Delaunay triangulations on orientable surfaces of low genus

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SoCG'16 - Boston

Outline

1. Introduction
2. 1- and 2- tori
3. Algorithm
4. This paper

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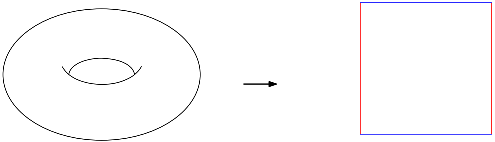
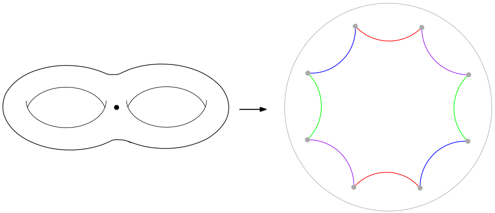
Which surfaces?

- 1 handle, flat torus



locally Euclidean metric

- 2 handles, Bolza surface



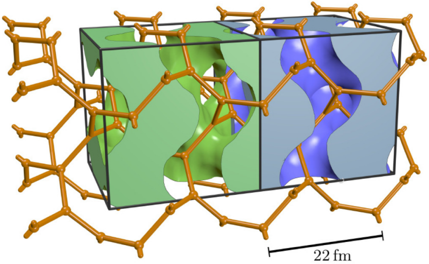
locally hyperbolic metric

Motivation

Applications - Examples

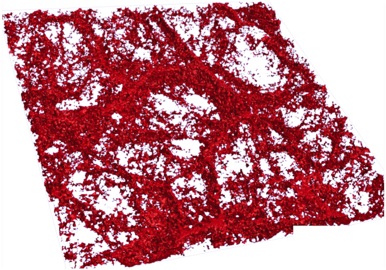
[3D] Flat torus

Tiny: Nuclear pasta



[Schuetrumpf, Klatt, Iida, Schröder-Turk *et al*]

Huge: Cosmic web



[van de Weijgaert *et al*]

Motivation

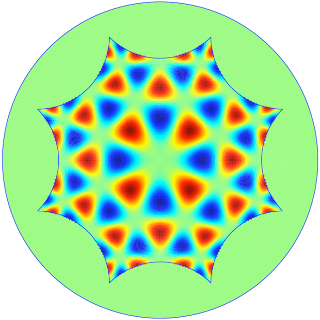
Applications - Examples

Bolza surface

Physics

[Sausset, Tarjus, Viot]

Neuromathematics



[Chossat, Faye, Faugeras]

Motivation

Algorithms / software for Delaunay triangulations

State-of-the-art:

- $\exists$ for the dD flat torus

  2d [Mazón, Recio], 3d [Dolbilin, Huson], dD [Caroli, *T.*]

  $\exists$ software 2d [Kruithof], 3d [Caroli, *T.*]

  CGAL

- $\nexists$ for the Bolza surface

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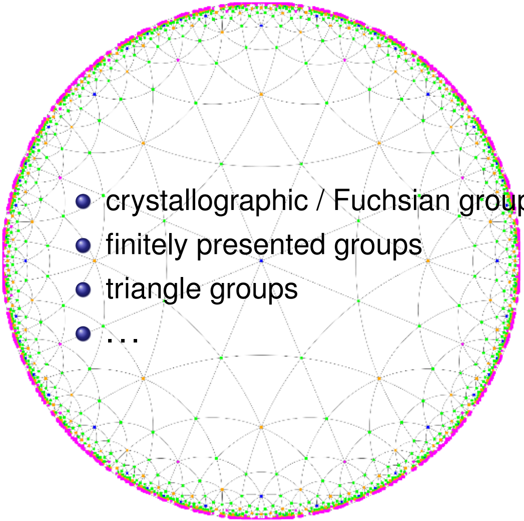
Goal:

Extend the standard incremental algorithm [Bowyer]

- easy to implement
- efficient in practice

Motivation

Beautiful groups



- crystallographic / Fuchsian groups
- finitely presented groups
- triangle groups
- ...

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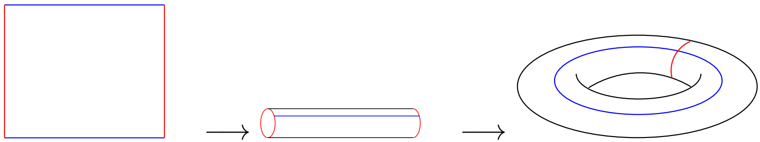
The Flat Torus

$\mathbb{T}^2 = \mathbb{R}^2 / G$

locally Euclidean metric

$G = < t_x, t_y >$



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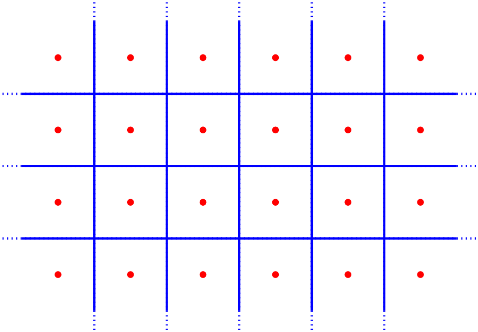
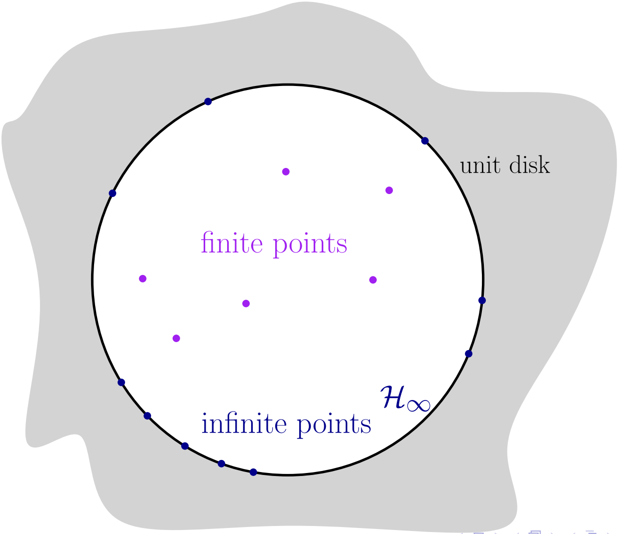


same $\forall p \in \mathbb{R}^2$

Hyperbolic plane $\mathbb{H}^2$

Poincaré disk conformal model



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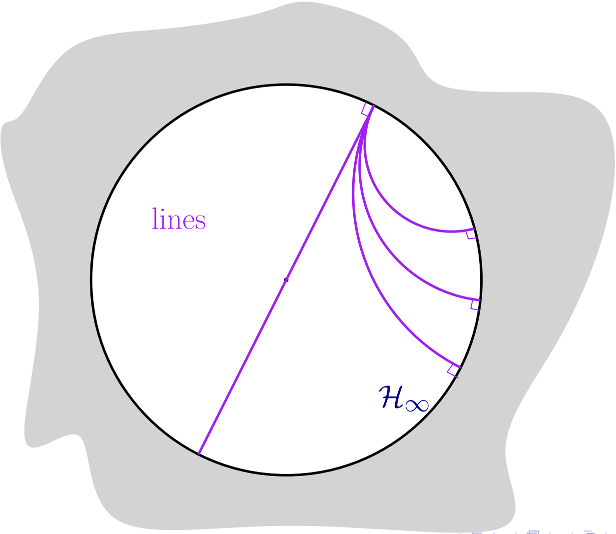

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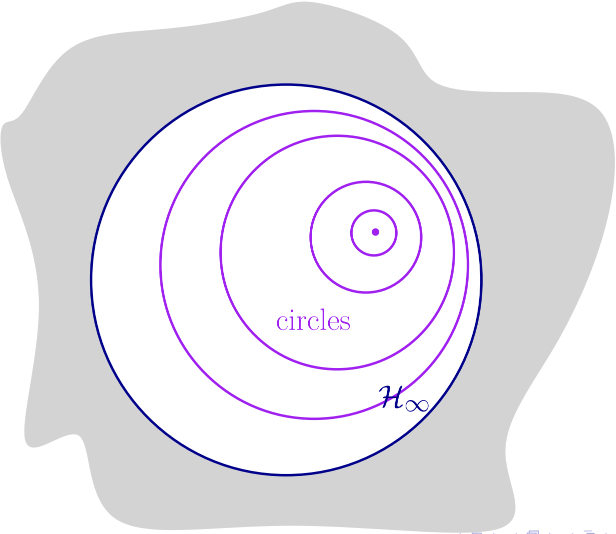
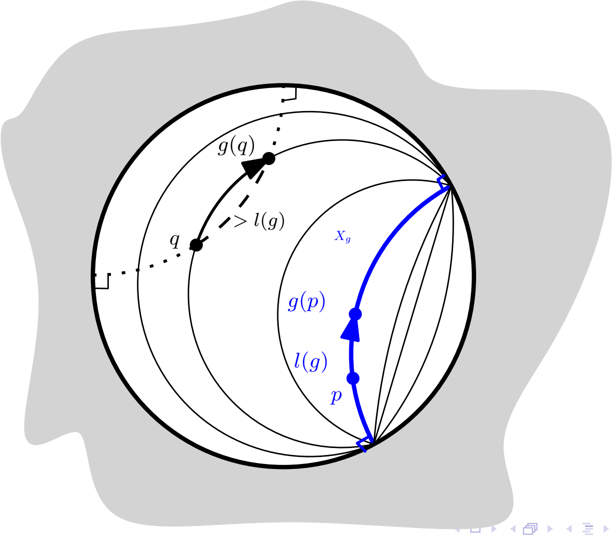


Hyperbolic plane $\mathbb{H}^2$

Hyperbolic translations



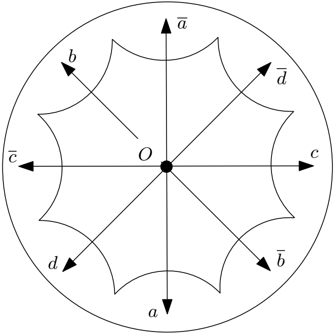
The Bolza surface

$\mathcal{M} = \mathbb{H}^2/\mathcal{G}$

locally hyperbolic metric

$\mathcal{G} = \langle a, b, c, d \mid abcd\overline{a}\overline{b}\overline{c}\overline{d} \rangle$

translations do not commute



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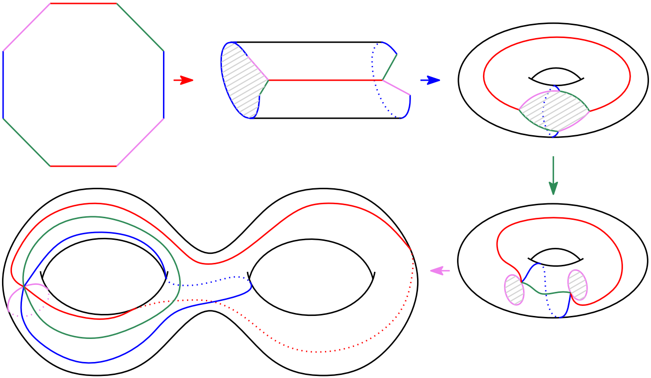

Thanks to Iordan Iordanov

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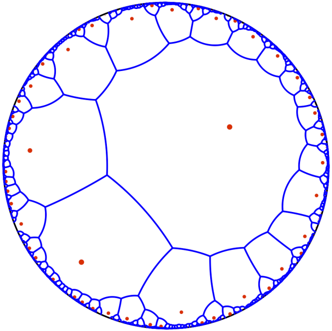

depends on p generic: 18 sides [Näätänen]

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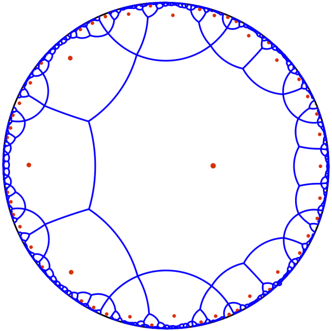

depends on p 14 sides [Näätänen]

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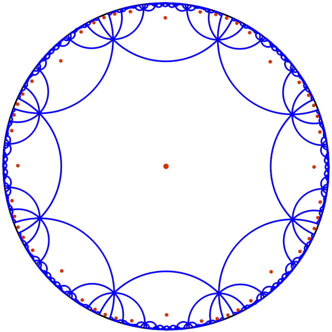

depends on p

8 sides [Näätänen]

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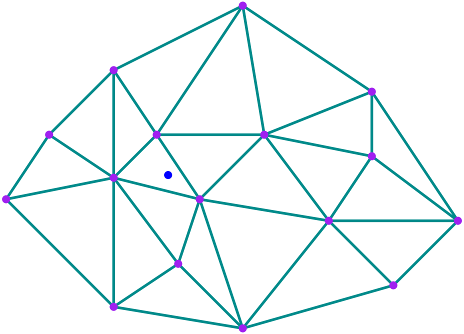
Incremental algorithm [Bowyer]

$\mathbb{R}^2$



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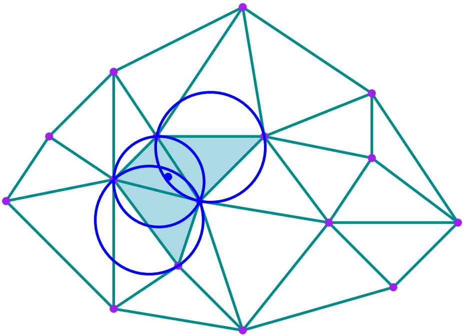

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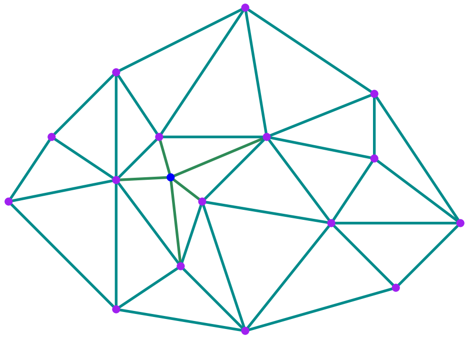

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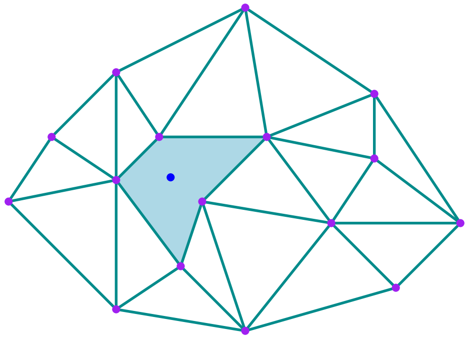
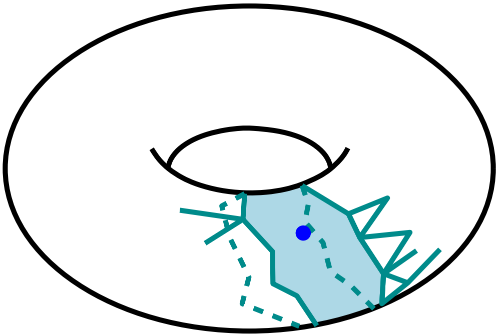


the conflict region is a topological disk

Incremental algorithm [Bowyer]

On a surface

the conflict region is not always a topological disk



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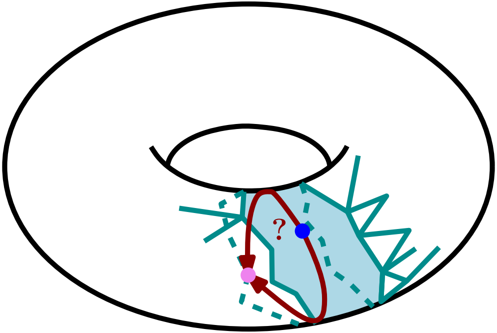

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Sufficient condition

M manifold,

systole(M) = least length of a non-contractible loop on M

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$\mathcal{P}$ set of points

If

$$\text{systole}(M) > 2 \cdot \text{diameter(largest empty disk)}(\mathcal{P})$$

then

the graph of edges of $DT_M(\mathcal{P})$ has no cycle of length ≤ 2

systole(M) $> 2 \cdot$ diameter(largest empty disk)($\mathcal{P}$)

Use a sequence of covering spaces M_k of M

$\simeq$ *a sequence of normal subgroups of* $\mathcal{G}$

$\rightsquigarrow$ increase systole

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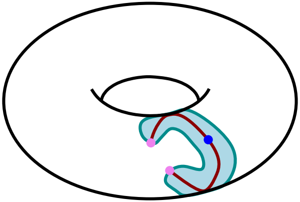

the conflict region is always a disk

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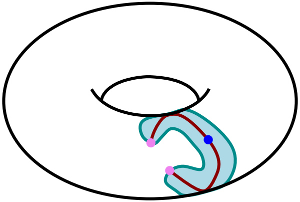

the conflict region is always a disk

Reduce the number of sheets while points are inserted

Covering spaces

- Tool: construction of 2^k-sheeted covering spaces
  construction of normal subgroups of $\mathcal{G}$ of index 2^k

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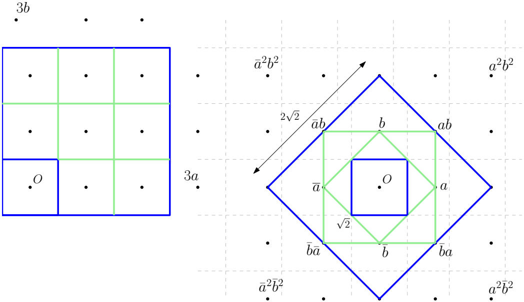

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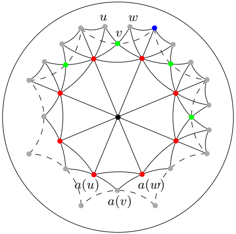
  - $\geq$ 32 sheets (argument: areas)
  - $\leq$ 128 sheets (GAP assisted proof)
  - practical approach: 1 sheet + 14 “dummy” points



typo on page 13: 48 $\rightarrow$ 32

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- general hyperbolic closed surfaces: existence

Future work

Bolza surface

- algebraic issues
- implementation
- tighten the gap $32 \leftrightarrow 128$

Higher genus

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Higher genus

Thank you for your attention