An introduction to computational geometry

Cours électif - École des mines - Nancy - 2011

Xavier Goaoc

Question 1: What is it all about?

It's about designing algorithmic solutions to geometric problems.

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  In a mathematical sense.

  Often based on geometric arguments.

  Often requires new geometric insight.

- ⋆ Proven efficient.

  In the sense of complexity theory.

  Complexity of algorithms are analyzed in an adequate model of computation.

  Understanding the complexity of the problems themselves is important.

- ⋆ Work in practice.

  Algorithms that are simple enough to be implemented.

  Implementations that handle degeneracies and finite precision arithmetic.

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Model of computation

Definition of the operations allowed in an algorithm and their cost.

Goal: estimate the ressources required by an algorithm

as a function of the input size.

Ex: execution time, memory space, number of I/O transfers, number of processors...

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Classical model in CG: Real RAM model

Allows manipulation of real (as in $\mathbb{R}$) numbers.

Input size $n \rightarrow$ complexity $f(n) = \max_{\text{input } |X|=n} f(X)$

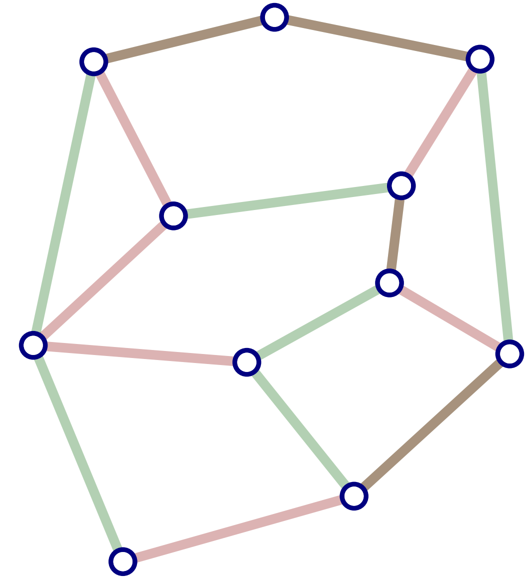
Care about asymptotic order of magnitude of f $(O(), \Omega(), \Theta())$.

Brute force does not scale well (or: why should we think?)

The "Travelling salesman problem".

Input: n cities and all inter-city distances.

Output: order on the cities that minimizes the distance travelled.



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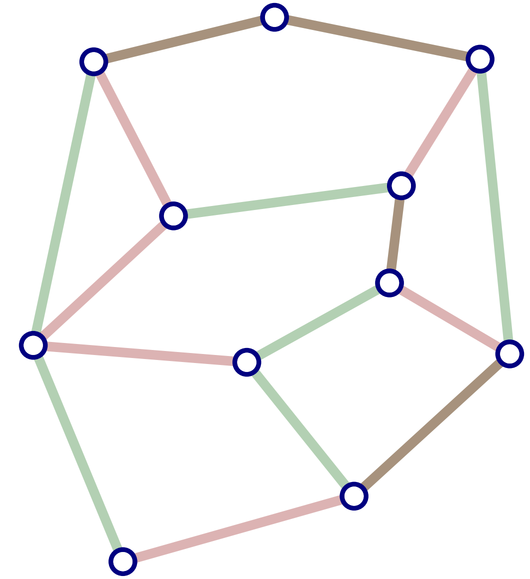

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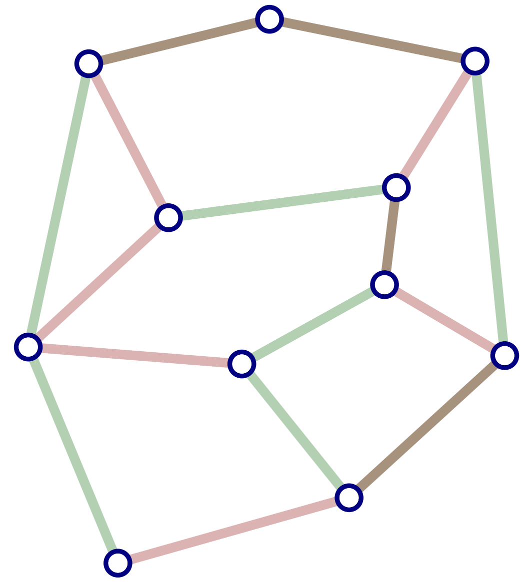

Brute-force approach: test all $n!$ orders and pick the best.

Assume your computer can process 10^{15} orders per second.

Generate the order, add-up the distances, compare to the current best...

A very generous over-estimation.

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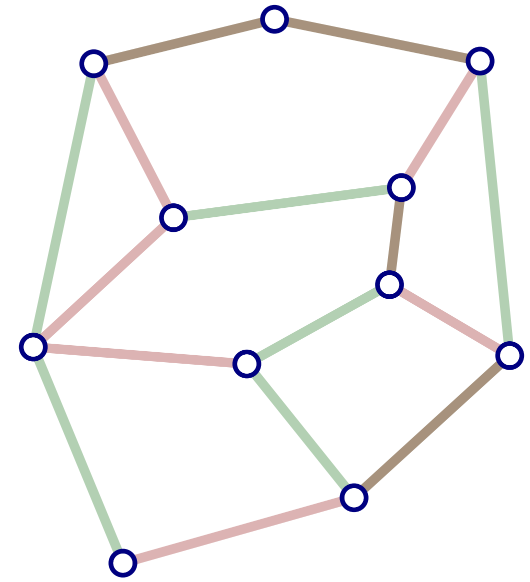

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Start the computation now. It will end...

in 30-60 min. for $n = 20$.

in two weeks for $n = 22$.

in twenty years for $n = 24$.

in four centuries for $n = 25$.

in the dark for $n = 30$.

Orders of magnitude

Sort by increasing asymptotic orders of magnitude:

$$n,\ 2^n,\ n^2,\ n!,\ \sqrt{n},\ \log n,\ \log^* n,\ 2^{n^2}$$

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$$\forall a, b, c, d, e \in \mathbb{R}_+$$

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Three classes of problems

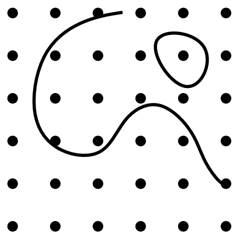
Undecidable: no algorithm will solve the problem. Ever.

NP-hard: conjectured unlikely that a polynomial-time algorithm exists.

Polynomial-time: solvable by an algorithm with complexity $O(n^c)$

for some constant c.

Hilbert's tenth problem



Input: a polynomial P in n variables with integer coefficients.

Output: yes if P has a integer solution, no otherwise.

Ex: $P(x_1, x_2, x_3) = x_1^2 + 3x_1x_2 - 2x_2^2 + 4x_3 + 3$

Tenth question in Hilbert's list of *Problèmes futurs des mathématiques*.

Raised in 1900. Algorithmic question before the age of computers.

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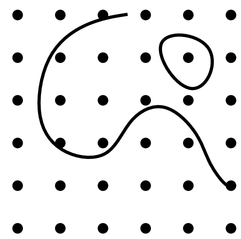

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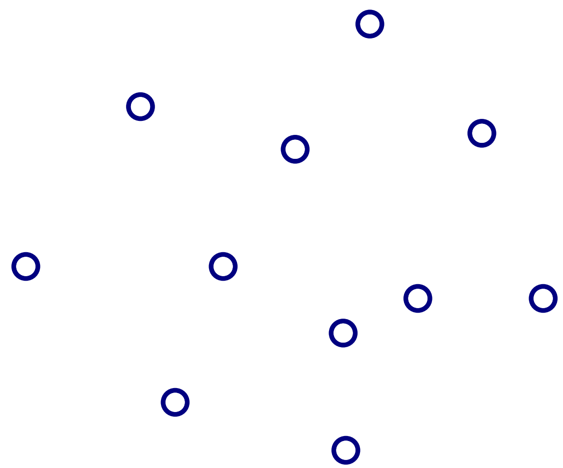
Given a set of points in the plane

find a triangulation of the convex hull

that minimizes the sum of edge lengths.



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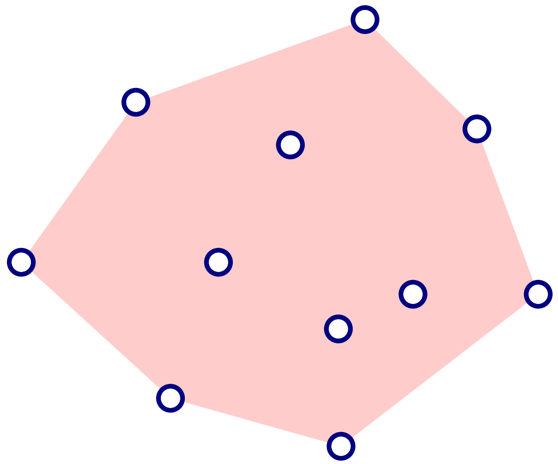

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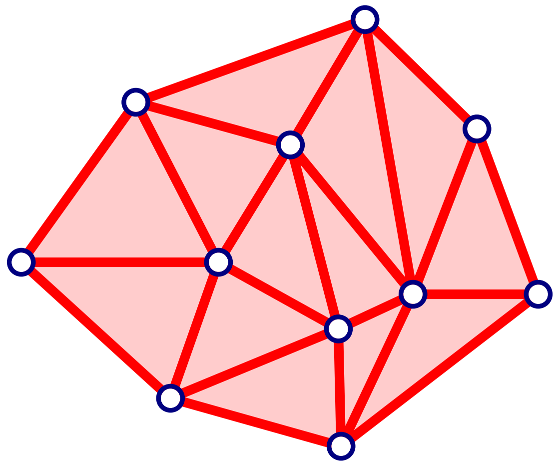

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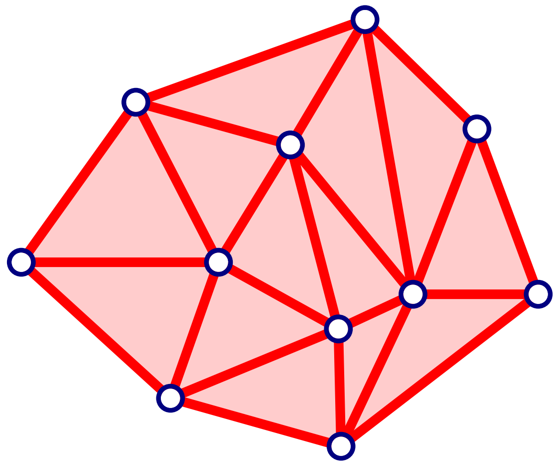

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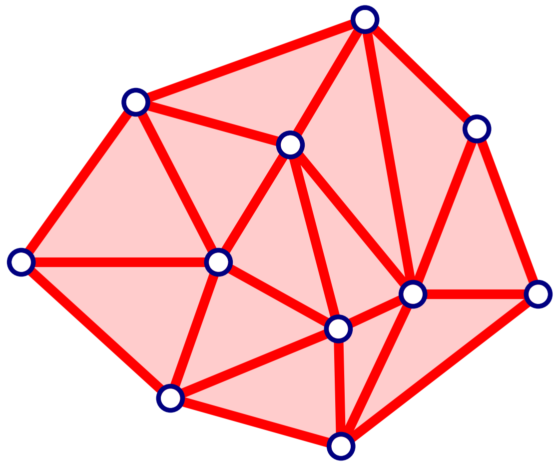

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NP-hard

Problems solvable in polynomial time

Algorithms for the same problem may have different complexities.

Ex: Merge sort has $\Theta(n \log n)$ complexity.

Bubble sort has $\Theta(n^2)$ complexity.

Quick sort has $\Theta(n^2)$ complexity but $O(n \log n)$ average-case complexity.

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This can have a drastic impact.

http://cg.scs.carleton.ca/∼morin/misc/sortalg/

Wrap-up: what is it about?

Algorithmic solutions to geometric problems.

Proofs of correctness and complexity bounds.

Beware of undecidable or NP-hard problems.

Asymptotic complexity matters in practice.

(Attention to degeneracy and numerical issues.)

Question 2 (getting started)

How to compute the intersections among n segments in 2D?

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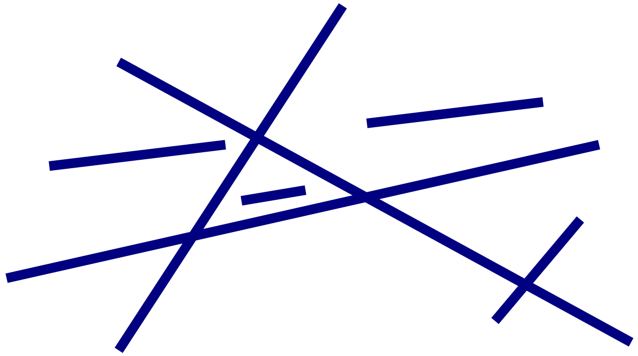

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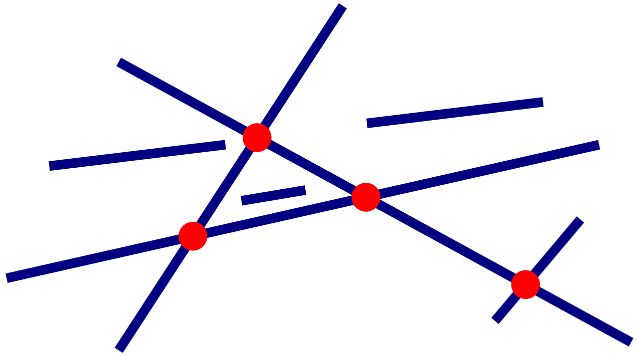

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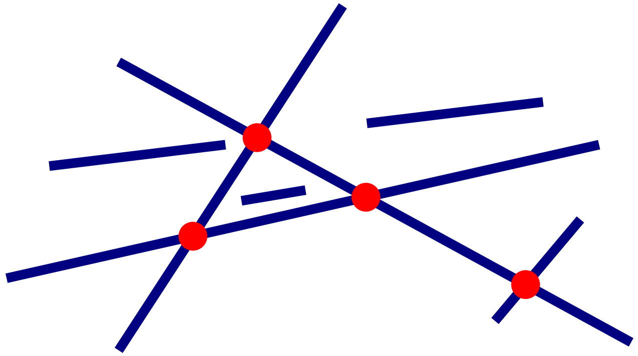
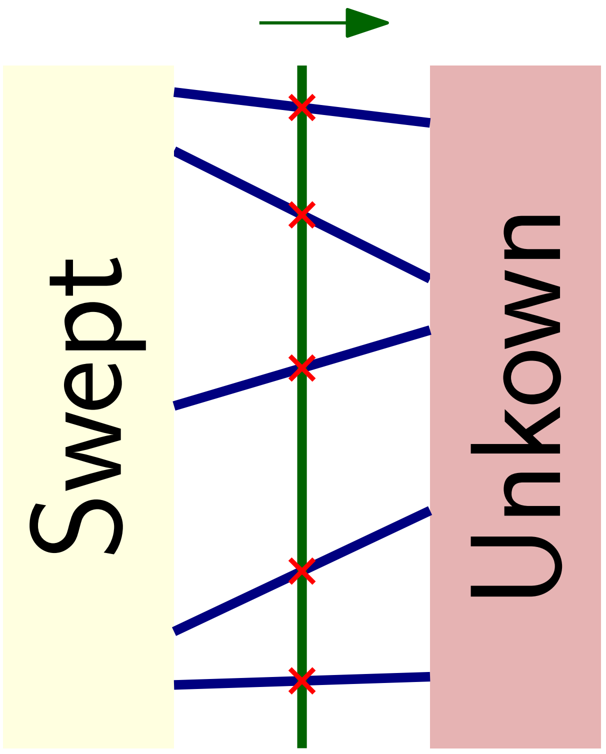


Any idea?

Use geometry to avoid testing unnecessary pairs.

Idea: sweep the plane using a line.



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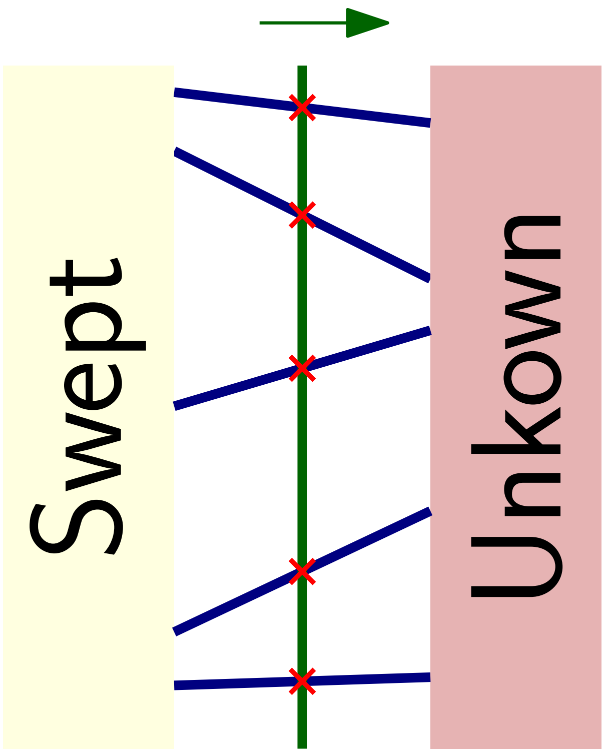

Principle:

Two segments that intersect must meet the sweep line consecutively before it reaches the intersection point.

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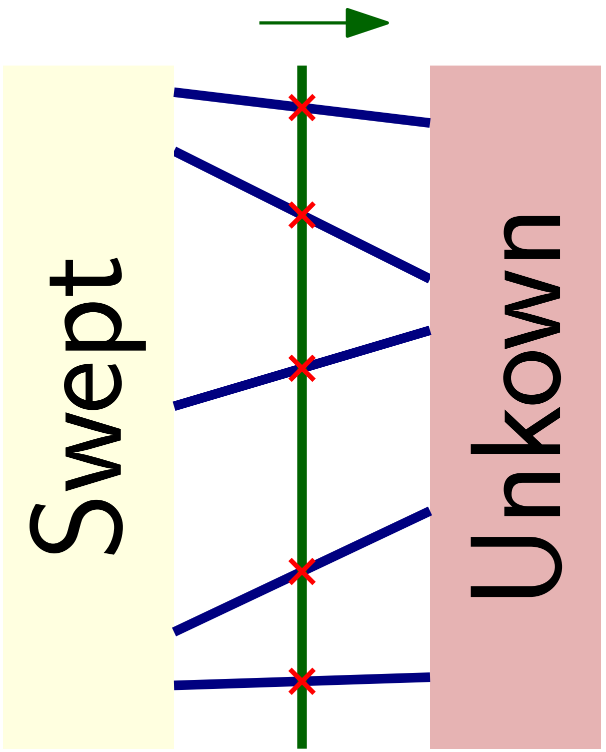

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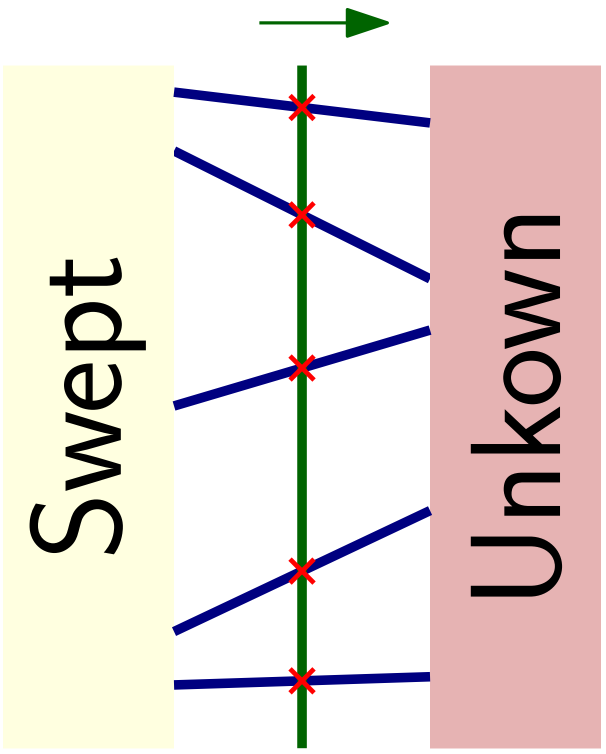

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Data structure? Predicates?

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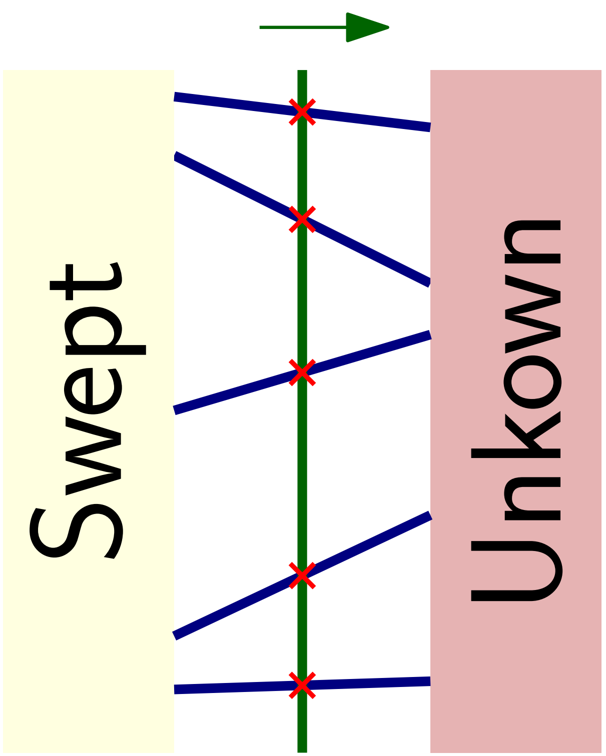

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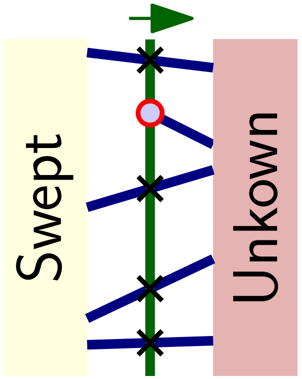

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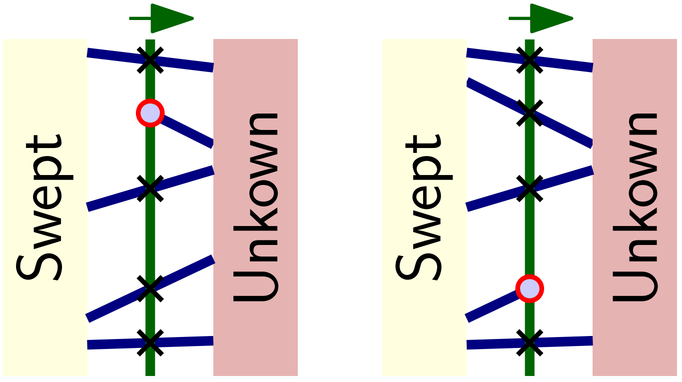

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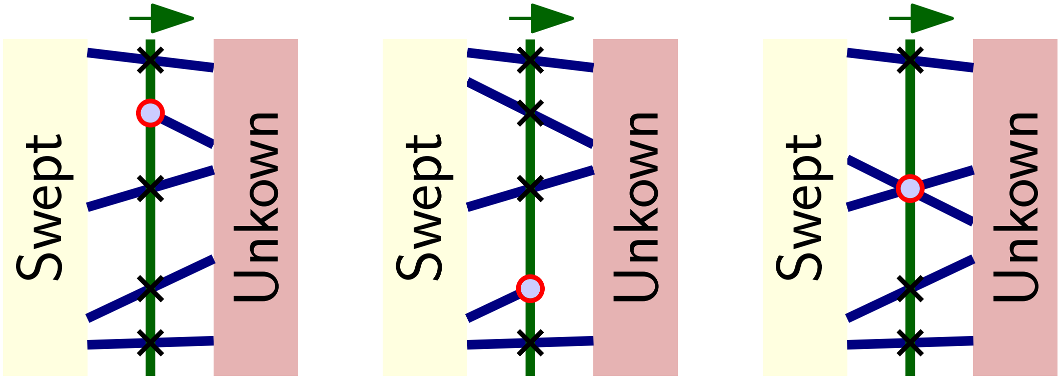

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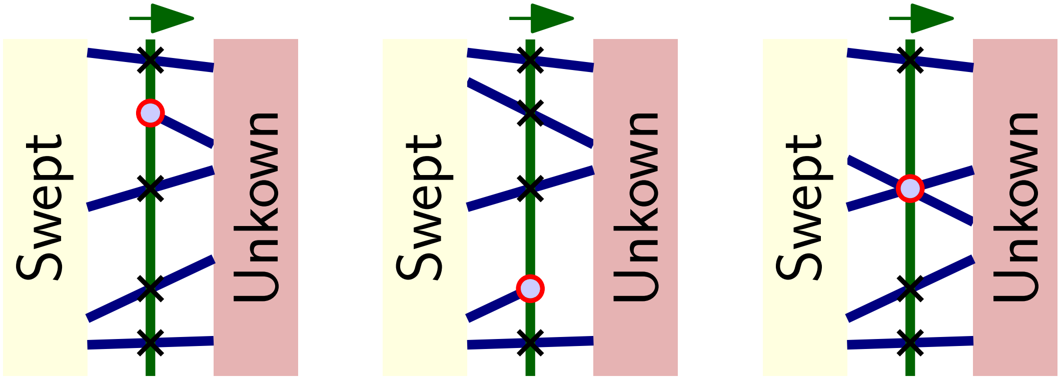
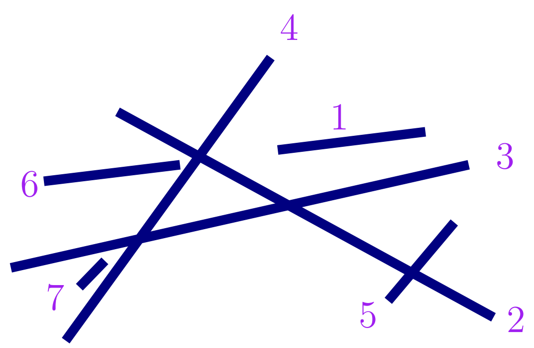


Data structures

Ordered list of segments intersected by the line.
Supports efficient insertion, deletion & exchange.

List of events sorted by x-coordinates.
Supports efficient insertion & deletion.

Algorithm



Insert the endpoints of all segments in Events.

Sweep $\leftarrow \emptyset$.

While Events $\neq \emptyset$

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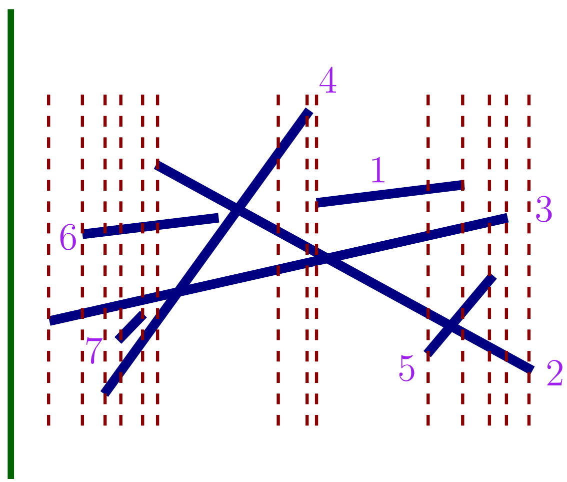

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Sweep$= \{\}$

Output$= \{\}$

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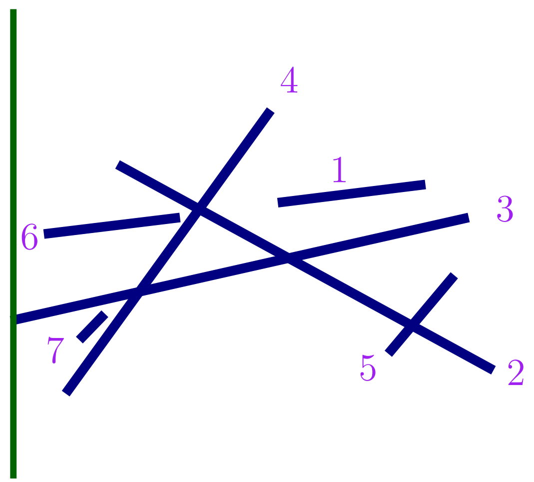

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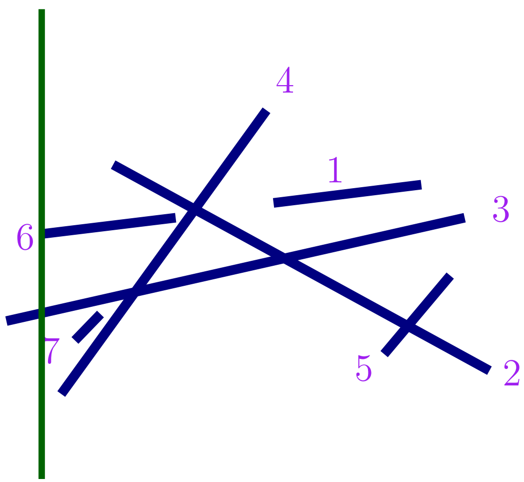
Sweep= $\{3\}$

Output= $\{\}$

Algorithm



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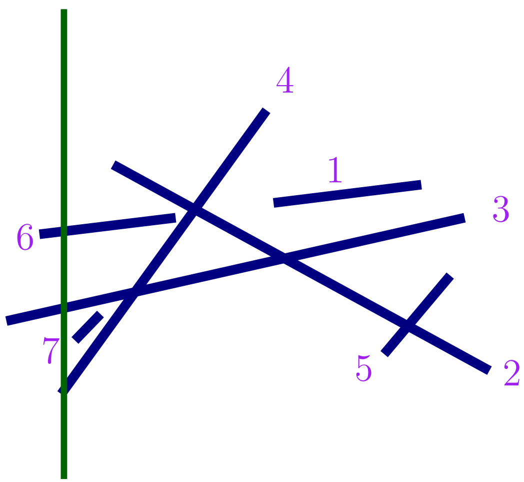
Sweep$= \{6, 3\}$

Output$= \{\}$

Algorithm



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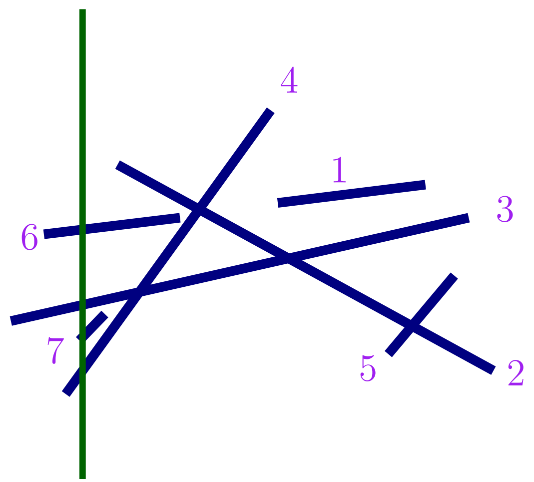
Sweep= $\{6, 3, 4\}$

Output= $\{(3, 4)\}$

Algorithm



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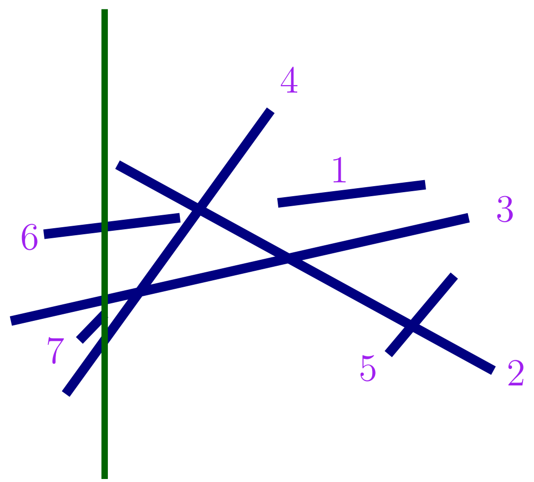

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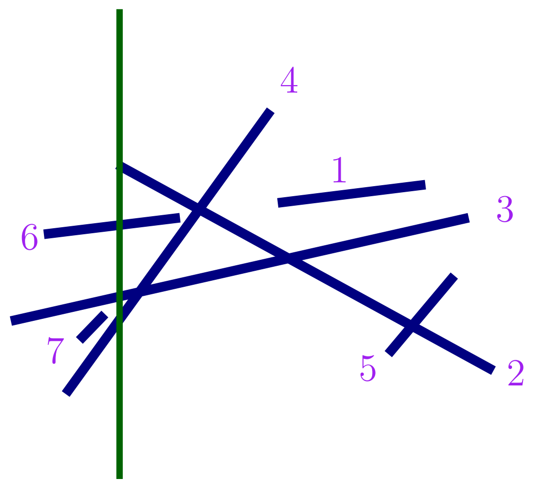

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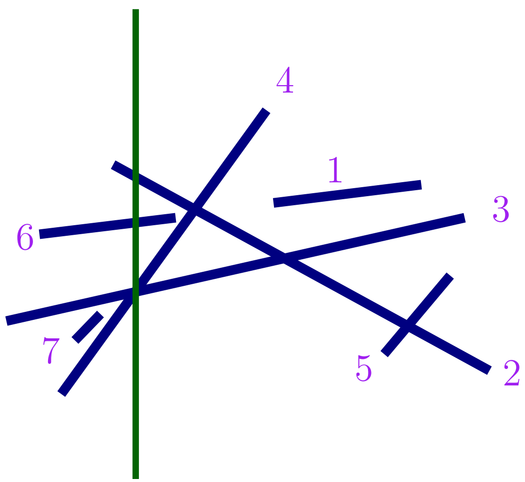

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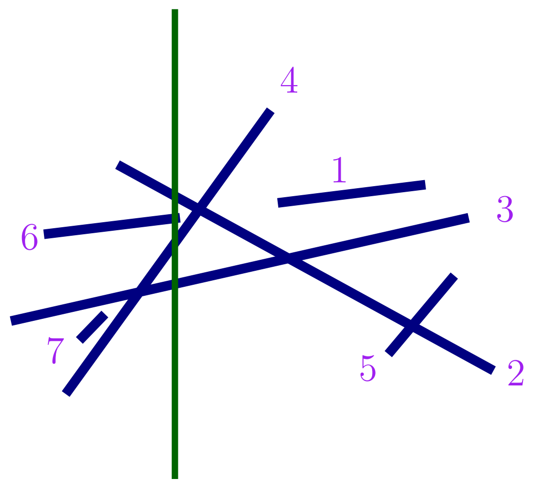

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Events: sorted list of events.

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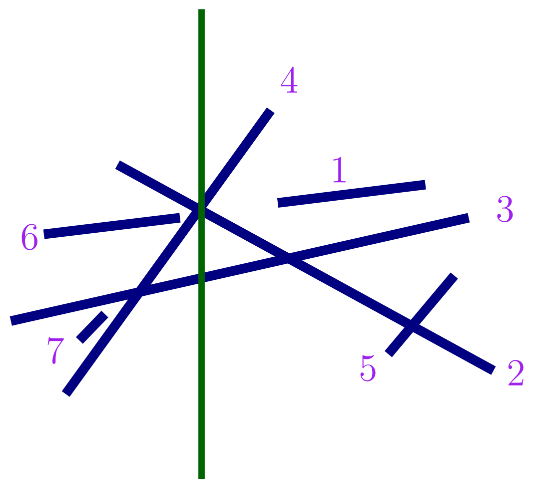
Events$= \{R_6, I_{2,4}, R_4, L_1, L_5, R_1, R_5, R_3, R_2\}$

Sweep$= \{2, 6, 4, 3\}$

Output$= \{(3,4), (2,4)\}$

Algorithm



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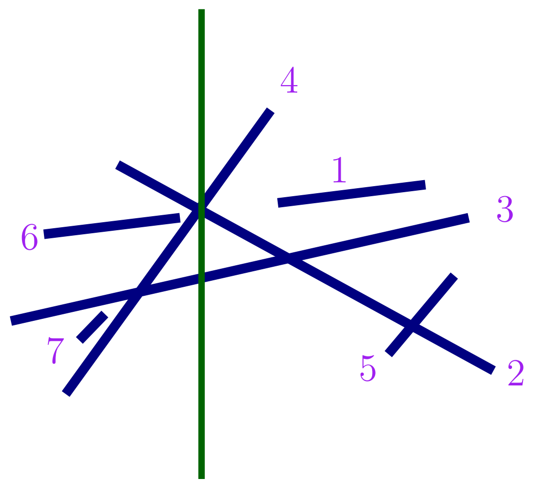

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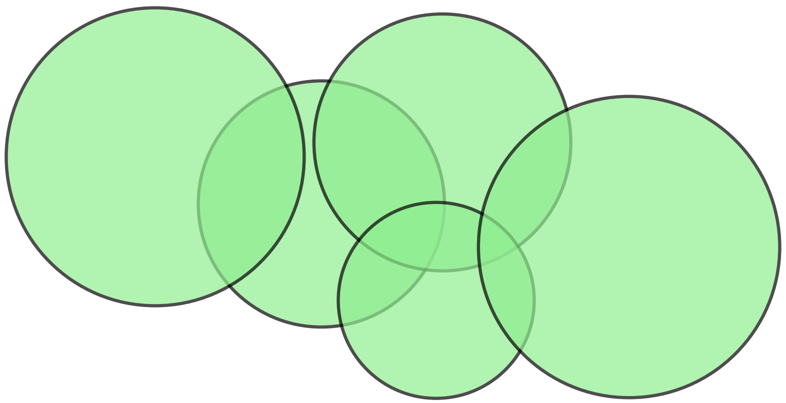
etc...

Correctness? Complexity?

Wrap-up: sweep algorithms

Generic principle, three predicates: x-extreme points, intersection, x-coordinate comparison.

Other objects (polygons, circles, algebraic curves, etc...), other spaces ($\mathbb{S}^2$, $\mathbb{R}^3$, $\mathbb{S}^1 \times \mathbb{S}^1$...).



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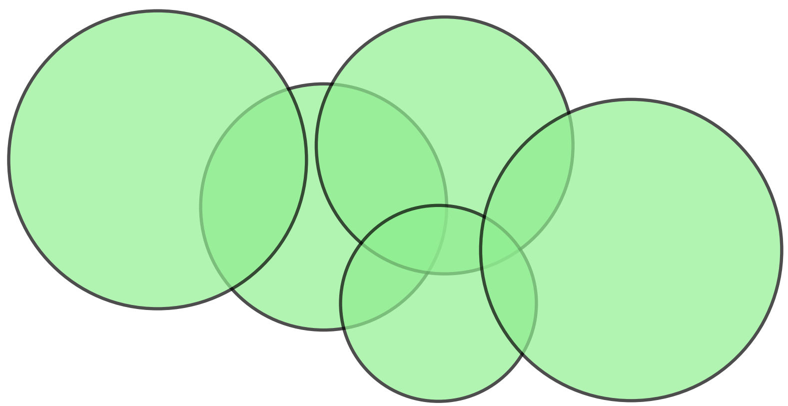

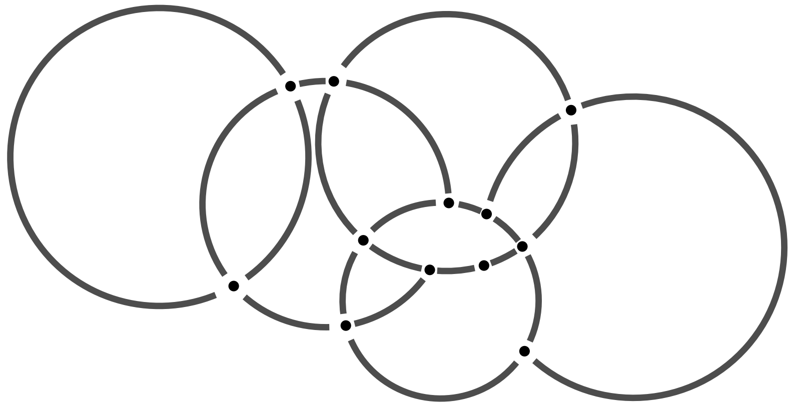

Computing arrangements of geometric objects.

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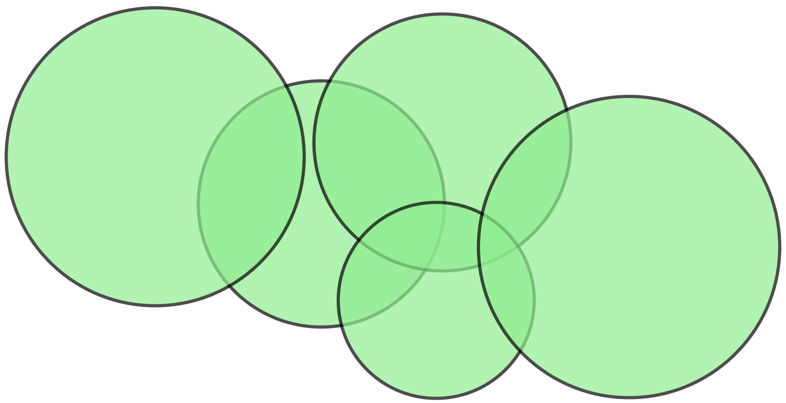

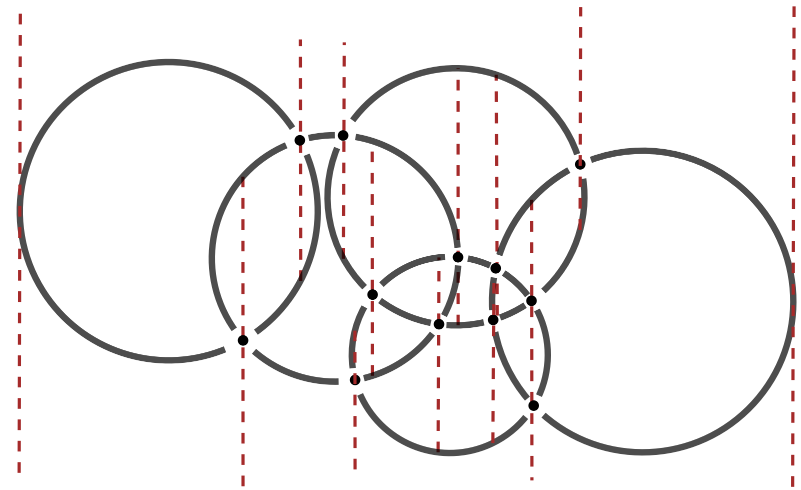

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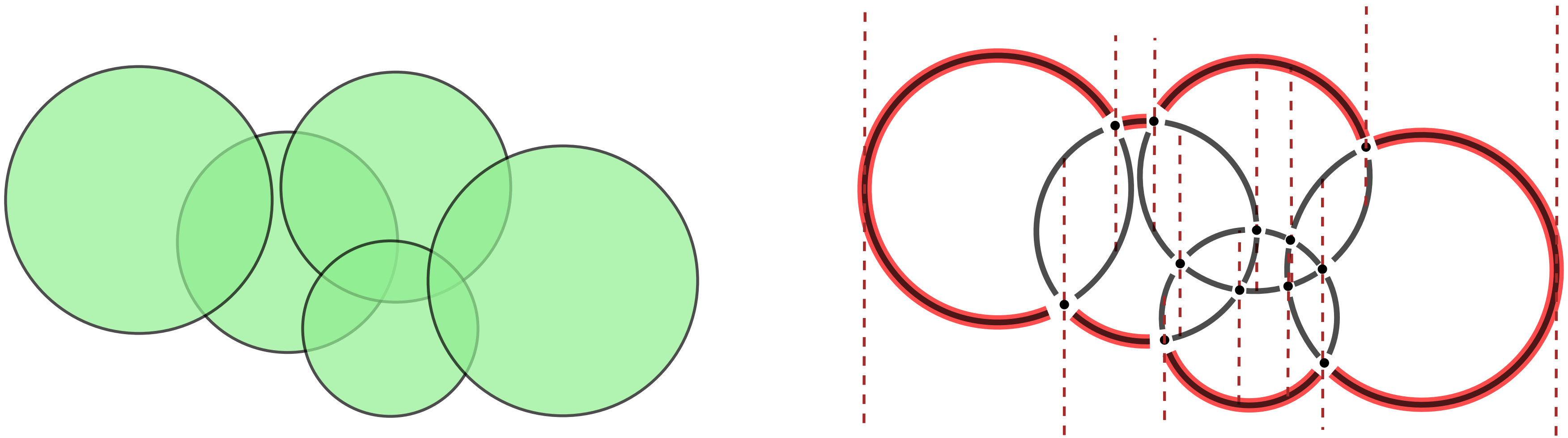

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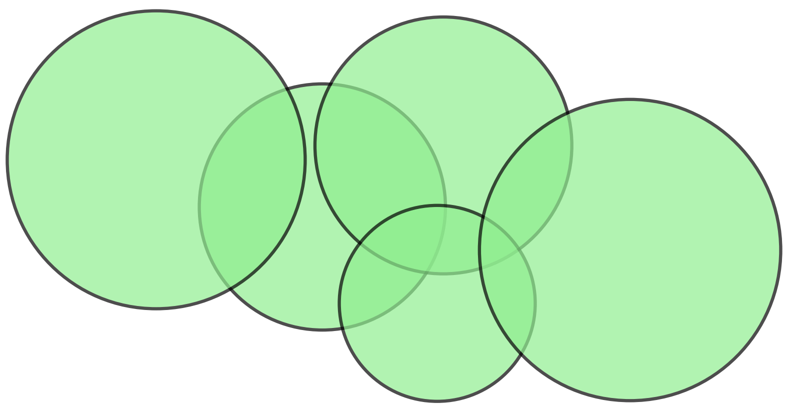

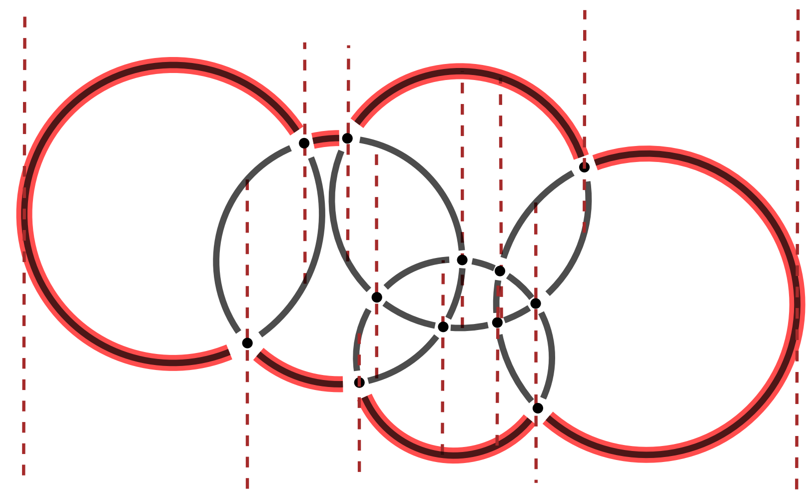

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All that in $O((n+k)\log n)$.

Question 3

How to triangulate a set of points?

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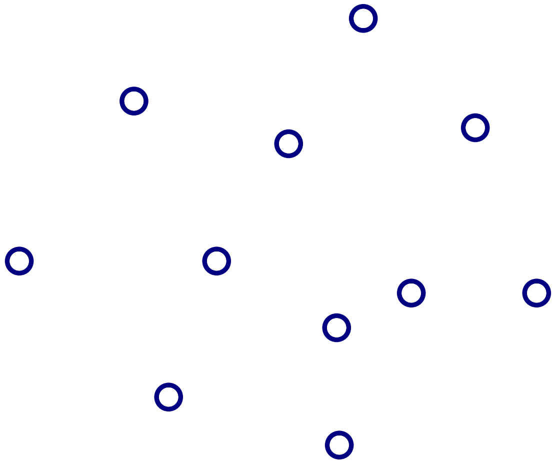

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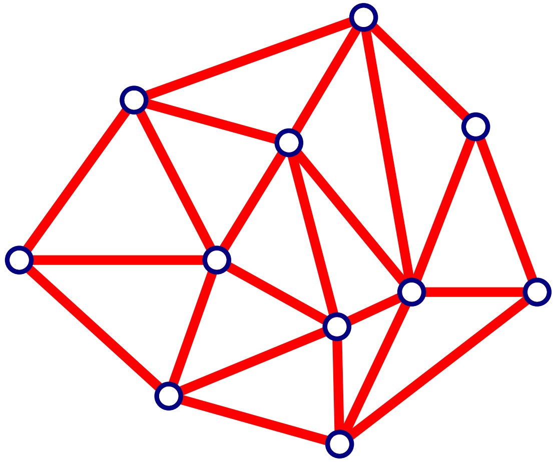

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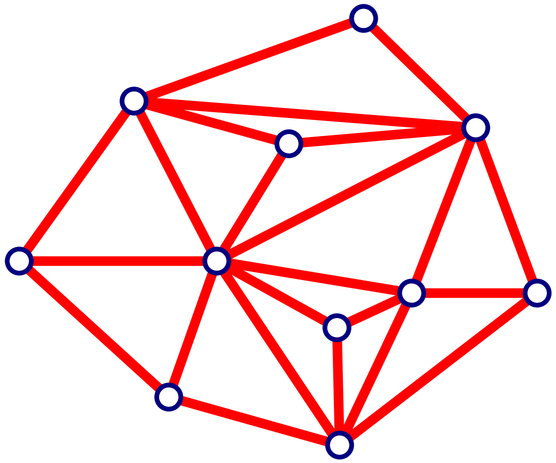

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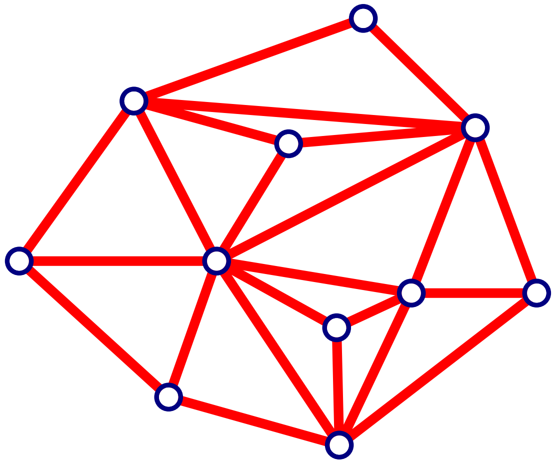

What is a *good* triangulation?

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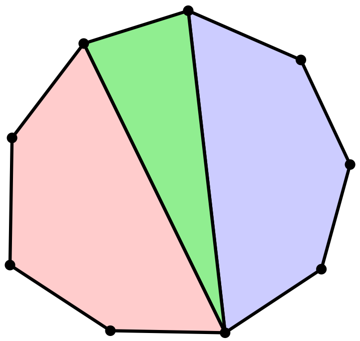

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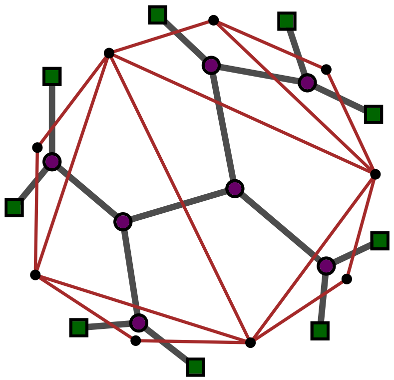

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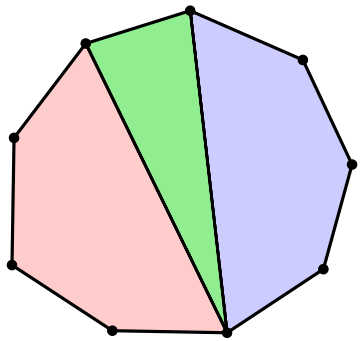

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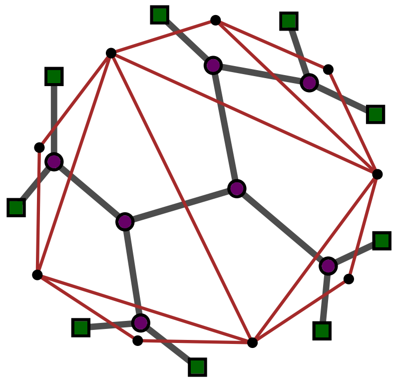

Grows fast: $C_n \sim \frac{4^n}{n\sqrt{\pi n}}$

First numbers: $3(1), 4(2), 5(5), 6(14), \ldots, 10(16796), \ldots, 20(6564120420)\ldots$

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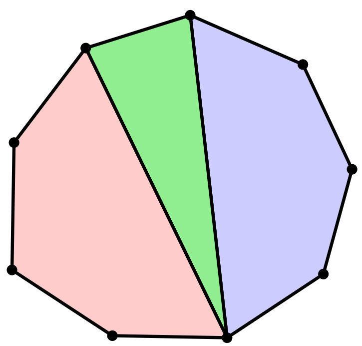

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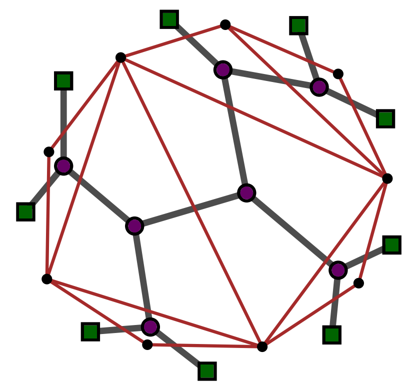

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For points in arbitrary position: $\Omega(8.48^n)$ [2007] and $O(30^n)$ [2009].

Number of edges and triangles

Theorem. Let P be a set of n points in the plane, not all collinear. Let k be the number of points in P that lie on the boundary of the convex hull of P. Any triangulation of P has $2n - 2 - k$ triangles and $3n - 3 - k$ edges.

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Idea?

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The average degree of a point is ≤ 6.

Which triangulation shall we compute?

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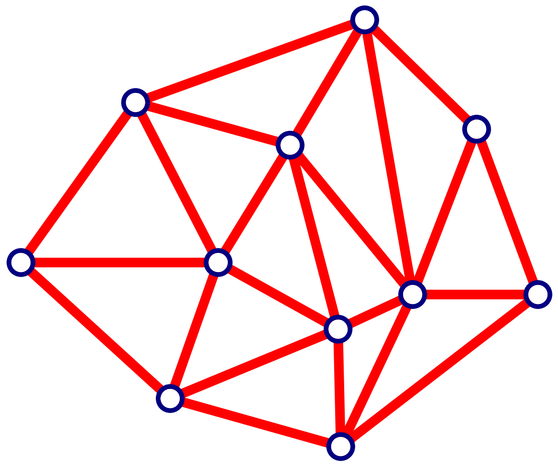

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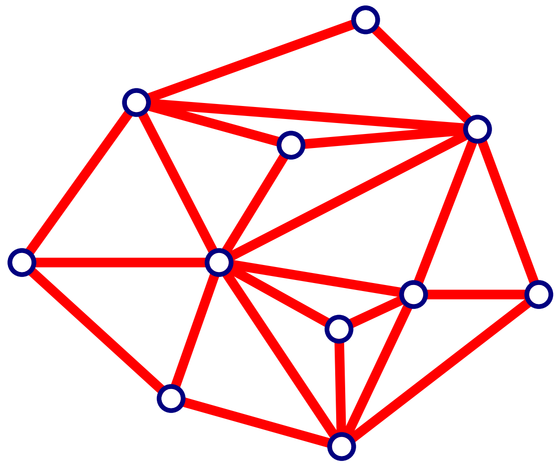

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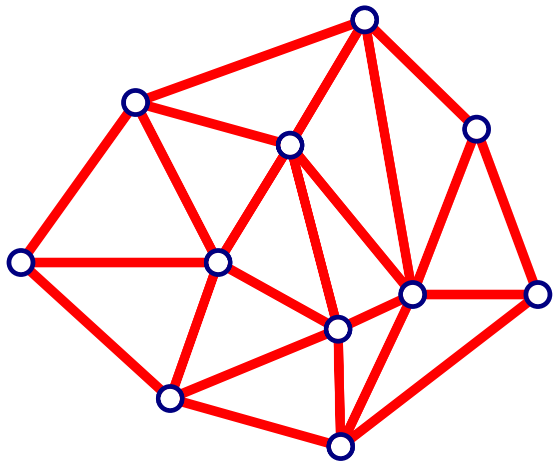

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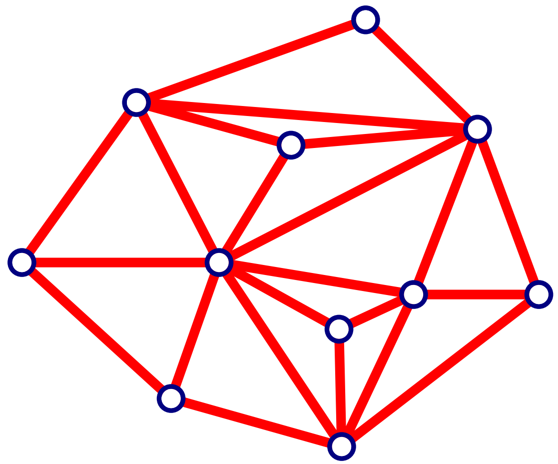
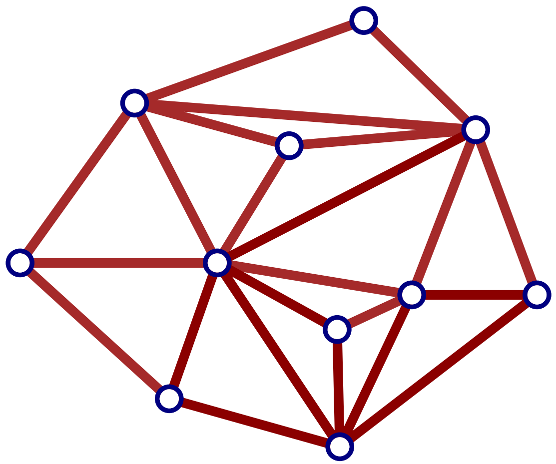


Associate to every triangulation the vector of angles sorted from smallest to largest.

Let's compute the triangulation with lexicographically smallest vector of angles.

Legal edges

Consider an edge of a triangulation incident to two triangles forming a convex quadrangle.



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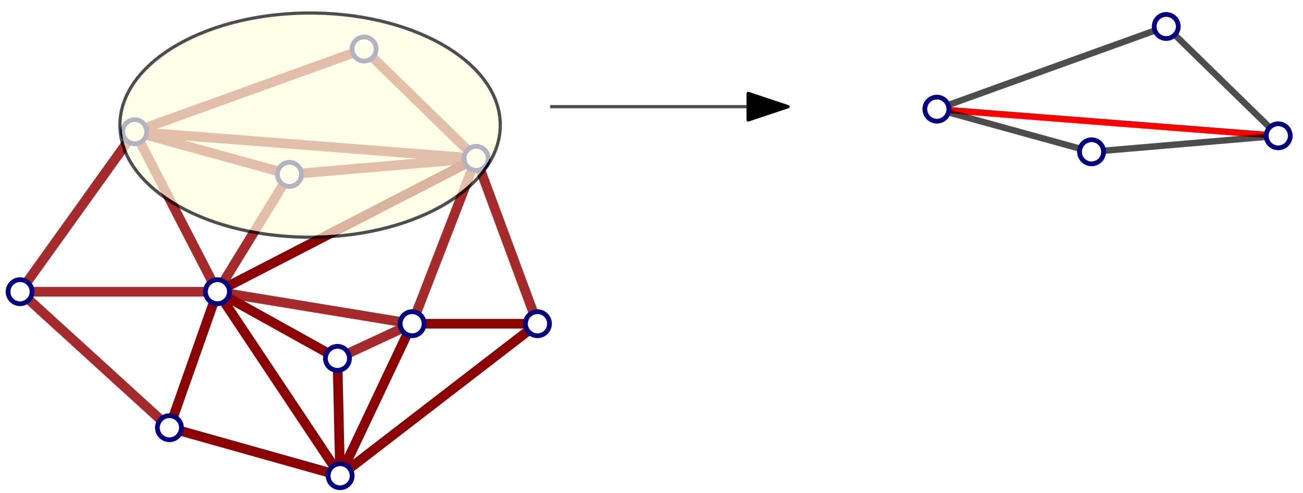

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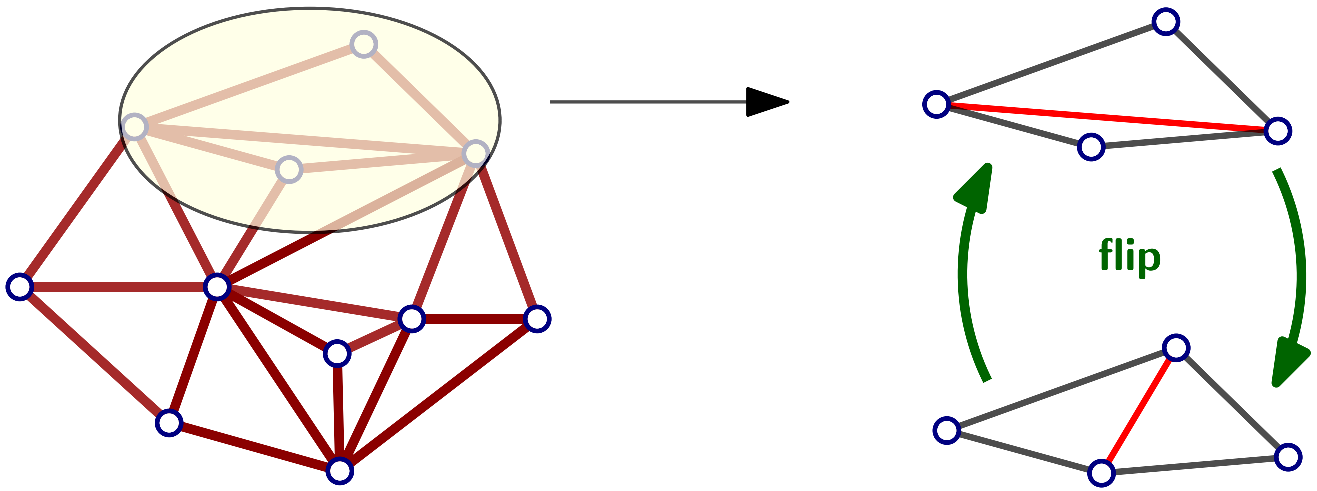
Exchanging the edge for the other diagonal of the quadrangle yields a new triangulation.

This is called "flipping" the edge.



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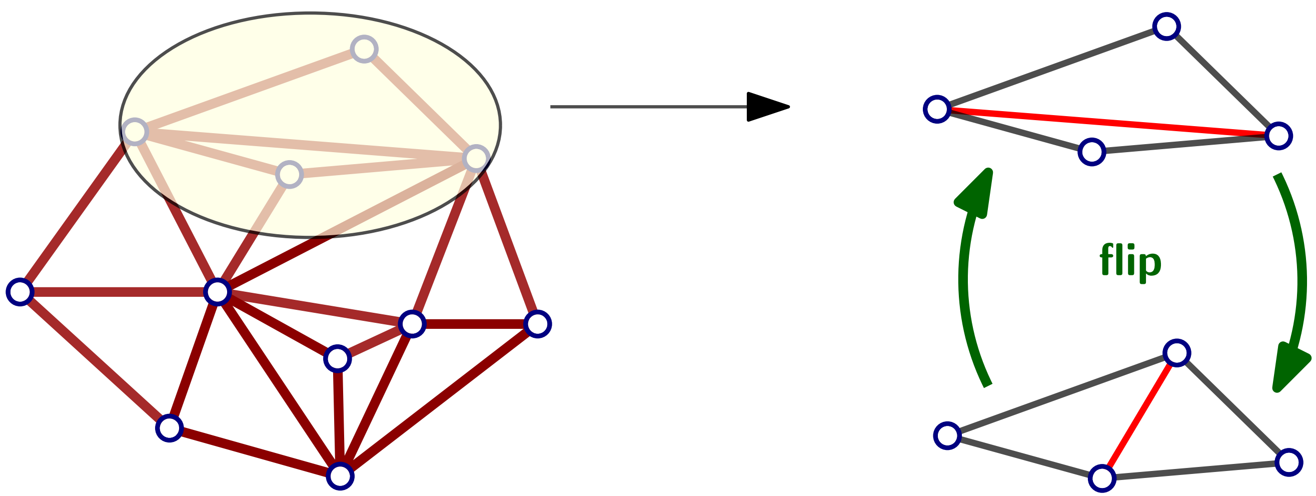

Call an edge illegal if it can be flipped and flipping it decreases the vector of angles.

Call a triangulation legal if it contains no illegal edge.

Computing a legal triangulation

Termination? Correctness? Complexity?

Start from any triangulation.

While there exists an illegal edge,

flip that edge.

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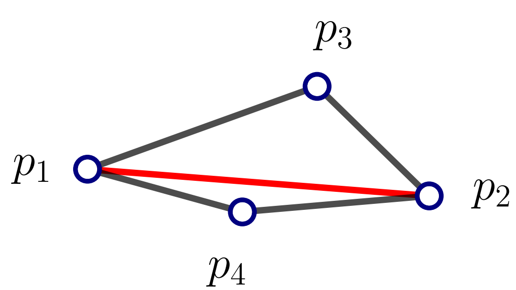
An elegant test for edge "illegality"



p_1p_2 is illegal

$\Leftrightarrow$

p_4 lies inside the circle circumscribed to $p_1p_2p_3$

$\Leftrightarrow$

p_3 lies inside the circle circumscribed to $p_1p_2p_4$

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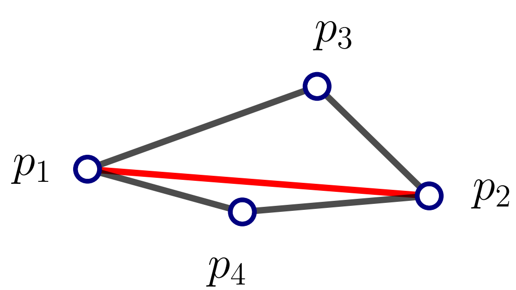

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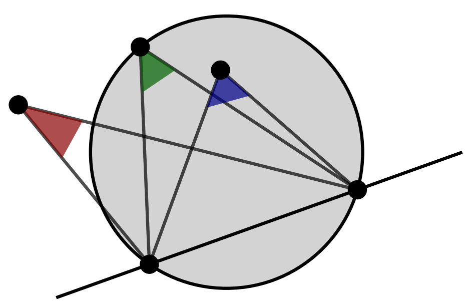
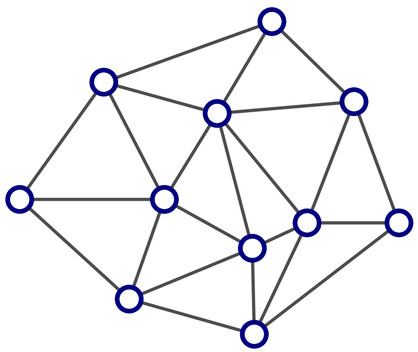
Main ingredient of the proof:



$\leq \quad \leq$

Delaunay triangulations

Let P be a set of n points in the plane.



A triangulation is a Delaunay triangulation
$\Leftrightarrow$ the interior of every triangle circumcircle is empty of points of P.

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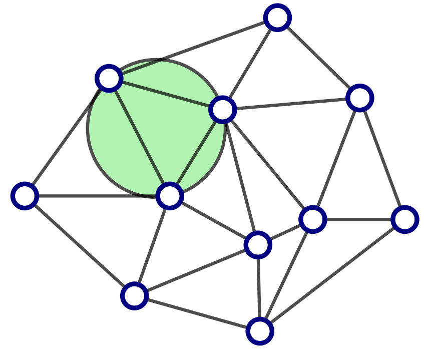

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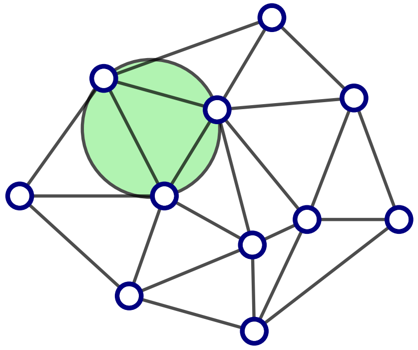

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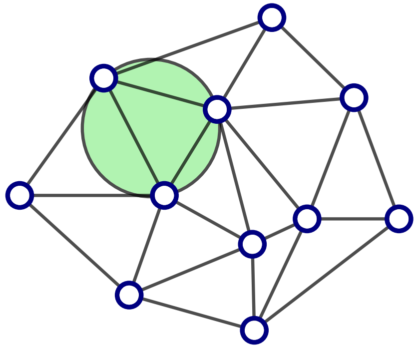

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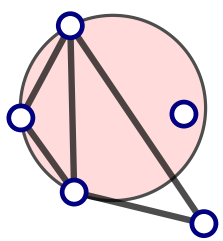
Proof: Argue that  is impossible in a legal triangulation.

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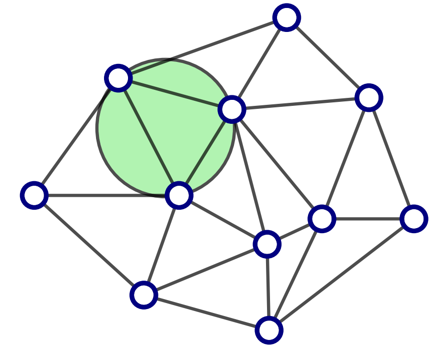

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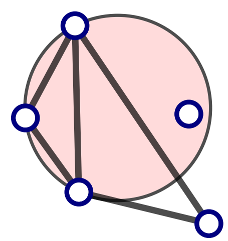
Proof: Argue that  is impossible in a legal triangulation.

Theorem. If no 4 points of P are cocircular then P has a unique Delaunay triangulation.

Theorem. All Delaunay triangulations of a point set P have the same minimal angle.

Incremental construction

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in the plane.

For simplicity we assume that P is contained in the triangle $p_1p_2p_3$.

Incremental algorithm:

Add the points one by one.

Maintain a Delaunay triangulation.

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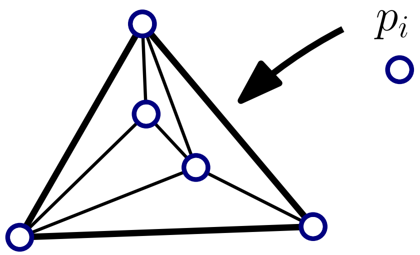

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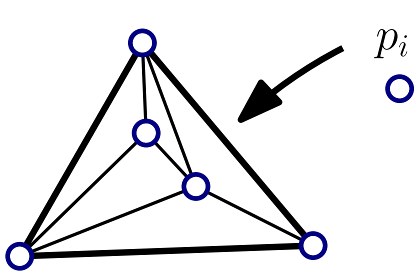
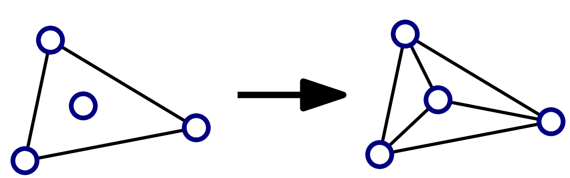


Triangle subdivision



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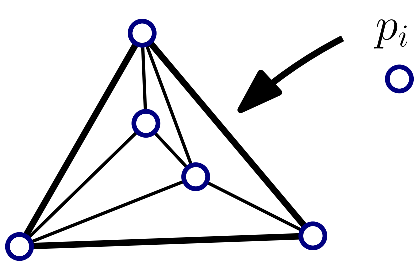
Add the points one by one.

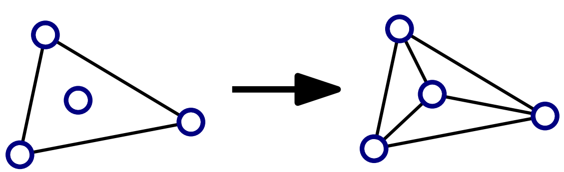
Maintain a Delaunay triangulation.

Three sub-problems.

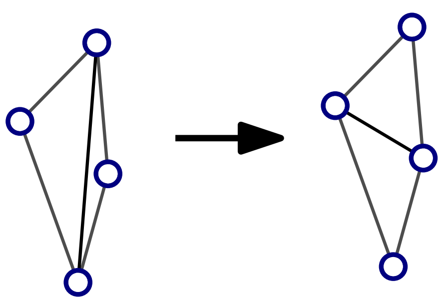
Point location



Triangle subdivision



Correction



Incremental construction

Let $P = \{p_1, \ldots, p_n\}$ be a set of n points in the plane.

For simplicity we assume that P is contained in the triangle $p_1p_2p_3$.

Incremental algorithm:

Add the points one by one.

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$T \leftarrow \{p_1p_2p_3\}$

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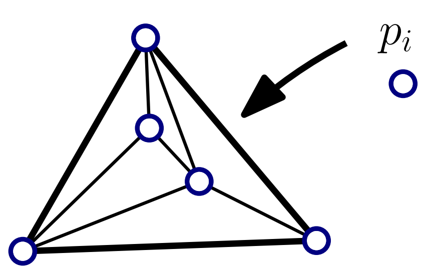

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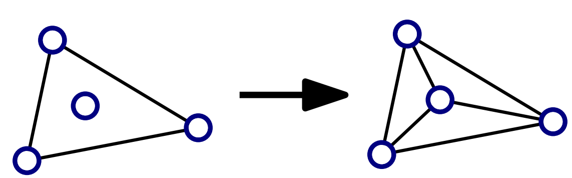

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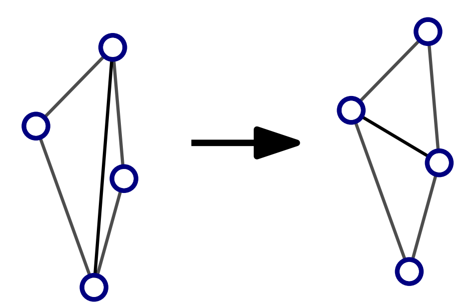

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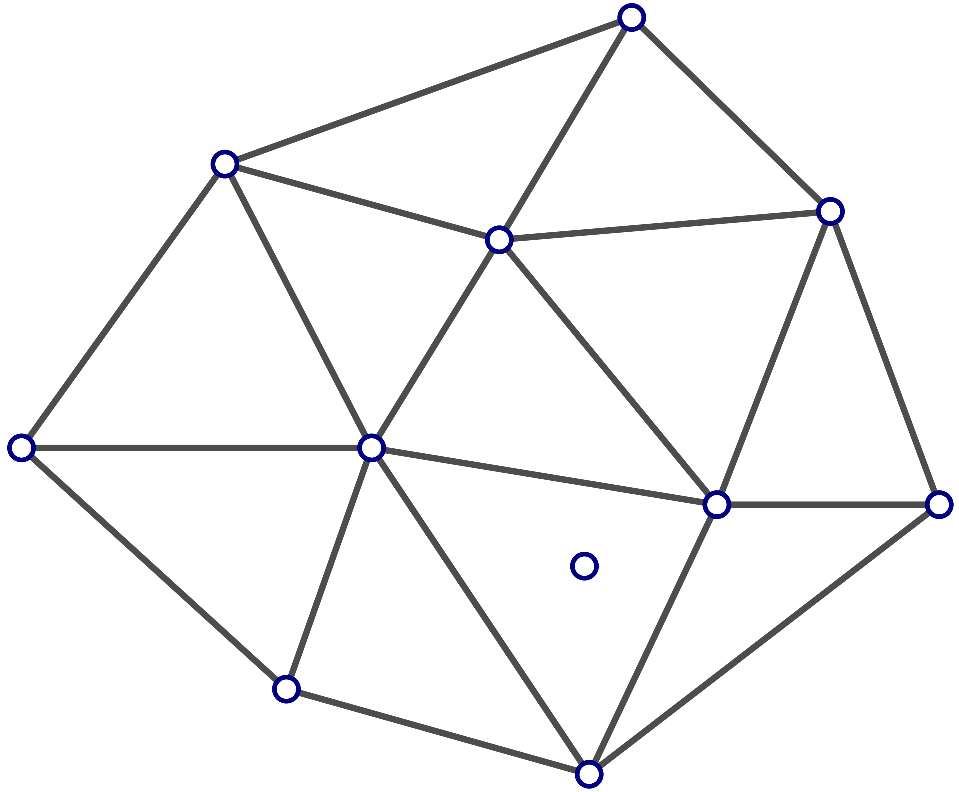

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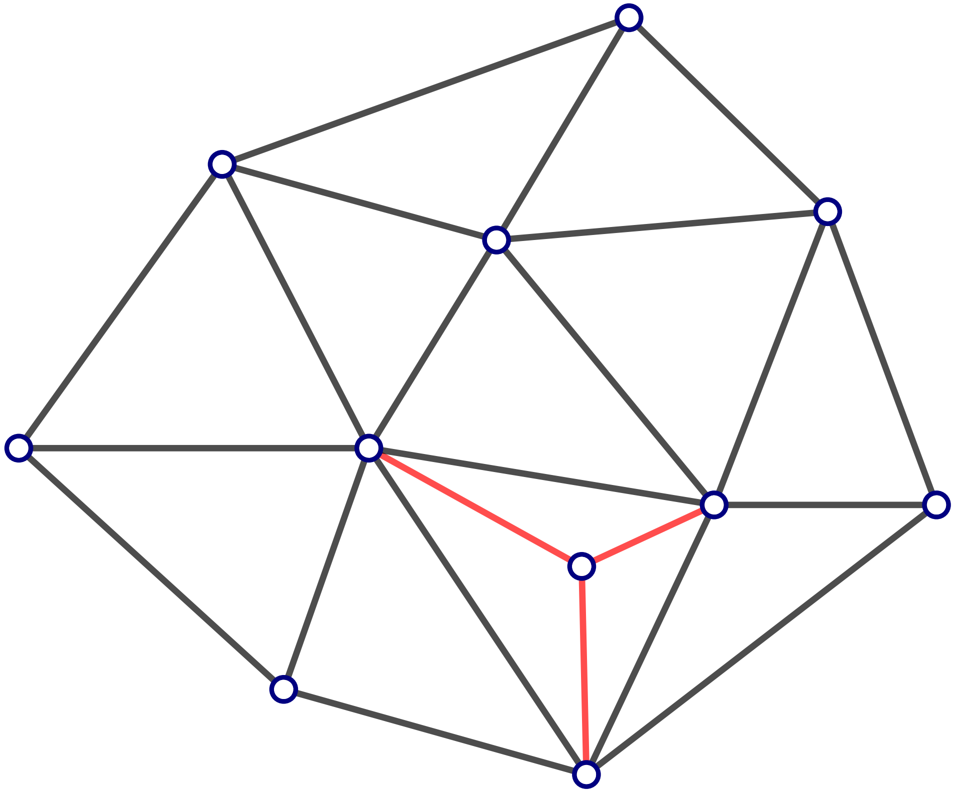

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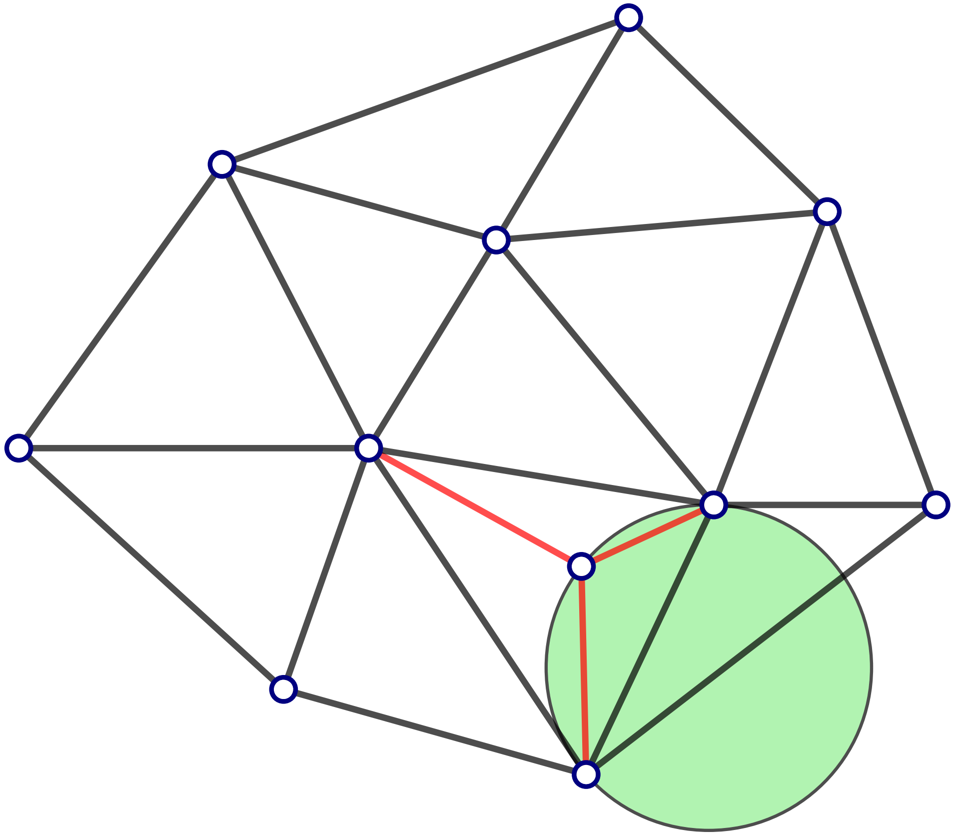

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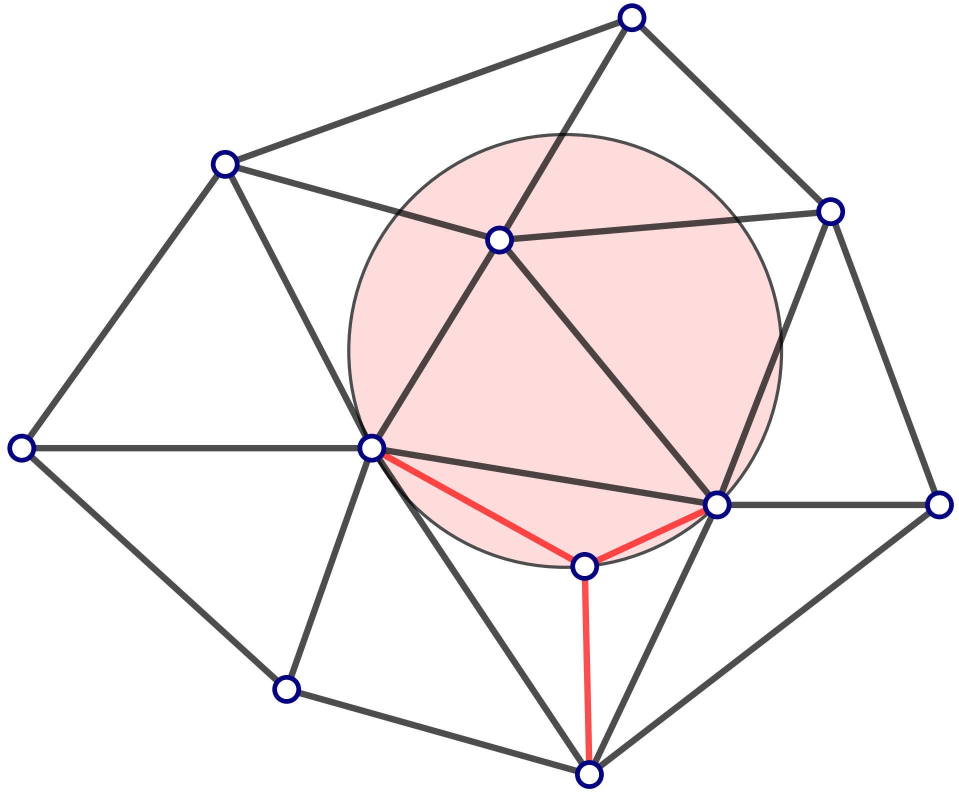

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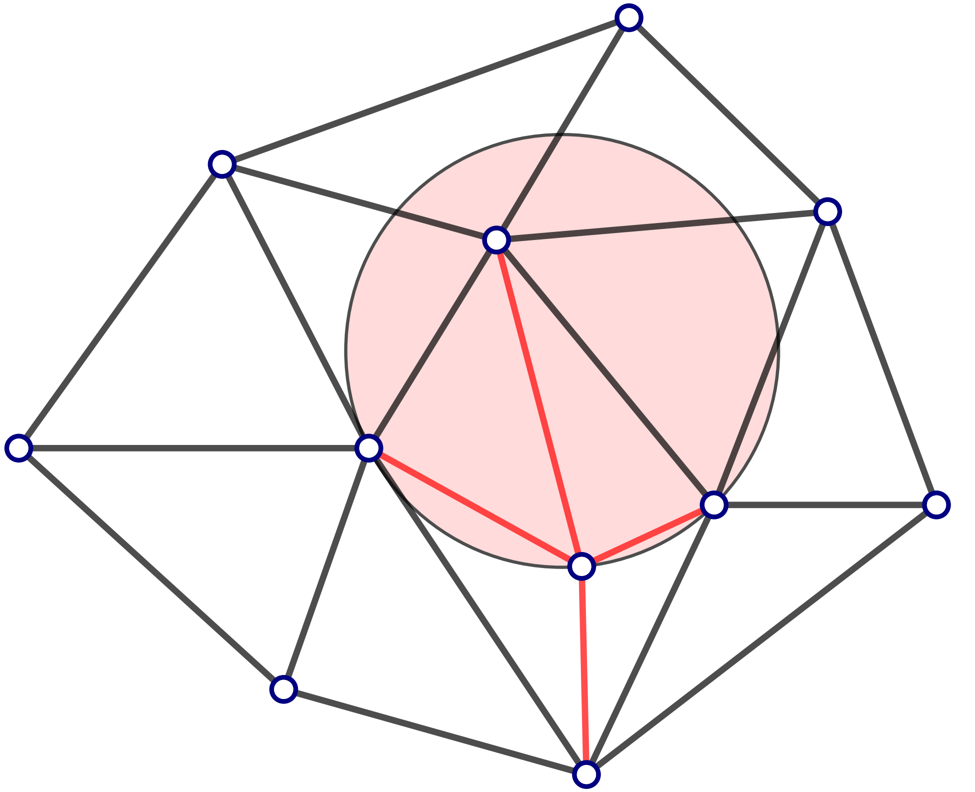

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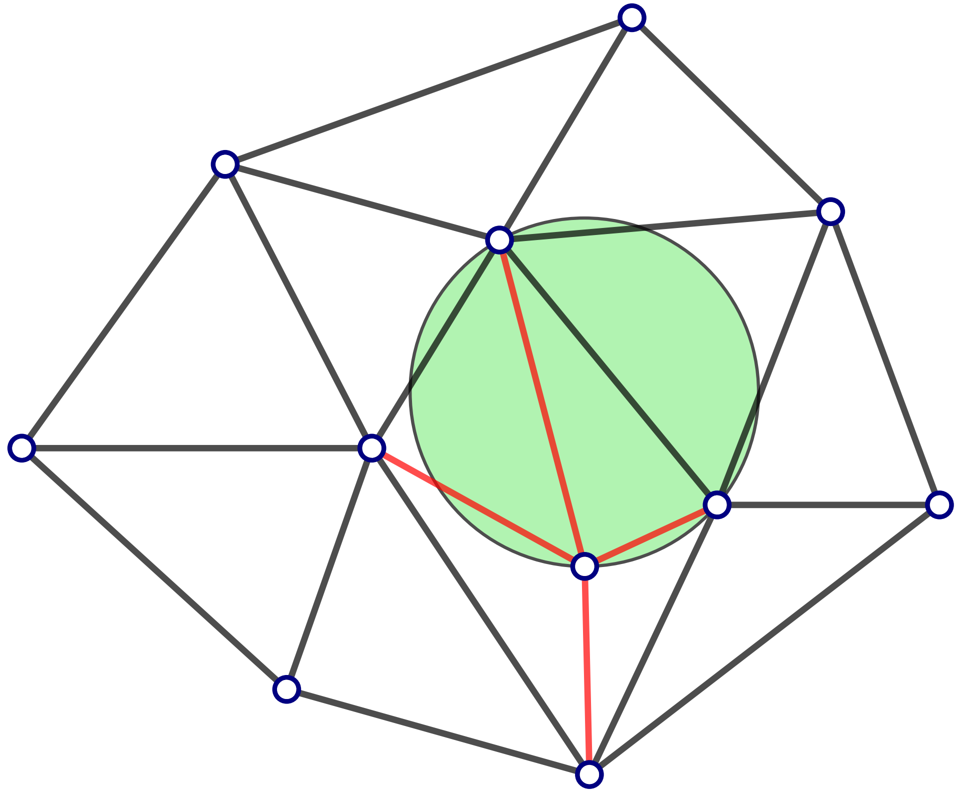

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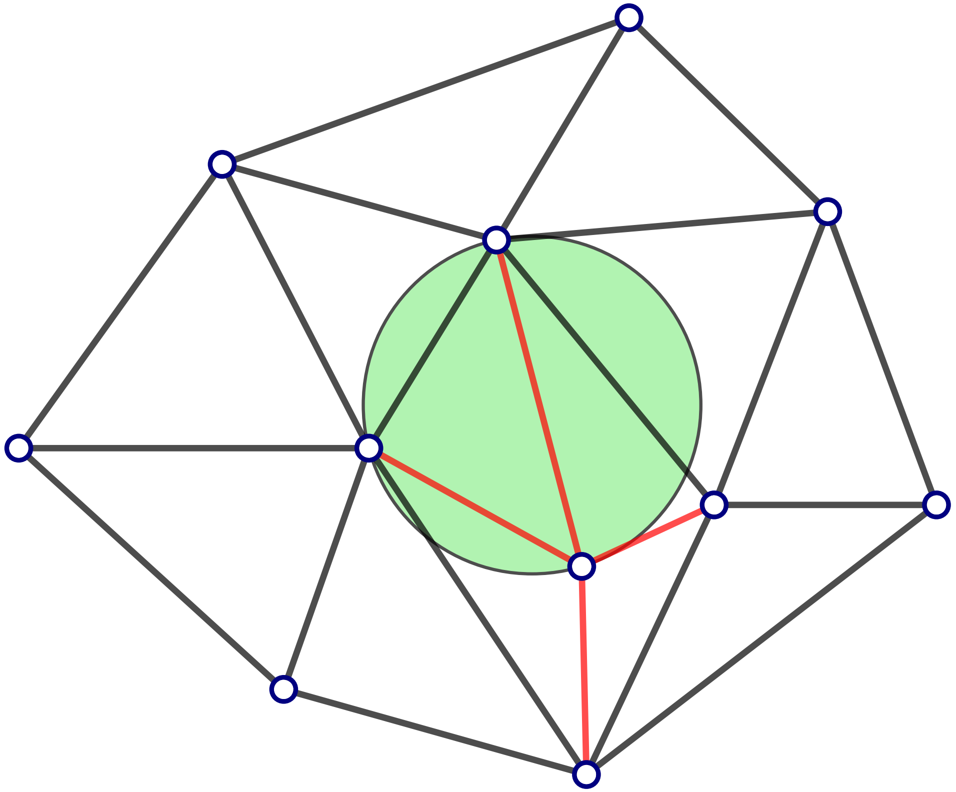

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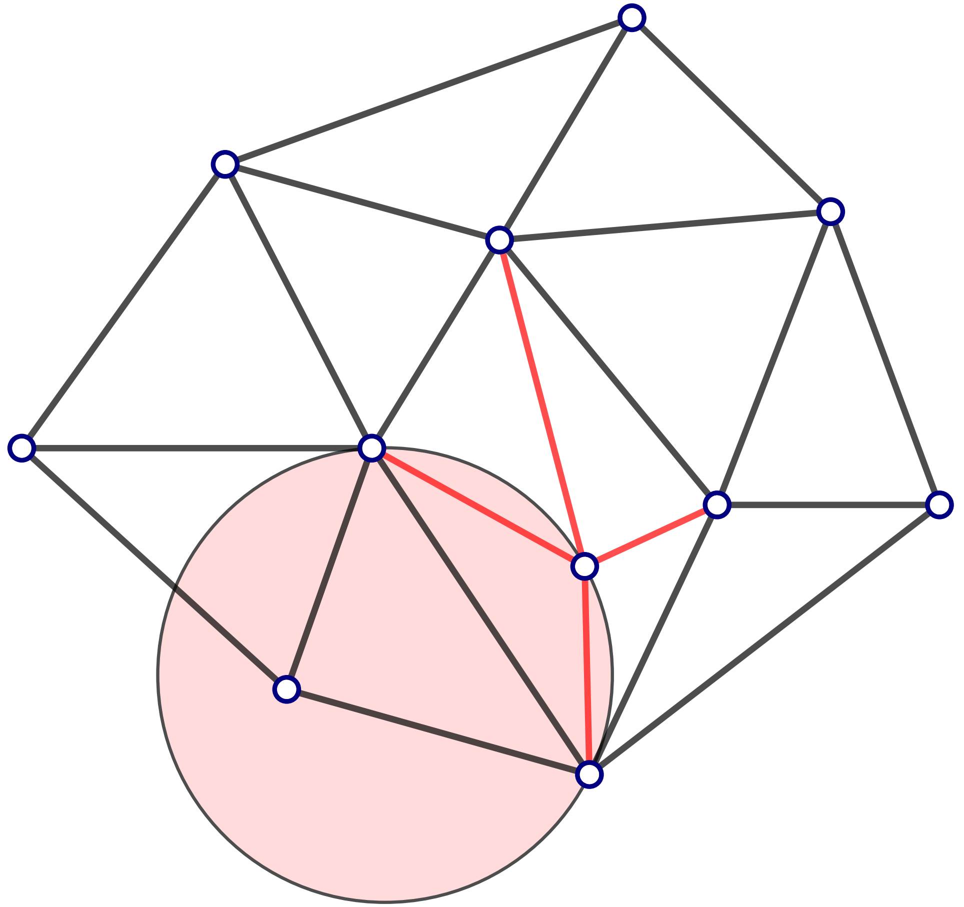

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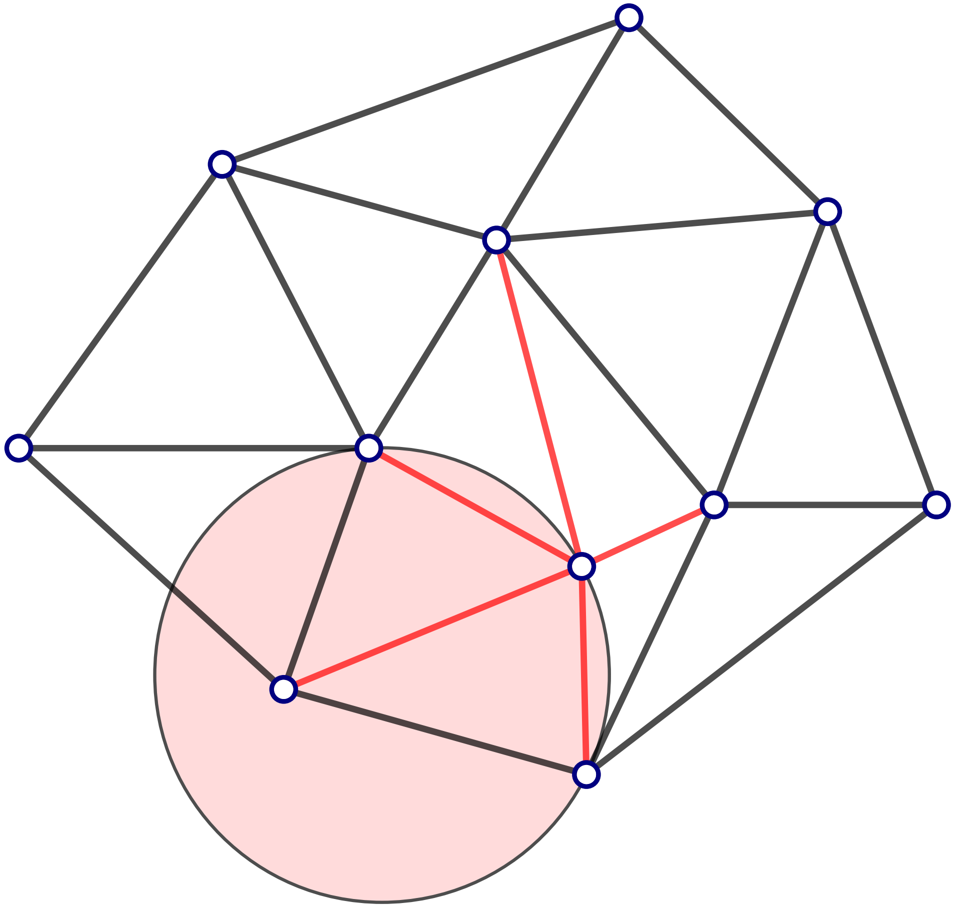

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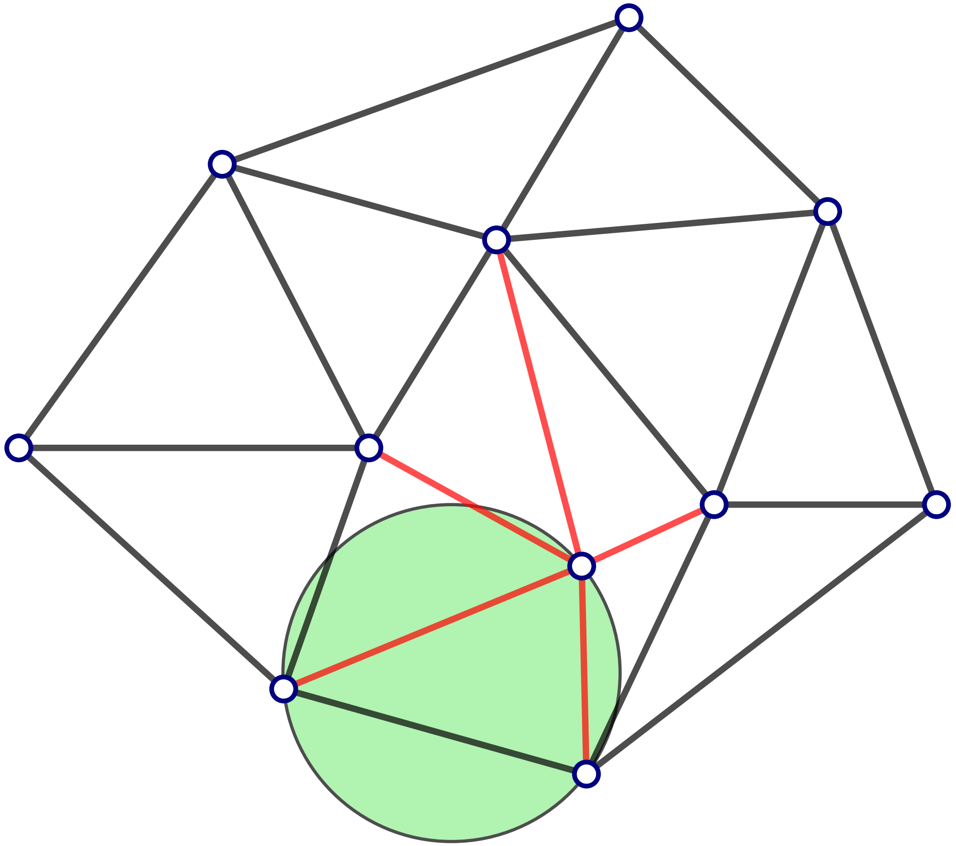

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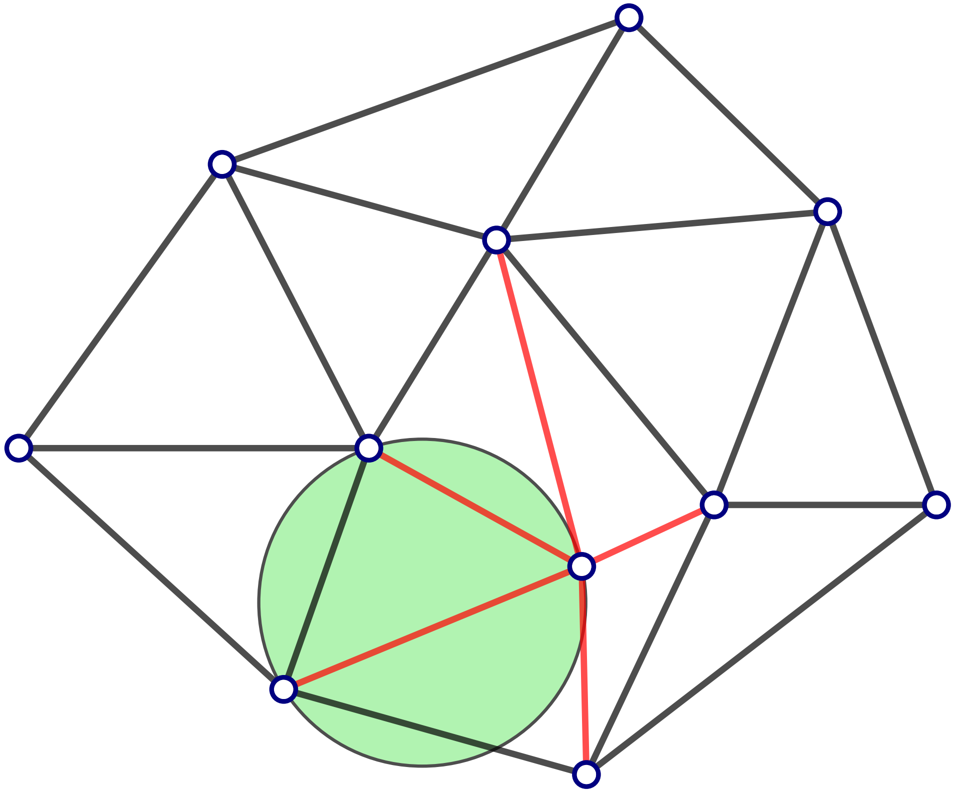

Each flip adds one edge to the new point.

Total cost is $O(d_i)$ where d_i is the degree of p_i in the Delaunay triangulation of $p_1, \ldots, p_i$.

Point location

We use the history of all triangles built to speed up point location.

We maintain a directed acyclic graph during triangle subdivision and edge flips.

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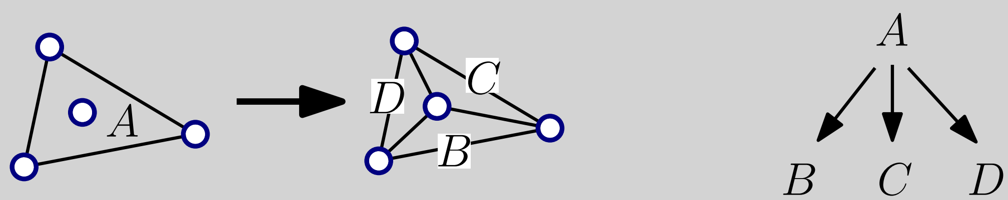

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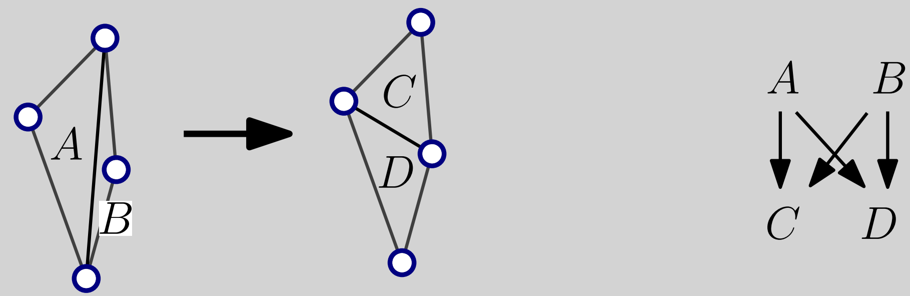

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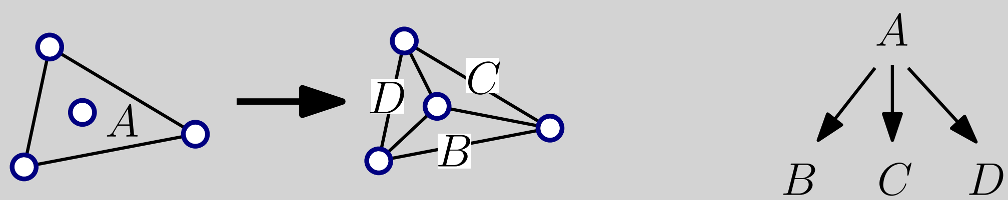

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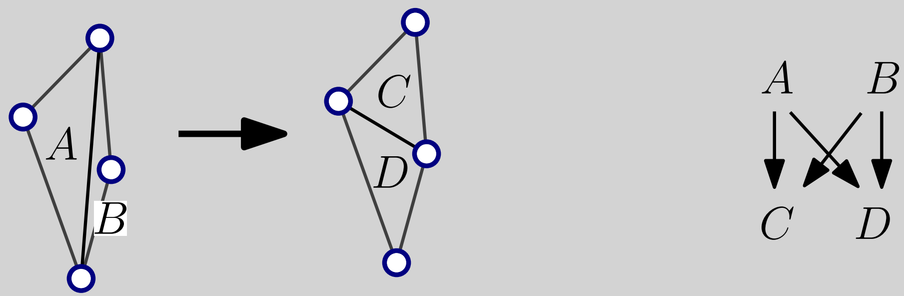

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then it belongs to the triangle to exactly one child of that node.

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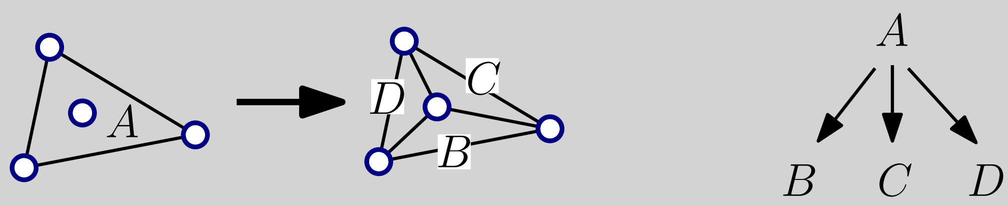

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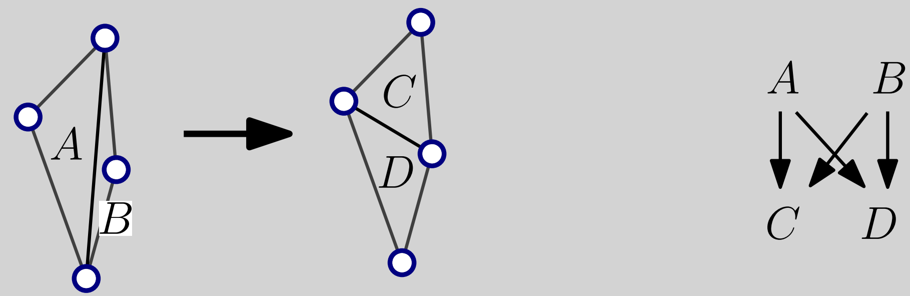

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We start from the root (any p_i belongs to $p_1p_2p_3$) and trickle down

until we find a triangle from the current triangulation (ie a sink of the DAG).

Complexity analysis

Cost $= O\left(\Sigma_i d_i + t_i\right)$.

$T \leftarrow \{p_1 p_2 p_3\}$

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- Insert p_i in T.
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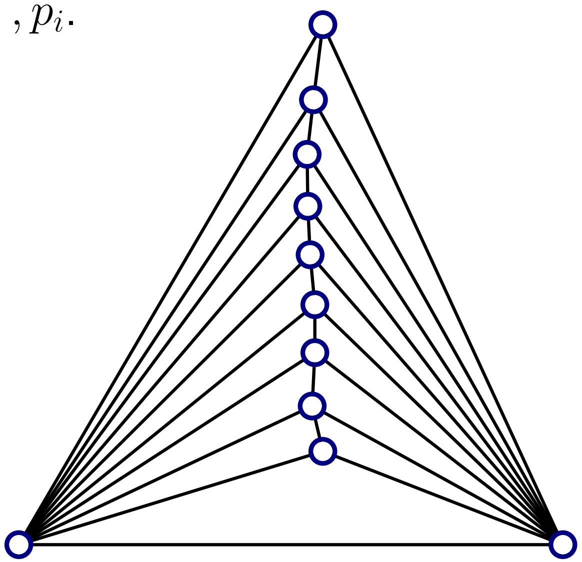
It can happen that $t_i = i - 1$ for all $i \Rightarrow \Omega(n^2)$ complexity.

The order in which the points are inserted is important.



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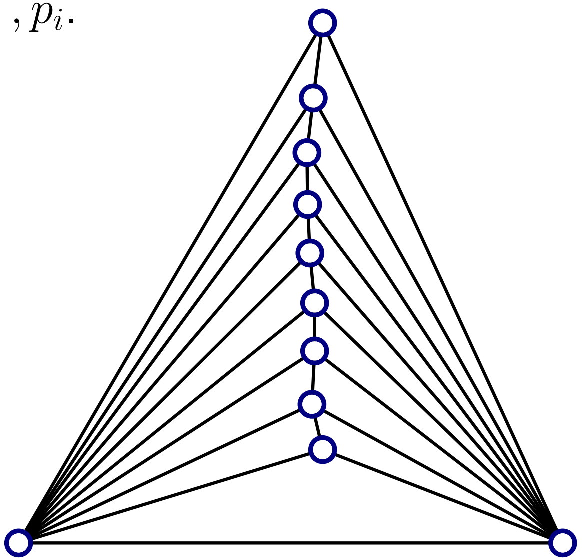

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Renumber the points $p_4, \ldots, p_n$ randomly.

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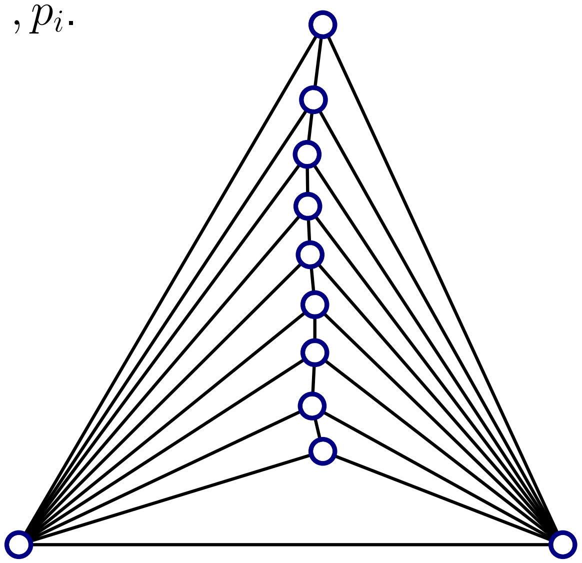

Randomized incremental construction - complexity analysis

We bound the expected complexity of the algorithm.

Expectation is taken with respect to random internal choices. The input is arbitrary.

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Use the bound on the average degree.

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Theorem. We can compute a Delaunay triangulation of n points in $\mathbb{R}^2$ in $O(n \log n)$ time.

Expected running time of a randomized algorithm.

Higher dimension

Delaunay triangulations generalize to arbitrary dimension (empty circumscribed hypersphere).

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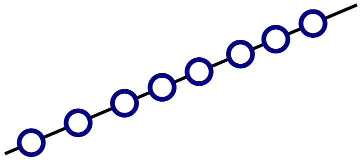

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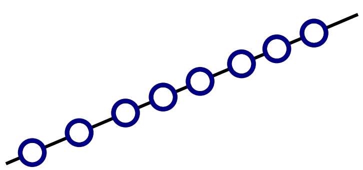

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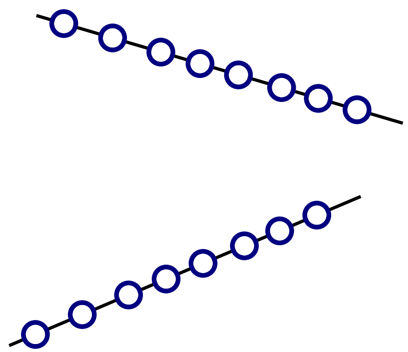

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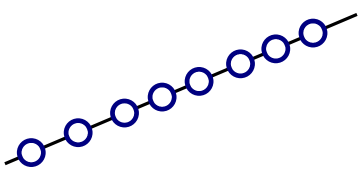

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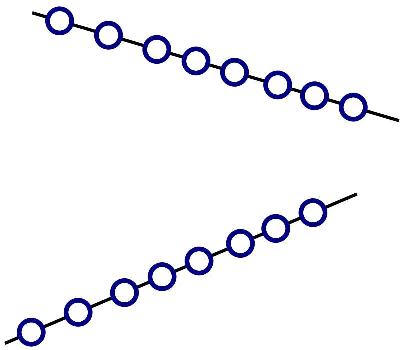

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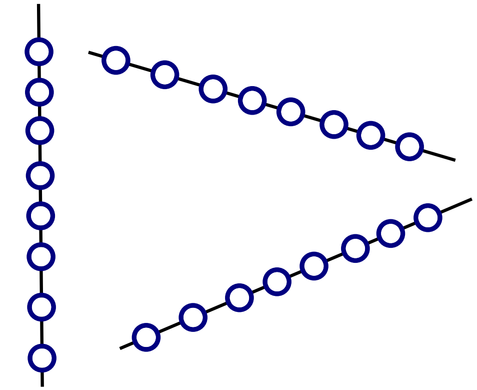

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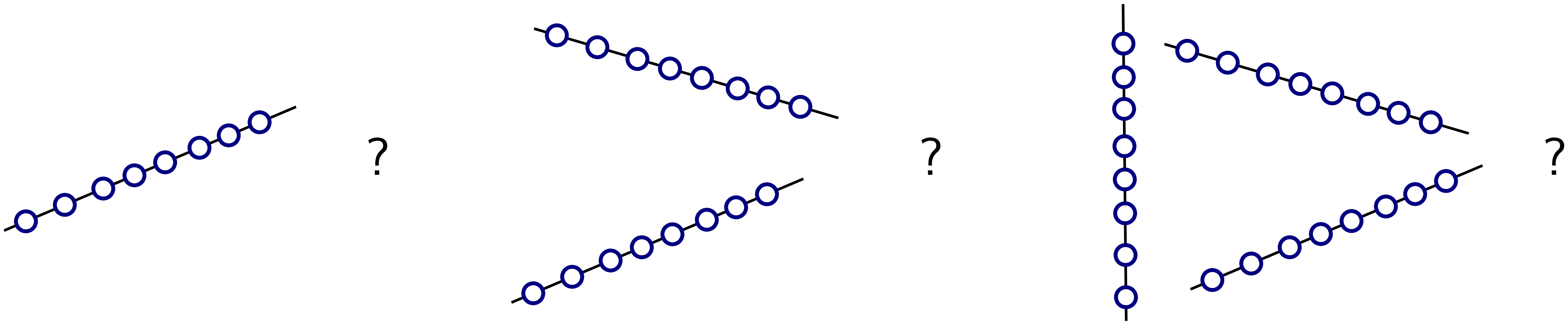


$\Theta\left(n^{\lceil \frac{d}{2} \rceil}\right)$ worst-case complexity in dimension $d \geq 3$.

Can be constructed in expected $O\left(n^{\lceil \frac{d}{2} \rceil}\right)$ time (similar approach).

Wrapping up: Delaunay triangulations

Particular triangulation with good geometric properties.

Efficient flip-based algorithm to compute it.

Optimized & flexible implementations available in CGAL.

Randomization is a powerful technique.

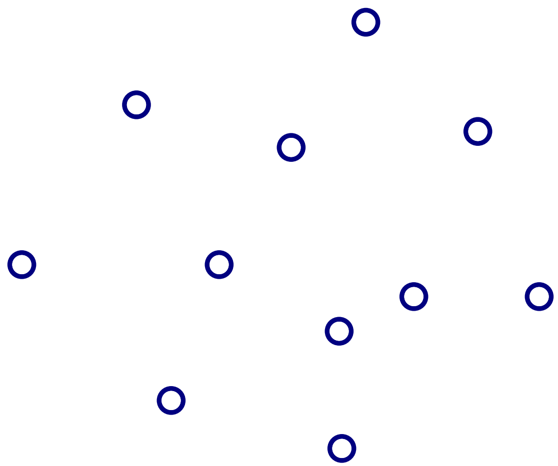
Generic setup: randomized incremental construction.

Many variations: higher dimension, constrained DT, etc...

Question 4

How do you find the nearest post-office?

Input:



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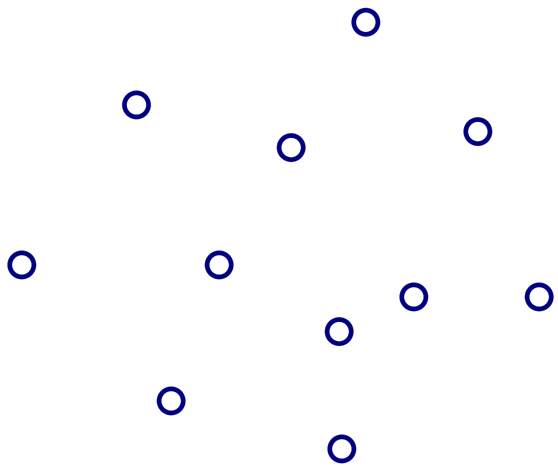

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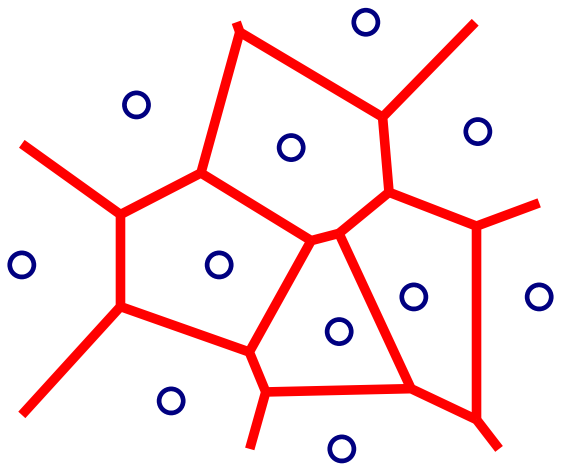
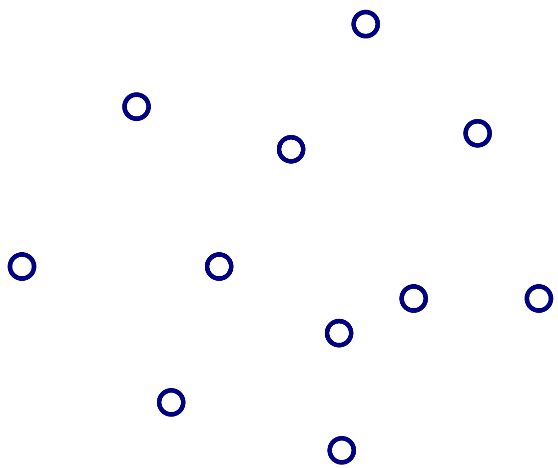


Voronoi diagram - definition

Given a family of sites $p_1, \dots p_n$ in a space with a distance.

Partition the space into regions $R_1, \dots R_n$.

$R_i =$ set of points closer to p_i than to p_j for any $j \neq i$.



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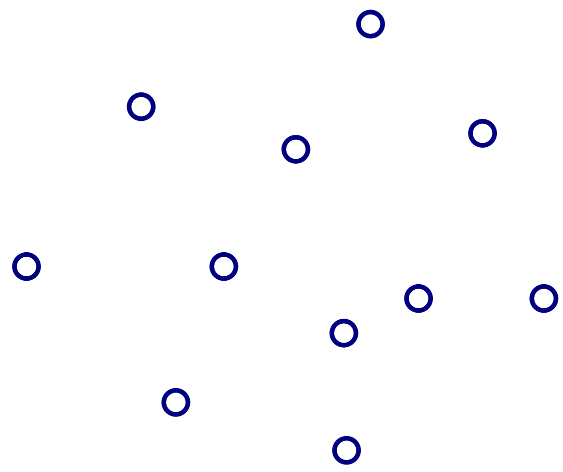

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The space can be $\mathbb{R}^2$, $\mathbb{R}^3$, a surface, etc...

The points can be points, disks, polygons, etc...

We focus on the case of sites in the plane.

Voronoi diagram - applications

Voronoi diagrams capture notions of area of influence appearing in many natural sciences.

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Can be used for natural neighbors interpolation (more later), facility positionning (Voronoi game)...

Structure of a Voronoi diagram

Bisector of p_i and p_j = locus of points at equal distance from p_i and p_j.

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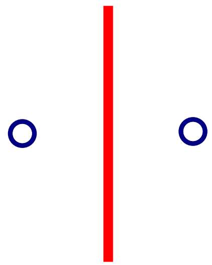

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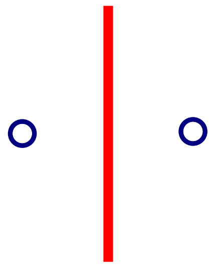

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The set of points closer to p_i than to p_j is a halfplane bounded by their bissector.

Each region R_i is an intersection of closed halfspaces: a convex (not necessarily bounded) polygon.

Structure of a Voronoi diagram

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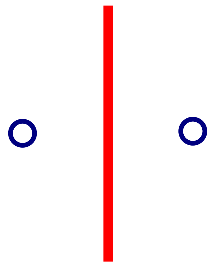
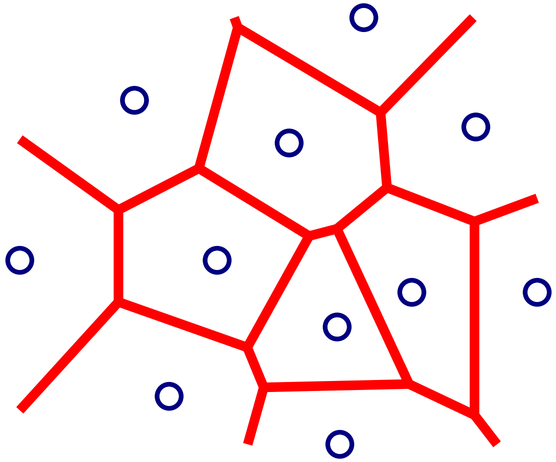


The set of points closer to p_i than to p_j is a halfplane bounded by their bissector.

Each region R_i is an intersection of closed halfspaces: a convex (not necessarily bounded) polygon.



Theorem. The Voronoi diagram of n points in the plane has at most $2n-5$ vertices and $3n-6$ edges/ray/lines.

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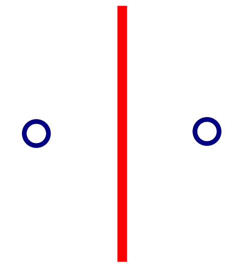

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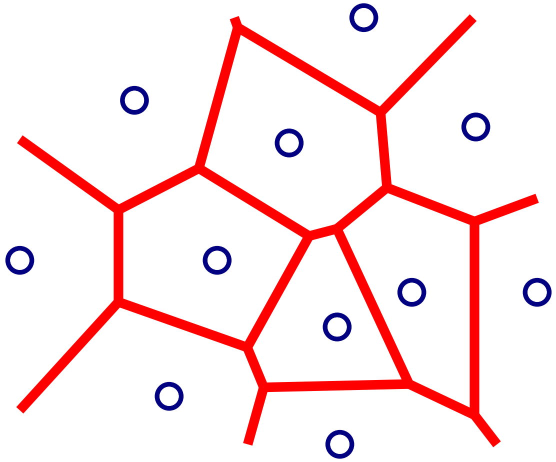

Theorem. The Voronoi diagram of n points in the plane has at most $2n-5$ vertices and $3n-6$ edges/ray/lines.

Proof. Let v and e be the number of vertices and edges of the VD.

Add a point at infinity where all rays meet.

Euler's formula: $(v+1)-e+n=2$.

Edge/vertex incidences + every vertex has degree ≥ 3.

$$2e \geq 3(v+1)$$

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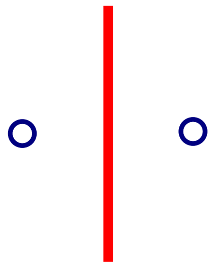

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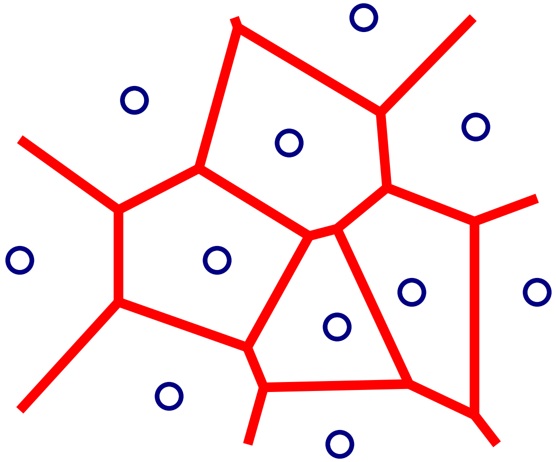

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The Voronoi diagram of n points has $O(n)$ complexity.

Direct algorithm

Naive algorithm to compute the Voronoi Diagram.

Complexity?

For $i = 1 \dots n$

Compute R_i as intersection of $n-1$ half-planes.

Reconnect everything...

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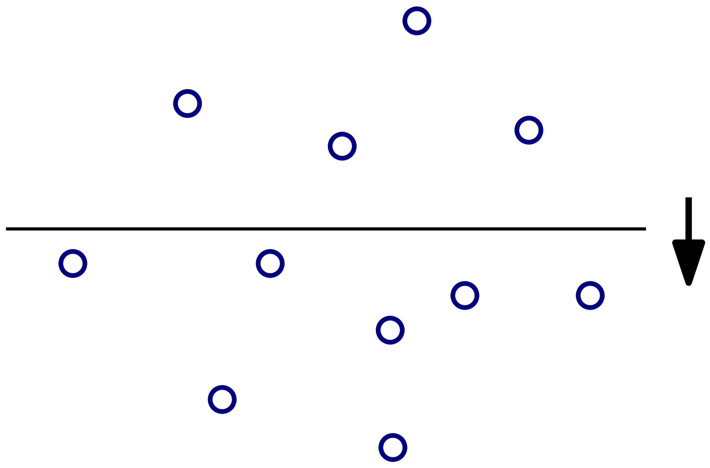

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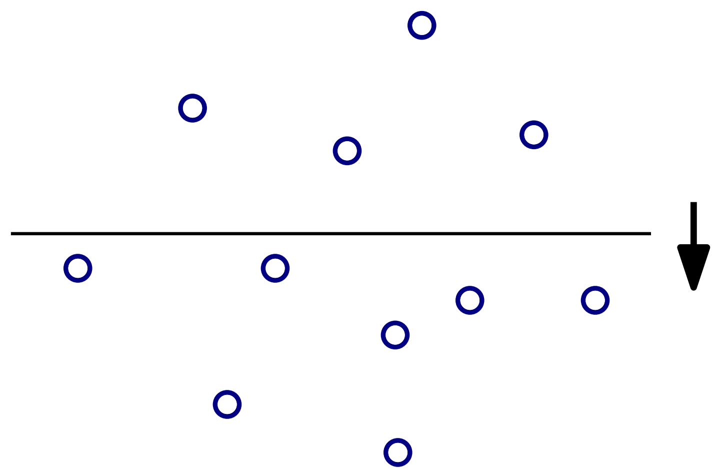

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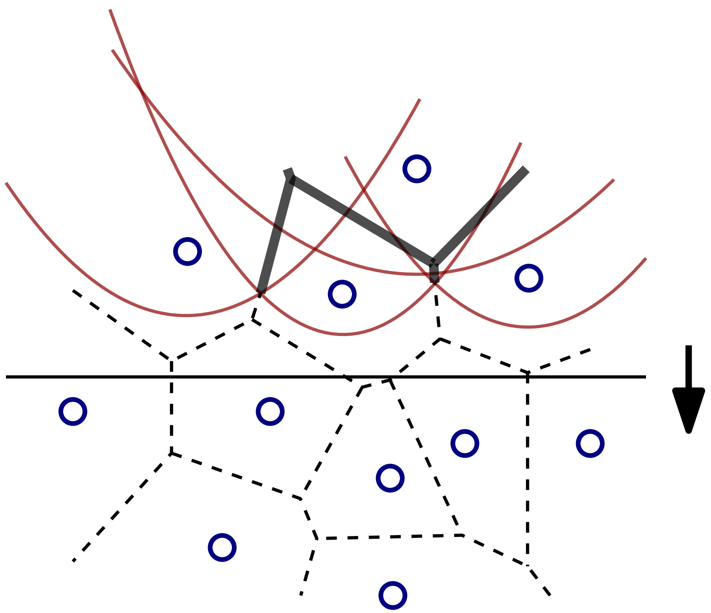

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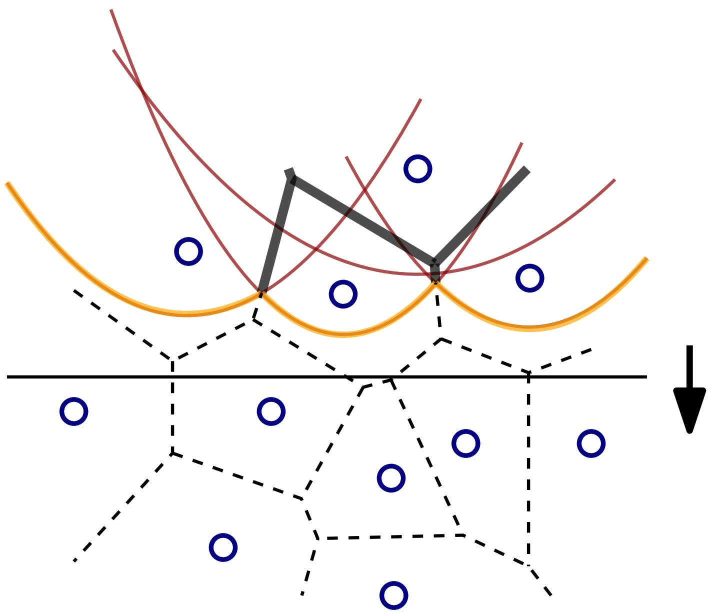

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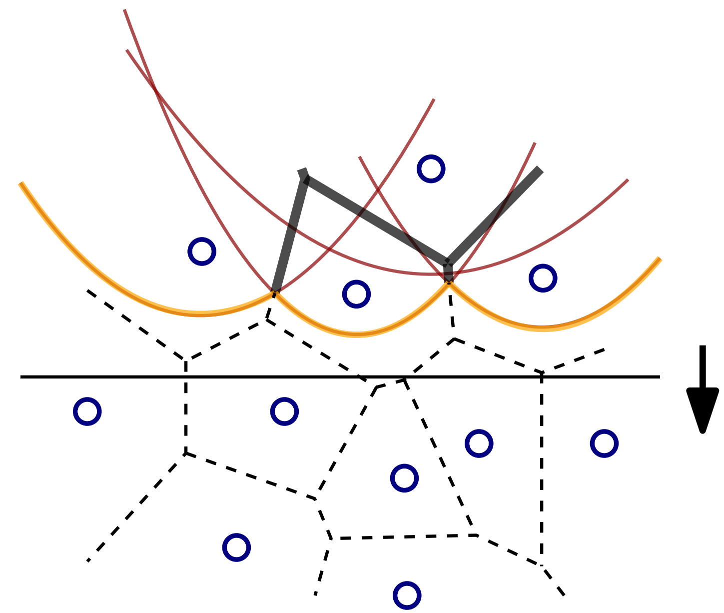

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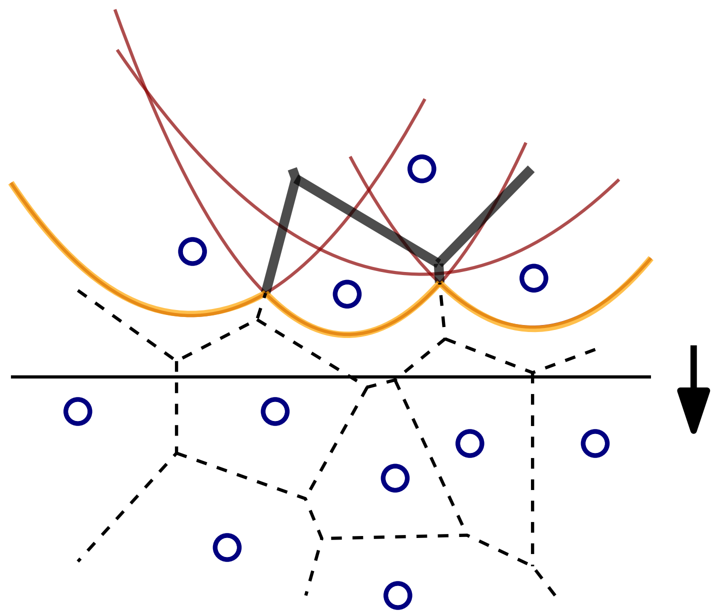
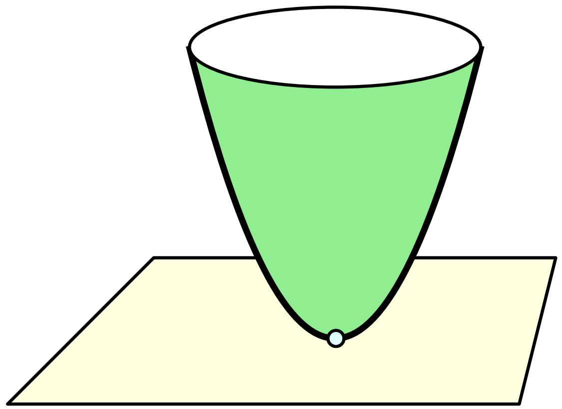


This algorithm takes $O(n \log n)$ time.

Fortune's algorithm.

Lifting $\mathbb{R}^2 \to \mathbb{R}^3$

Consider the unit paraboloid $(P) : z = x^2 + y^2$ in $\mathbb{R}^3$.



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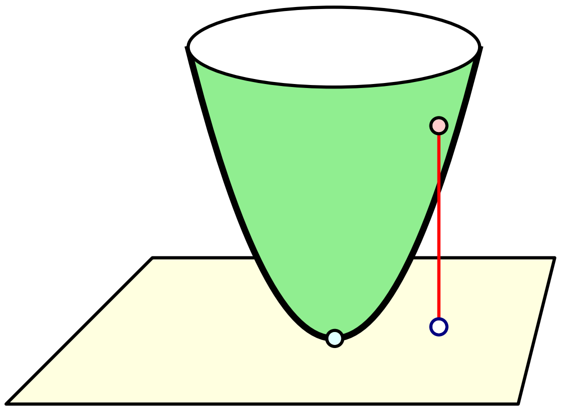

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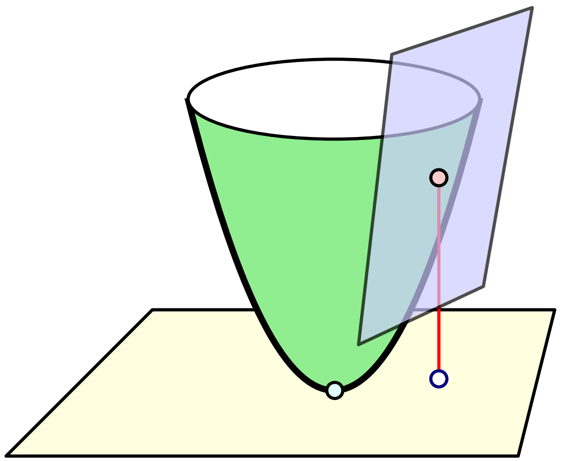

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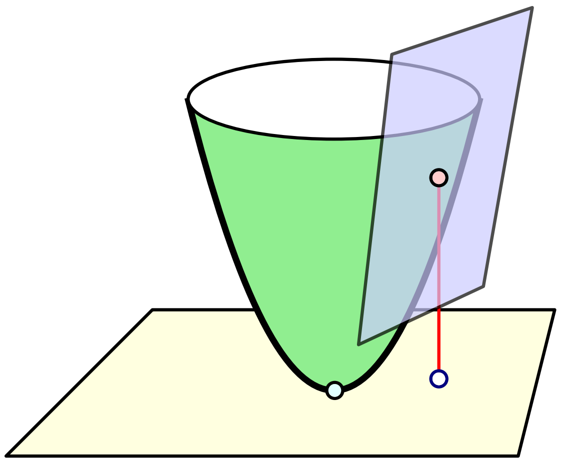

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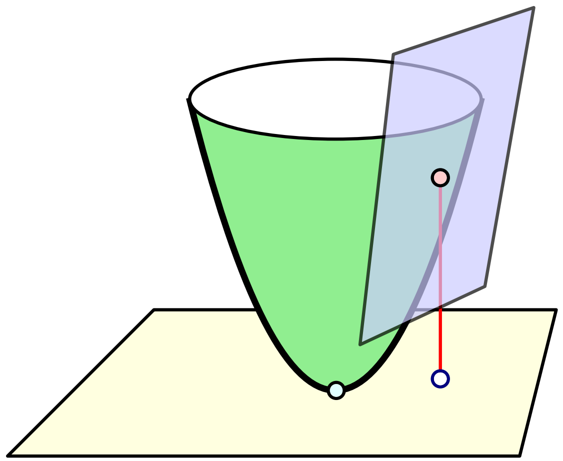

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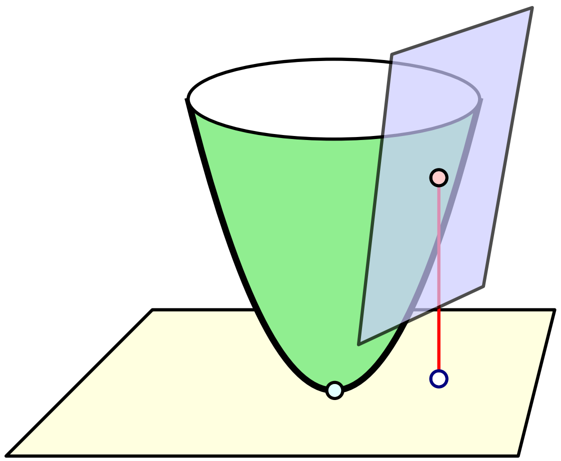

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Equation of $h(a) : 2x_a(x - x_a) + 2y_a(y - y_a) - z + x_a^2 + y_a^2 = 0$

$b'' = (b_x, b_y, 2b_x a_x + 2b_y a_y - (a_x^2 + a_y^2)) \Rightarrow b'b'' = ab^2$

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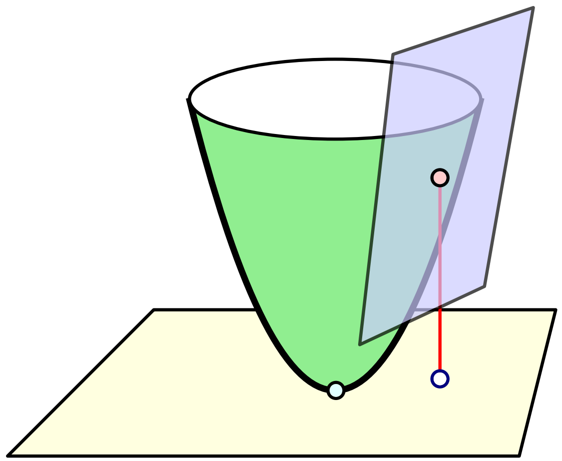

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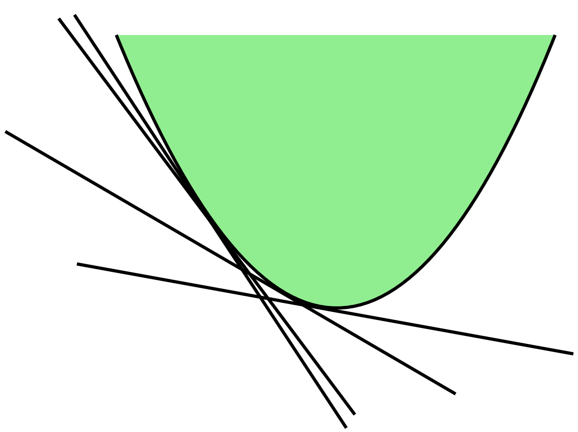

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Lifting each point of R_i to $h(p'_i)$

lifts the VD to the upper enveloppe of $h(p'_1), \ldots, h(p'_n)$

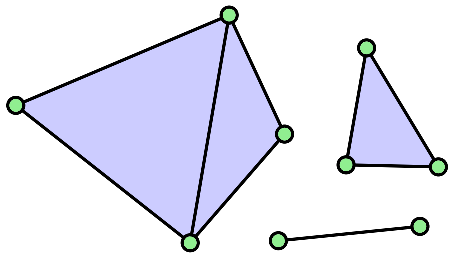
$\rightsquigarrow$ Compute the convex hull of n halfspaces in $\mathbb{R}^3$.

Incidences and combinatorial duality

Consider a set of vertices, edges and polygons.

A polygon P and an edge e are incident if e is on the boundary of P.

Same for edge/vertex and polygon/vertex incidences.



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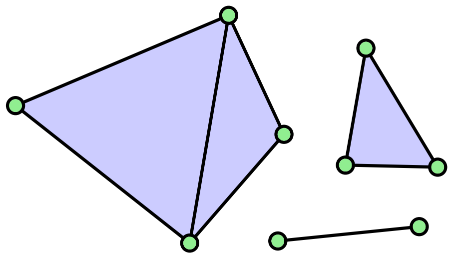

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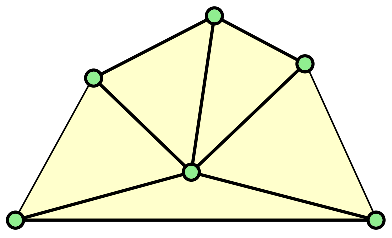

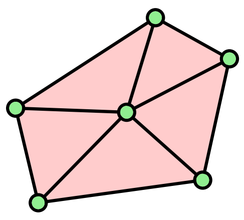

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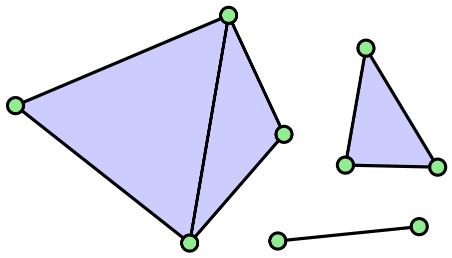

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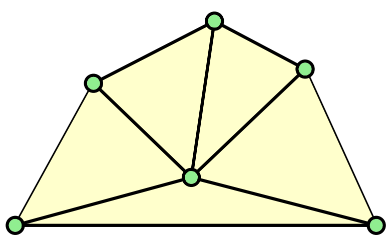

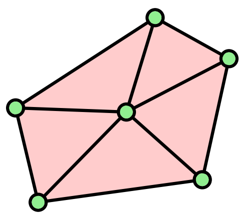

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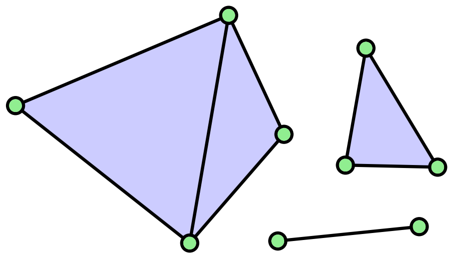

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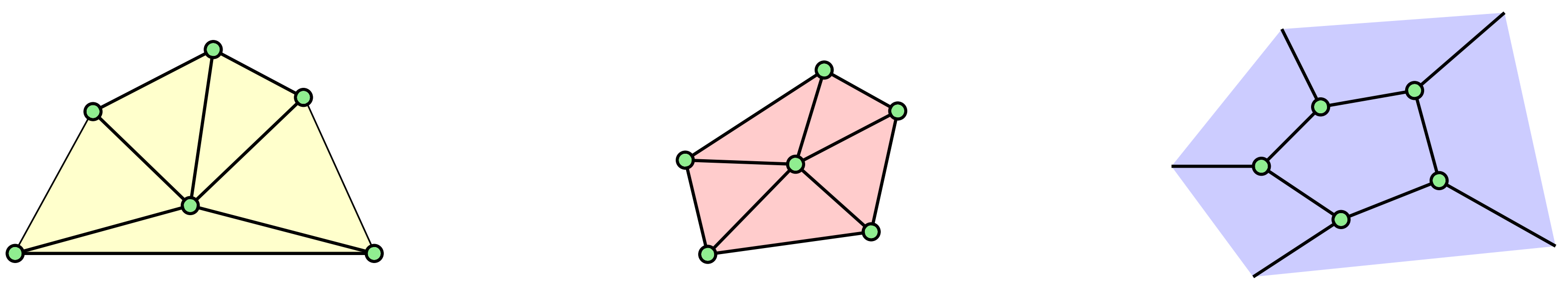

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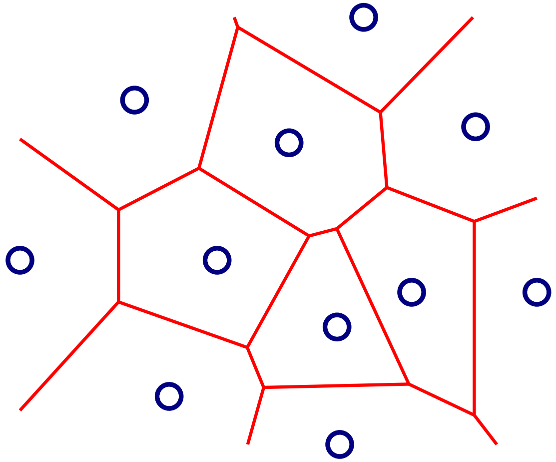
Voronoi & Delaunay



There is a combinatorial duality between
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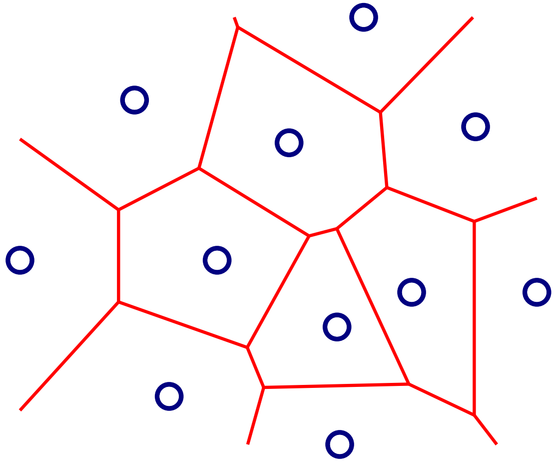
Voronoi & Delaunay



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Voronoi regions

sites

Voronoi & Delaunay

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A point on a Voronoi edge is at equal distance
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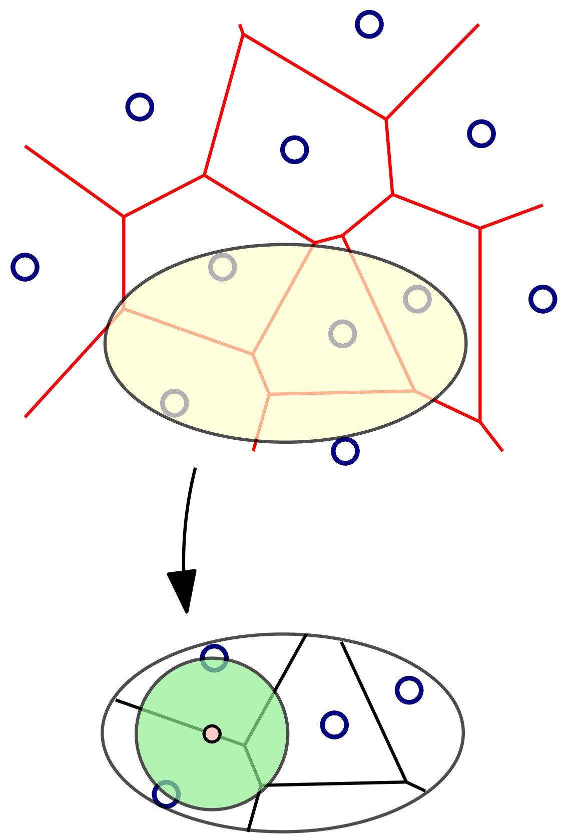

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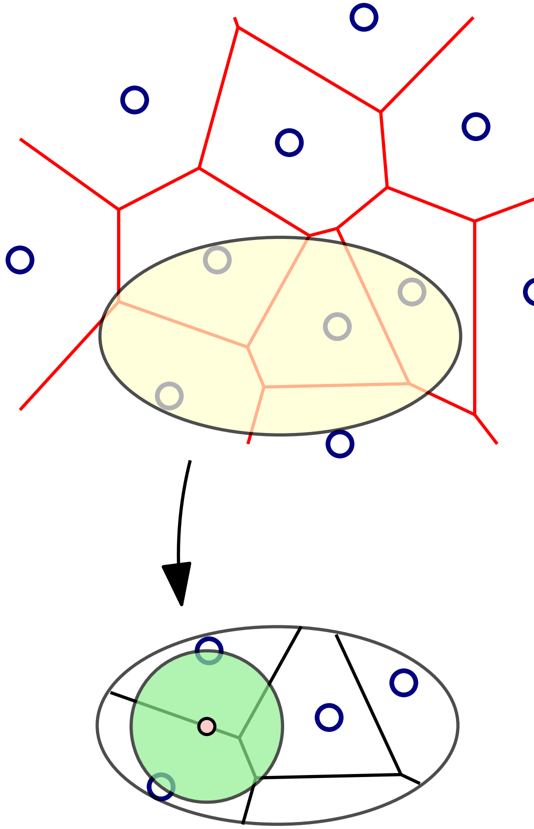

Voronoi regions	Voronoi edges
↕	↕
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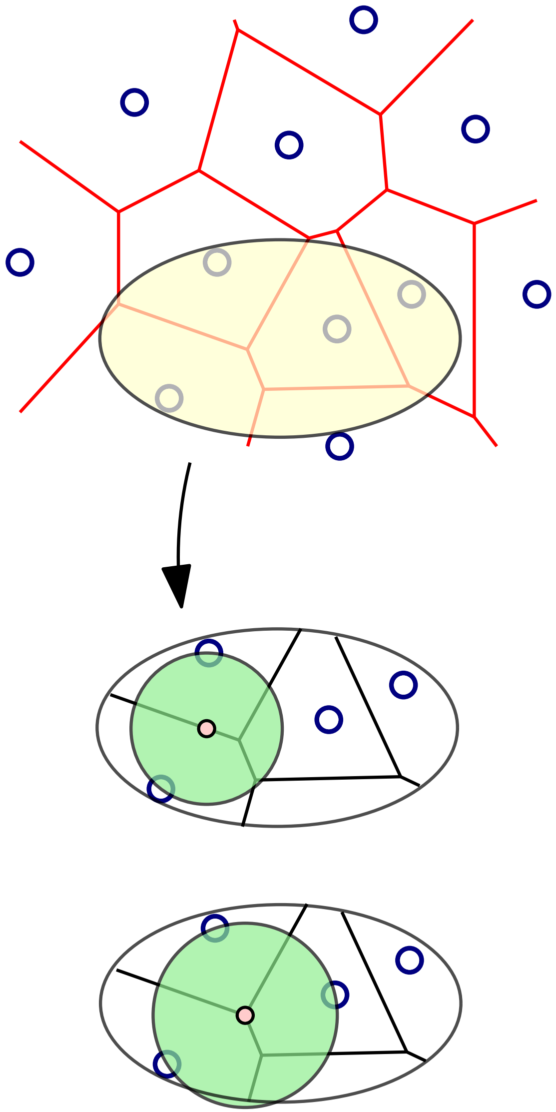

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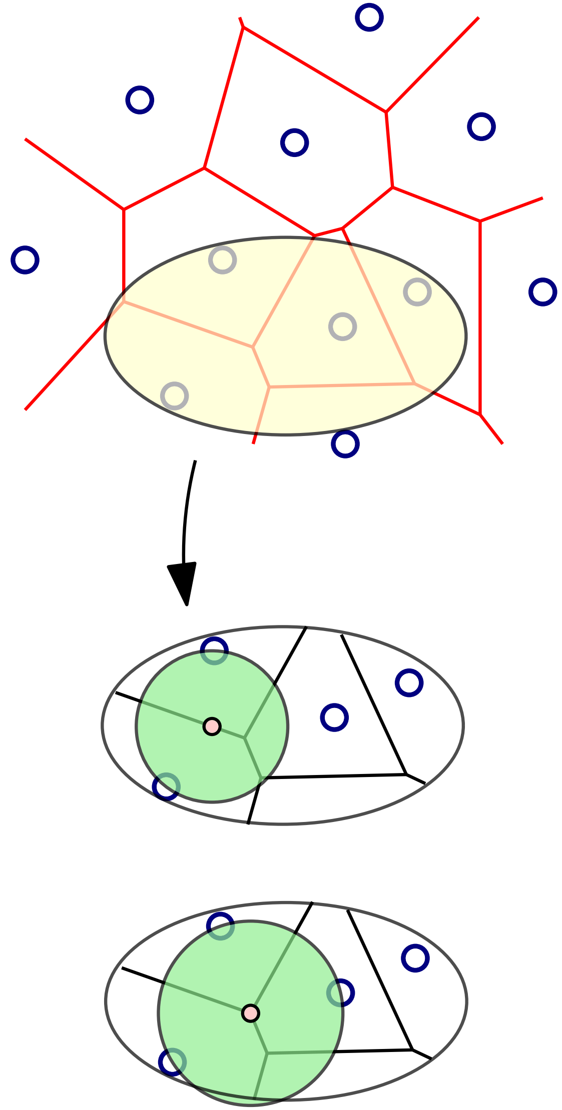
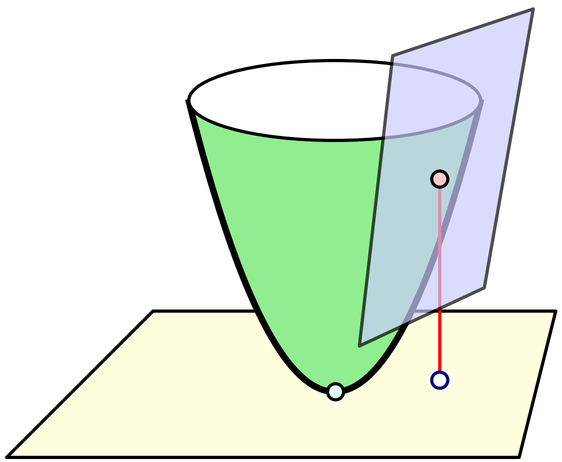


Voronoi regions	Voronoi edges	Voronoi vertices
↕	↕	↕
sites	Delaunay edges	(circumcenters of) Delaunay triangles

Lifting (bis)



The lift to the paraboloid maps every site p_i to

$$\begin{cases} \text{a point } p_i' \text{ on } (P), \\ \text{the plane } h(p_i) \text{ tangent to } (P) \text{ in } p_i. \end{cases}$$

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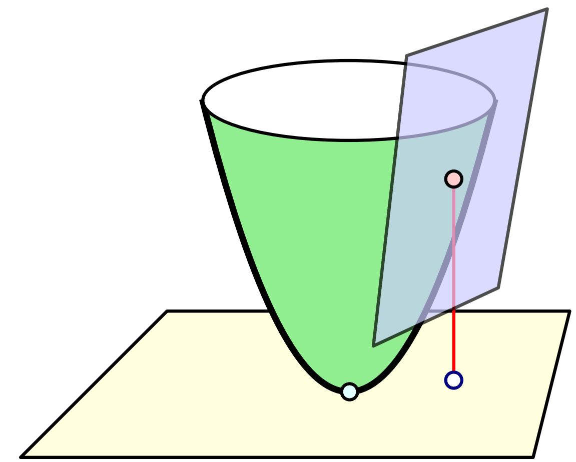

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The Voronoi diagram of $p_1, \ldots, p_n$ is the projection of the convex hull of $h(p_1'), \ldots, h(p_n')$

The Delaunay triangulation of $p_1, \ldots, p_n$ is the projection of the convex hull of $p_1', \ldots, p_n'$

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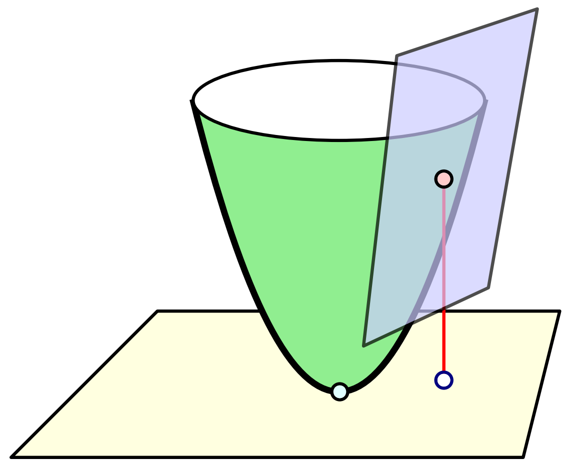

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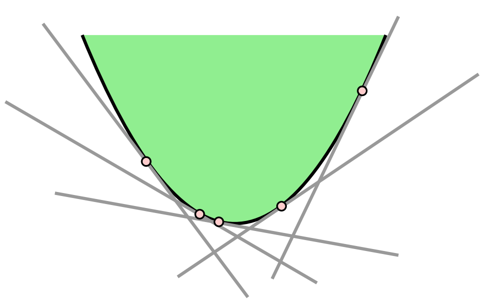

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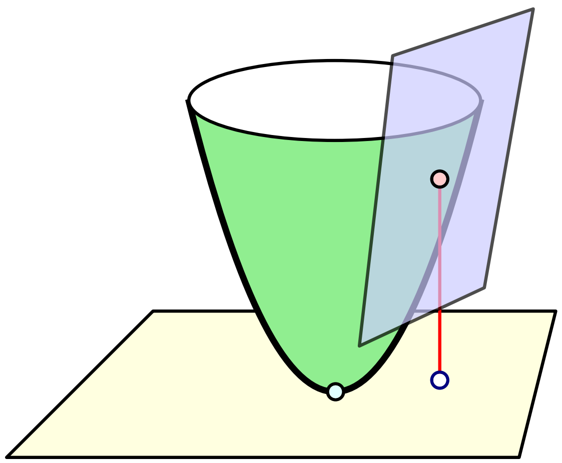

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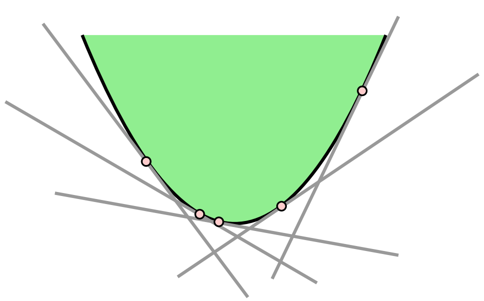
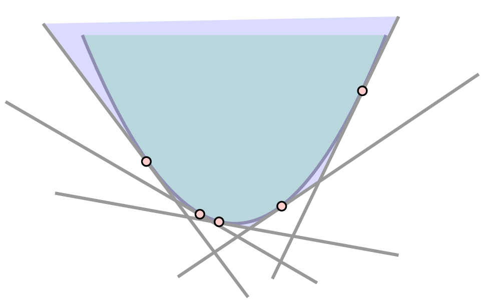
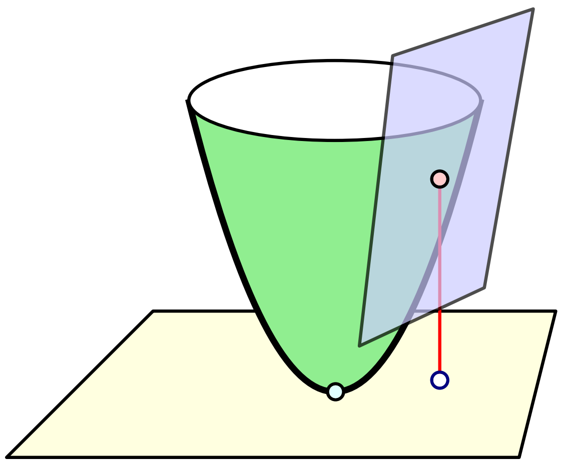




Lifting (bis)



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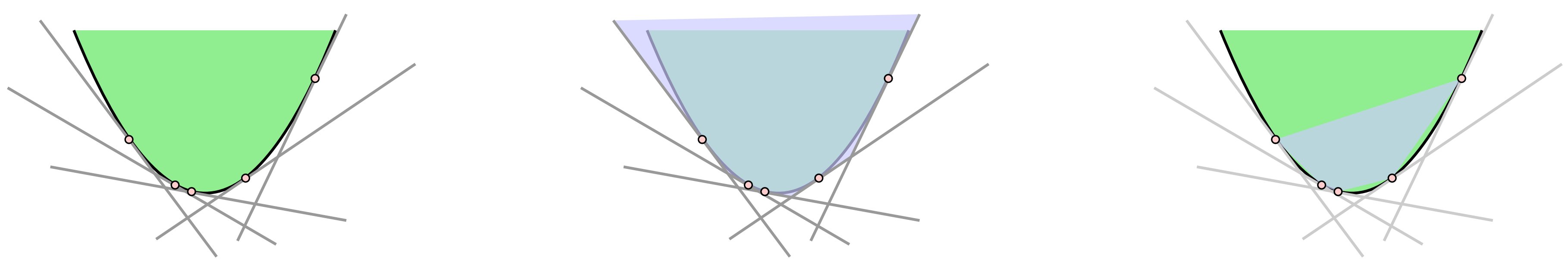

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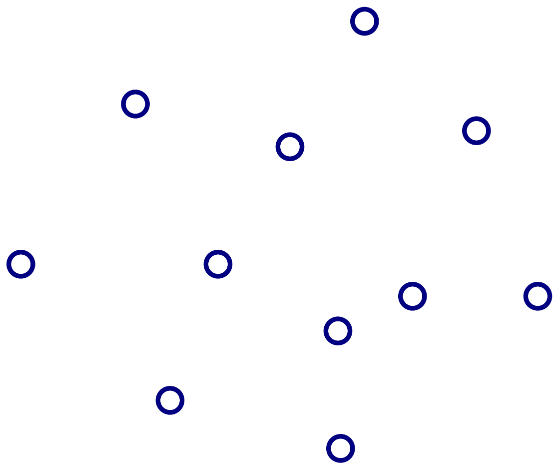

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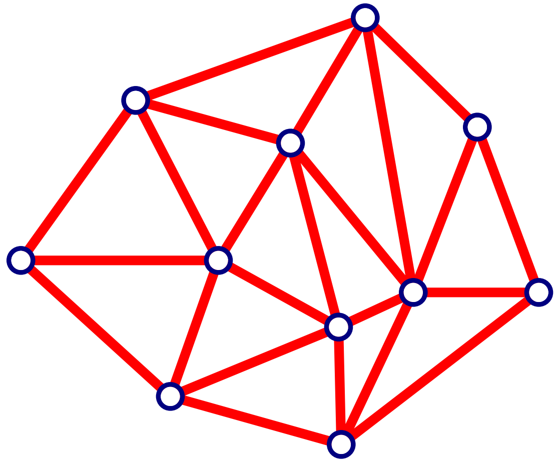

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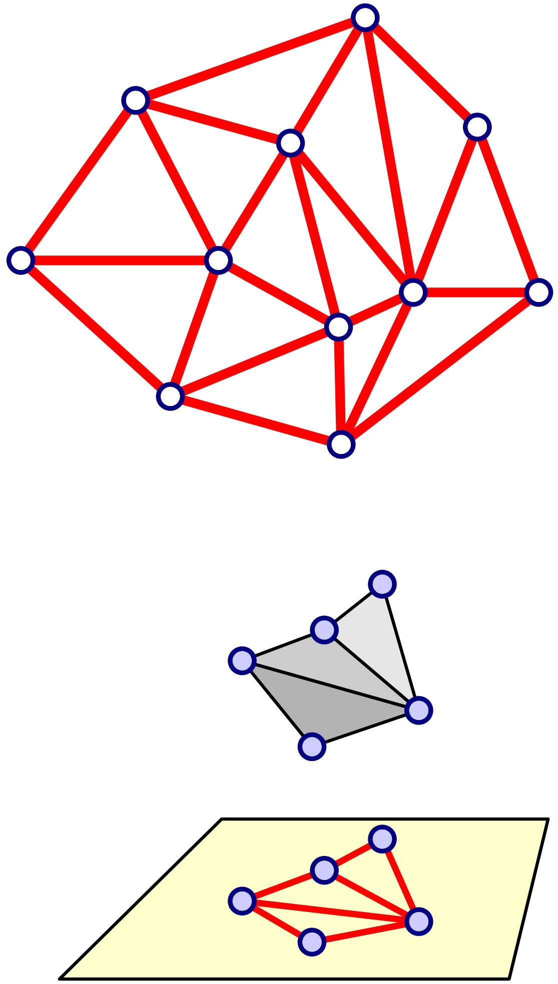

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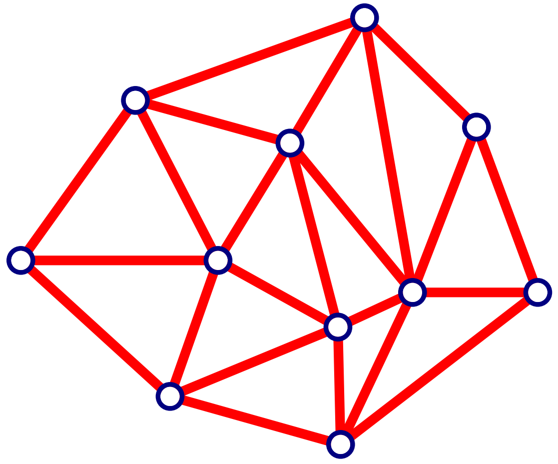

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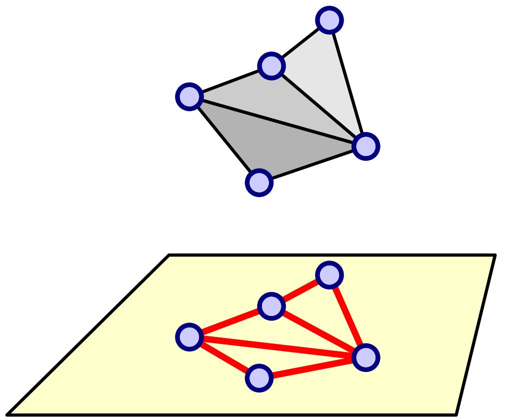

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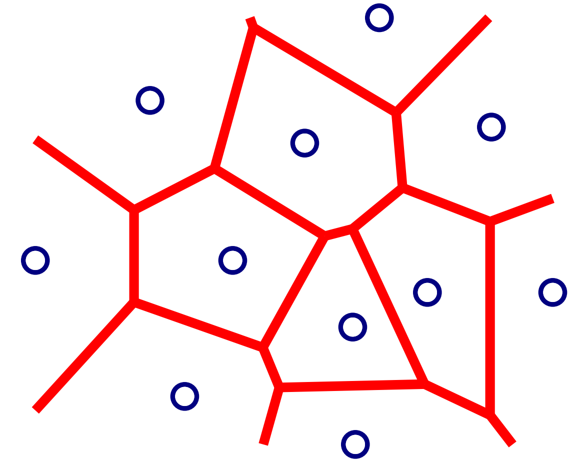

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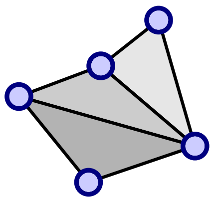

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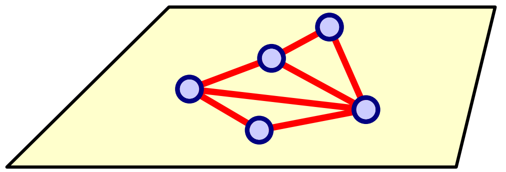

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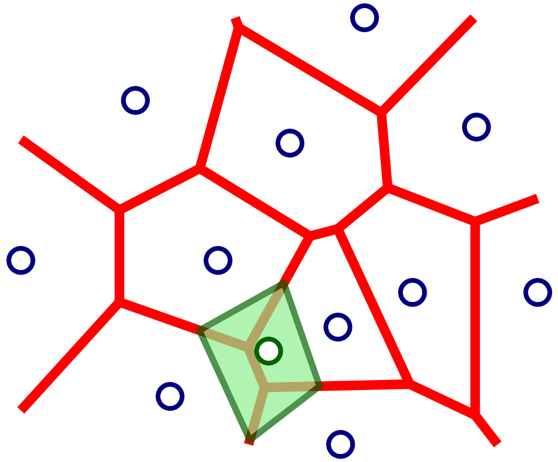

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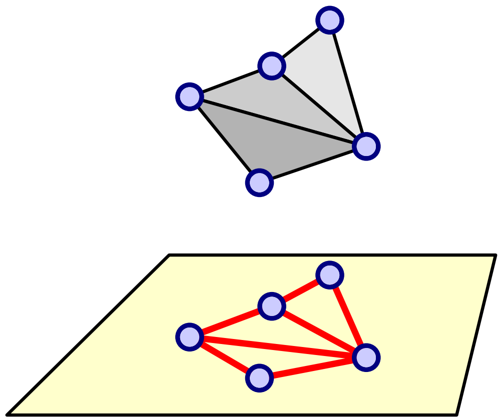

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$$\tilde{f}(p) = \Sigma_{i=1}^{n} w_i f(p_i)$$

$$w_i = \frac{|R(p) \cap R_i|}{|R_i|}$$

$R(p) =$ Voronoi region of p in $\{p_1, \ldots, p_n\} \cup \{p\}$.

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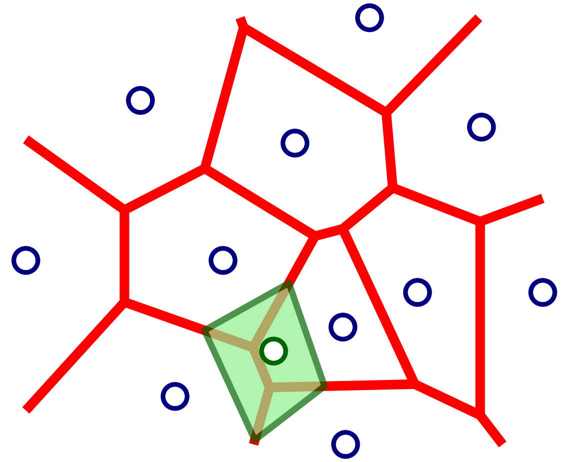

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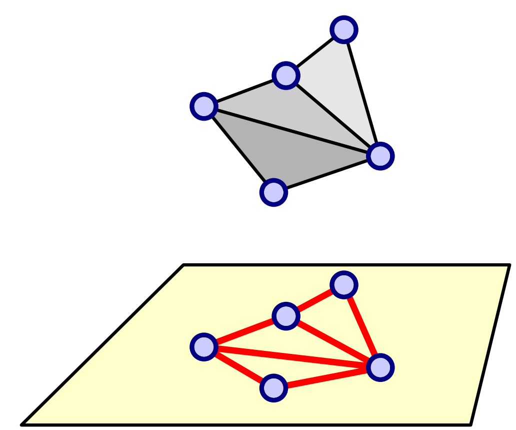

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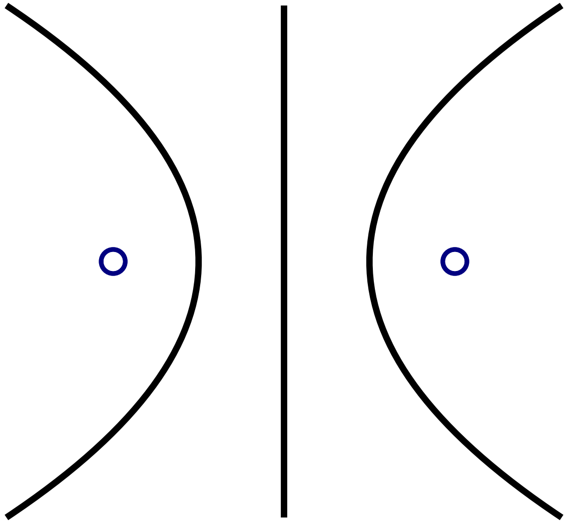
Now $\tilde{f}$ is smooth everywhere but in the points $p_1, \ldots p_n$.

Going a bit further

bisectors $\rightsquigarrow$ k-sector?



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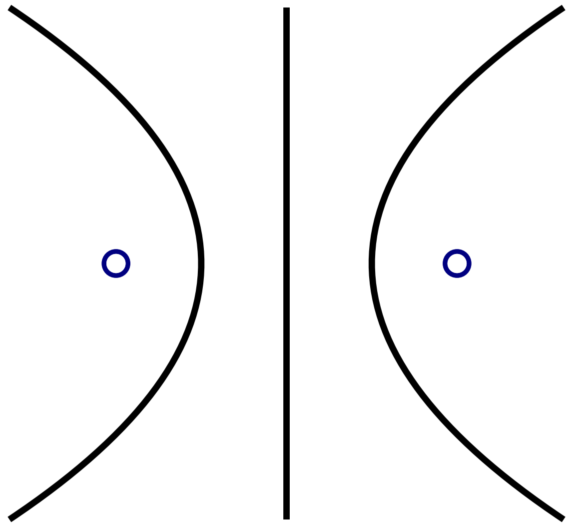

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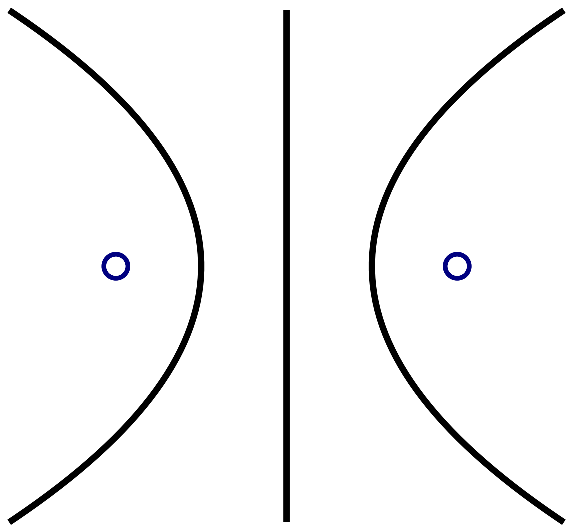

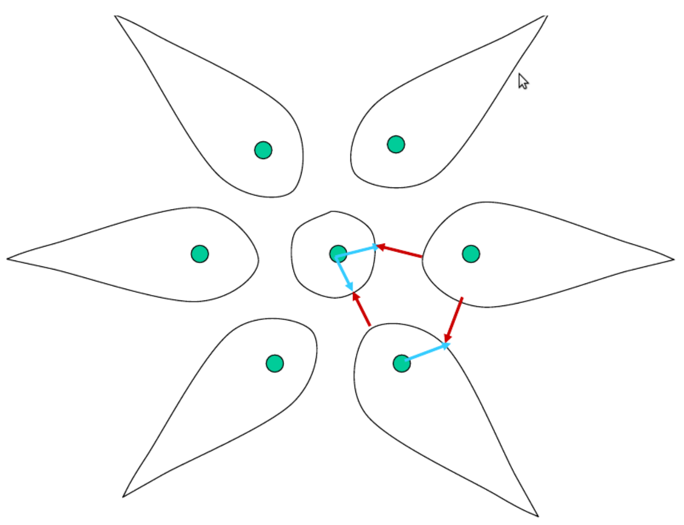

Voronoi diagram with a neutral region.

Wrapping-up: Voronoi diagram

Natural structure: decomposition of space according to the closest site.

"Natural enough" that it was rediscovered over and over.

(combinatorially and geometrically) dual to Delaunay triangulation.

Combinatorial / geometric transforms help.

Question 5

Why are geometric algorithms hard to implement correctly?

First issue: degeneracies

Many algorithms are described assuming general position of the input.

$$\left.\begin{array}{l}\text{No two points have the same } x\text{-coordinate.}\\ \text{No three segments intersect in the same point.}\\ \text{No four points lie on the same circle.}\end{array}\right\}\quad \text{properties that hold generically.}$$

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Can we handle degeneracies without treating each one separately?

Can we at least detect them efficiently?

Hardness of testing degeneracies

The 3-sum problem: Given n numbers, decide if three of them sum to 0.

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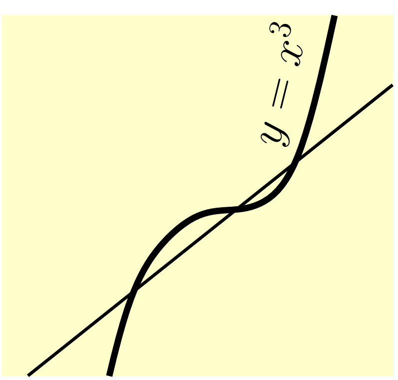

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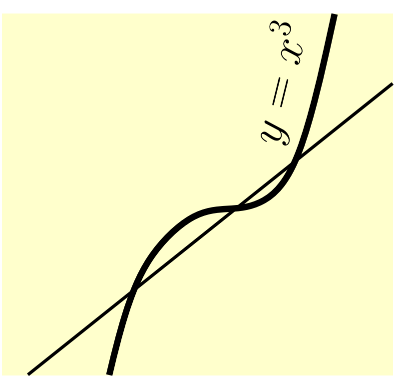

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Testing if $d+1$ points lie on a common hyperplane in $\mathbb{R}^d$ is $\lceil \frac{d}{2} \rceil$-sum hard.

Perturbing? Easier said than done

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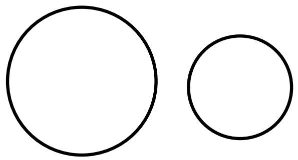
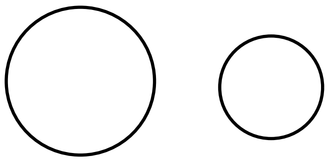
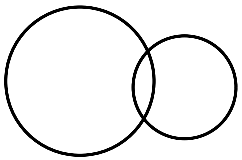
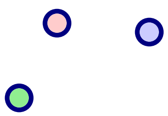
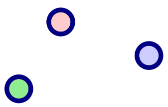
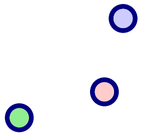
 can be perturbed in  but not in 

 can be perturbed in  but not in 

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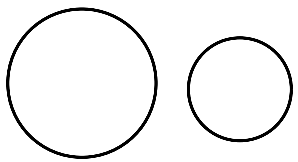
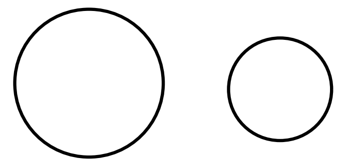
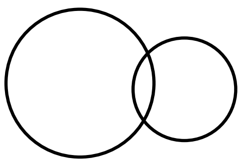
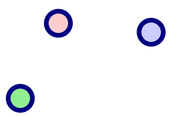
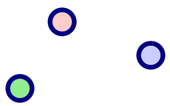
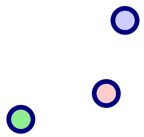
 can be perturbed in  but not in 

 can be perturbed in  but not in 

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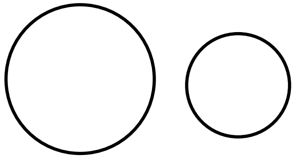
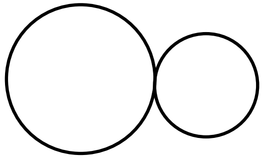
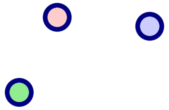
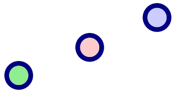
 should not be perturbed in 

 should not be perturbed in 

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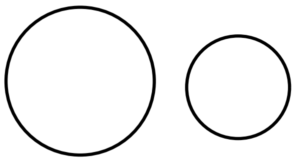
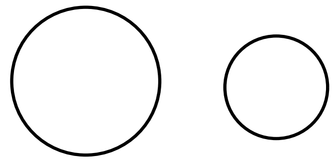
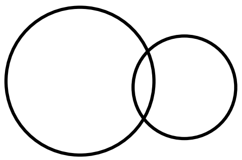
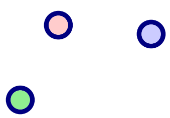
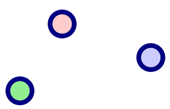
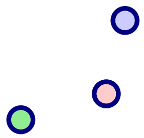
 can be perturbed in  but not in 

 can be perturbed in  but not in 

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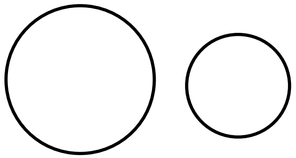
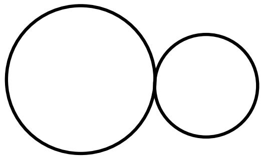
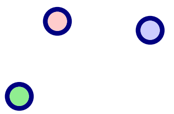
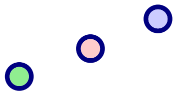
 should not be perturbed in 

 should not be perturbed in 

Bottom line: "Epsilon=10^-12" is not an option if we want any kind of guarantee.

Systematic approach: simulation of simplicity

Degeneracy correspond to vanishing of some of the polynomials evaluating the geometric predicates.

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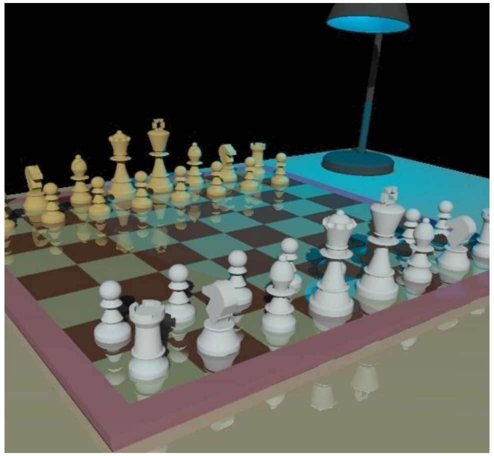
Partial perturbation: shearing $(x, y) \mapsto (x + ty, y)$

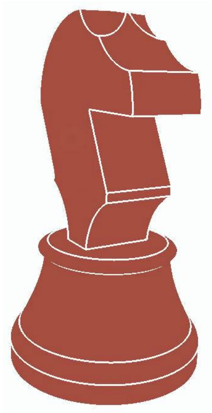
The hard way: zillions of cases

The problem is to find a way to tabulate all cases in a way where exhaustivity can be proven.

Example: intersection of two quadric surfaces in $\mathbb{R}^3$

Fundamental problem in solid modelling.





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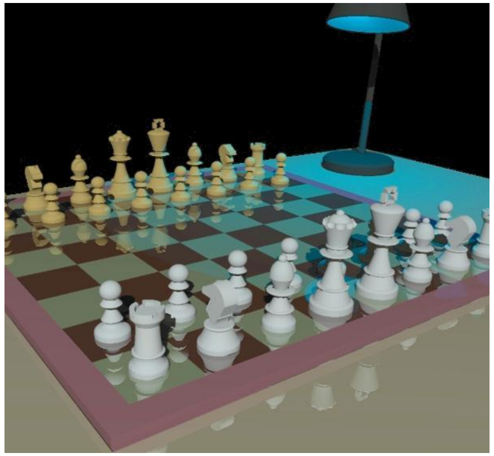

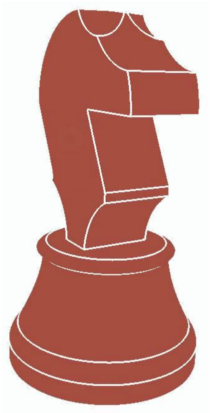

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A published algorithm missed several.

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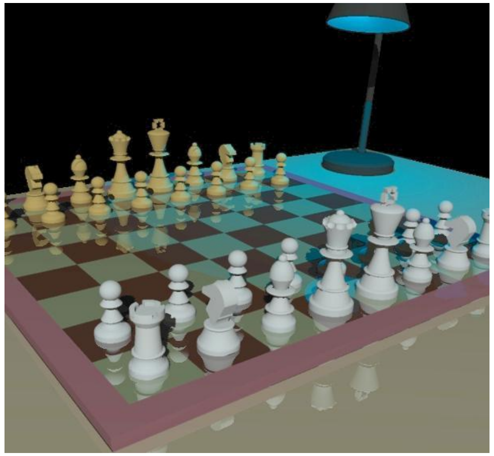

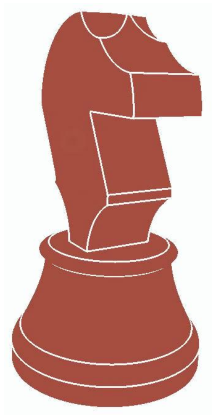

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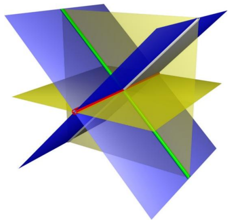

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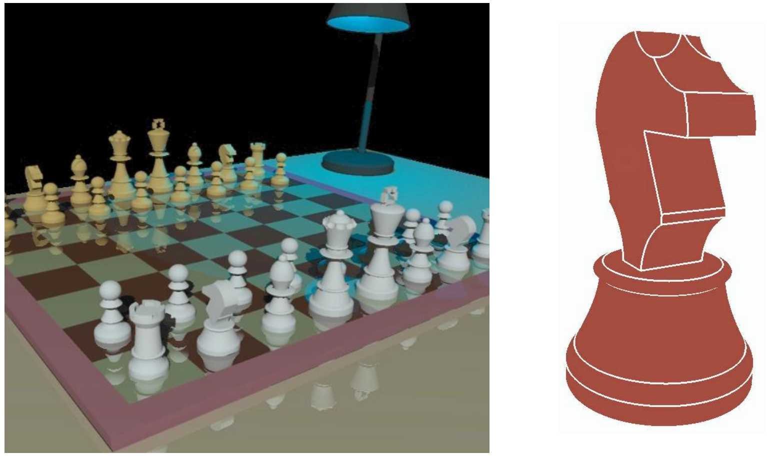

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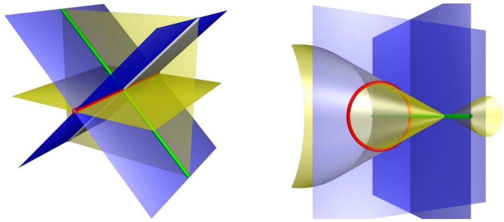

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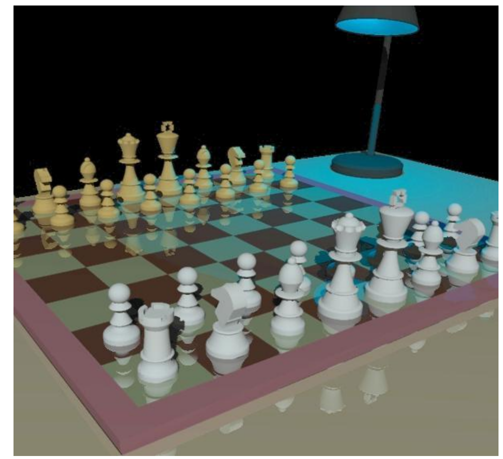

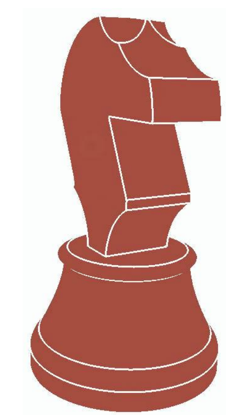

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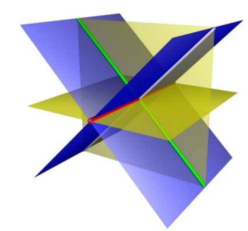

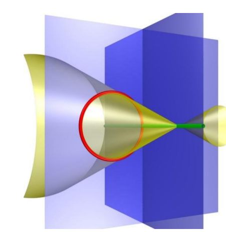

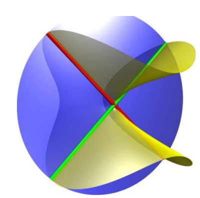

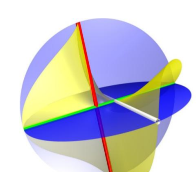

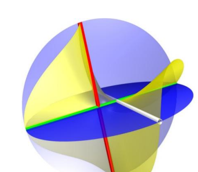

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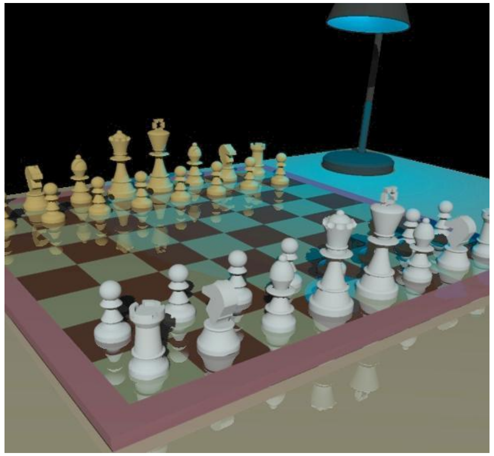

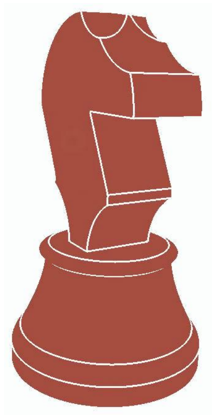

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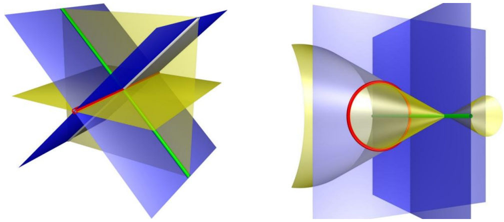

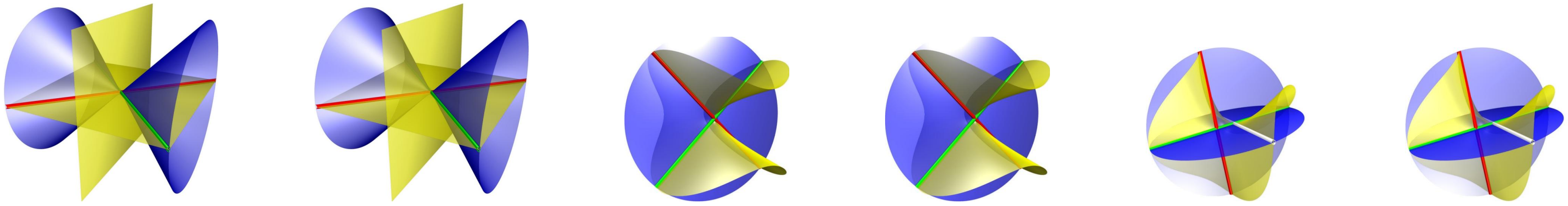

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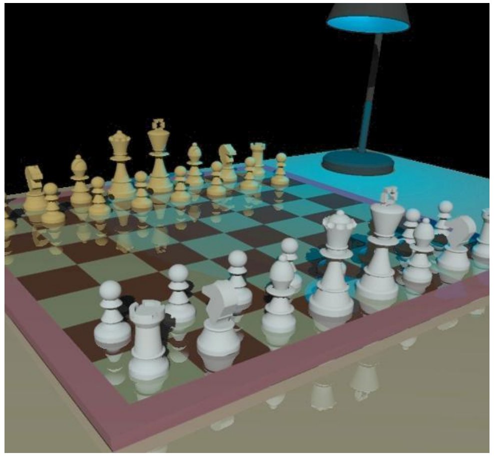

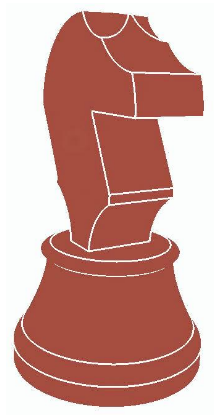

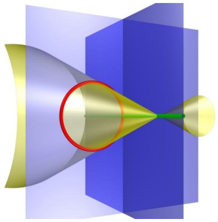

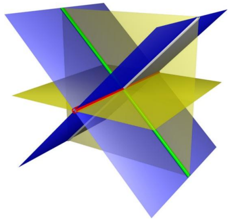

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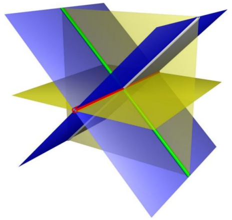

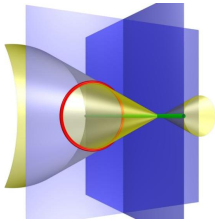

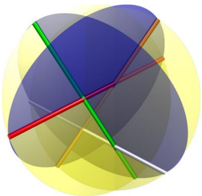

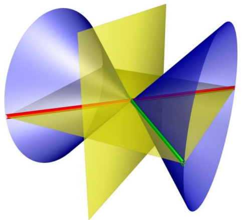

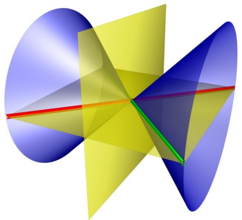

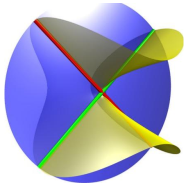

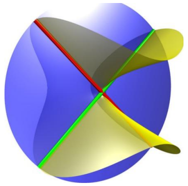

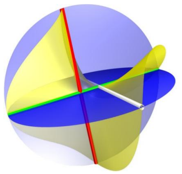

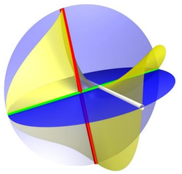

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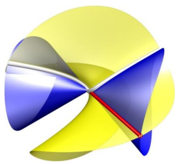

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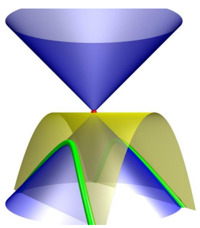

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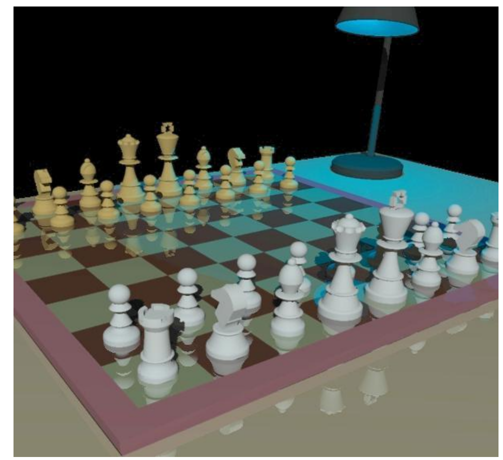

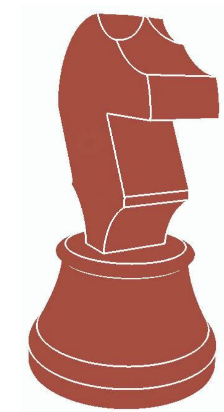

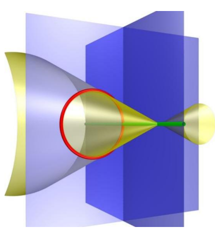

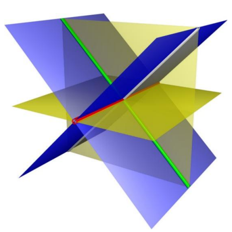

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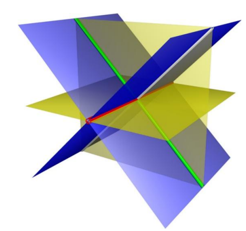

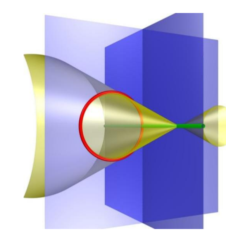

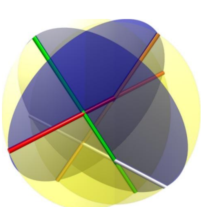

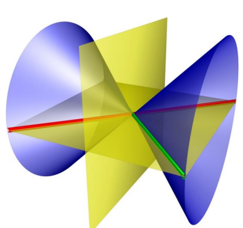

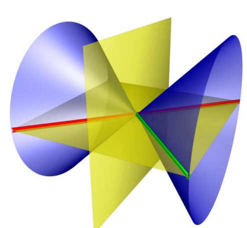

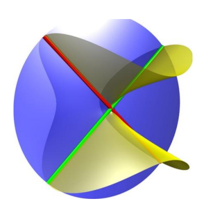

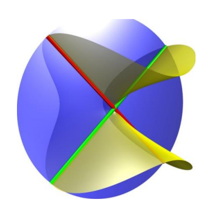

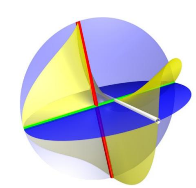

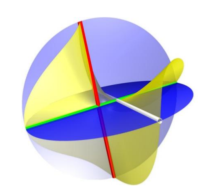

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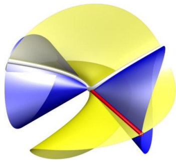

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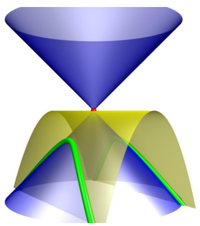

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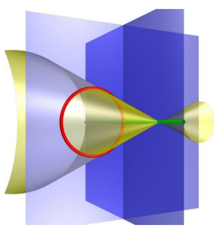

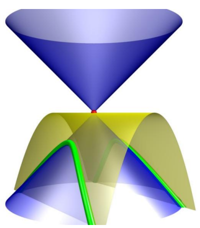

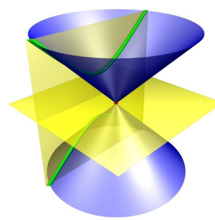

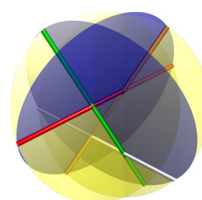

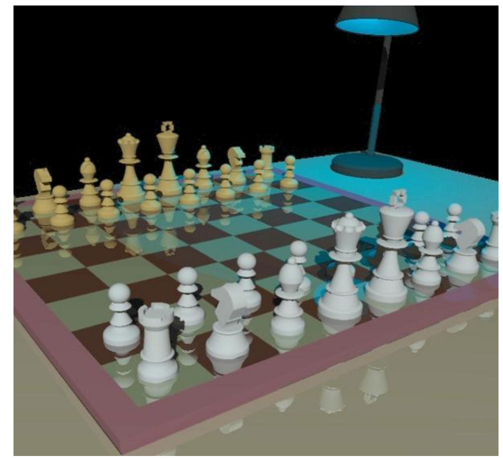

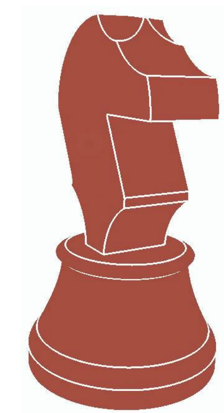

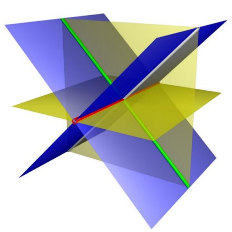

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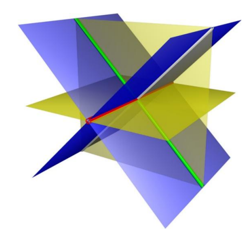

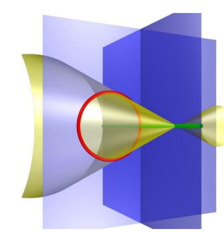

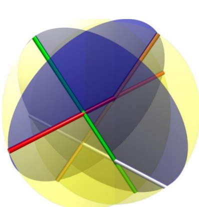

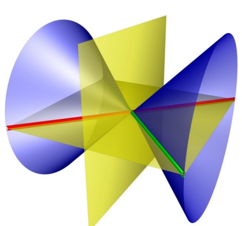

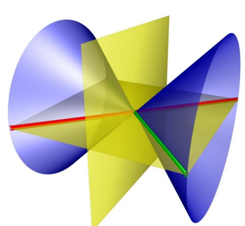

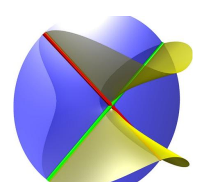

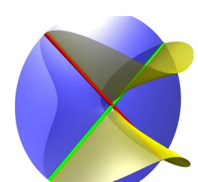

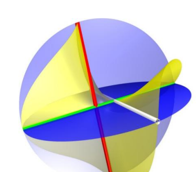

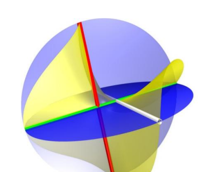

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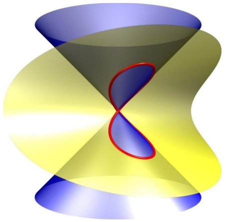

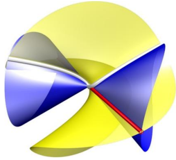

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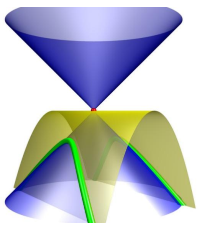

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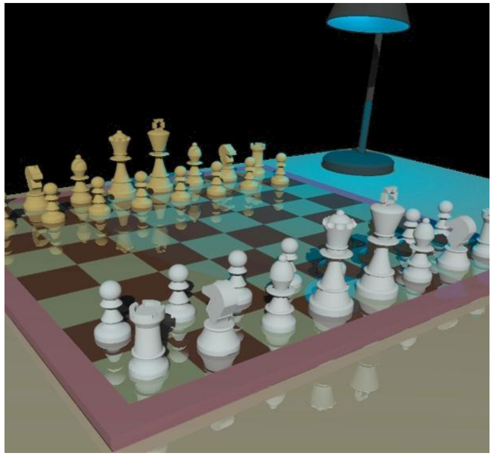

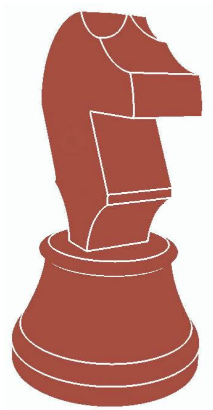

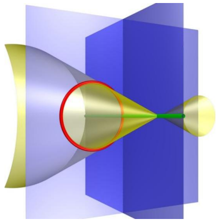

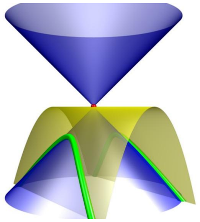

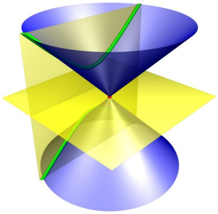

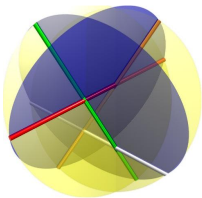

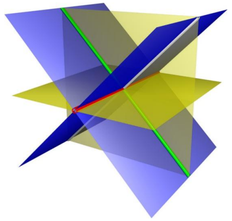

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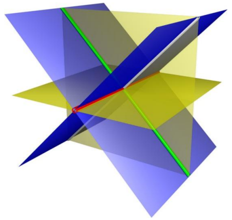

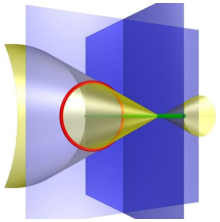

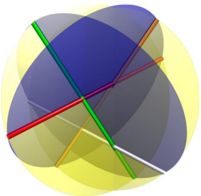

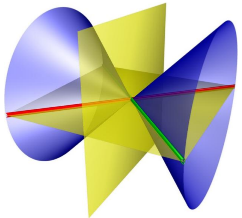

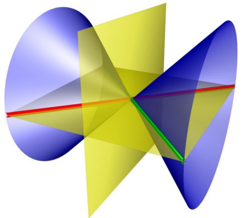

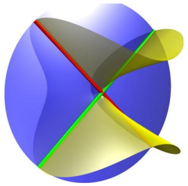

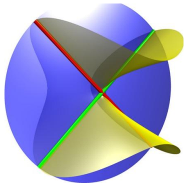

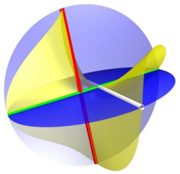

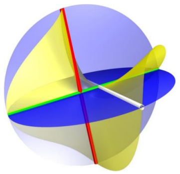

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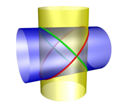 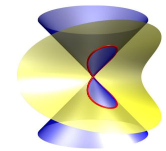 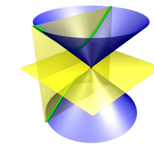 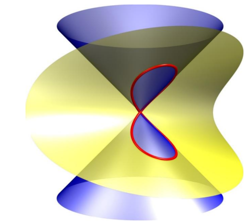

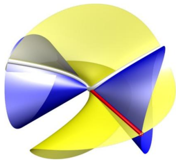

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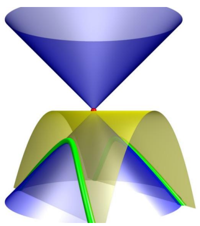

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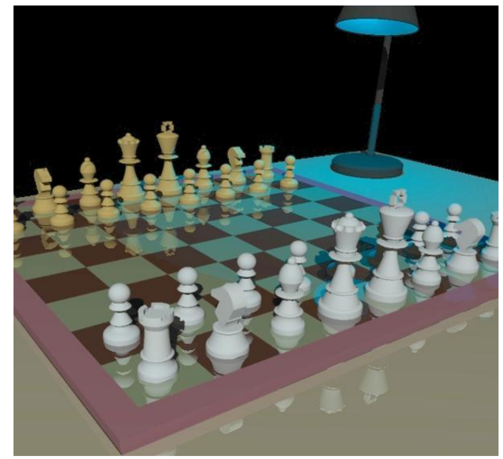 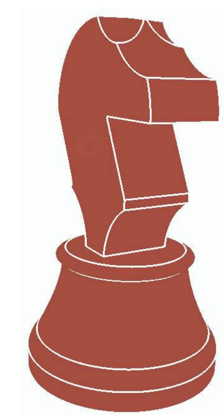

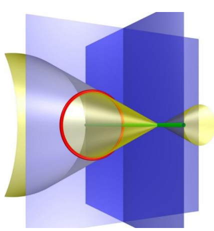 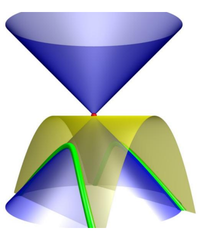 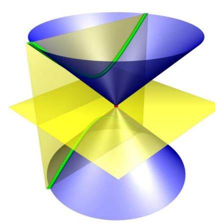 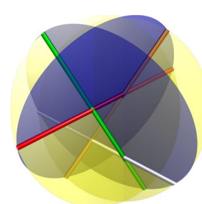

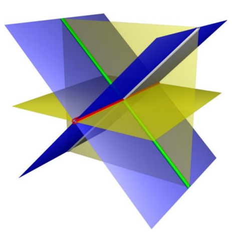

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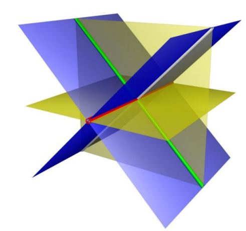 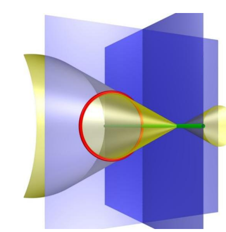

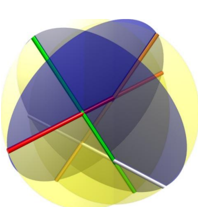 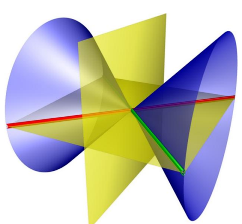 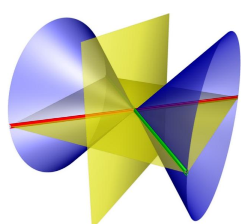 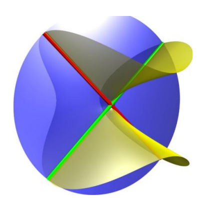 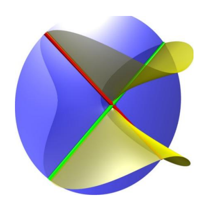 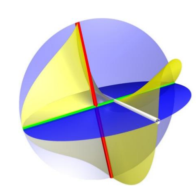 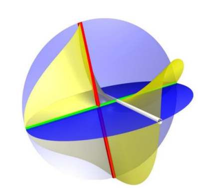

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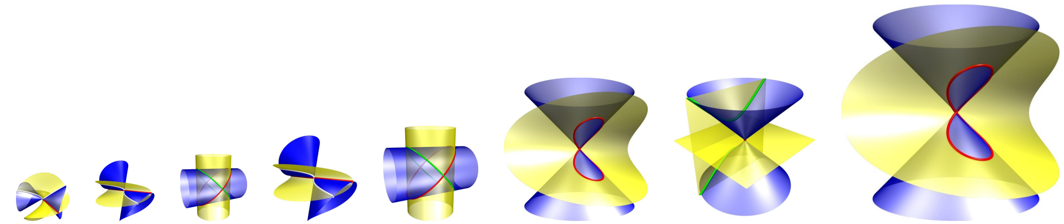

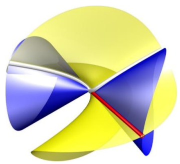

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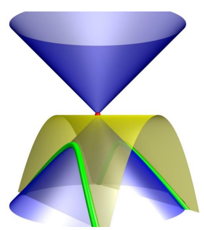

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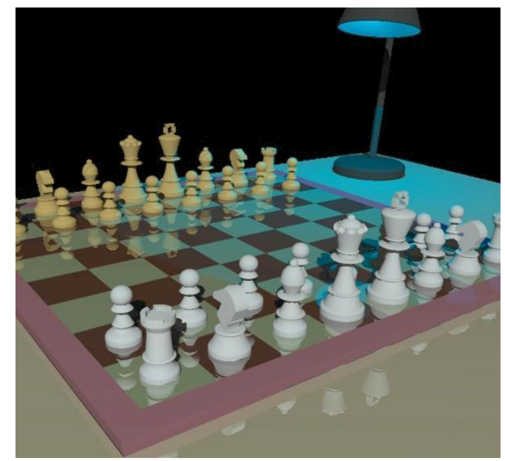

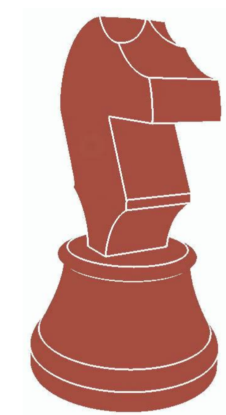

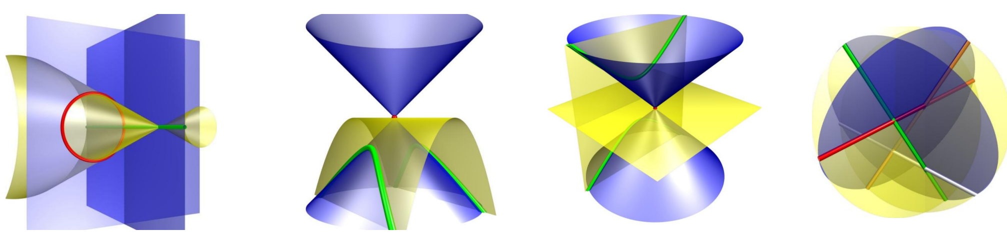

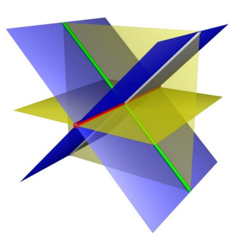

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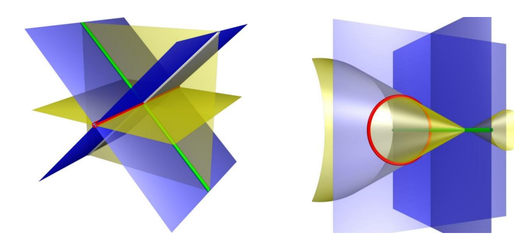

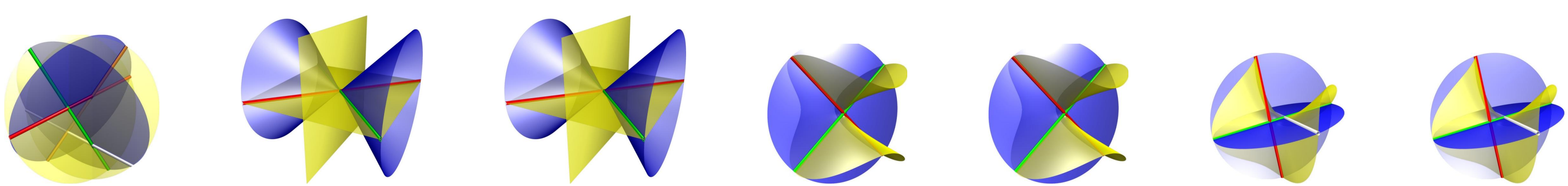

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Encoding of the orbits under these transformations

Work on pencils of quadrics.

First encoding with Segre's characteristic (discriminates between intersection types in $\mathbb{CP}^3$).

Characteristic polynomial of a pencil.

One quadric Q in $\mathbb{R}^3 \to$ symmetric 4×4 matrix M_Q.

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Result: Tabulation with over 40 cases, 26 intersection types in total, proof of exhaustivity.

Second issue: numerical rounding

The arithmetic on a computer uses bounded precision (32 bits, 64 bits, IEEE float norms, etc...).

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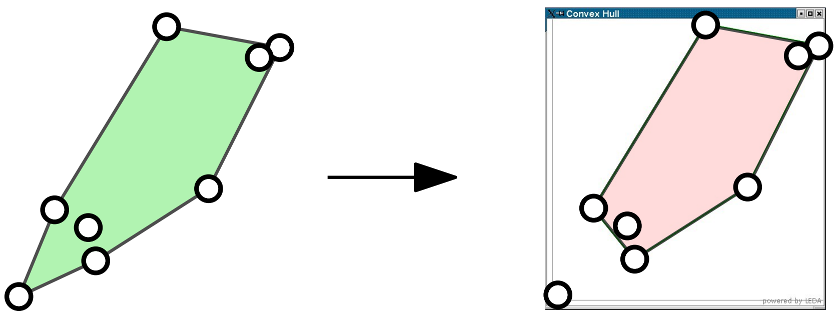

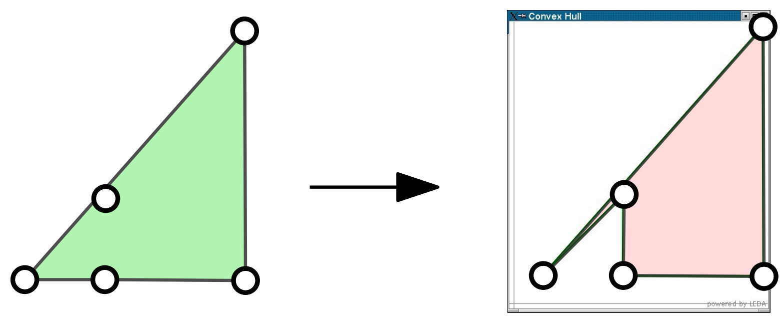

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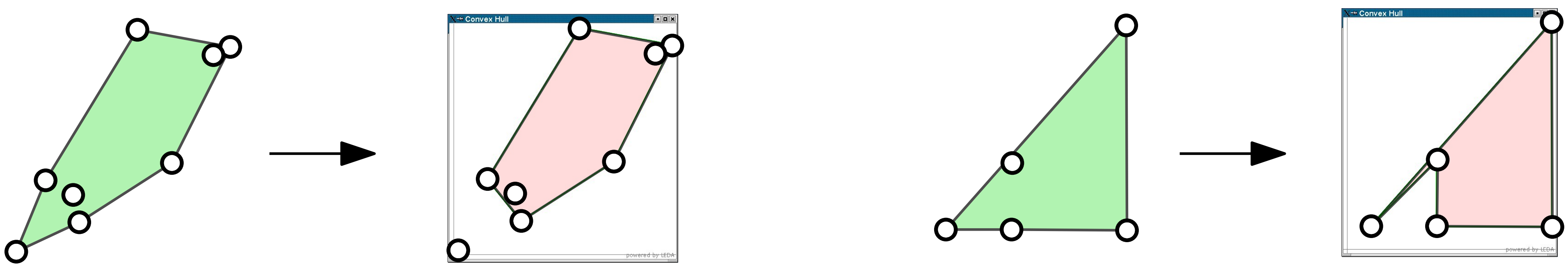
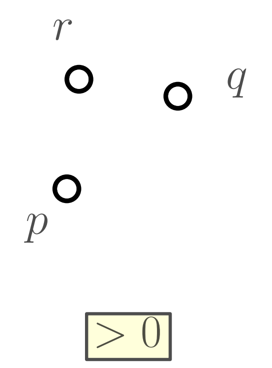


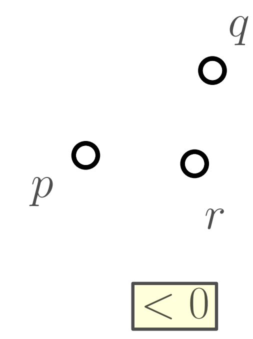
The problem: three points are nearly aligned, and the orientation predicates make inconsistent errors.

"Sometimes left, sometimes right".

A close look at that example

Orientation of (p, q, r) given by the sign of $\begin{vmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ 1 & 1 & 1 \end{vmatrix}$.





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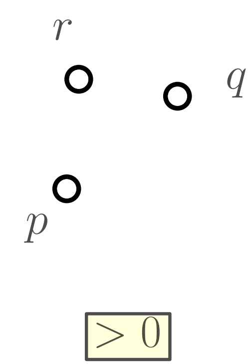

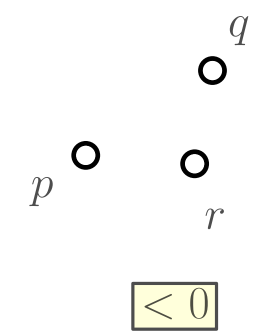


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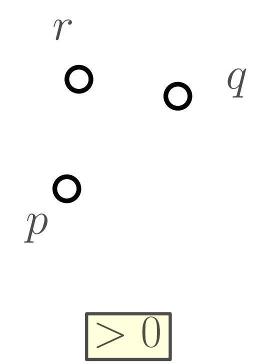

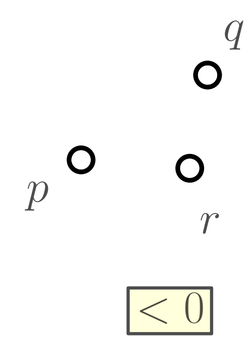


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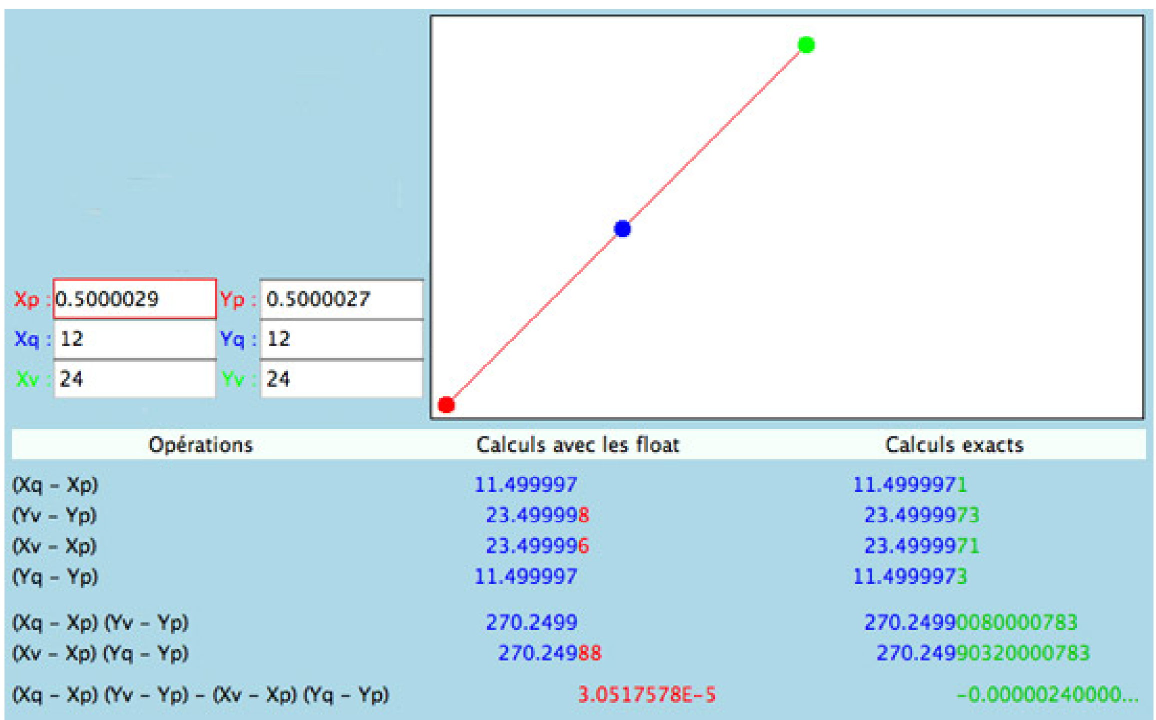

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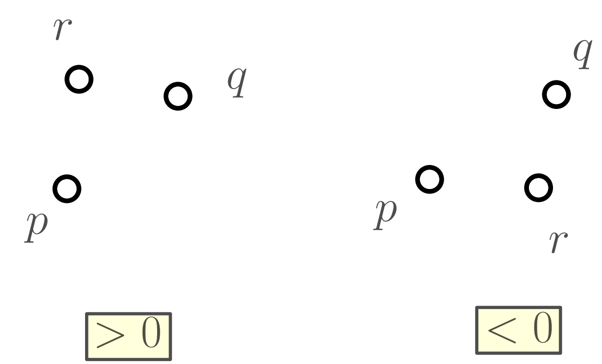


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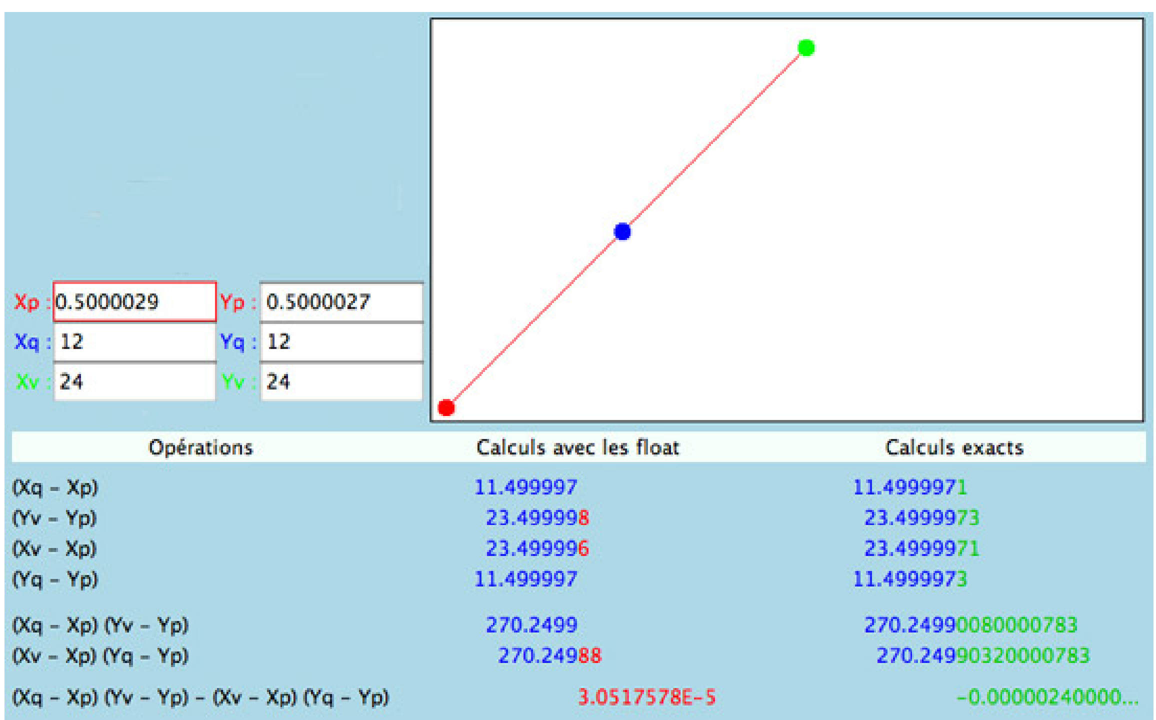

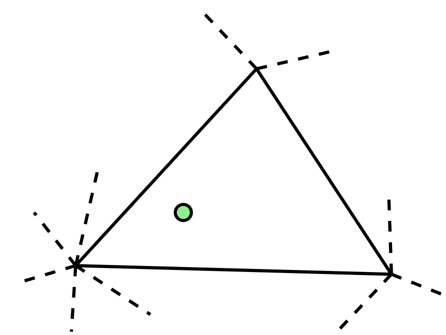

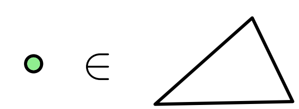

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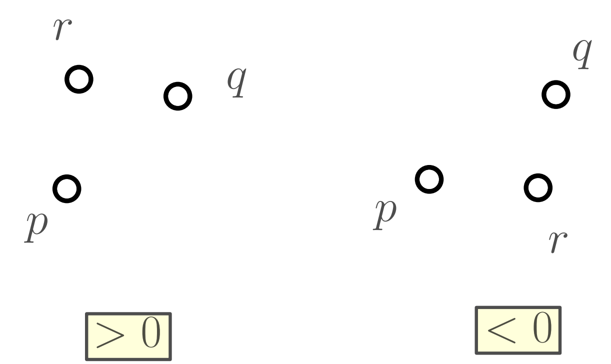


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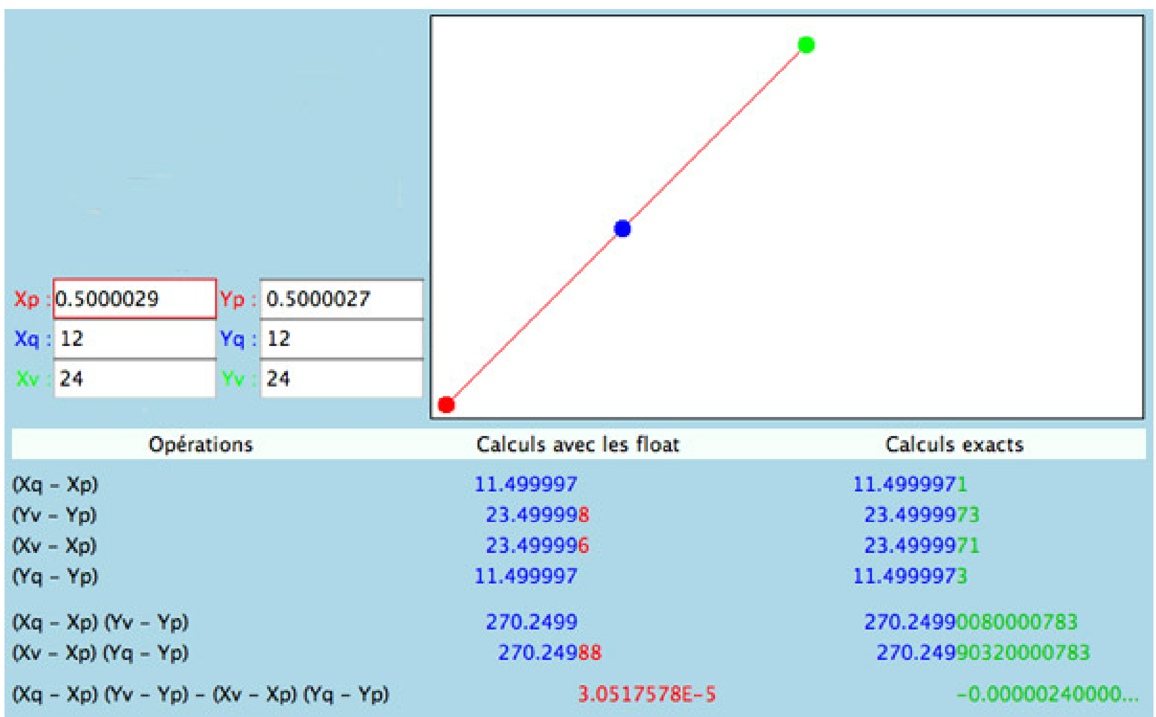

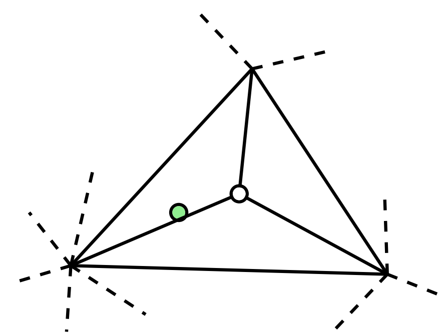

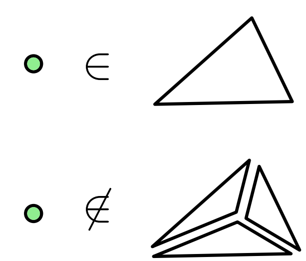

Consequences of numerical rounding

A "correct" code can make incorrect decisions. These errors are inconsistent.

Crash, infinite loops, smooth execution but wrong answer... which is the worse?

Can be hard to detect...

Interval arithmetic

Keep the precision bounded but keep track of the error.

A number is represented by an interval (reduced to a single element if precision is sufficient).

Define all operations on intervals.

$24 - 0.5000027 = 23.499998$ becomes $[24, 24] - [0.5000027, 0.5000027] = [23.49999, 23.50000]$.

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If the result of the computation is exactly 0 we will never have enough precision...
For those few cases, we need to be able to do the computations exactly.

Exact number types for integers, rational numbers, algebraic numbers.

Algebraic numbers

$\sqrt[5]{23}$ and $\sqrt[7]{25}$ are not integers, but we can still compare them exactly using integer arithmetic.

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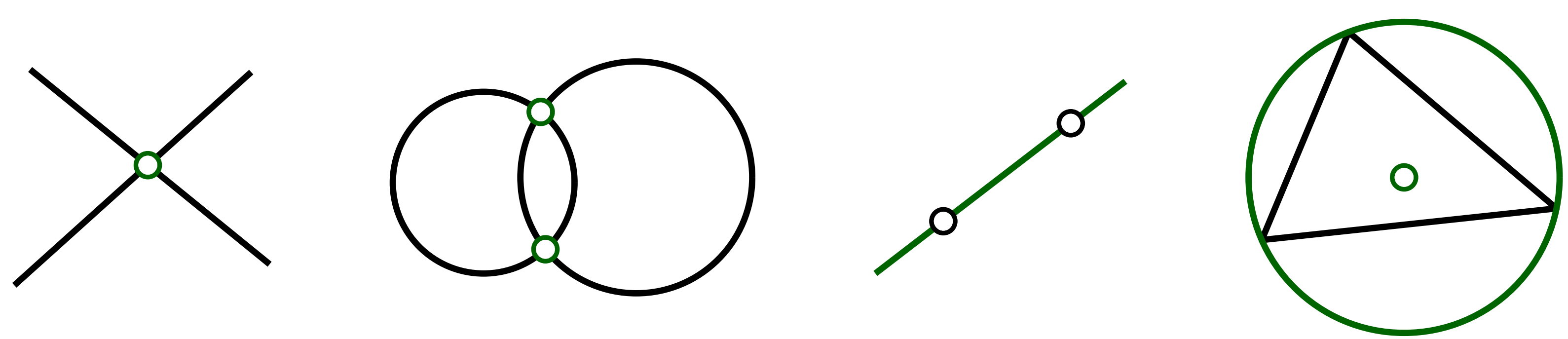
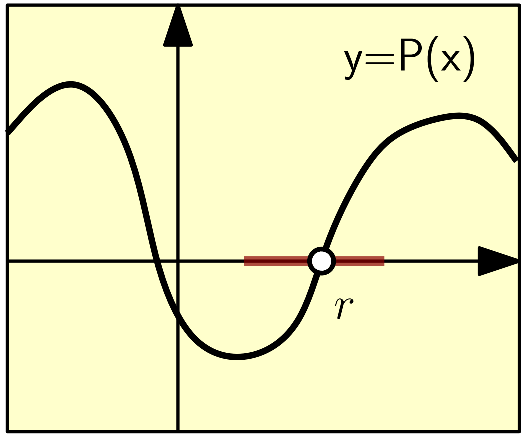
There are few algebraic numbers (ie countably many).

The result of most classical operations on geometric objects defined by integers
can be described using algebraic numbers.



Representing and manipulating algebraic numbers

An algebraic number can be represented by a polynomial (a family of integers) and an isolation interval.



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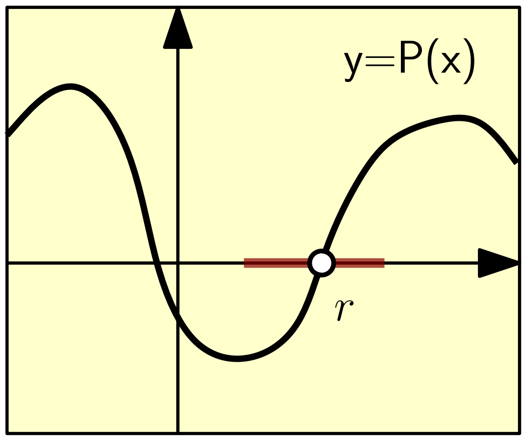

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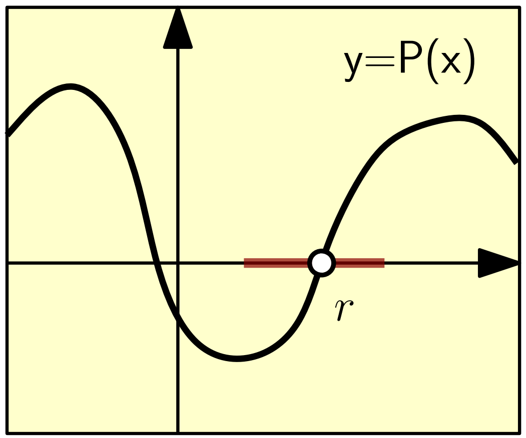

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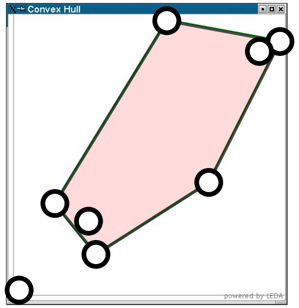
Implemented in the C/C++ CORE library.

Using algebraic numbers

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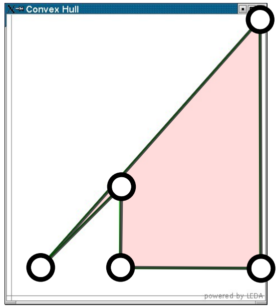
leads to



These problems can be avoided by using

```
Core::Expr xp,yp,xq,yq,xr,yr;
Orientation = sign((xq-xp)*(yr-yp)-(xr-xp)*(yq-yp));
```

Decision VS constructions

Distinguish between decision (for branching) and constructions.

Decisions are made by evaluating signs of polynomial in the input and can be filtered.

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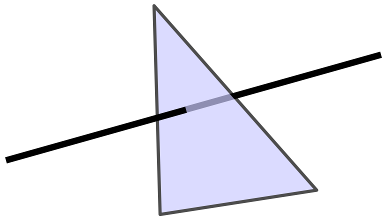

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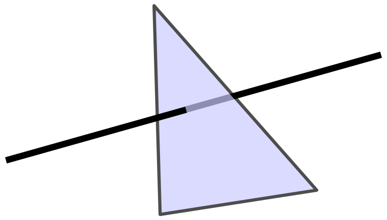

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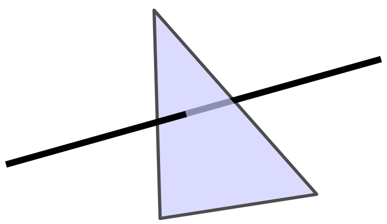

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Evaluate 3D orientations of quadruples of points

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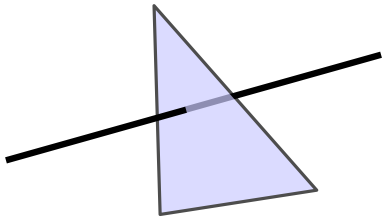

find intersection with plane, compute barycentric coordinates.
$\rightarrow$ evaluate the sign of polynomials of degree 6.

Evaluate 3D orientations of quadruples of points
$\rightarrow$ evaluate the sign of polynomials of degree 3.

Wrap-up: robustness

Treating degeneracies requires great care.

Numerical problems will arise.

If not treated properly, they produce crashes, infinite loops or wrong results.

Exact number types exist and are implemented. This is good enough for prototyping.

Reliability and efficiency are achieved by using good predicates

and filtering exact number type with interval arithmetic.

The End