3 Exercices 7 octobre 2021

3.1 Dessiner...

3.1 Correction:

3.2 Nearest neighbor graphs

$S$ a set of $n$ points. $q_0 \in S$. Let $q_1$ denote the nearest neighbor of $q_0$ in $S \setminus \{q_0\}$. Let $q_2$ denote the second nearest neighbor of $q_0$ in $S$, i.e., the nearest neighbor in $S \setminus \{q_0, q_1\}$. Similarly $q_i$ the $i^{th}$ nearest neighbor.

The directed nearest neighbor graph of $S$ is the graph whose vertices are the points in $S$ and $pq$ is an edge of the graph if $q$ is the nearest neighbor of $p$.

Fact: The degree of the nearest neighbor graph is $\leq 6$. (proof optional).

3.2.1 Nearest neighbor

Prove that $q_0 q_1$ is an edge of the Delaunay triangulation of $S$.

3.2.2 Second nearest neighbor

Prove that $q_0 q_2$ or $q_1 q_2$ is an edge of the Delaunay triangulation of $S$.

3.2.3 $k^{th}$ nearest neighbor

Prove that $\forall k \exists i < k$ such that $q_k q_i$ is an edge of the Delaunay triangulation of $S$.

3.2.4 Nearest neighbor graph

Write an algorithm that takes the Delaunay triangulation of $S$ and output the directed nearest neighbor graph of $S$.

You can write things like:

for $v$ enumerating all vertices of $DT(S)$,
for $w$ enumerating the neighbor of $v$ in $DT(S)$,
or output edge($v, w$),
or $v.color = red$ to add some information in a vertex (or edge or...)

What is the complexity of this algorithm?

3.2.5 Nearest neighbor graph

Write an algorithm that takes the Delaunay triangulation of $S$ and output the directed second nearest neighbor graph of $S$.

What is the complexity of this algorithm?
3.2 Correction:

3.2.1 Nearest neighbor
The disk centered at $q_0$ passing through $q_1$ contains only $q_0$, thus the disk of diameter $q_0q_1$, which is included in the previous one is empty. By the empty circle property, $q_0q_1$ is a Delaunay edge.

3.2.2 Second nearest neighbor
The disk $D_2$ centered at $q_0$ passing through $q_2$ contains only $q_0$ and $q_1$, thus we consider the two disks $Z_0$ and $Z_1$ passing through $q_2$ tangent to $D_2$ and respectively passing through $q_0$ and $q_1$. We have to cases:
- $Z_0 \subset Z_1 \subset D_2$ and $Z_0$ is empty, by the empty circle property, $q_0q_2$ is a Delaunay edge.
- $Z_1 \subset Z_0 \subset D_2$ and $Z_1$ is empty, by the empty circle property, $q_1q_2$ is a Delaunay edge.

3.2.3 $k^{th}$ nearest neighbor
The disk of center $q_0$ through $q_k$ verifies $D_k \cap S = \{q_0, q_1, \ldots q_{k-1}\}$. Consider the pencil of circles through $q_k$ tangent to $D_k$. The biggest empty circle of that pencil inside $D_k$ pass through a point inside $D_k$ that is some $q_i$ with $i < k$ and by the empty circle property, $q_iq_k$ is a Delaunay edge.

3.2.4 Nearest neighbor graph
for $u$ enumerating all vertices of $DT(S)$ 
\begin{align*}
  d &= \infty; \\
  &\text{for } w \text{ enumerating the neighbor of } u \text{ in } DT(S) \{ \\
  &\quad \text{if } ||uw|| < d \text{ then } \{nn = w; \ d = ||uw||; \} \\
  &\}\text{output edge}(u, nn),
\end{align*}

The inside loop costs $d^i(u)$, thus the total cost of the algorithm is $\sum_{u \in S} d^i(u) < 6n$.

3.2.5 Nearest neighbor graph
for $u$ enumerating all vertices of $DT(S)$ 
\begin{align*}
  d &= \infty; \\
  &\text{for } w \text{ enumerating the neighbor of } u \text{ in } DT(S) \{ \\
  &\quad \text{if } ||uw|| < d \text{ then } \{nn = w; \ d = ||uw||; \} \\
  &\}\text{for } w \text{ enumerating the neighbor of } u, nn \text{ in } DT(S) \{ \\
  &\quad \text{if } ||uw|| < d \text{ and } w \neq u \text{ then } \{sn = w; \ d = ||uw||; \} \\
  &\}\text{output edge}(u, sn),
\end{align*}

The cost is
\begin{align*}
  \sum_{u \in S} (d^i_{DT}(u) + d^i_{NN}(u, nn)) &= \sum_{u \in S} d^i_{DT}(u) + \sum_{u \in S} \sum_{v \in \{u, nn\}} d^v_{DT}(v) \\
  &= \sum_{u \in S} d^i_{DT}(u) + \sum_{v \in S} \sum_{u \text{ such that } v = u, nn} d^v_{DT}(v) \\
  &= \sum_{u \in S} d^i_{DT}(u) + \sum_{v \in S} d^v_{NN}(v) \cdot d^i_{DT}(v) \\
  &= \sum_{u \in S} d^i_{DT}(u) + \sum_{v \in S} 6d^v_{DT}(v) \leq 42n
\end{align*}