3 Probabilistic analyses: randomized algorithms, evenly distributed points.

3.1 Area of incident triangles

Let X be Poisson point process of intensity n in the plane. What is the expected area of the union of triangles incident to a point of X?

3.1 Correction:

There is n points per unit area and 2 triangles per point, thus the average size of a Delaunay triangle is $\frac{1}{2n}$. There are on average 6 triangles incident a point. Thus we expect $\frac{3}{n}$, BUT we are not allowed to multiply these two quantities that may be dependent.

Let's do a formal computation.

The area of the triangles incident to the origin in $DT(X\cap\{0\})$ is the same as the area of the triangles in conflict with the origin in $DT(X)$:

$$\mathbb{E}\left[\tfrac{1}{6}\sum_{p,q,t\in X^3}\mathbb{1}_{[pqt\in DT(X)]}\mathbb{1}_{[O\in Disk(pqt)]}\times area(pqt)\right]$$

$$= \tfrac{n^3}{6}\int_{(\mathbb{R}^2)^3}\mathbb{P}\left[X\cap B(pqt)=\emptyset\right]\mathbb{1}_{[O\in Disk(pqt)]}\times area(pqt)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}t$$

$$= \tfrac{n^3}{6}\int_0^\infty\int_0^r\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} e^{-n\pi R^2}r^2 area(\alpha_1\alpha_2\alpha_3)2r^3 area(\alpha_1\alpha_2\alpha_3)R\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}\theta\mathrm{d}R\mathrm{d}r$$

$$= \tfrac{n^3}{6}\int_0^\infty e^{-n\pi r^2}r^5\left(\int_0^r R\mathrm{d}R\right)\left(\int_0^{2\pi}\mathrm{d}\theta\mathrm{d}R\right)\mathrm{d}r\left(\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}2area(\alpha_1\alpha_2\alpha_3)^2\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\right)$$

$$= \tfrac{n^3}{6}\int_0^\infty e^{-n\pi r^2}r^5 2\pi\frac{r^2}{2}\mathrm{d}r\ \ 6\pi^3 = \tfrac{n^3}{6}\pi\frac{3}{n^4\pi^4}6\pi^3 = \frac{3}{n}$$

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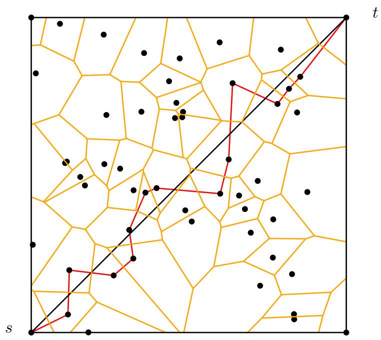
Maple computation:
> assume(n>0):with(LinearAlgebra):
> int( exp(-n*Pi*r^2)*r^7,r=0..infinity);
3/(n^4*Pi^4)
> int(int(int(1/2* Determinant([[        1,          1,          1],
                               [cos(alpha1),cos(alpha2),cos(alpha3)],
                               [sin(alpha1),sin(alpha2),sin(alpha3)]])^2,
    alpha1=0..2*Pi),alpha2=0..2*Pi),alpha3=0..2*Pi);
6*Pi^3
```

3.2 Voronoi Path

Let X be a set of n points inside the square $[0,1]^2$ and add to X the four corners of the square. Let $s=(0,0)$ et $t=(1,1)$. Imagine a point moving on the line segment from s to t, its nearest neighbor in X is $s=v_0$ at the beginning, then it changes to another point $v_1\in X$. We denote $s=v_0,v_1,v_2\ldots v_{k-1},v_k=t$ the sequence of nearest neighbors of the moving point. The polygonal line $v_0,v_1,v_2\ldots v_{k-1},v_k$ is called the Voronoi path.



3.2.1 Delaunay

Is v_iv_{i+1} a Delaunay edge? (Prove or give a counter-example).

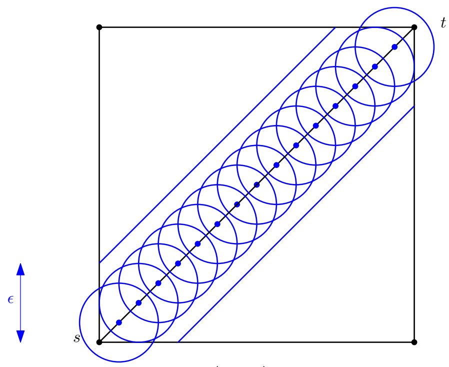
3.2.2 Length

Give if possible upper and lower bound on the length (the sum of the lengths of all relevant edges) of the Voronoi path. Draw examples for these bounds.

3.2.3 Strip

In this question, assume that X is a set of n random points in the square (plus the four corners). The lines $y = x + \epsilon$ and $y = x - \epsilon$ describe a strip around the diagonal of the square and we want to study the probability that the Voronoi path leave this strip.

We cover the diagonal with disks of radius $\frac{\epsilon}{2}$ centered at points $\left(\frac{j\epsilon}{4}, \frac{j\epsilon}{4}\right)$ with j integer in $\left[1, \frac{4}{\epsilon} - 1\right]$.

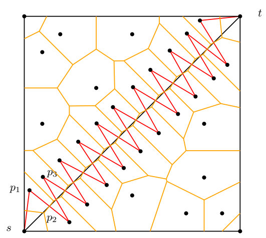


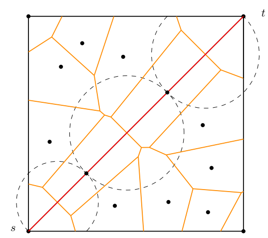
- Prove that at a point (x, x) on the diagonal the circle centered at (x, x) of radius ϵ contains at least one of the circle of the diagonal covering.
- Deduce an upper bound of the required probability. (The classical inequality $1 - t \le e^{-t}$ could be useful).
- Find some (smallest possible) value of ϵ, depending on n, such that this probability is exponentially small (as a function of n).

3.2 Correction:

3.2.1 Delaunay

When the nearest neighbor changes from v_i to v_{i+1}, the moving point is placed in w_i on the Voronoi edge between the Voronoi cells of v_i and v_{i+1}, thus v_iv_{i+1} is a Delaunay edge since Delaunay triangulation and Voronoi diagram are dual.





3.2.2 Length

Using a zig-zag, it is possible to design an example where the Voronoi path is as lengthy as we want. Using for X points on the diagonal, the length can be as short as $\sqrt{2}$ (see figures).

3.2.3 Strip

- The closest center of circle from (x, x) is at distance less than $\frac{\sqrt{2}\epsilon}{8}$. Using the triangle inequality a point inside this circle is at distance less than $\frac{\sqrt{2}\epsilon}{8} + \frac{\epsilon}{2} = 0.67\epsilon < \frac{\epsilon}{\sqrt{2}} = 0.707\epsilon$ from (x, x).
- The Voronoi path can exit the strip only if a circle of center (x, x) is empty and thus a circle of the covering is empty.

$$\begin{aligned}
Prob(\text{VPath exit the strip}) &\le Prob(\text{at least a circle of the covering is empty}) \\
&\le \sharp(C \in \text{covering}) \cdot Prob(C \text{ is empty}) \\
&\le \sharp(C \in \text{covering}) \cdot \left(1 - \frac{\text{circle area}}{\text{square area}}\right)^n \\
&\le \frac{4}{\epsilon} \cdot \left(1 - \frac{\pi \frac{\epsilon^2}{4}}{1}\right)^n \le \frac{4}{\epsilon} e^{-\frac{\pi}{4} n \epsilon^2} \le 4\sqrt{n} e^{-\frac{\pi}{4} \log n}
\end{aligned}$$

- Choosing $\epsilon = \sqrt{\frac{\log n}{n}}$

$$Prob(\text{VPath exit the strip}) \le 4n^{\frac{1}{2} - \frac{\pi}{4}} \simeq 4n^{-0.28}$$