Voronoi Diagrams and Delaunay Triangulations

Jean-Daniel Boissonnat

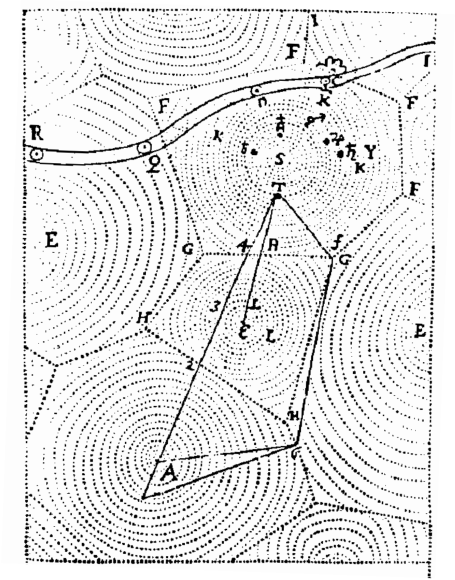
MPRI, Lecture 1, September 20, 2012

Outline

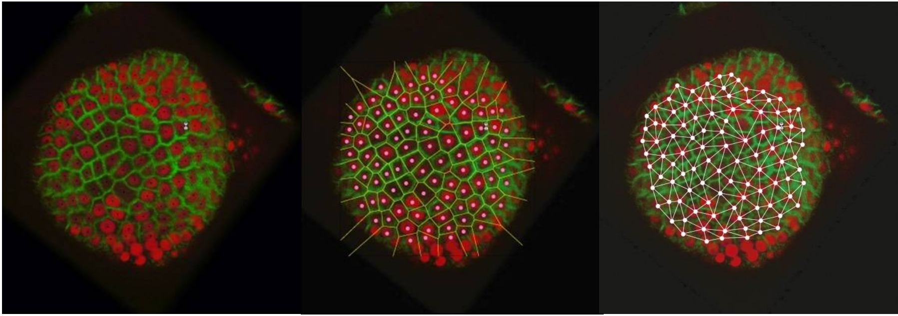
- Euclidean Voronoi diagrams
- Delaunay triangulations
- Convex hulls

Voronoi diagrams in nature

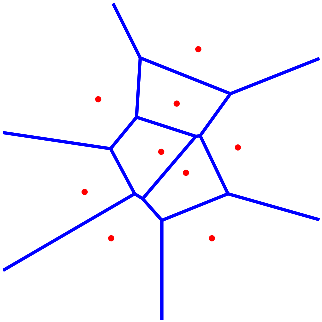
The solar system (Descartes)



Growth of merystem



Euclidean Voronoi diagrams



Voronoi cell $\qquad V(p_i) = \{x : \|x - p_i\| \leq \|x - p_j\|, \ \forall j\}$

Voronoi diagram $(\mathcal{P})$ $\qquad = \{$ cell complex whose cells are the $V(p_i)$ and their faces, $p_i \in \mathcal{P}$ $\}$

Polyhedra and cell complexes

Polyhedron

The intersection of a finite collection of half-spaces :

$$\mathcal{V} = \bigcap_{i \in I} \ h_i^+$$

Polyhedra and cell complexes

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Faces of a polyhedron

$$F_J = \bigcap_{j \in J} \; h_j^+ \; \bigcap_{i \in I \setminus J} \; h_i$$

Polyhedra and cell complexes

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Cell complex

A finite collection C of polyhedra called the faces of C such that

- $f \in C, g \subset f \Rightarrow g \in C$
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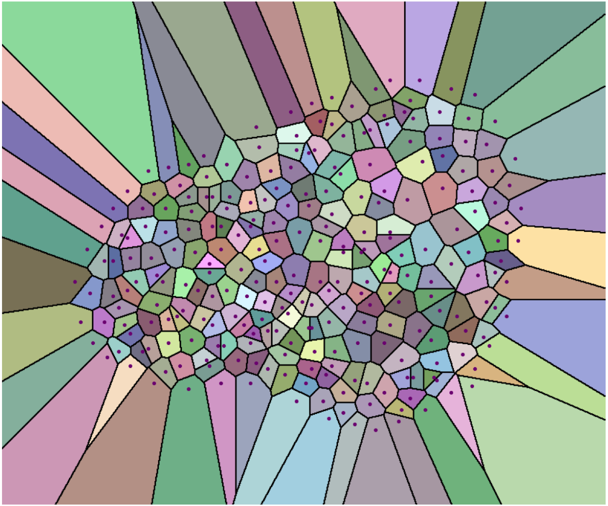
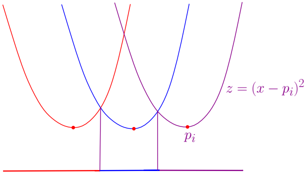
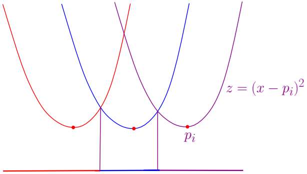


Voronoi diagrams and polytopes



Vor($p_1, \ldots, p_n$) is the minimization diagram of the n functions $\delta_i(x) = (x - p_i)^2$

Voronoi diagrams and polytopes



$\mathrm{Vor}(p_1, \ldots, p_n)$ is the minimization diagram of the n functions $\delta_i(x) = (x - p_i)^2$

$\arg\min(\delta_i) = \arg\max(h_i)$
where $h_{p_i}(x) = 2\, p_i \cdot x - p_i^2$

The minimization diagram of the δ_i is also the maximization diagram of the affine functions $h_i(x)$

Voronoi diagrams and polytopes

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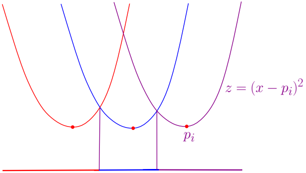

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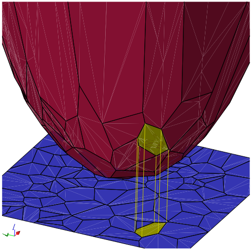

The faces of Vor$(\mathcal{P})$ are the projection of the faces of $\mathcal{V}(\mathcal{P}) = \bigcap_i \; h_{p_i}^+$
$h_{p_i}^+ = \{x : x_{d+1} > 2p_i \cdot x - p_i^2\}$

Voronoi diagrams and polytopes

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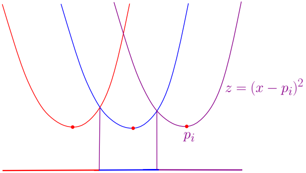

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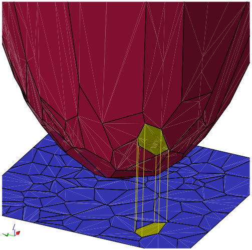

The faces of $\mathrm{Vor}(\mathcal{P})$ are the projection of the faces of $\mathcal{V}(\mathcal{P}) = \bigcap_i \, h_{p_i}^+$
$h_{p_i}^+ = \{x : x_{d+1} > 2p_i \cdot x - p_i^2\}$

Note !

$h_{p_i}(x) = 0$ is the hyperplane tangent to $\mathcal{Q} : x_{d+1} = x^2$ at (x, x^2)

Voronoi diagrams and polytopes

Lifting map

The faces of $\mathrm{Vor}(\mathcal{P})$ are the projection of the faces of the

polyhedron $\mathcal{V}(\mathcal{P}) = \bigcap_i \; h_{p_i}^{+}$

where h_{p_i} is the hyperplane tangent to paraboloid $\mathcal{Q}$
at the lifted point (p_i, p_i^2)

Voronoi diagrams and polytopes

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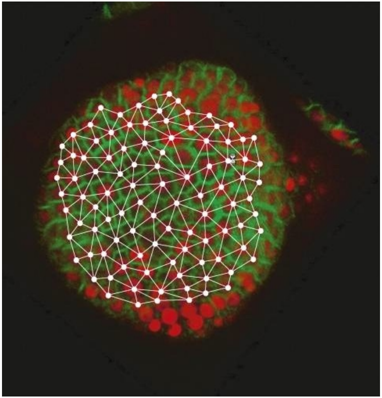
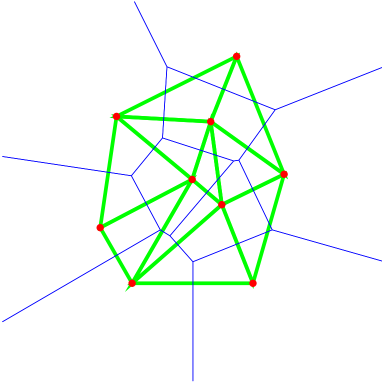
Corollaries

- The size of $\mathrm{Vor}(\mathcal{P})$ is the same as the size of $\mathcal{V}(\mathcal{P})$
- Computing $\mathrm{Vor}(\mathcal{P})$ reduces to computing $\mathcal{V}(\mathcal{P})$

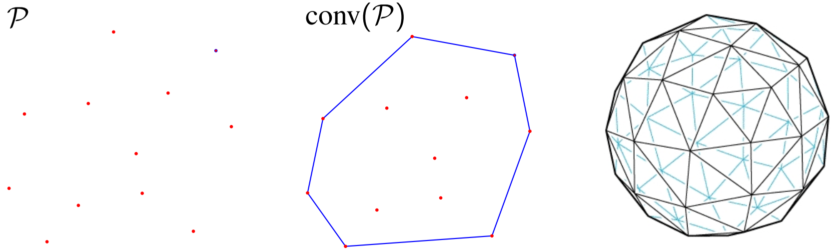
Delaunay Triangulations

Dual triangulation





Convex hull of a finite point set $\mathcal{P}$



Definition

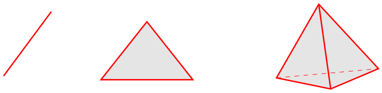
$$\mathrm{conv}(\mathcal{P}) = \{\sum \lambda_i p_i, \quad \lambda_i \geq 0, \quad \sum_i \lambda_i = 1\}$$

Geometric simplices

k-dimensional simplex (*k*-simplex for short)

The convex hull of $k + 1$ points that are affinely independent

1-simplex = line segment
2-simplex = triangle
3-simplex = tetrahedron



Geometric simplicial complexes

Definition

A finite collection of simplices C called the faces of C such that

- $\forall f \in C$, f is a simplex
- $f \in C$, $f \subset g \Rightarrow g \in C$
- $\forall f, g \in C$, either $f \cap g = \emptyset$ or $f \cap g \in C$

The dimension of the complex is the max dimension of its simplices

Abstract simplicial complexes

Given a finite set of points $\mathcal{P}$ (not necessarily from a Euclidean space) a subset $C = \{\sigma_1, ..., \sigma_m\}$ is a simplicial complex if

1. $\forall i, \quad \sigma_i \subset \mathcal{P}$
2. $\forall i,$ all the subsets of σ_i are in C
3. $\forall i, j, \quad \sigma_i \cap \sigma_j \in C$

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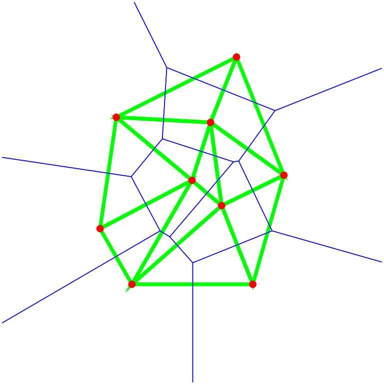
Theorem

Any simplicial complex of dimension k can be embedded in $\mathbb{R}^{2k+1}$

Nerve of the Voronoi diagram of $\mathcal{P} = \{p_1, ..., p_n\} \subset \mathbb{R}^d$

If j Voronoi cells $V(p_{i_1}), ..., V(p_{i_j})$ have a non empty intersection, $\mathrm{conv}(p_{i_1}, ..., p_{i_j})$ is a simplex of the Delaunay triangulation $\mathrm{Del}(P)$



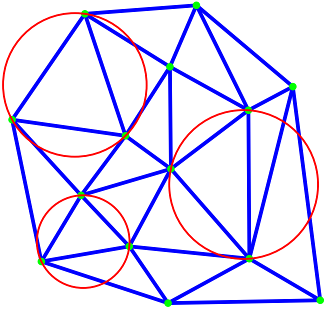
Note : $\mathrm{Del}(\mathcal{P})$ is not always embedded in $\mathbb{R}^d$

Empty circumballs

An (open) d-ball B circumscribing a simplex $\sigma \subset \mathcal{P}$ is called empty if

1. $\sigma \subset \partial B$
2. $B \cap \mathcal{P} = \emptyset$

$\mathrm{Del}(\mathcal{P})$ is the collection of simplices admitting an empty circumball



Generic point sets

$\mathcal{P} = \{p_1, p_2 \ldots p_n\}$ is said to be generic if $\nexists$ $d+1$ points of $\mathcal{P}$ lying on a same sphere

If $\mathcal{P}$ is generic, $t \subset \mathcal{P}$ is a Delaunay simplex iff
$\exists$ a sphere $\sigma_t = \{x, \sigma_t(x) = 0\}$ s.t.

$$\sigma_t(p) = 0 \ \forall p \in t$$
$$\sigma_t(q) > 0 \ \forall q \in \mathcal{P} \setminus t$$

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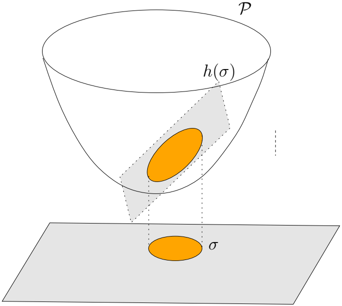
Theorem [Delaunay 1936]

If $\mathcal{P}$ is generic, $\mathrm{Del}(\mathcal{P})$ is embedded in $\mathbb{R}^d$

Proof of Delaunay's theorem



Linearization

$\sigma(x) = x^2 - 2c \cdot x + s, \quad s = c^2 - r^2$

$$\sigma(x) < 0 \Leftrightarrow \begin{cases} z < 2c \cdot x - s & (h_\sigma^-) \\ z = x^2 & (\mathcal{P}) \end{cases}$$

$$\Leftrightarrow \hat{x} = (x, x^2) \in h_\sigma^-$$

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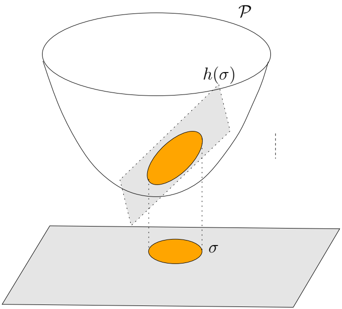

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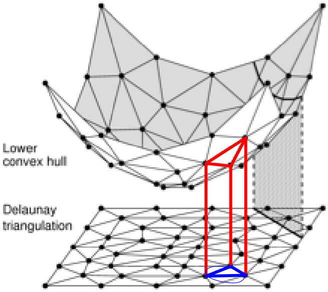
Proof of Delaunay's th.

t a simplex, σ_t its circumscribing sphere

$$t \in \mathrm{Del}(\mathcal{P}) \Leftrightarrow \forall i, \hat{p}_i \in h_{\sigma_t}^+$$

$$\Leftrightarrow \hat{t} \text{ is a face of } \mathrm{conv}^-(\hat{\mathcal{P}})$$

Proof of Delaunay's theorem



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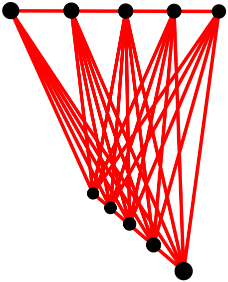
$\Leftrightarrow \hat{t}$ is a face of $\mathrm{conv}^-(\hat{\mathcal{P}})$

$$\boxed{\mathrm{Del}(\mathcal{P}) = \mathrm{proj}(\mathrm{conv}^-(\hat{\mathcal{P}}))}$$

Combinatorial complexity

The combinatorial complexity of the Delaunay triangulation diagram of n points of $\mathbb{R}^d$ is the same as the combinatorial complexity of a convex hull of n points of $\mathbb{R}^{d+1}$



Quadratic in $\mathbb{R}^3$

Constructing $\mathrm{Del}(\mathcal{P})$, $\mathcal{P} = \{p_1, ..., p_n\} \subset \mathbb{R}^d$

Algorithm

1. Lift the points of $\mathcal{P}$ onto the paraboloid $x_{d+1} = x^2$ of $\mathbb{R}^{d+1}$: $p_i \to \hat{p}_i = (p_i, p_i^2)$
2. Compute $\mathrm{conv}(\{\hat{p}_i\})$
3. Project the lower hull $\mathrm{conv}^-(\{\hat{p}_i\})$ onto $\mathbb{R}^d$

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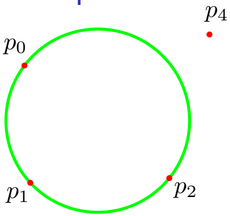
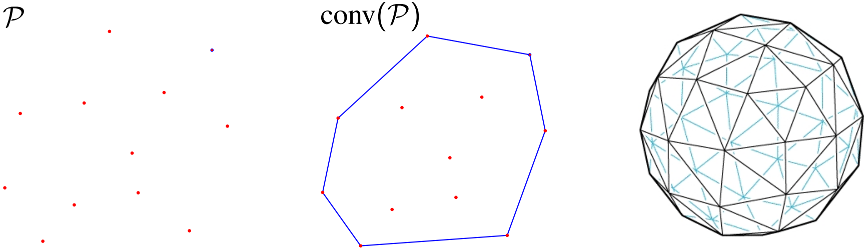
Main predicate



$$\texttt{insphere}(p_0, \ldots, p_{d+1}) = \texttt{orient}(\hat{p}_0, \ldots, \hat{p}_{d+1})$$
$$= \mathrm{sign}\begin{vmatrix} 1 & \ldots & 1 \\ p_0 & \ldots & p_{d+1} \\ p_0^2 & \ldots & p_{d+1}^2 \end{vmatrix}$$

Convex Hulls



Set of all possible convex combinations of points in $\mathcal{P}$

$$\sum \lambda_i p_i, \quad \lambda_i \geq 0, \quad \sum_i \lambda_i = 1$$

We call polytope the convex hull of a finite set of points

Cell complex

A finite collection of polytopal cells C called the faces of C such that

- $f \in C, g \subset f \Rightarrow g \in C$
- $\forall f, g \in C$, either $f \cap g = \emptyset$ or $f \cap g \in C$

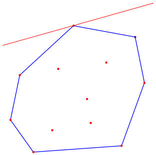
Simplicial complex

all faces are simplices

Facial structure of a polytope

Supporting hyperplane

$H \cap C \neq \emptyset$ and C is entirely contained in one of the two half-spaces defined by H



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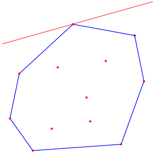
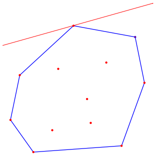


Faces

The faces of P are the polytopes $P \cap h$, h support. hyp.

Facial structure of a polytope



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The face complex

The faces of P form a cell complex C

General position

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A point set $\mathcal{P}$ is said to be in general position iff no subset of $k+2$ points lie in a k-flat

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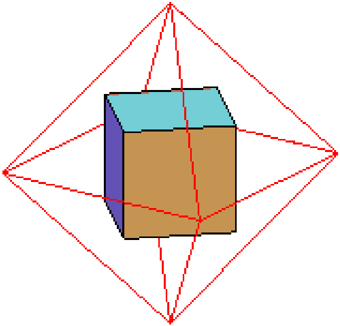
Boundary complex

If $\mathcal{P}$ is in general position, all the faces of $\mathrm{conv}(\mathcal{P})$ are simplices

The boundary of $\mathrm{conv}(\mathcal{P})$ is a simplicial complex

Two ways of defining polyhedra



Convex hull of n points

Intersection of n half-spaces

Duality between points and hyperplanes

hyperplane $h : x_d = a \cdot x' - b$ of $\mathbb{R}^d \longrightarrow$ point $h^* = (a, b) \in \mathbb{R}^{d-1} \times \mathbb{R}$

point $p = (p', p_d) \in \mathbb{R}^d \longrightarrow$ hyperplane $p^* \subset \mathbb{R}^d$
$= \{(a, b) \in \mathbb{R}^d : b = p' \cdot a - p_d\}$

The mapping *

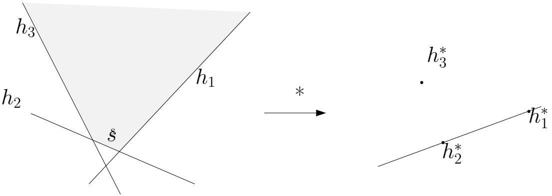
- preserves incidences :

$$\begin{array}{rcl} p \in h & \Longleftrightarrow & p_d = a \cdot p' - b \Longleftrightarrow b = p' \cdot a - p_d \Longleftrightarrow h^* \in p^* \\ p \in h^+ & \Longleftrightarrow & p_d > a \cdot p' - b \Longleftrightarrow b > p' \cdot a - p_d \Longleftrightarrow h^* \in p^{*+} \end{array}$$

- is an involution and thus is bijective : $h^{**} = h$ and $p^{**} = p$

Duality between polytopes

Let $h_1, \ldots, h_n$ be n hyperplanes de $\mathbb{R}^d$ and let $P = \cap h_i^+$



A vertex s of P is the intersection of $k \geq d$ hyperplanes $h_1, \ldots, h_k$ lying above all the other hyperplanes

$\Longrightarrow$ s^* is a hyperplane $\ni h_1^*, \ldots, h_k^*$
supporting P^*=conv$^-(h_1^*, \ldots, h_k^*)$

General position

s is the intersection of d hyperplanes

$\Longrightarrow$ s^* supports a $(d-1)$-face (simplex) de P^*

More generally and under the general position assumption,

if f is a $(d-k)$-face of P and $\mathrm{aff}(f) = \cap_{i=1}^{k} h_i$

$$p \in f \quad \Leftrightarrow \quad \begin{array}{l} h_i^* \in p^* \text{ for } i = 1, \ldots, k \\ h_i^* \in p^{*+} \text{ for } i = k+1, \ldots, n \end{array}$$

$$\Leftrightarrow \quad \begin{array}{l} p^* \text{support. hyp. of } P^* = \mathrm{conv}(h_1^*, \ldots, h_n^*) \\ p^* \ni h_1^*, \ldots, h_k^* \end{array}$$

$$\Leftrightarrow \quad f^* = \mathrm{conv}(h_1^*, \ldots, h_k^*) \text{ is a } (k-1)-\text{face of } P^*$$

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\end{aligned}
$$

Duality between P and P^*

- The correspondence between the faces of P and P^* is involutive and therefore bijective
- It reverses inclusions : $\forall f, g \in P, f \subset g \Rightarrow g^* \subset f^*$

Algorithmic consequences

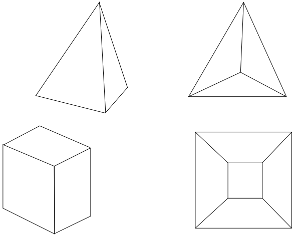
- Computing the intersection of n upper half-spaces or the lower convex hull of n points are equivalent problems
- Depending on the application, the primal or the dual setting may be more appropriate

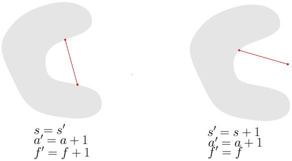
Euler formula for 3-polytopes

The numbers of vertices *s*, edges *a* and facets *f* of a polytope of $\mathbb{R}^3$ satisfy

$$s - a + f = 2$$

Schlegel diagram





Euler formula for 3-polytopes : $s - a + f = 2$

Incidences edges-facets

$$2a \geq 3f \quad \Longrightarrow \quad \begin{array}{c} a \leq 3s - 6 \\ f \leq 2s - 4 \end{array}$$

with equality when all facets are triangles

Beyond the 3rd dimension

Upper bound theorem

[McMullen 1970]

If P is the intersection of n half-spaces of $\mathbb{R}^d$

$$\text{nb faces of } P = \Theta(n^{\lfloor \frac{d}{2} \rfloor})$$

Beyond the 3rd dimension

Upper bound theorem

[McMullen 1970]

If P is the intersection of n half-spaces of $\mathbb{R}^d$

$$\text{nb faces of } P = \Theta(n^{\lfloor \frac{d}{2} \rfloor})$$

General position

- all vertices of P are incident to d edges (in the worst-case) and have distinct x_d
- the convex hull of $k < d$ edges incident to a vertex p is a k-face of P
- any k-face is the intersection of $d - k$ hyperplanes defining P

Proof of the upper bound th.

Bounding the number of vertices

1. $\geq \lceil \frac{d}{2} \rceil$ edges incident to a vertex p are in $h_p^+ : x_d \geq x_d(p)$ or in h_p^-
   - $\Rightarrow$ p is a x_d-max or x_d-min vertex of at least one $\lceil \frac{d}{2} \rceil$-face of P
   - $\Rightarrow$ # vertices of $P \leq 2 \times \#\ \lceil \frac{d}{2} \rceil$-faces of P

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Bounding the total number of faces

The number of faces incident to p depends on d but not on n

Representation of a convex hull

Adjacency graph (AG) of the facets

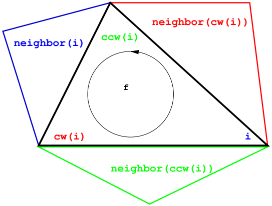
In general position, all the facets are $(d-1)$-simplexes

⇒ **Vertex**

Face* *v_face*

Face

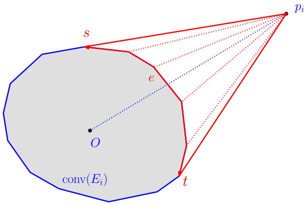
Vertex* *vertex*[*d*]
Face* *neighbor*[*d*]



Incremental algorithm

$\mathcal{P}_i$: set of the i points that have been inserted first

$\mathrm{conv}(\mathcal{P}_i)$: convex hull at step i



$f = [p_1, ..., p_d]$ is a red facet iff its supporting hyperplane separates p_i from $\mathrm{conv}(\mathcal{P}_i)$

$\iff \texttt{orient}(p_1, ..., p_d, p_i) \times \texttt{orient}(p_1, ..., p_d, O) < 0$

$$\mathrm{orient}(p_0, p_1, ..., p_d) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ p_0 & p_1 & \ldots & p_d \end{vmatrix} = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_{01} & x_{11} & \ldots & x_{d1} \\ \vdots & \vdots & \ldots & \vdots \\ x_{0d} & x_{1d} & \ldots & x_{dd} \end{vmatrix}$$

Update of $\mathrm{conv}(\mathcal{P}_i)$

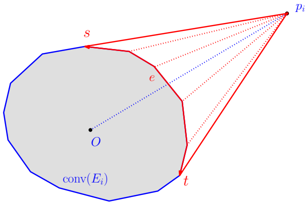
red facet = facet whose supporting hyperplane separates o and p_{i+1}

horizon : $(d-2)$-faces shared by a blue and a red facet

Update $\mathrm{conv}(\mathcal{P}_i)$:

1. find the red facets
2. remove them and create the new facets

   $[p_{i+1}, g], \forall g \in$ horizon



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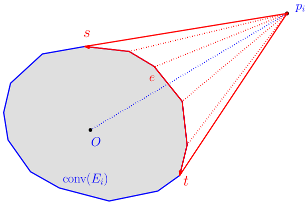
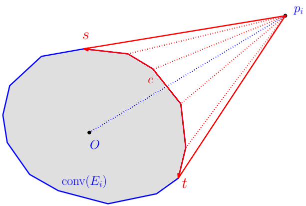


Complexity

proportional to the nb of red facets

Complexity analysis

- update proportionnal to the number of red facets
- # new facets $= |\mathrm{conv}(i, d-1)|$
  $= O(i^{\lfloor \frac{d-1}{2} \rfloor})$
- fast locate : insert the points in lexicographic order and search a 1st red facet in $\mathrm{star}(p_{i-1})$
  (which necessarily exists)



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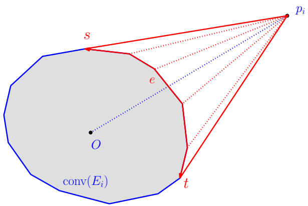
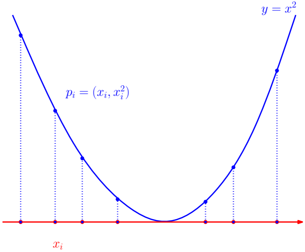


$$T(n,d) = O(n \log n) + \sum_{i=1}^{n} i^{\lfloor \frac{d-1}{2} \rfloor})$$
$$= O(n \log n + n^{\left\lfloor \frac{d+1}{2} \right\rfloor})$$

Worst-case optimal in even dimensions

Lower bound



$\mathrm{conv}(\{p_i\}) \Longrightarrow \mathrm{tri}(\{x_i\})$

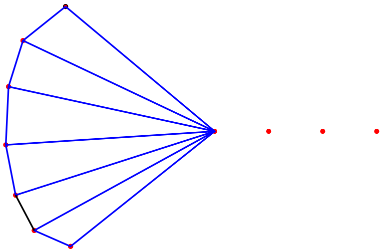
the orientation test reduces to 3 comparisons

$$\mathrm{orient}(p_i, p_j, p_k) = \begin{vmatrix} x_i - x_j & x_i - x_k \\ x_i^2 - x_j^2 & x_i^2 - x_k^2 \end{vmatrix}$$

$$= (x_i - x_j)(x_j - x_k)(x_k - x_i)$$

$\Longrightarrow$ Lower bound : $\Omega(n \log n)$

Lower bound for the incremental algorithm



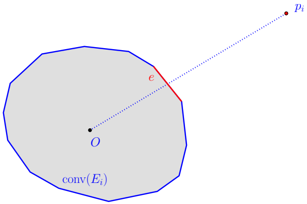
No incremental algorithm can compute the convex hull of n points of $\mathbb{R}^3$ in less than $\Omega(n^2)$

Randomized incremental algorithm

o a point inside $\mathrm{conv}(\mathcal{P})$

$\mathcal{P}_i$: the set of the first i inserted points

$\mathrm{conv}(\mathcal{P}_i)$: convex hull at step i



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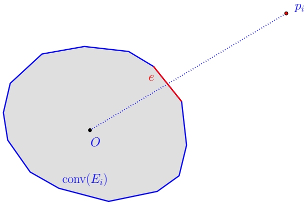

Conflict graph

bipartite graph $\{p_j\} \times \{\text{facets of } \mathrm{conv}(\mathcal{P}_i)\}$

$$p_j \dagger f \iff j > i \text{ } (p_j \text{ not yet inserted}), \; f \cap op_j \neq \emptyset$$

Randomized analysis

Hyp. : points are inserted in random order

Randomized analysis

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Notations

R : random sample of size r of $\mathcal{P}$

$F(R) = \{$ subsets of d points of $R\}$

$F_0(R) = \{$ elements of $F(R)$ with 0 conflict in $R\}$

(i.e. $\in \mathrm{conv}(R)$)

$F_1(R) = \{$ elements of $F(R)$ with 1 conflict in $R\}$

$C_i(r, \mathcal{P}) = E(|F_i(R)|)$

(expectation over all random samples $R \subset \mathcal{P}$ of size r)

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(expectation over all random samples $R \subset \mathcal{P}$ of size r)

Lemma

$$C_1(r, \mathcal{P}) = C_0(r, \mathcal{P}) = O(r^{\lfloor \frac{d}{2} \rfloor})$$

Proof of the lemma : $C_1(r,\mathcal{P}) = C_0(r,\mathcal{P}) = O(r^{\lfloor \frac{d}{2} \rfloor})$

$R' = R \setminus \{p\}$

$$\begin{array}{lll} f \in F_0(R') \text{ if } & f \in F_1(R) \text{ and } p \dagger f & (\text{proba} = \frac{1}{r}) \\ \text{or} & f \in F_0(R) \text{ and } R' \ni \text{the } d \text{ vertices of } f & (\text{proba} = \frac{r-d}{r}) \end{array}$$

Taking the expectation,

$$\begin{aligned} C_0(r-1,R) &= \frac{1}{r}\,|F_1(R)| + \frac{r-d}{r}\,|F_0(R)| \\ C_0(r-1,\mathcal{P}) &= \frac{1}{r}\,C_1(r,\mathcal{P}) + \frac{r-d}{r}\,C_0(r,\mathcal{P}) \\ C_1(r,\mathcal{P}) &= d\,C_0(r,\mathcal{P}) - r\,(C_0(r,\mathcal{P}) - C_0(r-1,\mathcal{P})) \\ &\leq d\,C_0(r,\mathcal{P}) \\ &= O(r^{\lfloor \frac{d}{2} \rfloor}) \end{aligned}$$

Randomized analysis 1

Updating the convex hull + memory space

Expected number $N(i)$ of facets created at step i

$$
\begin{aligned}
N(i) &= \sum_{f \in F(\mathcal{P})} \mathrm{proba}(f \in F_0(\mathcal{P}_i)) \times \frac{d}{i} \\
&= \frac{d}{i}\, O\left(i^{\lfloor \frac{d}{2} \rfloor}\right) \\
&= O(n^{\lfloor \frac{d}{2} \rfloor - 1})
\end{aligned}
$$

Expected total number of created facets = $O(n^{\lfloor \frac{d}{2} \rfloor})$

$O(n)$ if $d = 2, 3$

Randomized analysis2

Updating the conflict graph

Cost proportional to the total number of conflicts between facets that have been created and points not yet inserted

$N(i,j) =$ expected number of conflicts $f \dagger p_j$
f face of $\mathrm{conv}(\mathcal{P}_i)$ created at step i
$j > i$ (p_j has not been inserted yet)

$\mathcal{P}_i$: a random subset of $\mathcal{P}$
p_j : a random point of $\mathcal{P} \setminus \mathcal{P}_i$
$\mathcal{P}_i^+ = \mathcal{P}_i \cup \{p_j\}$: a random subset of $i+1$ points of $\mathcal{P}$

$$N(i,j) = \sum_{f \in F(\mathcal{P})} \mathrm{proba}(f \in F_1(\mathcal{P}_i^+)) \times \frac{d}{i} \times \frac{1}{i+1} = \frac{d\, C_1(i+1)}{i\,(i+1)}$$

Expected total cost of updating the conflict graph

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} N(i,j) = O(n \log n + n^{\lfloor \frac{d}{2} \rfloor})$$

Theorem

- The convex hull of n points of $\mathbb{R}^d$ can be computed in time $O(n \log n + n^{\lfloor \frac{d}{2} \rfloor})$ using $O(n^{\lfloor \frac{d}{2} \rfloor})$ space
- The same bounds hold for computing the intersection of n half-spaces of $\mathbb{R}^d$
- The randomized algorithm can be derandomized

  [Chazelle 1992]
- The same results hold for Voronoi diagrams and Delaunay triangulations provided that $d \rightarrow d+1$

You know my methods. Apply them !

Sherlock Holmes