2 Exercices octobre 2020

2.1 Dessiner...

2.1 Correction:
2.2 Nearest neighbors

$S$ a set of $n$ points.

Denote $NN(q)$ the nearest neighbor of $q$ in $S \setminus \{q\}$ ($q$ may belong to $S$ or not).

For a point $q$, cut the plane in four quadrants using the two lines through $q$ of slopes 1 and -1, and call these quadrants North, West, South, and East. Denote $NN_N(q)$, $NN_W(q)$, $NN_S(q)$, and $NN_E(q)$ the nearest neighbor of $q$ in $S \setminus \{q\}$ in each quadrant.

2.2.1 Nearest neighbor

For $q \in S$, is $NN(q)$ a neighbor of $q$ in the Delaunay triangulation of $S$? Prove it or draw a counter example.

If the answer is no, then prove that there exist $w$ a Delaunay neighbor of $q$ such that the distance $\|qw\|$ is not larger than $\alpha \cdot \|qNN(q)\|$ for some constant $\alpha$. Give the best value for $\alpha$.

2.2.2 Oriented nearest neighbor

For $q \in S$, is $NN_N(q)$ a neighbor of $q$ in the Delaunay triangulation of $S$? Prove it or draw a counter example.

If the answer is no, then prove that there exist $w$ a Delaunay neighbor of $q$ in the half-plane above $q$ such that the distance $\|qw\|$ is not larger than $\alpha \cdot \|qNN_N(q)\|$ for some constant $\alpha$. Give the best value for $\alpha$.

2.2.3 Nearest neighbor path

Let $T$ be a triangulation of $S$, $p$ be a point (not in $S$) and $q \in S$. Define the sequence $(q_i)_{i \in \mathbb{N}}$ by $q_0 = q$ and $q_{i+1}$ the closest point of $p$ amongst the neighbors of $q_i$ in the triangulation $T$, and $q_i$ itself (assume no ties).

What are the possible behavior of this sequence? Does it have a limit? If yes, what is this limit?

If $T$ is not any triangulation but the Delaunay triangulation, what are the possible behavior of this sequence? Does it have a limit? If yes, what is this limit?
2.2 Correction:

2.2.1 Nearest neighbor

The circle of center $q$ passing through $NN(q)$ enclose no other point of $S$ than $q$ (since $NN(q)$ is the closest neighbor). The disk of diameter $qNN(q)$ is enclosed in the previous one and has $q$ and $NN(q)$ on its boundary, thus this disk witnesses that $qNN(q)$ is an edge of the Delaunay triangulation.

2.2.2 Oriented nearest neighbor

$NN_N(q)$ is not always a neighbor of $q$ in the Delaunay triangulation of $S$, see above figure.

Consider the intersections points of the circle $C_1$ of center $q$ through $NN(q)$ and the North-West ray and the North-East ray. Then consider the circle $C_2$ through $q$ and these two points. $C_2$ must enclose $NN(q)$ since it enclose the part of $C_3$ in the North quadrant where $NN(q)$ lies. Thus $C_2$ is not empty and must contain a Delaunay neighbor of $q$. Looking at the largest distance between a point inside $C_2$ and $q$ we get $\alpha = \sqrt{2}$.

The left picture provide an example proving $\alpha$ is tight.

2.2.3 Nearest neighbor path

The sequence is stationary. Actually, the distance $\|pq_i\|$ is positive and decreasing and, since $S$ is finite, can take a finite number of values. Thus it has a limit. Since we assume that no two points of $S$ are at the same distance of $p$, the sequence is stationary at some point $q_{\text{limit}}$. See example above.

If the triangulation is the Delaunay triangulation, then the limit is $NN(q)$ because if the circle $C$ of center $p$ through $q_{\text{limit}}$ contains a point $w$, then we can consider the maximal empty disk passing through $q_{\text{limit}}$ tangent to $C$ inside $C$. This circle pass through a point $w'$ of $S$ ($w' = w$ or another point inside $C$ better than $w$) witnessing an edge $q_{\text{limit}}w'$ of the Delaunay triangulation proving that $q_{\text{limit}}$ has a neighbor closest to $p$ establishing a contradiction.
2.3 Diameter

Let \( S \) be a set of points in the plane. The diameter of \( S \) is the pair of point in \( S^2 \) that realizes the largest distance (assume no degenracies).

The collision problem is: “given a two set of \( n \) real numbers in some interval, determine if if the intersection of the two sets is non empty”

**Theorem:** The collision problem has an \( \Omega(n \log n) \) lower bound in the real-RAM model.

2.3.1 Diameter lower bound

Prove that the diameter problem has an \( \Omega(n \log n) \) lower bound in the real-RAM model.

Hint: design a stupid algorithm for the collision problem for a set numbers in \([0, \pi/2]\).

2.3 Correction:

Stupid algorithm for the collision problem:

Let \( \alpha_i \) and \( \beta_i \) be two sets of \( n \) numbers in \([0, \pi/2]\).

Create \( 2n \) points \( p_i = (\cos \alpha_i, \sin \alpha_i) \) and \( q_i = (-\cos \beta_i, -\sin \beta_i) \).

Solve the diameter problem on this set of points.

If the diameter \( p_i q_j \) has length 2 answer that \( \alpha_i = \beta_j \) as a witness of non empty intersection.

If the length is strictly less than 2, then answer “empty intersection”.

The complexity of this algorithm is linear plus the complexity of solving the diameter problem, thus the diameter problem cannot be solved faster than \( n \log n \) without contradicting the lower bound on the collision problem.