From triangles to curves

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Warning

- focus on practical methods
- non exhaustive, biased

mostly (not only)
Warning

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- non exhaustive, biased

mostly (not only)

“Commercial”: ECG book coming out soon...
Warning

- focus on practical methods
- non exhaustive, biased
- mostly (not only)

Advice to people having some knowledge of Computer Algebra: you may leave the room
- non technical, superficial...
Circles are never far from triangles
Construction of curves from lines

Parabola: smooth connection between line segments
Construction of curves from lines

Parabola: smooth connection between line segments
Construction of curves from lines
Triangles and curves

[Florence, 1997]
Triangular period
Curves already appear for linear input

Bisecting curve

2D line segments
arcs of parabolas
Curves already appear for linear input

Voronoi diagram
2D line segments
arcs of parabolas

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Curves already appear for linear input

Voronoi diagram

3D line segments

patches of quadric surfaces
More generally:

manipulations of *algebraic curves and surfaces*
More generally:

manipulations of *algebraic curves and surfaces*

Only considered here

*Exact Geometric Computation*  

[Yap][... ]
Why we should not be afraid of Computer Algebra

- useful
- interesting
- not so hard to understand
Why we should not be afraid of Computer Algebra

- useful
- interesting
- not so hard to understand
- people are nice
Why we should not be afraid of Computer Algebra

trying to convince myself...

- useful
- interesting
- not so hard to understand (?)
- some people are nice
One tool: Resultant

Resultant of a system of polynomial equations

= necessary and sufficient condition such that it has a root.
One tool: Resultant

Resultant of a system of polynomial equations

= necessary and sufficient condition such that it has a root.

How to compute the resultant? hard problem
Sylvester resultant

Univariate case

\[
\begin{cases}
   P &= a_0 x^m + \cdots + a_m \\
   Q &= b_0 x^n + \cdots + b_n
\end{cases}
\]

\(a_0 \neq 0, b_0 \neq 0, m > n,\)

coefficients in a field \(\mathbb{K}\) (algebraically closed).
Sylvester resultant

\[
\begin{align*}
\{ & P = a_0 x^m + \cdots + a_m \\
& Q = b_0 x^n + \cdots + b_n
\}
\end{align*}
\]

Sylvester resultant =

\[
\begin{pmatrix}
a_0 & b_0 \\
0 & a_0 \\
& \ddots \\
& & a_1 \\
& & & b_1 \\
& & & & \ddots \\
& & & & & b_0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
a_0 \\
a_1 \\
& \ddots \\
& & a_m \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
& b_0 \\
& b_1 \\
& & \ddots \\
& & & b_n \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
& & & & a_m \\
& & & & b_n \\
\end{pmatrix}
\]
Sylvester resultant

\[
\begin{align*}
P &= a_0 x^m + \cdots + a_m \\
Q &= b_0 x^n + \cdots + b_n
\end{align*}
\]

Sylvester resultant =

\[
\begin{array}{cccc}
a_0 & a_1 & \cdots & b_0 \\
a_1 & a_0 & \cdots & b_1 \\
\vdots & \vdots & \ddots & \vdots \\
a_m & \cdots & \cdots & b_n \\
\end{array}
\]

= 0 iff

\( P \) and \( Q \) have a common root in \( \mathbb{K} \).
Demystifying Resultant - I

\[
\begin{align*}
\left\{ \begin{array}{c}
ax + by - c &= 0 \\
 dx + ey - f &= 0
\end{array} \right.
\end{align*}
\]

seen as: \(x\) unknown, \(y\) parameter
Demystifying Resultant - I

\[
\begin{align*}
\begin{cases}
ax + by - c &= 0 \\
 dx + ey - f &= 0
\end{cases}
\end{align*}
\]

seen as: \(x\) unknown, \(y\) parameter

Sylvester Resultant \(= \begin{vmatrix} a & d \\ by - c & ey - f \end{vmatrix} \)
Demystifying Resultant - I

\[
\begin{align*}
{\begin{align*}
ax + by - c &= 0 \\
dx + ey - f &= 0
\end{align*}}
\]

seen as: \(x\) unknown, \(y\) parameter

Sylvester Resultant

\[
\begin{vmatrix}
a & d \\
by - c & ey - f
\end{vmatrix}
\]

\[= a(ey - f) - d(by - c)\]

Boils down to eliminate \(x\)
$p, q, s$ three points in the plane, $t$ a fourth point.

Is $t$ lying on the circle $C_{pq} s$?
Demystifying Resultant - II

$C_{pq}$ on $C_{pq}$

$C_{pq}$ center $(x_c, y_c)$ radius $r$

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

iff

$$\begin{align*}
2x_p x_c + 2y_p y_c + (r^2 - x_c^2 - y_c^2) - (x_p^2 + y_p^2) &= 0 \\
2x_q x_c + 2y_q y_c + (r^2 - x_c^2 - y_c^2) - (x_q^2 + y_q^2) &= 0 \\
2x_s x_c + 2y_s y_c + (r^2 - x_c^2 - y_c^2) - (x_s^2 + y_s^2) &= 0 \\
2x_t x_c + 2y_t y_c + (r^2 - x_c^2 - y_c^2) - (x_t^2 + y_t^2) &= 0
\end{align*}$$
Demystifying Resultant - II

\[ \begin{align*}
2x_pX + 2y_pY + R - (x_p^2 + y_p^2)Z &= 0 \\
2x_qX + 2y_qY + R - (x_q^2 + y_q^2)Z &= 0 \\
2x_sX + 2y_sY + R - (x_s^2 + y_s^2)Z &= 0 \\
2x_tX + 2y_tY + R - (x_t^2 + y_t^2)Z &= 0
\end{align*} \]

iff \[ \begin{align*}
\text{has a non-trivial solution } (X, Y, R, Z) \text{ and}
\end{align*} \]

\[ \begin{align*}
X/Z &= x_c \\
Y/Z &= y_c \\
R/Z &= r^2 - x_c^2 - y_c^2.
\end{align*} \]
Demystifying Resultant - II

iff

\[
\begin{align*}
2x_p x_c + 2y_p y_c + (r^2 - x_c^2 - y_c^2) - (x_p^2 + y_p^2) &= 0 \\
2x_q x_c + 2y_q y_c + (r^2 - x_c^2 - y_c^2) - (x_q^2 + y_q^2) &= 0 \\
2x_s x_c + 2y_s y_c + (r^2 - x_c^2 - y_c^2) - (x_s^2 + y_s^2) &= 0 \\
2x_t x_c + 2y_t y_c + (r^2 - x_c^2 - y_c^2) - (x_t^2 + y_t^2) &= 0
\end{align*}
\]

iff

\[
\begin{vmatrix}
   x_p & y_p & 1 & x_p^2 + y_p^2 \\
   x_q & y_q & 1 & x_q^2 + y_q^2 \\
   x_s & y_s & 1 & x_s^2 + y_s^2 \\
   x_t & y_t & 1 & x_t^2 + y_t^2 \\
\end{vmatrix} = 0
\]
Demystifying Resultant - II

iff

\[
\begin{align*}
2x_px_c + 2y_py_c + (r^2 - x_c^2 - y_c^2) - (x_p^2 + y_p^2) &= 0 \\
2x_qx_c + 2y_qy_c + (r^2 - x_c^2 - y_c^2) - (x_q^2 + y_q^2) &= 0 \\
2x_sx_c + 2y_sy_c + (r^2 - x_c^2 - y_c^2) - (x_s^2 + y_s^2) &= 0 \\
2x_tx_c + 2y_ty_c + (r^2 - x_c^2 - y_c^2) - (x_t^2 + y_t^2) &= 0
\end{align*}
\]

iff

\[
\begin{vmatrix}
x_p & y_p & 1 & x_p^2 + y_p^2 \\
x_q & y_q & 1 & x_q^2 + y_q^2 \\
x_s & y_s & 1 & x_s^2 + y_s^2 \\
x_t & y_t & 1 & x_t^2 + y_t^2
\end{vmatrix} = 0
\]

= resultant of the system

Allows to eliminate $x_c, y_c, r^2$
Resultant

- Resultant often used in simple cases without noticing
Resultant

- Resultant often used in simple cases without noticing
- **Linear algebra** helps solve non-linear problems
Digression on algebraic degree

One measure of **efficiency** and **precision** of a **predicate**: algebraic degree
If predicate = sign of a resultant

Resultant has minimal degree $\Rightarrow$ optimal predicate?
If predicate = sign of a resultant

Resultant has minimal degree $\rightarrow$ optimal predicate?

No:
  - methods often return a multiple of the resultant
    $\rightarrow$ resultant hard to compute
Digression on algebraic degree

If predicate = sign of a resultant

Resultant has **minimal degree** $\rightarrow$ optimal predicate?

No:

- methods often return a multiple of the resultant
  $\rightarrow$ resultant hard to compute
- the resultant may be factored
  $\rightarrow$ predicate can have a lower degree
Digression on algebraic degree

If predicate = sign of a resultant

Resultant has minimal degree $\implies$ optimal predicate?

No:

- methods often return a multiple of the resultant
  $\implies$ resultant hard to compute
- the resultant may be factored
  $\implies$ predicate can have a lower degree
- a factor may be $P^2 + Q^2$
  $\implies$ the degree does not mean so much
Digression on algebraic degree

- filtering techniques used for efficiency
  → maybe not such an interesting measure?
Digression on algebraic degree

- Degree of a **predicate** → not trivial
- Degree of an **algorithm** → depends on the algebraic expressions of predicates
- Degree of a geometric **problem** → ?

Digression ↦ thread
Another tool: Sturm sequences

\[ \mathcal{P} = P_0, P_1, \ldots, P_d \in \mathbb{R}[X] \]

\[ \alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\} \]

\[ \text{Var}(\mathcal{P}; \alpha) = \text{number of sign variations in the sequence} \]
\[ P_0(\alpha), P_1(\alpha), \ldots, P_d(\alpha) \]

\[ \text{Var}(\mathcal{P}; \alpha, \beta) = \text{Var}(\mathcal{P}; \alpha) - \text{Var}(\mathcal{P}; \beta) \]
Sturm sequences

\( P, Q \in K[X] \) signed remainder sequence of \( P \) and \( Q = \) sequence \( S(P, Q) : P_0, P_1, \ldots, P_k \)

\[
\begin{align*}
P_0 &= P \\
P_1 &= Q \\
P_2 &= -\text{Rem}(P_0, P_1) \\
&\vdots \\
P_k &= -\text{Rem}(P_{k-2}, P_{k-1}) \\
P_{k+1} &= -\text{Rem}(P_{k-1}, P_k) = 0
\end{align*}
\]

where \( \text{Rem}(A, B) = \text{remainder of the Euclidean division of } A \text{ by } B \)
Sturm sequences

Sturm sequence of $P = S(P, P')$ of signed reminders of $P$ and $P'$

$\text{Var}(S(P, P'); \alpha, \beta)$

is the number of roots of $P$ in the interval $[\alpha, \beta]$
Sturm sequences for dummies

\[ P = aX^2 + bX + c \]

\[ \alpha \beta \]

\[ P' \]

\[ \Delta = 2aX + b \]

\[ P = P' \]

\[ (X^2 + b)^2 - (b^2 - 4a) \]

\[ P_0 = P, \quad P_1 = P', \quad P_2 = \Delta \text{ if } \Delta > 0 \]

\[ \alpha = -\infty, \quad \beta = +\infty \]

\[ \text{Var}(P; \alpha) = 2, \quad \text{Var}(P; \beta) = 0 \]

2 roots
Sturm sequences for dummies by a dummy

\[ P = aX^2 + bX + c \]

\[ P' = 2aX + b \]
\[ P = P' \cdot \left( \frac{X}{2} + \frac{b}{4a} \right) - \left( \frac{b^2}{4a} - c \right) \]
Sturm sequences for dummies by a dummy

\[ P = aX^2 + bX + c \]

\[ P' = 2aX + b \]

\[ P = P'.\left(\frac{X}{2} + \frac{b}{4a}\right) - \left(\frac{b^2}{4a} - c\right) \]

\[ P_0 = P, \quad P_1 = P', \quad P_2 = \Delta \]

if \( \Delta > 0 \)

\[ \alpha = -\infty, \quad \beta = +\infty \]

\[ \text{Var}(P; \alpha) = 2 \]

\[ \text{Var}(P; \beta) = 0 \]

2 roots
Sturm sequences

Sequence $S(P, P'Q)$ of signed reminders of $P$ and $P'Q$ counts the number of roots of $P$ at which $Q$ is positive.

Sturm sequences allow to compare roots of $P$ and $Q$.
Comparing intersection points

signs of polynomial expressions
Comparing intersection points

signs of polynomial expressions

comparison of algebraic numbers
Comparing intersection points

signs of polynomial expressions

comparison of algebraic numbers

Sturm sequences

signs of polynomial expressions
Arithmetic filters for sign computations:

Approximate evaluation $P^a(x)$
+ Error $\varepsilon$

$|P^a(x)| > \varepsilon$

? 

y 

n

sign ($P(x)$) = sign ($P^a(x)$) 

Exact computation

Exact geometric computation $\neq$ Exact arithmetics
Practical efficiency

Comparison of algebraic numbers of degree 2:

\[ J + P_1 > 0 \]
Case 1, 2, 3
\[ P_1 < 0 \]
Case 3, 4, 5
\[ \neg + \]
\[ P_3 > 0 \]
Case 3, 4
\[ K + P_2 > 0; P_3 < 0 \]
Case 4, 5
\[ \neg + \]
\[ P_2 > 0; P_3 < 0 \]
Case 4, 5
\[ K + \]
\[ P_2 > 0; P_3 < 0 \]
Case 4, 5
\[ \neg + \]
\[ P_1 > 0 \]
Case 1, 2, 3
\[ \neg + \]
\[ P_3 < 0 \]
Case 2, 3
\[ D + \]
\[ P_3 < 0 \]
Case 2, 3
\[ \neg + \]
\[ P_4 > 0 \]
Case 3a
\[ \neg + \]
\[ P_4 > 0 \]
Case 3a
\[ l_1 > l_2 \]
Case 1, 2
\[ l_1 < l_2 \]
Case 3b
\[ l_1 > l_2 \]
Case 3b
\[ l_1 < l_2 \]
Case 3b

polynomial expressions pre-computed
static Sturm sequences
Algebra is not just “computations”
it has a meaning...!

\[ K = 0 \iff l_1, r_1, l_2, r_2 \text{ harmonic division} \]
Algebra is not just “computations”
it has a meaning...!

\[ K = 0 \iff l_1, r_1, l_2, r_2 \text{ harmonic division} \]

- Geometric interpretation in more complicated cases...?
Algebra is not just “computations”
it has a meaning...!

\[ K = 0 \iff l_1, r_1, l_2, r_2 \text{ harmonic division} \]

- Geometric interpretation in more complicated cases...?
- Optimal degree...?
Open Source Project

www.cgal.org
Open Source Project
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Release 3.2 soon
Open Source Project

www.cgal.org

Release 3.2 soon

Exclusive news: Out before Microsoft new OS!
Open Source Project
www.cgal.org

Release 3.2 soon

new: 2D Circular Kernel
manipulations of circular arcs
Open Source Project

www.cgal.org

Release 3.2 soon

- **new**: 2D Circular Kernel
  manipulations of circular arcs
- Arrangement package **redesigned**
- ...
Intersection of two quadrics $Q_S$ and $Q_T$

Levin’s pencil method

- find a "good" quadric in the pencil $Q_R(\lambda) = \lambda S - T$
  - $\lambda$ root of degree 3 pol.
- Diagonalize $R(\lambda)$.
  - Eigenvalues = roots of degree 2 pol. $\in \mathbb{Q}(\lambda)$.
  - Normalize eigenvectors.
- Plug the parameterization of $Q_R(\lambda)$ in $Q_T$.
  - Degree 2 in one of the parameters. Solve

"good" = simple ruled

\[
\begin{array}{cccc}
  x & x & x & \\
  x & x & x & \\
  x & x & x & \\
  \vdots & \vdots & \vdots & \\
  \ldots & \ldots & \ldots & \\
\end{array}
\]

principal subdeterminant = 0
Intersection of two quadrics $Q_S$ and $Q_T$

Levin’s pencil method
- find a “good” quadric in the pencil $Q_R(\lambda) = \lambda S - T$
  - $\lambda$ root of degree 3 pol.
- Diagonalize $R(\lambda)$.
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  - Normalize eigenvectors.
- Plug the parameterization of $Q_R(\lambda)$ in $Q_T$.
  - Degree 2 in one of the parameters. Solve

Improvement
- work in $\mathbb{P}^3$
- Relax the constraint on $Q_R(\lambda)$
  - Rational, ruled.
- Apply Gauss reduction of the quadratic form:
  - $P^T R P$ diagonal.
  - Rational transformation.
- Plug the parameterization in $Q_T$.
  - Degree 2 in one of the parameters. Solve
Intersection of quadrics

Levin’s pencil method

\[ \sqrt{\sqrt{\sqrt{\sqrt{}}}} \]

New parameterization
- rational when it exists,
  involves \( \sqrt{\text{pol.}} \) otherwise.
- quasi-optimal in \( \sqrt{\cdot} \).

Implemented

© Dupont et al
Planar arrangement of curves of degree 4
a curve can have 6 singular points
Sort out (upper, lower) → arrangement on each quadric

Surfacic approach
Arrangement of quadrics
Sweeping approach

Sweeping plane:
Trapezoidal map of evolving conics

Volumic approach:
vertical decomposition
Arrangement of quadrics
Sweeping approach

Events:

- new quadric
- features in the map intersect
Arrangement of quadrics
Sweeping approach

Events:
- new quadric
- features in the map intersect

$x$ solution of

$\exists \ y, z_1, z_2 \ s.t. \ \left\{ \begin{array}{l}
Q_i(x, y, z_1) = 0 \\
Q_j(x, y, z_1) = 0 \\
Q_k(x, y, z_2) = 0 \\
Q_l(x, y, z_2) = 0
\end{array} \right.$

$x$ in an extension field of degree 16
Arrangement of quadrics
Sweeping approach

Events:
- new quadric
- features in the map intersect

$x$ solution of

$\exists \ y, z_1, z_2 \ s.t. \ \begin{cases} Q_i(x, y, z_1) = 0 \\ Q_j(x, y, z_1) = 0 \end{cases} \ \text{and} \ \begin{cases} Q_k(x, y, z_2) = 0 \\ Q_l(x, y, z_2) = 0 \end{cases}$

$x$ in an extension field of degree 16

Comparison of events:
difference of events in an extension field of degree 256…

- Optimal degree…?
Apollonius diagram
Additively weighted Voronoi diagram

Weighted points $\sigma_i = (p_i, r_i)$, $p_i \in \mathbb{R}^2, r_i \in \mathbb{R}$

$$\delta_i(x) = \|x - p_i\| - r_i$$
Apollonius diagram
Additively weighted Voronoi diagram

Weighted points $\sigma_i = (p_i, r_i), \ p_i \in \mathbb{R}^2, r_i \in \mathbb{R}$

$$\delta_i(x) = \|x - p_i\| - r_i$$

$C_i \subset \mathbb{R}^3: \ x_3 = \|x - p_i\| - r_i$

$\iff \quad (x_3 + r_i)^2 = (x - p_i)^2 \quad x_3 + r_i > 0$ (half-cone)
Apollonius diagram
Additively weighted Voronoi diagram

Weighted points $\sigma_i = (p_i, r_i)$, $p_i \in \mathbb{R}^2$, $r_i \in \mathbb{R}$

$$\delta_i(x) = \|x - p_i\| - r_i$$

$C_i \subset \mathbb{R}^3 : x_3 = \|x - p_i\| - r_i$

$$\iff (x_3 + r_i)^2 = (x - p_i)^2 \quad x_3 + r_i > 0 \quad \text{half-cone}$$

Apollonius diagram =
lower envelope of the half-cones.
Bisector of $\sigma_i$ and $\sigma_j =$
projection of a plane conic section $C_i \cap C_j$. 
Apollonius diagram
Additively weighted Voronoi diagram

Weighted points $\sigma_i = (p_i, r_i)$, $p_i \in \mathbb{R}^2$, $r_i \in \mathbb{R}$

$$\delta_i(x) = \|x - p_i\| - r_i$$

$C_i \subset \mathbb{R}^3 : x_3 = \|x - p_i\| - r_i$

$$\iff (x_3 + r_i)^2 = (x - p_i)^2 \quad x_3 + r_i > 0 \quad \text{half-cone}$$

Apollonius diagram =
lower envelope of the half-cones.

Bisector of $\sigma_i$ and $\sigma_j =$
projection of a plane conic section $C_i \cap C_j$.

$\Sigma_i$ sphere $\subset \mathbb{R}^3$, center $(p_i, r_i)$ radius $\sqrt{2}r_i$
Apollonius diagram
Additively weighted Voronoi diagram

Weighted points $\sigma_i = (p_i, r_i), \ p_i \in \mathbb{R}^2, r_i \in \mathbb{R}$

$$\delta_i(x) = \|x - p_i\| - r_i$$

$C_i \subset \mathbb{R}^3 : \ x_3 = \|x - p_i\| - r_i$

$$\iff (x_3 + r_i)^2 = (x - p_i)^2 \quad x_3 + r_i > 0 \quad \text{half-cone}$$

Apollonius diagram =
lower envelope of the half-cones.

Bisector of $\sigma_i$ and $\sigma_j =$
projection of a plane conic section $C_i \cap C_j$.

$\Sigma_i$ sphere $\subset \mathbb{R}^3$, center $(p_i, r_i)$ radius $\sqrt{2r_i}$

$X_i$ projection of $x$ onto $C_i$

$x \in A(\sigma_i) \iff \|x - p_i\| - r_i < \|x - p_j\| - r_j \ (\forall j)$

$\iff \text{pow}(X_i, \Sigma_i) < \text{pow}(X_i, \Sigma_j)$
Apollonius diagram
Additively weighted Voronoi diagram

Weighted points $\sigma_i = (p_i, r_i)$, $p_i \in \mathbb{R}^2$, $r_i \in \mathbb{R}$

$$\delta_i(x) = \|x - p_i\| - r_i$$

$C_i \subset \mathbb{R}^3 : \quad x_3 = \|x - p_i\| - r_i$
$$\iff (x_3 + r_i)^2 = (x - p_i)^2 \quad x_3 + r_i > 0 \quad \text{half-cone}$$

Apollonius diagram =
lower envelope of the half-cones.
Bisector of $\sigma_i$ and $\sigma_j =$
projection of a plane conic section $C_i \cap C_j$.

$\Sigma_i$ sphere $\subset \mathbb{R}^3$, center $(p_i, r_i)$ radius $\sqrt{2}r_i$
$A(\sigma_i) =$ projection of the intersection of
the half-cone $C_i$ with the power region of $\Sigma_i$

Same in $\mathbb{R}^d$
Tricky predicates
Degree 16
Tricky predicates
Degree 16

- Implementation degree 20:
  degree 16 requires $\sim 100$ times as many arithmetic operations...
- Optimal degree...?
Challenges

- *theoretical*: questions on degree...
Challenges

- **theoretical**: questions on degree...
- Robust (Exact?) computation on higher degree curves and surfaces
Challenges

- **theoretical**: questions on degree...
- Robust (Exact?) computation on higher degree curves and surfaces
- Improvement of *practical efficiency* for low degree curves
  CAD-VLSI (circular arcs):
    ~ 10 times slower than industrial non-robust code
  good start!
Challenges

- Applications to **Structural biology**
  Manipulations of a large number of spheres
  (low degree surfaces...)

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ευχαριστώ

Material taken from:

- Greece
  National University of Athens
  University of Crete

- Germany
  Max-Planck Institut für Informatik
  Universität des Saarlandes

- Israel
  Tel-Aviv University

- France
  Loria
  INRIA Sophia Antipolis
Ευχαριστώ

Merci, ô astre souverain... Hé, ô Soleil... Tu as entendu ma prière... Voici que mes rayons déclinent...

Mais... Mais, ma parole, d'après vous... Que se passe-t-il... Est-ce que je deviens fou, moi aussi... L'est de la sorcellerie...

Un eclipse... ! Un eclipse... ! Un eclipse...

Au lieu aussi...

O Soleil, puissant astre du jour, je t'en conjure, sois clément... Aie pitié de tes fils et que la lumière réapparaisse!

Oui - ou - ou - ou!

Par Pachacamac! le soleil lui obéit... Vite! vite! qu'ont-ils délivré à l'instant !