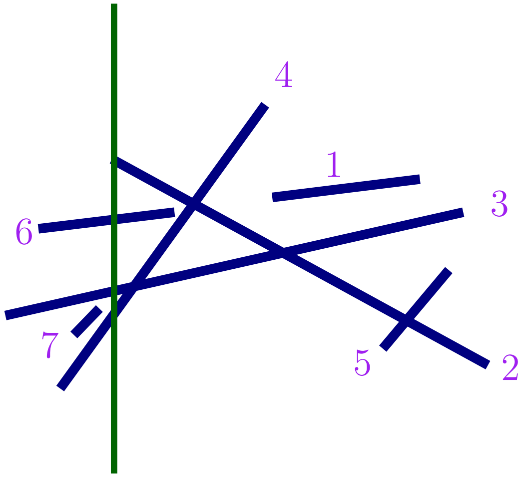

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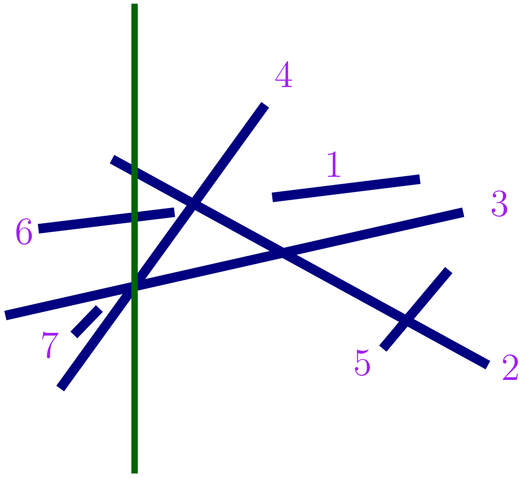

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Events: sorted list of events.

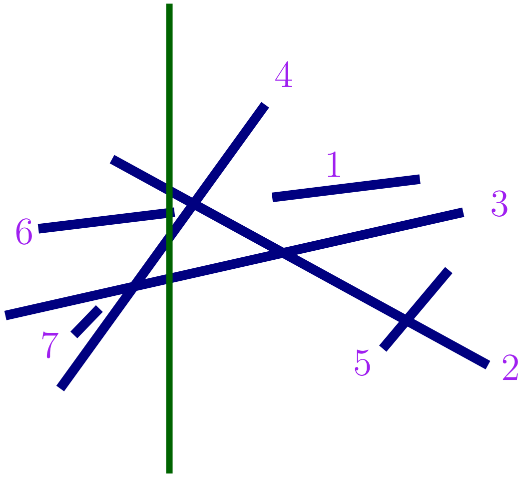
Sweep: sorted list of segments intersecting the sweep line.

Events= $\{I_{4,3}, R_6, R_4, L_1, L_5, R_1, R_5, R_3, R_2\}$

Sweep= $\{2, 6, 4, 3\}$

Output= $\{(3, 4)\}$

Algorithm



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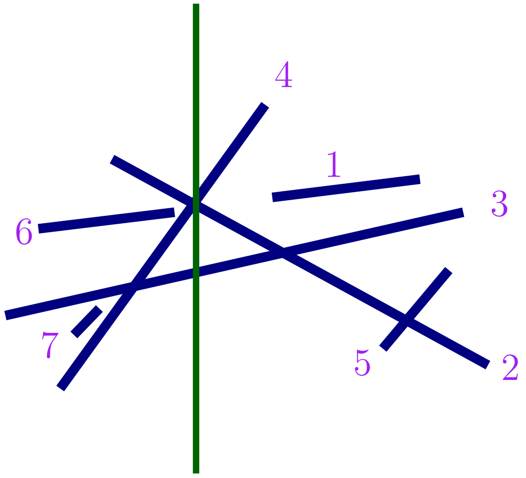

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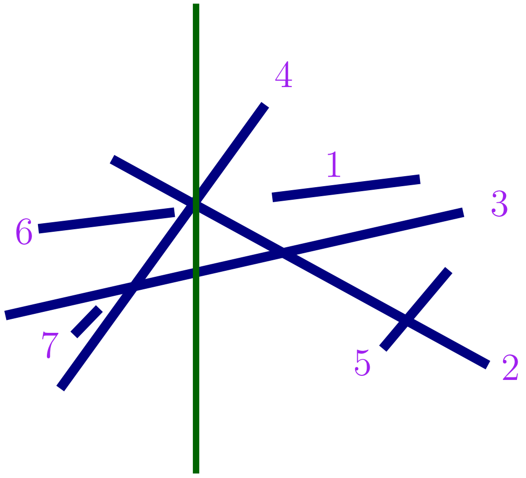

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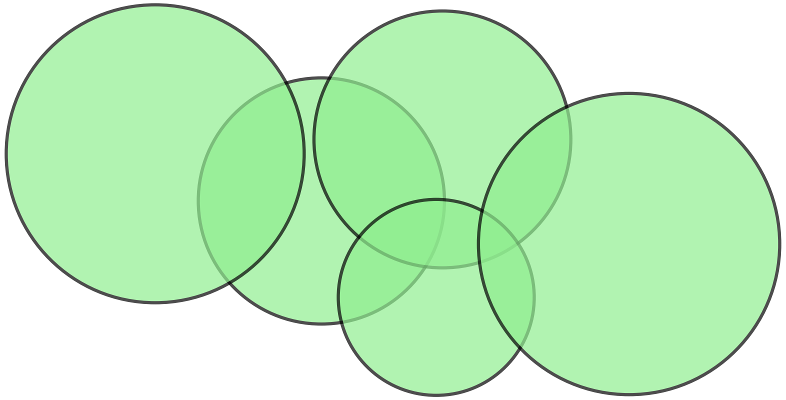
etc...

Correctness? Complexity?

Wrap-up: sweep algorithms

Generic principle, three predicates: x-extreme points, intersection, x-coordinate comparison.

Other objects (polygons, circles, algebraic curves, etc...), other spaces ($\mathbb{S}^2$, $\mathbb{R}^3$, $\mathbb{S}^1 \times \mathbb{S}^1$...).



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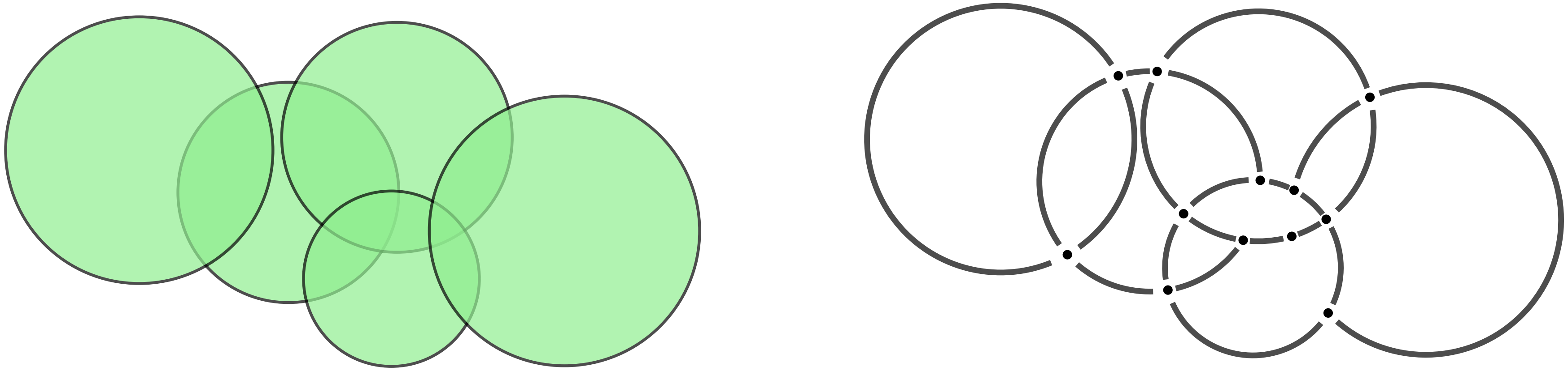

Computing arrangements of geometric objects.

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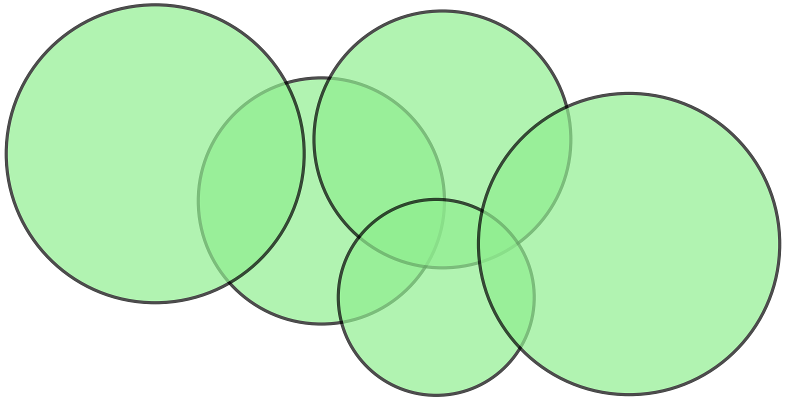

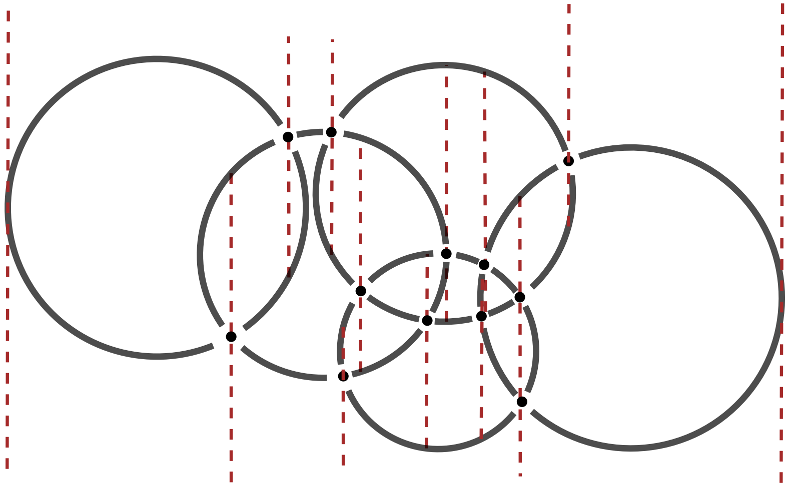

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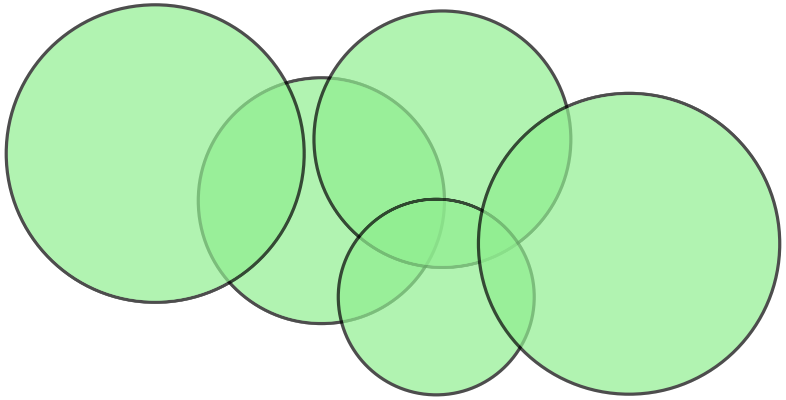

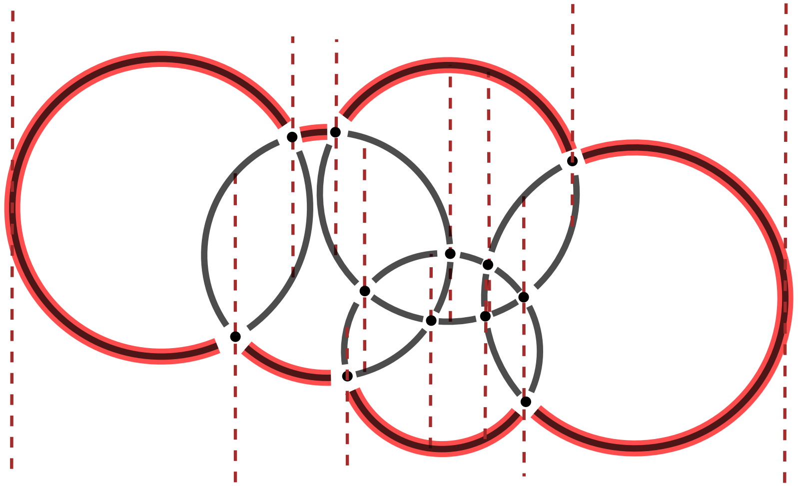

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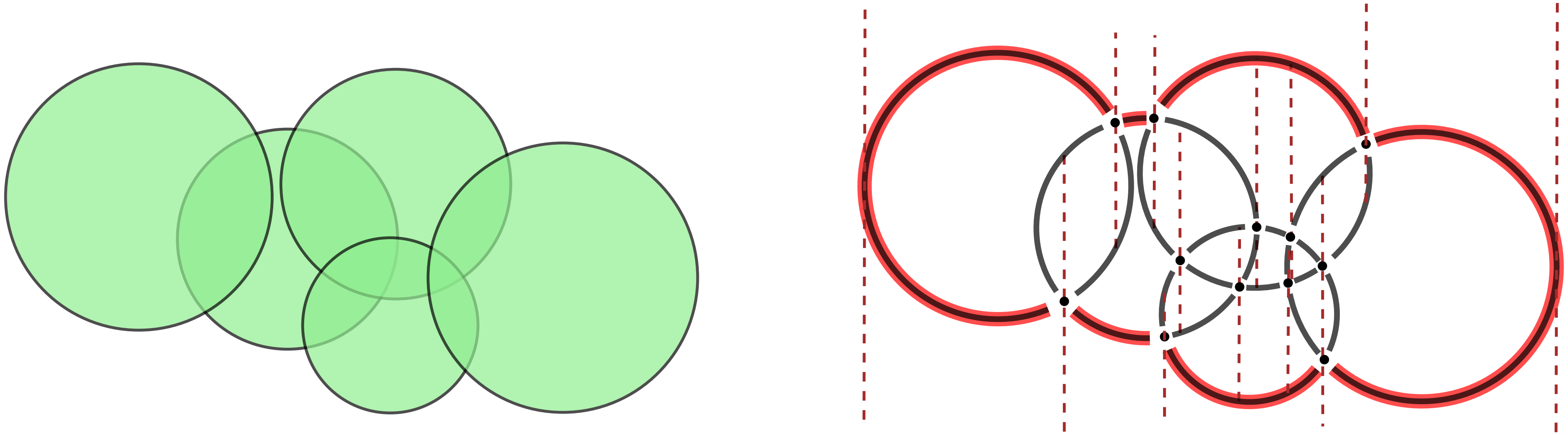

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All that in $O((n+k)\log n)$.

Question 3

Why are geometric algorithms hard to implement correctly?

First issue: degeneracies

Many algorithms are described assuming general position of the input.

$$\left.\begin{array}{l}\text{No two points have the same } x\text{-coordinate.}\\ \text{No three segments intersect in the same point.}\\ \text{No four points lie on the same circle.}\end{array}\right\} \quad \text{properties that hold generically.}$$

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Can we handle degeneracies without treating each one separately?

Can we at least detect them efficiently?

Hardness of testing degeneracies

The 3**-sum problem**: Given n numbers, decide if three of them sum to 0.

What is the best algorithm you can come-up with?

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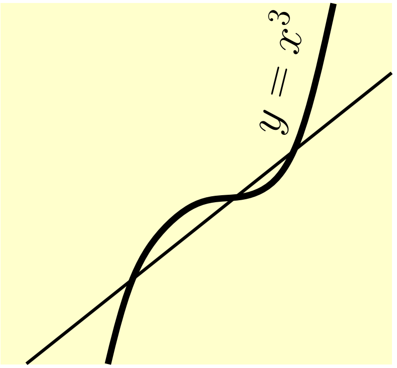
$t_i + t_j + t_k = 0 \Leftrightarrow p_i, p_j, p_k$ are aligned.

p, q and r are aligned $\Leftrightarrow \begin{vmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ 1 & 1 & 1 \end{vmatrix} = 0.$



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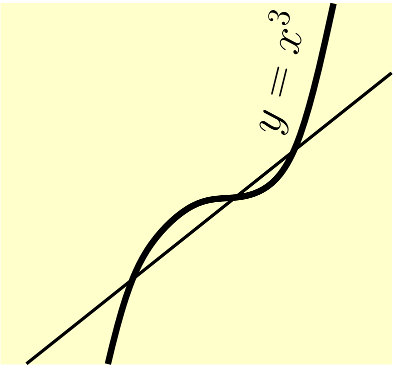

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Testing if $d+1$ points lie on a common hyperplane in $\mathbb{R}^d$ is $\lceil \frac{d}{2} \rceil$-sum hard.

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In principle, perturbing the points eliminate degeneracies.

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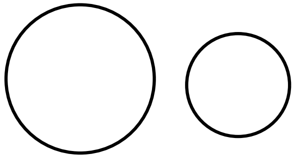
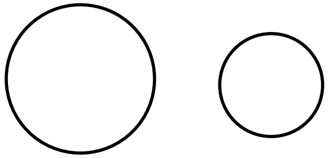

can be perturbed in

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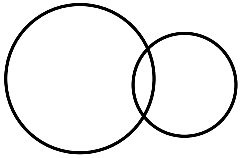

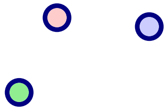
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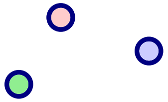
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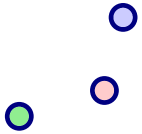

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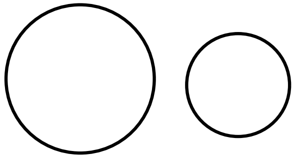

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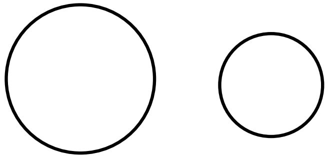

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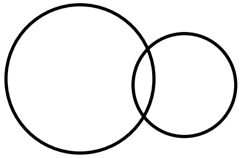

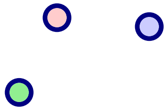

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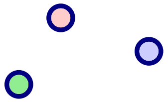

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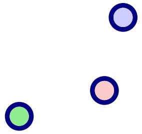

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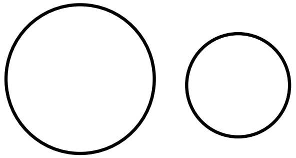

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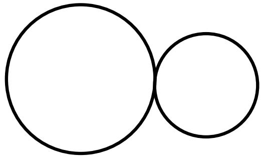

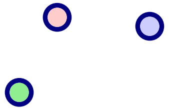

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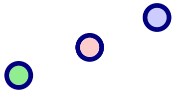

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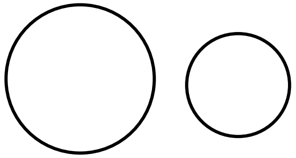

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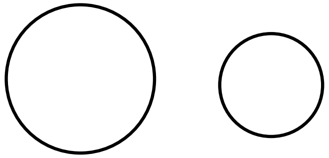

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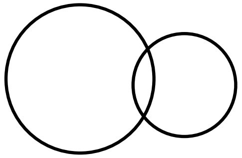

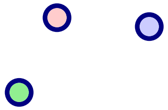

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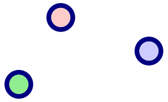

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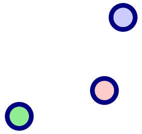

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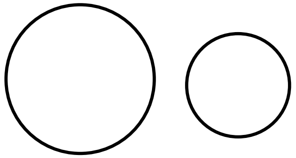

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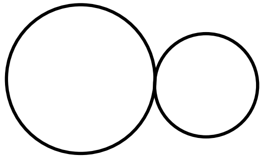

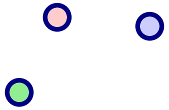

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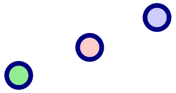

Bottom line: "Epsilon=10^-12" is not an option if we want any kind of guarantee.

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The input we are interested in is the value when $t = 0$.
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Make all computations for "$t > 0$ sufficiently small" then take $\lim_{t\to 0}$.

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Choose the functions so that the relevant polynomials do not identically vanish.

Example: convex hull computation, point-in-polygon.

Predicates are x*-coordinates comparison* and *orientation*.

Replacing p_i by $p_i + (t^{2^{2i}}, t^{2^{2i+1}})$ handles all degeneracies for these predicates.

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Partial perturbation: shearing $(x, y) \mapsto (x + ty, y)$

Second issue: numerical rounding

The arithmetic on a computer uses bounded precision (32 bits, 64 bits, IEEE float norms, etc...).

Small errors will be made in computations.

Not to mention that processors make errors from time to time...

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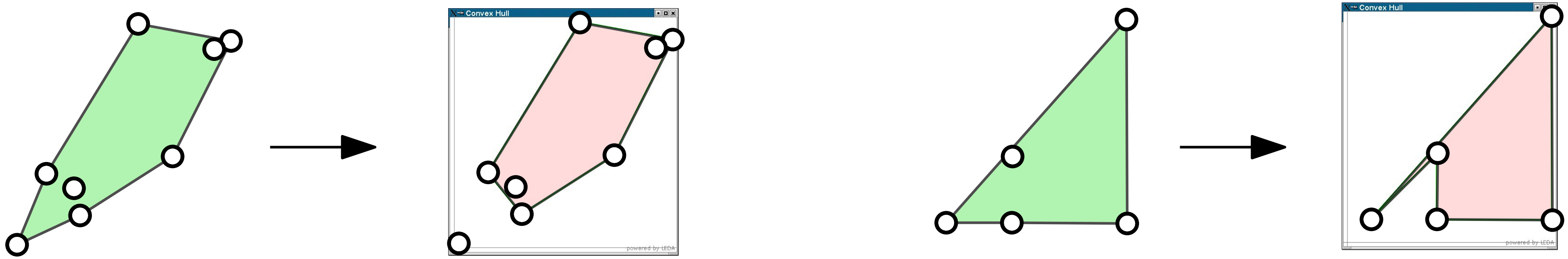

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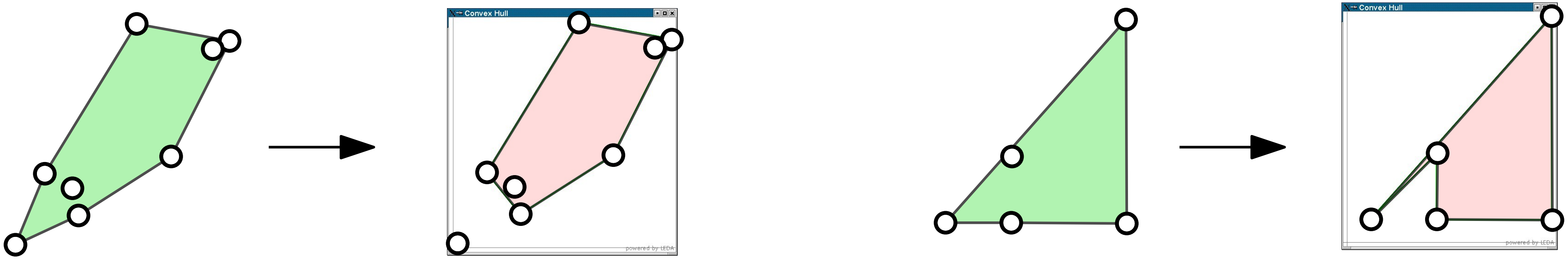
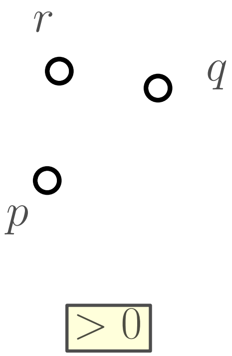


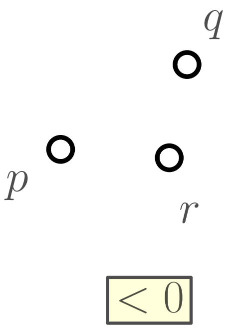
The problem: three points are nearly aligned, and the orientation predicates make inconsistent errors.

"Sometimes left, sometimes right".

A close look at that example

Orientation of (p, q, r) given by the sign of $\begin{vmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ 1 & 1 & 1 \end{vmatrix}$.





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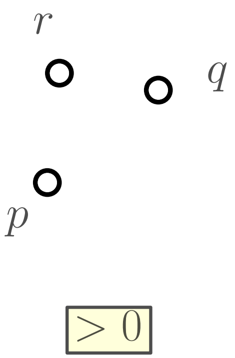

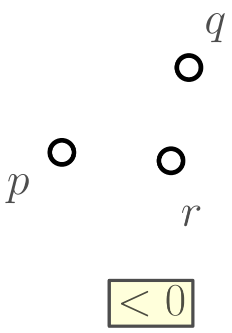

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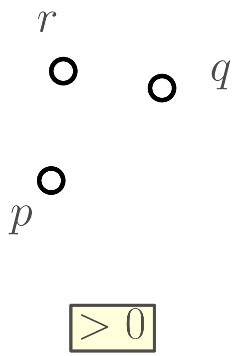

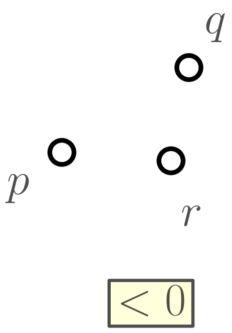

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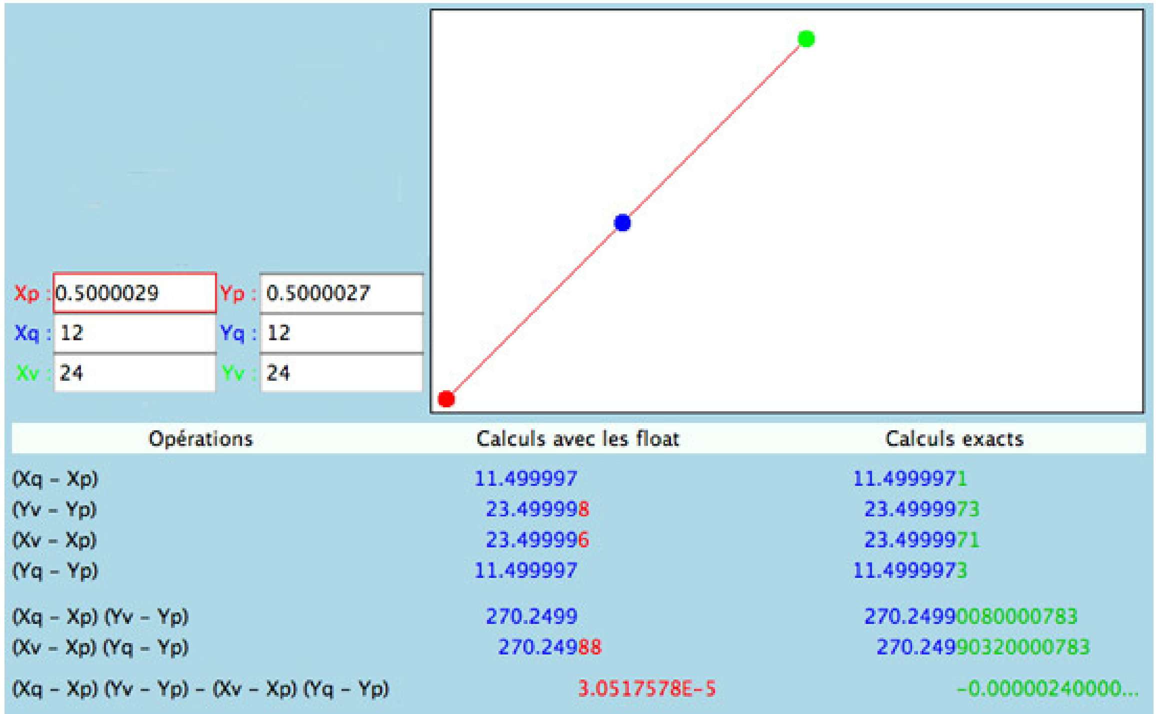

Xp : 0.5000029
Yp : 0.5000027
Xq : 12
Yq : 12
Xv : 24
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(Xq - Xp)	11.499997	11.4999971
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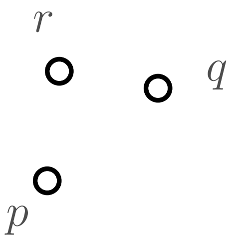

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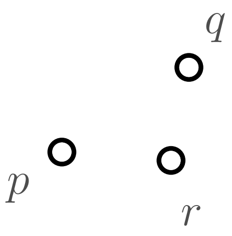

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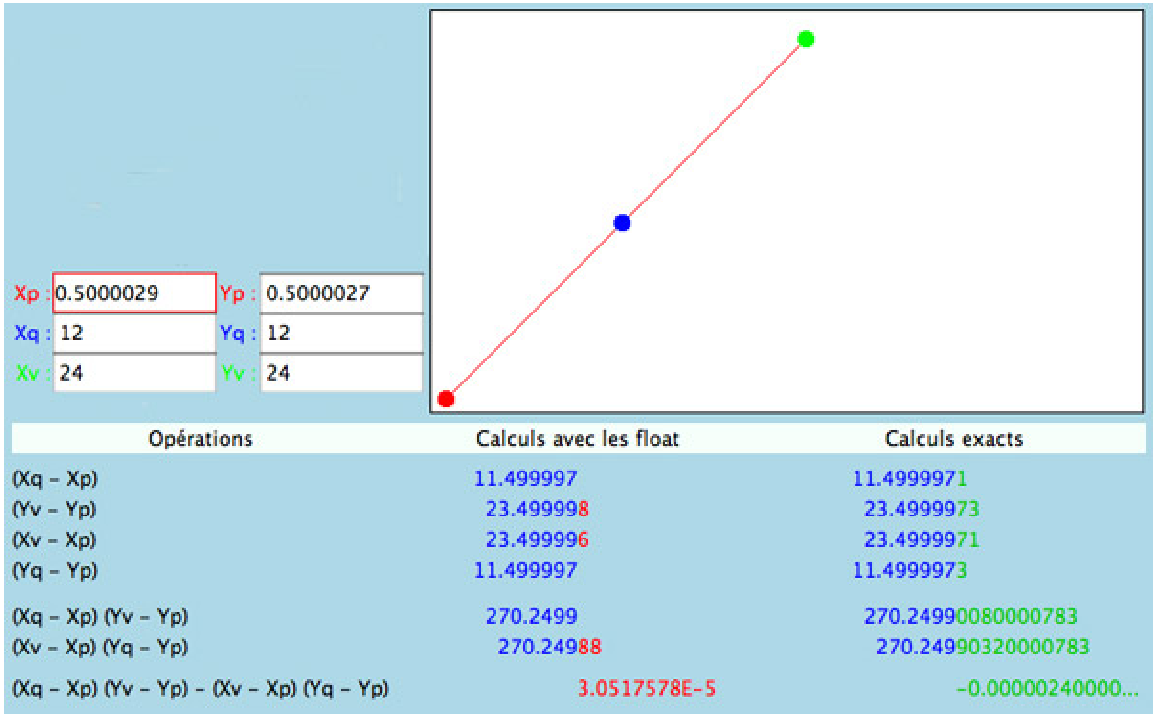

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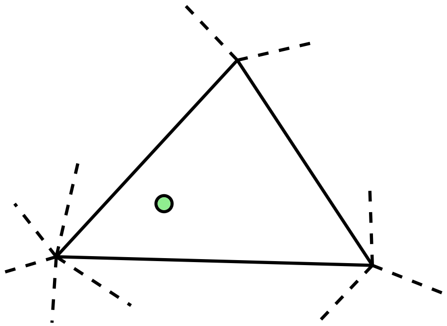

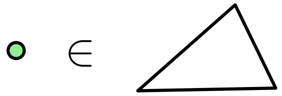

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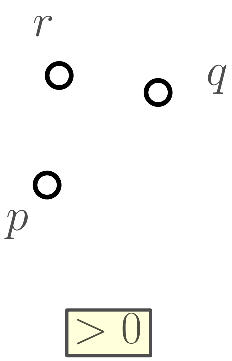

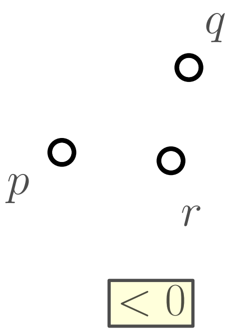

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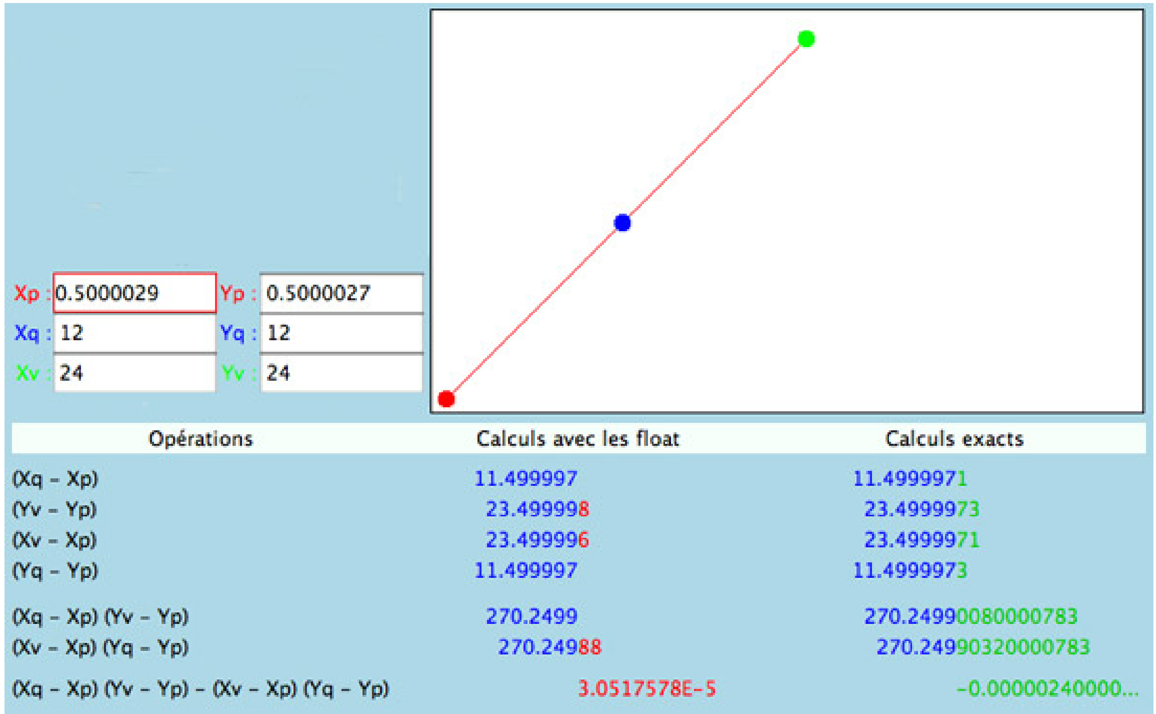

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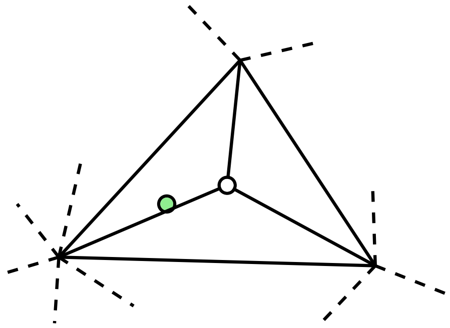

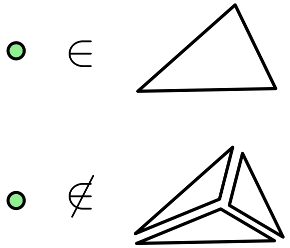

Consequences of numerical rounding

A ”correct” code can make incorrect decisions. These errors are inconsistent.

Crash, infinite loops, smooth execution but wrong answer... which is the worse?

Can be hard to detect...

Interval arithmetic

Keep the precision bounded but keep track of the error.

A number is represented by an interval (reduced to a single element if precision is sufficient).

Define all operations on intervals.

$$24 - 0.5000027 = 23.4999973 \sim 23.499998$$

becomes $[24, 24] - [0.5000027, 0.5000027] = [23.49999, 23.50000]$.

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If the result of the computation is exactly 0 we will never have enough precision...
For those few cases, we need to be able to do the computations exactly.

Exact number types for integers, rational numbers, algebraic numbers.

Algebraic numbers

$\sqrt[5]{23}$ and $\sqrt[7]{25}$ are not integers, but we can still compare them exactly using integer arithmetic.

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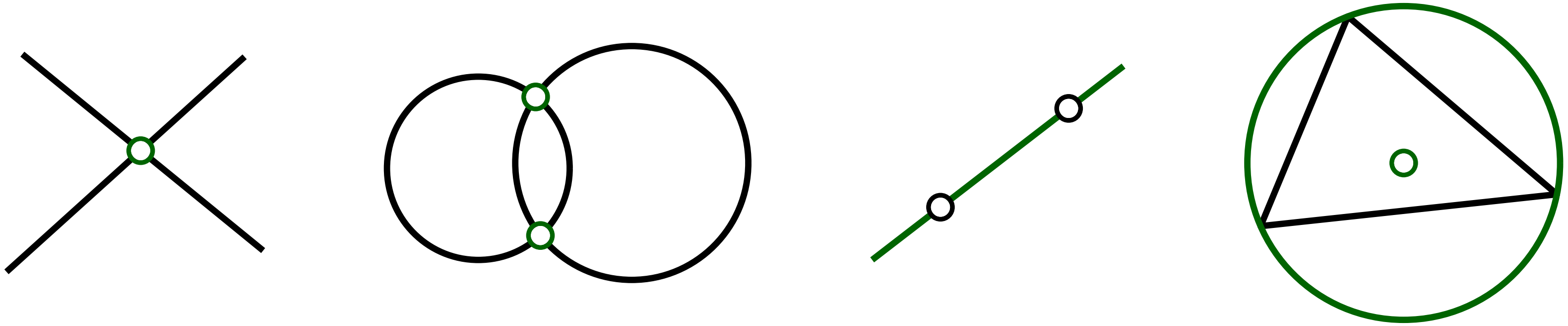
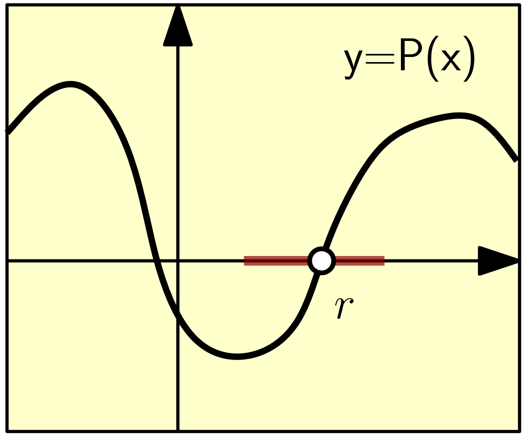
There are few algebraic numbers (ie countably many).

The result of most classical operations on geometric objects defined by integers
can be described using algebraic numbers.



Representing and manipulating algebraic numbers

An algebraic number can be represented by a polynomial (a family of integers) and an isolation interval.



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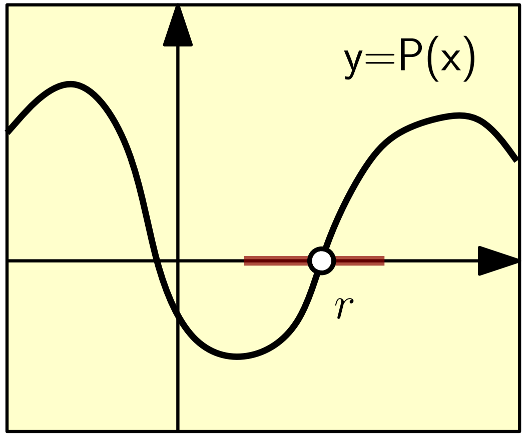

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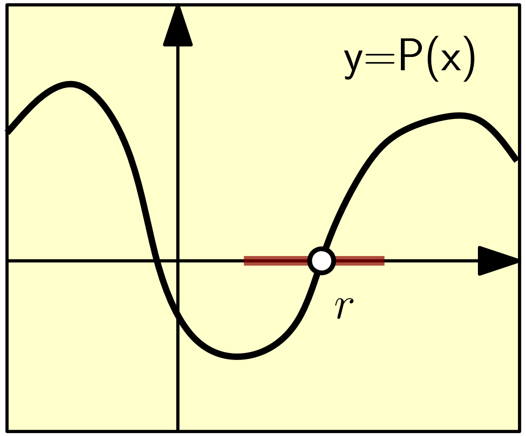

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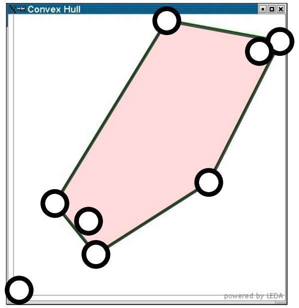
Implemented in the C/C++ CORE library.

Using algebraic numbers



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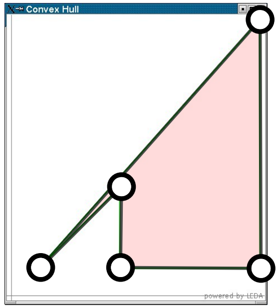
leads to



These problems can be avoided by using

```
Core::Expr xp,yp,xq,yq,xr,yr;
Orientation = sign((xq-xp)*(yr-yp)-(xr-xp)*(yq-yp));
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Decision VS constructions

Distinguish between decision (for branching) and constructions.

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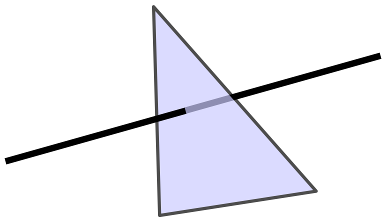

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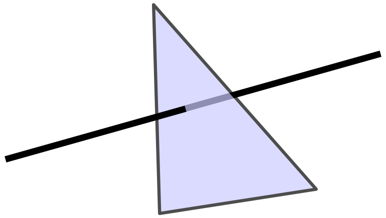

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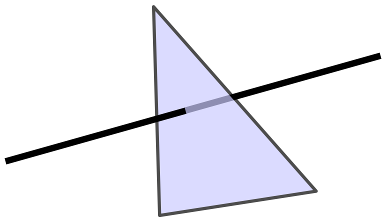

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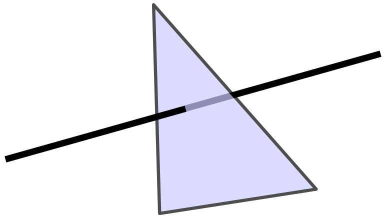

find intersection with plane, compute barycentric coordinates.
→ evaluate the sign of polynomials of degree 6.

Evaluate 3D orientations of quadruples of points
→ evaluate the sign of polynomials of degree 3.

Wrap-up: robustness

Treating degeneracies requires great care.

Numerical problems will arise.

If not treated properly, they produce crashes, infinite loops or wrong results.

Exact number types exist and are implemented. This is good enough for prototyping.

Reliability and efficiency are achieved by using good predicates
and filtering exact number type with interval arithmetic.

The End