Triangulating the Real Projective Plane

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Projét Géométrica
Definition

The real projective plane $\mathbb{P}^2$ is a set of points in $\textit{one-to-one correspondence}$ with the lines of a vector space $\mathbb{V}^3$ in $\mathbb{R}^3$, with the points in $\mathbb{P}^2$ linearly dependent iff the corresponding lines of $\mathbb{V}^3$ are linearly dependent.
What is the Real Projective Plane?

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Figure: The sphere model of $\mathbb{P}^2$
What is the Real Projective Plane?

- $p = (x, y, z) = \lambda(x, y, z)$.

Triangulating the Real Projective Plane
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![Triangulating the Real Projective Plane](image)
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Suppose \( \mathcal{M} \) is a map on \( \mathbb{P}^2 \) and \( e \) is an edge of \( \mathcal{M} \).
Definition

A triangulation of $\mathbb{P}^2$ is a simplicial complex such that each face is bounded by a 3-cycle.

Suppose $\mathcal{M}$ is a map on $\mathbb{P}^2$ and $e$ is an edge of $\mathcal{M}$.

- **Contraction of $e$ in $\mathcal{M}$** is to remove $e$ and identify its two endpoints.
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- *Contraction of $e$* in $\mathcal{M}$ is to remove $e$ and identify its two endpoints.
- Contraction is allowed only if the resulting graph $\mathcal{H}$ is a simplicial complex.
A \textit{triangulation} of $\mathbb{P}^2$ is a simplicial complex such that each face is bounded by a 3-cycle.

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- \textit{Contraction} of $e$ in $\mathcal{M}$ is to remove $e$ and identify its two endpoints.
- Contraction is allowed only if the resulting graph $\mathcal{H}$ is a simplicial complex.
- If $\mathcal{M}$ has no contractible edge, then $\mathcal{M}$ is called \textit{irreducible}. 

\textbf{Triangulating the Real Projective Plane}
The real projective plane $\mathbb{P}^2$ admits exactly two irreducible triangulations.
Theorem (Barnette, 1982)

The real projective plane $\mathbb{P}^2$ admits exactly two irreducible triangulations.

![The two irreducible triangulations of $\mathbb{P}^2$](image)

**Figure:** The two irreducible triangulations of $\mathbb{P}^2$
Our Goal

Design an algorithm:

Input: A point set $P = \{p_1, p_2, \ldots, p_n\}$.

Output: A triangulation of $P$ if one exists.

Basic predicates needed:
- When does a triangulation of $P$ exist?
- An orientation test to distinguish "interior" from "exterior" of a triangle.
- Point location.

Triangulating the Real Projective Plane
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No computational results on \( \mathbb{P}^2 \) exist!
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Outline of the Talk

- Design an “in-triangle” test.
- Constructing a triangulation of $\mathbb{P}^2$
- Point location
- Algorithm
- Conclusion and Open Problems
What’s really “inside”?

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- The interior of a triangle can be unambiguously defined if we associate a distinguishing plane with it.
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- The real projective plane with a disk cut out is a \textit{Möbius band}.
- The interior of a triangle can be unambiguously defined if we associate a \textit{distinguishing plane} with it.
We define a many-one mapping \( s : \mathbb{P}^2 \rightarrow \mathbb{R}^3 \) from points in \( \mathbb{P}^2 \) to points in \( \mathbb{R}^3 \) as follows:

\[
\begin{align*}
s(x, y, z) &= \begin{cases} 
(1, xz, yz) & \text{if } z \neq 0, \\
(0, 1, x) & \text{if } z = 0, x \neq 0, \\
(0, 0, 1) & \text{if } z = 0, x = 0.
\end{cases}
\end{align*}
\]

Here, \( s_i = s(x_i, y_i, z_i) \) for \( i = 0, 1, 2 \), and \( s = s(x, y, z) \).
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We define a many-one mapping $s : \mathbb{P}^2 \rightarrow \mathbb{R}^3$ from points in $\mathbb{P}^2$ to points in $\mathbb{R}^3$ as follows:

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\end{cases}$$

Suppose $a = (x_0, y_0, z_0)$, $b = (x_1, y_1, z_1)$, $c = (x_2, y_2, z_2)$ and $p = (x, y, z)$, $p$ lies inside $\triangle abc$ if:
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$$\text{sign} s_0 + \text{sign} s_1 + \text{sign} s_2 = \pm 3$$

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and \( p = (x, y, z) \), \( p \) lies inside \( \triangle abc \) if:

\[
\text{sign} \left| \begin{array}{c}
s_0 \\
s_1 \\
s_2
\end{array} \right| + \text{sign} \left| \begin{array}{c}
s_1 \\
s_2 \\
s_0
\end{array} \right| = \pm 3
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Here \( s_i = s(x_i, y_i, z_i) \) for \( i = 0, 1, 2 \), and \( s = s(x, y, z) \).
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$p$ lies on the boundary of $\triangle abc$ if:

$$\text{sign} \left| \begin{array}{c} s_0 \\ s_1 \\ s \end{array} \right| + \text{sign} \left| \begin{array}{c} s_1 \\ s_2 \\ s \end{array} \right| + \text{sign} \left| \begin{array}{c} s_2 \\ s_0 \\ s \end{array} \right| = \pm 2$$

Triangulating the Real Projective Plane
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$$M = \begin{bmatrix}
0 & \frac{-(\beta^2+\gamma^2)}{\sqrt{(\beta^2+\gamma^2)(\alpha^2+\beta^2+\gamma^2)}} & \frac{\alpha}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} \\
\frac{\gamma}{\sqrt{\beta^2+\gamma^2}} & \frac{\alpha\beta}{\sqrt{(\beta^2+\gamma^2)(\alpha^2+\beta^2+\gamma^2)}} & \frac{\beta}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} \\
\frac{-\beta}{\sqrt{\beta^2+\gamma^2}} & \frac{\alpha\gamma}{\sqrt{(\beta^2+\gamma^2)(\alpha^2+\beta^2+\gamma^2)}} & \frac{\gamma}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}}
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0 & -1 & 0
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$$
\mathcal{M} = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
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\end{bmatrix}
$$
We intend to design an incremental algorithm that first constructs a triangulation of $P_2$. Adds a point $p \in P$. Identifies the enclosing triangle $\triangle abc$ of $p$ and "grows" the triangulation by making $p$ adjacent to $a$, $b$, and $c$.

Questions:
How to get the "initial" triangulation of $P_2$?
Does there exist a subset $S \subset P$ which can be used to construct this initial triangulation?
Some Motivation

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A New Idea!

**Definition**

An *incidence structure* \( I = \{ \mathcal{P}, \mathcal{L}, \Pi \} \) consists of a set of points \( \mathcal{P} \), a set of lines \( \mathcal{L} \), and a set of incidences \( \Pi \) between points in \( \mathcal{P} \) and lines in \( \mathcal{L} \).
A New Idea!

Triangulating the Real Projective Plane
A New Idea!

\[ \mathcal{P} = \{1, 2, 3, 4, 5, 6, 7\} \]
A New Idea!

\[ \mathcal{P} = \{1, 2, 3, 4, 5, 6, 7\} \]
\[ \mathcal{L} = \{a = 1261, b = 3253, c = 1451, d = 3463, e = 7247, f = 7317\} \]
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$\Pi = \{1c, 1d, 1f, 2a, 2b, 2e, 3b, 3d, 3f, 4c, 4d, 4e, 5b, 5c, 6a, 6d, 7e, 7f\}$
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Triangulating the Real Projective Plane
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Lemma

If among every set of four points in $\mathcal{P}$ at least three are collinear, then at least $(n - 1)$ points in $\mathcal{P}$ are collinear.
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Corollary

If no set of $(n - 1)$ points in $\mathcal{P}$ are collinear, then there exists a set of four points no three of which are collinear.
A New Idea!

Such a set of four points is called a $K_4$-quadrangulation.
A New Idea!

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**Lemma**

A $K_4$-quadrangulation can be used to construct a triangulation of $\mathbb{P}^2$. 

Triangulating the Real Projective Plane
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![Diagram](image.png)
A New Idea!

- It now becomes possible to associate a distinguishing plane with each triangle.

Triangulating the Real Projective Plane
A New Idea!

- It now becomes possible to associate a *distinguishing plane* with each triangle.
- The procedure described is still incomplete because the “pseudo-points" $p, q, r$ may not be in $\mathcal{P}$.
Definition
The set of triangles incident to exactly one pseudo-point is called a region.
A New Idea!

Definition

The set of triangles incident to exactly one pseudo-point is called a *region*.
Lemma

If there exists a $K_4$-quadrangulation $\mathcal{A}$ such that at least two points in $\mathcal{P}$ lie in different regions of $\mathcal{A}$, then it is possible to triangulate $\mathbb{P}^2$. 

Triangulating the Real Projective Plane
Lemma

If there exists a $K_4$-quadrangulation $A$ such that at least two points in $P$ lie in different regions of $A$, then it is possible to triangulate $\mathbb{P}^2$.

Such a $K_4$-quadrangulation is called a canonical set.
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A set of points $S$ is in *general position* if no three points in $S$ are collinear.
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A set of points $S$ is in general position if no three points in $S$ are collinear.

Lemma

If at least six points in $\mathcal{P}$ are in general position, then there always exists a canonical set.
Triangulating the Real Projective Plane
Triangulating the Real Projective Plane
A New Idea!
Computing triangulations of $\mathbb{P}^2$

**Theorem**

Given a point set $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ with at least six points in general position, it is always possible to construct a triangulation of $\mathbb{P}^2$. 

Triangulating the Real Projective Plane
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### Algorithm

1: Find a set $S = \{1, 2, 3, 4, 5, 6\}$ of six points such that no three points in $S$ are collinear.
## Algorithm

1. **Find a set** $S = \{1, 2, 3, 4, 5, 6\}$ **of six points such that no three points in** $S$ **are collinear.**
2. **Construct a projective triangulation with the set** $S$. **Associate distinguishing planes with every triangle of the triangulation.**
### Algorithm

1. **Find a set** $S = \{1, 2, 3, 4, 5, 6\}$ **of six points such that no three points in** $S$ **are collinear.**
2. **Construct a projective triangulation with the set** $S$. **Associate distinguishing planes with every triangle of the triangulation.**
3. **for all points** $p \in \mathcal{P} \setminus S$ **do**
4. **Identify the triangle** $\triangle abc$ **in which** $p$ **lies.**
5. **Make** $p$ **adjacent to the vertices** $a$, $b$ **and** $c$. **Make the distinguishing plane of** $\triangle apb$, $\triangle bpc$, **and** $\triangle cpa$ **the same as that for** $\triangle abc$.  
6. **end for**
Computing triangulations of $\mathbb{P}^2$

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5: Make $p$ adjacent to the vertices $a, b$ and $c$. Make the distinguishing plane of $\triangle apb, \triangle bpc, \text{ and } \triangle cpa$ the same as that for $\triangle abc$.
6: end for
7: return (triangulation of $\mathbb{P}^2$).
Computing triangulations of $\mathbb{P}^2$

- Step 1 takes $O(n^2)$ steps.
- Step 2 takes $O(1)$ steps.
- "Walking" in a triangulation takes $O(n)$ steps.
- The `for` loop takes a total of $O(n^2)$ steps.
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- Worst-case time complexity is $O(n^2)$ steps.
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Problems like the *Minimum Weight Triangulation* (NP-Hard for 2-d, [1]) or the *Minmax Length Triangulation* (solvable in $O(n^2)$ in 2-d, [2]) now have meaning on the real projective plane.


Questions?

Triangulating the Real Projective Plane
Lemma

If among every set of four points in $P$ at least three are collinear, then at least $(n - 1)$ points in $P$ are collinear.
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