Fast high-resolution drawing of algebraic curves and surfaces

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Overview

1. Implicit curve drawing
2. Previous work
3. Our approach
4. Fast multipoint evaluation
5. Algorithms
6. Experiments
Implicit curve drawing
Scientific visualization

Some scientific visualization applications:

- modeling
- medical imaging
- mechanism design

Goal: build an intuition and get an understanding of the data
Implicit curve drawing problem

General problem

Discrete representation of an implicit curve on a fixed grid

- **Input:**
  - function $F$
  - resolution $N$
  - visualization window

Implicit curve defined as the solution set

$$\{(x, y) \in \mathbb{R}^2 \mid F(x, y) = 0\}$$

- **Output:** drawing (set of pixels)
Implicit curve drawing problem

Our focus

Discrete representation of an algebraic curve on a fixed grid

- **Input:**
  - bivariate polynomial $P$ of partial degree $d$
  - resolution $N$
  - window $[-1, 1] \times [-1, 1]$

  Algebraic curve defined as the solution set

  $$\{(x, y) \in \mathbb{R}^2 \mid P(x, y) = 0\}$$

- **Output:** drawing (set of pixels)

**Goal:** fast high-resolution drawing of high degree algebraic curves

- $d \approx 100 \quad \rightarrow \quad d^2 \approx 10,000$ monomials
- $N \approx 1,000$
Correctness of the drawing

For numerical reasons, there may be some:

- **False negative** pixels
Correctness of the drawing

For numerical reasons, there may be some:

- **False negative** pixels
- **False positive** pixels
Previous work
Marching squares

The idea

2D variant of the widely used marching cubes algorithm [Lorensen & Cline, 1987]
Implicit curve defined by $P(X, Y) = 0$
Marching squares

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Marching squares

Complexity

Complexity (number of elementary operations)
Naive evaluation

\[ \theta(d^2N^2) \]

\( d \) partial degree
\( N \) resolution of the grid

Arithmetic complexity of the marching squares

With partial evaluation of \( P(x, y) \), assuming \( d < N \)

\[ \theta(dN^2) \]

Slow for high resolutions...
Can we have an algorithm in \( O(dN) \)?
Adaptive subdivision

Local refinements of the grid
Adaptive subdivision

Local refinements of the grid
Adaptive subdivision

Local refinements of the grid
Adaptive subdivision

Local refinements of the grid
Adaptive subdivision

Local refinements of the grid
Methods providing topological correctness

Adaptive 2D subdivision with interval arithmetic

- [Snyder, 1992]
- [Plantinga & Vegter, 2004]
- [Burr et al., 2008]
- [Lin & Yap, 2011]

- ...

Cylindrical algebraic decomposition (CAD)

- [Gonzalez-Vega & Necula, 2002]
- [Eigenwillig et al., 2007]
- [Alberti et al., 2008]
- [Cheng et al., 2009]
- [Kobel & Sagraloff, 2015]
- [Diatta et al., 2018]

- ...

https://isotop.gamble.loria.fr/
Our approach
A prerequisite

Interval arithmetic

For $I = [l, \bar{l}]$ and $J = [j, \bar{j}]$,

- $I + J = [l+j, \bar{l}+\bar{j}]$
- $I - J = [l-\bar{j}, \bar{l}-j]$
- $\ldots$
A prerequisite
Interval arithmetic

For \( I = [l, \bar{l}] \) and \( J = [j, \bar{j}] \),

- \( I + J = [l + j, \bar{l} + \bar{j}] \)
- \( I - J = [l - j, \bar{l} - \bar{j}] \)
- \( \ldots \)

Evaluation of the function \( f(X) = X^2 - X = (X - 1)X \) on the interval \([0, 2]\)

- \([0, 2]^2 - [0, 2] = [0, 4] - [0, 2] = [-2, 4]\)
- \(([0, 2] - 1) \cdot [0, 2] = [-1, 1] \cdot [0, 2] = [-2, 2]\)
Interval arithmetic

Inclusion property

\[ P(X) = 2X^3 - X^2 - 1.5X + 0.75 \]

How to compute \( P(I) \) for \( I = [-1, 1] \)?

\[ P(I) = [-0.75, 1.06 \ldots] \]
Interval arithmetic

Inclusion property

\[ P(X) = 2X^3 - X^2 - 1.5X + 0.75 \]

How to compute \( P(I) \) for \( I = [-1, 1] \)?

\[ \square P(I) = 2[-1, 1]^3 - [-1, 1]^2 - 1.5[-1, 1] + 0.75 \]
\[ = [-5.25, 5.25] \]

\[ P(I) = [-0.75, 1.06 \ldots] \]
Interval arithmetic
Inclusion property

\[ P(X) = 2X^3 - X^2 - 1.5X + 0.75 \]

How to compute \( P(I) \) for \( I = [-1, 1] \)?

\( \square P(I) = 2[-1, 1]^3 - [-1, 1]^2 - 1.5[-1, 1] + 0.75 \)
\( \quad = [-5.25, 5.25] \)

With Horner’s scheme:

\( \square P(I) = ((2[-1, 1] - 1)[-1, 1] - 1.5)[-1, 1] + 0.75 \)
\( \quad = [-3.75, 5.25] \)

\[ P(I) = [-0.75, 1.06] \ldots \]

\( P(I) \subseteq \square P(I) \)
Interval arithmetic

Convergence property

**Convergence at a point**

With $x \in [a, b]$

$\lim_{[a,b] \to [x,x]} \Box P([a, b]) = P(x)$
Our approach: guaranteed intersection with the grid

Marching squares

Adaptive subdivision

New approach: evaluation along fibers

⇒ Make it fast and provide some guarantees
Two algorithms

Edge drawing
- *evaluation in X*
  - Chebyshev nodes
  - multipoint evaluation with IDCT
- *subdivision in Y*
  - naive root finding method

Guarantees
- False positive and false negative pixels

Pixel drawing
- *evaluation in X*
  - Chebyshev nodes
  - multipoint evaluation with IDCT
  - Taylor approximation
- *subdivision in Y*
  - naive root finding method

Guarantees
- False positive pixels only
Subdivisions along a fiber

\[ P(x_k, Y) = \sum a_j Y^j \]
Subdivisions along a fiber

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Subdivisions along a fiber

\[ P(x_k, Y) = \sum a_j Y^j \]
Subdivisions along a fiber

\[ P(x_k, Y) = \sum a_j Y^j \]
Subdivisions along a fiber

\[ P(x_k, Y) = \sum a_j Y^j \]
An example

\[ X^2 + Y^2 - 1 = 0 \]

Resolution \( N = 64 \)
Pixel lighting
Edge drawing

Detect a crossing between two consecutive nodes of the grid
Light the adjacent pixels
Pixel lighting
Edge drawing

- Detect a crossing between two consecutive nodes of the grid
Pixel lighting
Edge drawing

- Detect a crossing between two consecutive nodes of the grid
- Light the adjacent pixels
Detect a crossing in pixel of the grid
Light that pixel
False positive and false negative pixels

Edge drawing

Some incorrect pixels:

- **False negative** when a connected component lies inside of a pixel
False positive and false negative pixels

Edge drawing

Some incorrect pixels:

- **False negative** when a connected component lies inside of a pixel
- **False positive** when the evaluation on an edge of a pixel is close to zero

That occurs for a segment $S$ when

$$0 \in \square P(S) + [-E, E]$$

Certification of segments that are not crossed:

$$0 \not\in \square P(S) + [-E, E]$$

$\Downarrow$

$$0 \not\in P(S)$$
False positive and false negative pixels

Pixel drawing

Some incorrect pixels:

- **False negative** when a connected component lies inside of a pixel
- **False positive** when the evaluation on an edge of a pixel is close to zero

That occurs for a segment $S$ when

$$0 \in \Box P(S) + [-E, E]$$

Certification of segments that are not crossed:

$$0 \notin \Box P(S) + [-E, E]$$

\[\downarrow\]

$$0 \notin P(S)$$
Fast multipoint evaluation
A prerequisite to fast multipoint evaluation
Chebyshev polynomials

**Definition**

The Chebyshev polynomials \(( T_k)\) verify \(\forall k \in \mathbb{N}, T_k(\cos \theta) = \cos(k\theta)\)

The first three Chebyshev polynomials

\[
\begin{align*}
\cos(0 \cdot \theta) &= 1 & T_0 &= 1 \\
\cos(1 \cdot \theta) &= \cos(\theta) & T_1 &= X \\
\cos(2 \cdot \theta) &= 2 \cos(\theta)^2 - 1 & T_2 &= 2X^2 - 1
\end{align*}
\]
A prerequisite to fast multipoint evaluation

Chebyshev polynomials

**Definition**

The Chebyshev polynomials \( T_k \) verify \( \forall k \in \mathbb{N}, T_k(\cos \theta) = \cos(k\theta) \)

**Lemma**

An arbitrary polynomial \( p \) of degree \( d \) can be written in terms of the Chebyshev polynomials:

\[
p(X) = \sum_{k=0}^{d} \alpha_k T_k(X)
\]
A prerequisite to fast multipoint evaluation

Chebyshev polynomials

Definition

The Chebyshev polynomials \(( T_k)\) verify \(\forall k \in \mathbb{N}, T_k(\cos \theta) = \cos(k\theta)\)

Lemma

An arbitrary polynomial \(p\) of degree \(d\) can be written in terms of the Chebyshev polynomials:

\[
p(X) = \sum_{k=0}^{d} \alpha_k T_k(X)
\]

Lemma

For \(N \in \mathbb{N}\), a polynomial \(p\) of degree \(d\) can be evaluated on the Chebyshev nodes \((c_n)_{0 \leq n \leq N-1}\) using the IDCT:

\[
(p(c_n))_{0 \leq n \leq N-1} = \frac{1}{2}(\alpha_0, \ldots, \alpha_0) + \text{IDCT}( (\alpha_k)_{0 \leq k \leq N-1} )
\]
A prerequisite to fast multipoint evaluation

Chebyshev nodes

**Definition**

For \( N \in \mathbb{N} \), the Chebyshev nodes are

\[
c_n = \cos\left(\frac{2n + 1}{2N} \pi\right), \quad n = 0, \ldots, N - 1
\]

They are the roots of \( T_N \)
Inverse Discrete Cosine Transform

Inverse Discrete Cosine Transform (IDCT): \( \alpha_k \rightarrow x_n \)

\[
x_n = \frac{1}{2} \alpha_0 + \sum_{k=1}^{N-1} \alpha_k \cos \left[ \frac{\pi k (2n + 1)}{2N} \right]
\]

\[
\begin{align*}
\text{IDCT} & \quad \text{linear transformation} \\
(\alpha_k) & \rightarrow (V_k) \quad \text{FFT} \quad (v_k) \rightarrow (x_k)
\end{align*}
\]

⇒ Fast thanks to the Fast Fourier Transform (FFT) algorithm in \( O(N \log_2 N) \)

[Makhoul, 1980]
Inverse Discrete Cosine Transform (IDCT): $\alpha_k \rightarrow x_n$

$$x_n = \frac{1}{2} \alpha_0 + \sum_{k=1}^{N-1} \alpha_k \cos \left[ \frac{\pi k (2n + 1)}{2N} \right]$$

\[\text{IDCT}\]

\[
\begin{array}{ccc}
(\alpha_k) & \text{linear transformation} & (V_k) \\
\downarrow \text{FFT} & & \downarrow \text{linear transformation} \\
(v_k) & & (x_k)
\end{array}
\]

⇒ Fast thanks to the Fast Fourier Transform (FFT) algorithm in $O(N \log_2 N)$

[Makhoul, 1980]

$$p(c_n) = \sum_{k=0}^{N-1} \alpha_k T_k \left( \cos \left( \frac{2n + 1}{2N} \pi \right) \right)$$
Inverse Discrete Cosine Transform

Inverse Discrete Cosine Transform (IDCT): \( \alpha_k \rightarrow x_n \)

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\]

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[Maikhoul, 1980]

\[
p(c_n) = \sum_{k=0}^{N-1} \alpha_k T_k \left( \cos \left( \frac{2n + 1}{2N} \pi \right) \right) = \sum_{k=0}^{N-1} \alpha_k \cos \left[\frac{\pi k(2n + 1)}{2N}\right]
\]
Inverse Discrete Cosine Transform

Inverse Discrete Cosine Transform (IDCT): \( \alpha_k \rightarrow x_n \)

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x_n = \frac{1}{2} \alpha_0 + \sum_{k=1}^{N-1} \alpha_k \cos \left[ \frac{\pi k(2n + 1)}{2N} \right]
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\( \Rightarrow \) Fast thanks to the Fast Fourier Transform (FFT) algorithm in \( O(N \log_2 N) \)

[Makhoul, 1980]

\[
p(c_n) = \frac{1}{2} \alpha_0 + \frac{1}{2} \alpha_0 + \sum_{k=1}^{N-1} \alpha_k \cos \left[ \frac{\pi k(2n + 1)}{2N} \right]
\]

\[
(p(c_n))_{0 \leq n \leq N-1} = \frac{1}{2} (\alpha_0, \ldots, \alpha_0) + \text{IDCT}((\alpha_k)_{0 \leq k \leq N-1})
\]
Error of the IDCT

[Makhoul, 1980] and [Brisebarre et al., 2020, Theorem 3.4] yield

**Theorem (H., Moroz, Pouget, 2022)**

Assume radix-2, precision-$p$ arithmetic, with rounding unit $u = 2^{-p}$. Let $\hat{x}$ be the computed $2^n$-point IDCT of $\alpha \in \mathbb{C}^{2^n}$, and let $x$ be the exact value. Then

$$\|\hat{x} - x\|_\infty = n\|\alpha\|_\infty O(u).$$

**Table: IDCT error bounds for $p = 53$ (double precision)**

<table>
<thead>
<tr>
<th>$N = 2^n$</th>
<th>1,024</th>
<th>2,048</th>
<th>4,096</th>
<th>8,192</th>
<th>16,384</th>
<th>32,768</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|\hat{x} - x|<em>\infty / |\alpha|</em>\infty$</td>
<td>7.97e-15</td>
<td>8.84e-15</td>
<td>9.72e-15</td>
<td>1.06e-14</td>
<td>1.15e-14</td>
<td>1.23e-14</td>
</tr>
</tbody>
</table>
Algorithms
General idea: edge enclosure

Illustration

\[ P(X, Y) = \sum \left( \sum a_{i,j}X^i \right) Y^j = \sum p_j(X)Y^j \]

\[ p_j(X) = \sum a_{i,j}X^i = \sum \alpha_{i,j} T_i(X) \]

\[ (p_j(c_n))_{0 \leq n \leq N-1} = \frac{1}{2} (\alpha_{0,j}, \ldots, \alpha_{0,j}) + \text{IDCT}((\alpha_{k,j})_{0 \leq k \leq N-1}) \]
General idea: edge enclosure

Illustration

\[ P(c_n, Y) = \sum p_j(c_n) Y^j \]
General idea: edge enclosure

Illustration

\[ P(c_3, Y) = \sum p_j(c_3) Y^j \]
General idea: edge enclosure

Illustration

\[ P(c_3, Y) = \sum p_j(c_3) Y^j \]
General idea: edge enclosure

Illustration

\[ P(c_3, Y) = \sum p_j(c_3) Y^j \]
General idea: edge enclosure

Illustration

\[ P(c_3, Y) = \sum p_j(c_3) Y^j \]
An edge enclosing algorithm

IDCT multipoint evaluation in $X$

at $c_0, c_1 \ldots$

subdivision in $Y$

IDCT multipoint evaluation of the partial polynomials of $P(X, Y) = \sum p_j(X) Y^j$
An edge enclosing algorithm

IDCT multipoint evaluation in $X$
   at $c_0, c_1 \ldots$

IDCT multipoint evaluation of the partial polynomials of $P(X, Y) = \sum p_j(X)Y^j$

subdivision in $Y$
An edge enclosing algorithm

\[ P(c_0, Y) \quad P(c_1, Y) \quad P(c_2, Y) \quad P(c_3, Y) \quad P(c_4, Y) \]

IDCT multipoint evaluation in \( X \) at \( c_0, c_1 \ldots \)

subdivision in \( Y \)

IDCT multipoint evaluation of the partial polynomials of \( P(X, Y) = \sum p_j(X) Y^j \)
General idea: pixel enclosure

Illustration

\[ P(I, Y) = \sum p_j(I) Y^j \]
General idea: pixel enclosure

Illustration

\[ P(I, Y) = \sum p_j(I) Y^j \]
General idea: pixel enclosure

Illustration

\[ P(I, Y) = \sum p_j(I) Y^j \]
General idea: pixel enclosure

Illustration

\[ P(I, Y) = \sum p_j(I) Y^j \]
A pixel enclosing algorithm

IDCT multipoint evaluation in $X$ around $c_0, c_1 \ldots$ subdivision in $Y$
A pixel enclosing algorithm

IDCT multipoint evaluation + 
Taylor approximation in \( X \)

subdivision in \( Y \)

Taylor expansion of the partial polynomials of \( P(X, Y) = \sum p_j(X) Y^j \)

\[
\left| p(c_n + r) - \left( p(c_n) + rp'(c_n) + \cdots + \frac{r^m}{m!} p^{(m)}(c_n) \right) \right| \leq \max_{l_{cn}} \left| p^{(m+1)} \right| \frac{|r|^{(m+1)}}{(m + 1)!}
\]
A pixel enclosing algorithm

\[ P(\mathcal{I}_c^0, Y) \]
\[ P(\mathcal{I}_c^1, Y) \]
\[ P(\mathcal{I}_c^2, Y) \]
\[ P(\mathcal{I}_c^3, Y) \]
\[ P(\mathcal{I}_c^4, Y) \]
\[ P(\mathcal{I}_c^n, I_1) \]
\[ P(\mathcal{I}_c^n, I_2) \]

IDCT multipoint evaluation +
Taylor approximation in \( X \)

subdivision in \( Y \)

Taylor expansion of the partial polynomials of \( P(X, Y) = \sum p_j(X) Y^j \)

\[
\left| p(c_n + r) - \left( p(c_n) + rp'(c_n) + \cdots + \frac{r^m}{m!} p^{(m)}(c_n) \right) \right| \leq \max_{l_{c_n}} \left| p^{(m+1)} \right| \frac{|r|^{(m+1)}}{(m + 1)!}
\]
A pixel enclosing algorithm

IDCT multipoint evaluation +
Taylor approximation in $X$

subdivision in $Y$

Taylor expansion of the partial polynomials of $P(X, Y) = \sum p_j(X) Y^j$

$$\left| p(c_n + r) - \left( p(c_n) + rp'(c_n) + \cdots + \frac{r^m}{m!} p^{(m)}(c_n) \right) \right| \leq \max_{l_c_n} p^{(m+1)} \left| \frac{r}{(m+1)!} \right|$$
## Complexities

### Arithmetic complexities

<table>
<thead>
<tr>
<th>Operation</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multipoint evaluation and subdivision</td>
<td>$O(d^3 + dN \log_2(N) + dNT)$</td>
</tr>
<tr>
<td>Multipoint Taylor approximation and subdivision</td>
<td>$O(md^3 + mN \log_2(N) + dNT)$</td>
</tr>
</tbody>
</table>

$d$ partial degree  
$N$ resolution  
$T$ maximum number of nodes of the subdivision trees over all vertical fibers / stripes  

With a constant number of branches in the window, we expect $T = O(\log_2(N))$
Experiments
Pixel classification

- crossed: blue
- not crossed: white
- undecided: yellow
Drawing for two families of polynomials

Experiments on smooth curves $\rightarrow$ random polynomials

$\xi_{i,j}$: random coefficients in $[-100, 100]$

**Kac polynomial**

$$P(X, Y) = \sum_{i+j=0}^{d} \xi_{i,j} X^i Y^j$$

**Kostlan-Shub-Smale (KSS) polynomial**

$$P(X, Y) = \sum_{i+j=0}^{d} \sqrt{\frac{d!}{i! j! (d-i-j)!}} \xi_{i,j} X^i Y^j$$
Drawing for two families of polynomials

Figure: Kac polynomial of degree $d = 110$ at a resolution $N = 1,024$, $\frac{b}{b+y} = 24\%$
Drawing for two families of polynomials

Figure: KSS polynomial of degree $d = 40$ at a resolution $N = 1,024$, $\frac{b}{b+y} = 19\%$
Comparison to state-of-the-art software

Our methods

- edge drawing → curve enclosing edges  
  → false positive and false negative
- pixel drawing → curve enclosing pixels  
  → false positive

Some similar methods

- scikit / NumPy → marching squares  
  → false negative
- MATLAB → could not find the method used  
  → false negative?
- Implicit Equations → 2D adaptive subdivision  
  → false positive

A topologically correct method

- Isotop → cylindrical algebraic decomposition
Timing

Comparison for a polynomial

Computation times for a **Kac** polynomial of degree 40 (in seconds)
Timing
Comparison for a polynomial

Computation times for a Kac polynomial of degree 40 (in seconds)

scikit: $O(dN^2)$

no guarantee
slow when $d$ and $N$ are large

Our methods: $O(dNT)$

as expected $T = O(\log_2(N))$

guarantees
fast when $d$ and $N$ are large
Output for a singular curve

Curve: $\text{dfold}_{8,1}$ from Challenge 14 of Oliver Labs[13][37] ($d = 18$)
Conclusion

Contributions

- Two algorithms
  - enclosure of the edges
  - enclosure of the pixels
- Fast implicit curve and surface algorithms for high resolutions: faster than marching squares and marching cubes
- Better guarantees on the drawing than marching squares
- Ability to handle high degrees \((d > 20)\) and high resolutions \((N > 1000)\)

Future work

- Can the thickness of the drawing be controlled?
- Could we have a faster subdivision with other root finding methods?
- Can the multipoint evaluation improve Plantinga and Vegter’s algorithm?
Timing

A CAD approach: Isotop

Figure: Computation times for a Kac polynomials (in seconds)