Randomized algorithms for Delaunay triangulations

Poisson Delaunay triangulation
Randomized algorithms for Delaunay triangulations

- Randomized backward analysis of binary trees
- Randomized incremental construction of Delaunay
- Jump and walk
- The Delaunay hierarchy
- Biased randomized incremental order
- Chew algorithm for convex polygon

Poisson Delaunay triangulation

- Poisson distribution
- Slivnyak-Mecke formula
- Blaschke-Petkanschin variables substitution
- Stupid analysis of the expected degree
- Straight walk expected analysis
- Catalog of properties
Sorting
Sorting
Sorting
Binary tree
Sorting
Binary tree
Sorting

Binary tree
Sorting

Binary tree
Sorting

Binary tree
Sorting
Binary tree

4 - 6
Sorting

Binary tree

```
        8
       / \
      4   7
       /   /
     14   12
```
Sorting

Binary tree

4 - 8
Sorting

Binary tree
Sorting

Binary tree

\[
\begin{align*}
&\infty, 1] \\
1 & \rightarrow 4 \\
& \rightarrow 8 \\
& \rightarrow 7 \\
11 & \rightarrow 14 \\
& \rightarrow 12
\end{align*}
\]
Sorting

Binary tree

\[
\begin{align*}
\text{] } & - \infty, 1] \\
\text{] } & 1, 4] \\
\text{] } & 11
\end{align*}
\]
Sorting

Binary tree
Sorting

1

time

5 - 1

new drawing
Sorting

new drawing

time

5 - 2
Sorting

new drawing

time

5 - 3
Sorting

1

2

3

4

5 - 4

new drawing
Sorting

new drawing
Sorting

time

new drawing
Sorting

new drawing

1
2
3
4
5
6
7
time

∞ 1 4 7 8 11 12 14 ∞
Sorting

1
2
3
4
5
6
7

new drawing

time
Sorting

time

6 - 1
Sorting
Sorting
Sorting

$k$

time

6 - 4
Sorting

Localisation

$k$

$n$

time

6 - 6

∞
Sorting

Localisation

time

n

k

6 - 7

∞  ∞
Sorting

Localisation

\( k \)

\( n \)

\( \text{time} \)
$\mathbb{E}[\#\text{visited nodes}] \in O\left(\frac{2}{k}\right)$
Sorting

$\mathbb{E} \left[ \# \text{visited nodes} \right] \mathcal{O} \left( \frac{2}{k} \right)$

Total insertion: $\sum_{k} \frac{2}{k} \approx 2 \log n$
Localisation

Total insertion: \( \sum_k \frac{2}{k} \approx 2 \log n \)

Total construction: \( \sum_k 2 \log k \approx 2n \log n \)

\[ \mathbb{E} [\# \text{visited nodes}] \mathcal{O} \left( \frac{2}{k} \right) \]
Sorting

[\[-\infty, \infty[\] - new drawing

conflict graph

7 - 1
Sorting

\[ -\infty, 8 \rightbracket \quad \rightbracket 8, \infty \]
Sorting

\[ -\infty, 8 \) \hspace{1cm} \) 8, \infty \]

\[ -\infty, 4 \) \hspace{1cm} \) 4, 8 \)

\[ 8, 14 \) \hspace{1cm} \) 14, \infty \)
## Sorting

<table>
<thead>
<tr>
<th>Unbalanced binary tree</th>
<th>History graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quicksort</td>
<td>Conflict graph</td>
</tr>
</tbody>
</table>

$O(n \log n)$  

Same analysis

- Backwards analysis
- Analyse last insertion and sum
- Last object is a random object
Randomization

Backwards analysis for Delaunay triangulation
Delaunay triangulation

\# of triangles during incremental construction?
Delaunay triangulation

\# of triangles during incremental construction?
Delaunay triangulation

# of triangles during incremental construction?

# triangles created/incident to last point?
Delaunay triangulation

# of triangles during incremental construction?

# triangles created/incident to last point?

Last point?
\frac{1}{n} \sum_{i=1}^{n} d^\circ(p_i) \leq 6
\[
\frac{1}{n} \sum_{i=1}^{n} d^\circ(p_i) \leq 6
\]

\[
\sum 6 = 6n
\]
Alternative analysis

Triangle $\Delta$ with $j$ stoppers
Alternative analysis

Triangle $\triangle$ with $j$ stoppers

Probability that it exists in the triangulation of a sample of size $\alpha n$

$$\approx \alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4} \alpha^3 \quad \text{if} \quad 2 \leq j \leq \frac{1}{\alpha}$$
Alternative analysis

Triangle \( \Delta \) with \( j \) stoppers

Probability that it exists in the triangulation of a sample of size \( \alpha n \)

\[
\sim \alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha) \frac{1}{\alpha} \geq \frac{1}{4} \alpha^3 \quad \text{if} \ 2 \leq j \leq \frac{1}{\alpha}
\]

Size of the triangulation of the sample

\[
= \sum_{j=0}^{n} \mathbb{P} [\Delta \text{ with } j \text{ stoppers is there}] \times \#\Delta \text{ with } j \text{ stoppers}
\]
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists in the triangulation of a sample of size $\alpha n$

$$\approx \alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4} \alpha^3$$
if $2 \leq j \leq \frac{1}{\alpha}$

Size of the triangulation of the sample

$$= \sum_{j=0}^{n} P[\Delta \text{ with } j \text{ stoppers is there}] \times \#\Delta \text{ with } j \text{ stoppers}$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \#\Delta \text{ with } j \text{ stoppers}$$
Alternative analysis

Triangle $\triangle$ with $j$ stoppers

Probability that it exists in the triangulation of a sample of size $\alpha n$

\[
\approx \alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4} \alpha^3 \quad \text{if } 2 \leq j \leq \frac{1}{\alpha}
\]

Size of the triangulation of the sample

\[
= \sum_{j=0}^{n} P[\text{$\triangle$ with } j \text{ stoppers is there}] \times \#\text{$\triangle$ with } j \text{ stoppers}
\]

\[
\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \#\text{$\triangle$ with } j \text{ stoppers} = \alpha^3 \#\text{$\triangle$ with } \leq \frac{1}{\alpha} \text{ stoppers}
\]
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists in the triangulation of a sample of size $\alpha n$

$$\simeq \alpha^3 (1 - \alpha)^j \geq \alpha^3 \left(1 - \frac{1}{\alpha}\right) \geq \frac{1}{4} \alpha^3 \quad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Size of the triangulation of the sample

$$= \sum_{j=0}^{n} \mathbb{P} [\Delta \text{ with } j \text{ stoppers is there}] \times \#\Delta \text{ with } j \text{ stoppers} \quad \geq \frac{1}{\alpha} \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \#\Delta \text{ with } j \text{ stoppers} = \alpha^3 \#\Delta \text{ with } \leq \frac{1}{\alpha} \text{ stoppers} = O(\alpha n)$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists in the triangulation of a sample of size $\alpha n$

$$\simeq \alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4} \alpha^3 \quad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Size of the triangulation of the sample

$$= \sum_{j=0}^{n} \mathbb{P} [\Delta \text{ with } j \text{ stoppers is there}] \times \#\Delta \text{ with } j \text{ stoppers} = O(\alpha n)$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \#\Delta \text{ with } j \text{ stoppers} = \alpha^3 \#\Delta \text{ with } \leq \frac{1}{\alpha} \text{ stoppers}$$

Size (order $\leq k$ Voronoi) $\leq \frac{\alpha n}{\alpha^3} = nk^2$
Alternative analysis

Triangle $\triangle$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \cdot \frac{2}{j+2} \cdot \frac{1}{j+1}$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

\[
= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}
\]

\# of created triangles

\[
= \sum_{j=0}^{n} \mathbb{P} [\Delta \text{ with } j \text{ stoppers appears}] \times \# \Delta \text{ with } j \text{ stoppers}
\]
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}$$

# of created triangles

$$= \sum_{j=0}^{n} \mathbb{P} [\Delta \text{ with } j \text{ stoppers appears}] \times \# \Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} (\mathbb{P} [\Delta \text{ with } j] - \mathbb{P} [\Delta \text{ with } j + 1]) \times \# \Delta \text{ with } \leq j \text{ stoppers}$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$\begin{align*}
= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}
\end{align*}$$

$\#$ of created triangles

$$\begin{align*}
= \sum_{j=0}^{n} P[\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers}
\end{align*}$$

$$\begin{align*}
= \sum_{j=0}^{n} (P[\Delta \text{ with } j] - P[\Delta \text{ with } j + 1]) \times \#\Delta \text{ with } \leq j \text{ stoppers}
\end{align*}$$

$$\begin{align*}
\approx \sum_{j=0}^{n} \frac{18}{j^4} \times nj^2 = O(n \sum \frac{1}{j^2}) = O(n)
\end{align*}$$
Alternative analysis

Triangle $\triangle$ with $j$ stoppers

Conflict graph / History graph

It remains to analyze conflict location
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \cdot \frac{2}{j+2} \cdot \frac{1}{j+1}$$

\# of conflicts occurring

$$= \sum_{j=0}^{n} j \times P[\text{\#\ of\ } \Delta\ with\ j\ stoppers\ appears] \times \#\Delta\ with\ j\ stoppers$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}$$

$\#$ of conflicts occurring

$$= \sum_{j=0}^{n} j \times P[\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} j \times (P[\Delta \text{ with } j] - P[\Delta \text{ with } j + 1]) \times \#\Delta \text{ with } \leq j \text{ stoppers}$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}$$

\# of conflicts occurring

$$= \sum_{j=0}^{n} j \times \mathbb{P} [\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} j \times (\mathbb{P} [\Delta \text{ with } j] - \mathbb{P} [\Delta \text{ with } j+1]) \times \#\Delta \text{ with } \leq j \text{ stoppers}$$

$$\approx \sum_{j=0}^{n} j \times \frac{18}{j^4} \times n j^2 = O(n \sum_{j=0}^{n} \frac{1}{j}) = O(n \log n)$$
History graph
History graph
History graph
History graph

Stepfather

Father
History graph

(Delaunay tree)

Stepfather

Father
History graph

(Delaunay tree)

Stepfather

Father

if conflict there was a conflict with the father or the stepfather or both
Conflict graph

14 - 2
Conflict graph
Conflict graph
Conflict graph
Conflict graph
Conflict graph
Walk
Walk

15 - 2
Walk
Walk

Complexity $O(n)$
Walk

Complexity $O(n)$

Better bounds for random points

Teaser probability lecture
Jump and walk
Jump and walk
Jump and walk
Jump and walk

Hopefully shorter walk

Designed for random points

$O\left(\sqrt[3]{n}\right)$ expected location time
Jump and walk (no distribution hypothesis)
Jump and walk (no distribution hypothesis)

\[ \mathbb{E} \left[ \text{\# of } \bullet \text{ in } \bigcirc \right] = \frac{n}{k} \]
Jump and walk (no distribution hypothesis)

\[ \mathbb{E} \left[ \text{\# of in} \right] = \frac{n}{k} \]

Walk length = \( O \left( \frac{n}{k} \right) \)

choose \( k = \sqrt{2n} \)
Jump and walk (no distribution hypothesis) Delaunay hierarchy

\[ \mathbb{E} [\# \text{ of } \bullet \text{ in } \circled{\bullet}] = \frac{n}{k} \]

Walk length = \( O \left( \frac{n}{k} \right) \)

choose \( k = \sqrt{n} \)
Jump and walk (no distribution hypothesis) Delaunay hierarchy

$$\mathbb{E} \left[ \text{# of } \bullet \text{ in } \bigcirc \right] = \frac{n}{k} \quad \frac{n}{k_1}$$

Walk length = $O \left( \frac{n}{k} \right)$

choose $k = \sqrt{n}$
Jump and walk (no distribution hypothesis) Delaunay hierarchy

\[ \mathbb{E} [\# \text{ of } \bullet \text{ in } \bigcirc] = \frac{n}{k} \]

\[ \frac{n}{k_1} + \frac{k_1}{k_2} \]

Walk length = \( O \left( \frac{n}{k} \right) \)

choose \( k = \sqrt{n} \)

Diagram of Delaunay hierarchy with nodes and edges.
Jump and walk (no distribution hypothesis) Delaunay hierarchy

\[ \mathbb{E} \left[ \# \text{ of } \bullet \text{ in } \bigcirc \right] = \frac{n}{k} \]

Walk length = \( O \left( \frac{n}{k} \right) \)

choose \( k = \sqrt[3]{n} \)
Jump and walk (no distribution hypothesis) Delaunay hierarchy

\[ E[\text{# of } \bullet \text{ in } \bigcirc] = \frac{n}{k} \]

Walk length = \( O\left(\frac{n}{k}\right) \)

choose \( k = \sqrt{n} \)

choose \( \frac{k_i}{k_{i+1}} = \alpha \)

\[ \frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots \]
Jump and walk (no distribution hypothesis) Delaunay hierarchy

\[ \mathbb{E} \left[ \text{\# of } \bullet \text{ in } \bigcirc \right] = \frac{n}{k} \]

Walk length = \( O \left( \frac{n}{k} \right) \)

Choose \( k = \sqrt{n} \)

Choose \( \frac{k_i}{k_{i+1}} = \alpha \)

Point location in \( O(\alpha \log_\alpha n) \)
Jump and walk (no distribution hypothesis)

\( \mathbb{E} [\# \text{ of } \bullet \text{ in } \bigcirc] = \frac{n}{k} \)

Delaunay hierarchy

Walk length = \( O \left( \frac{n}{k} \right) \)

choose \( k = \sqrt{n} \)

choose \( \frac{k_i}{k_{i+1}} = \alpha \)

point location in \( O(\alpha \log_\alpha n) \)

point location in \( O(\sqrt{\alpha} \log_\alpha n) \)
Randomization

How many randomness is necessary?

If the data are not known in advance

shuffle locally
Randomization

Drawbacks of random order

non locality of memory access

data structure for point location

Hilbert sort
Drawbacks of random order

- non locality of memory access
- data structure for point location

Hilbert sort

Walk should be fast

Last point is not at all a random point

no control of degree of last point
23 - 6
Triangle $\triangle$ with $j$ stoppers
Triangle $\triangle$ with $j$ stoppers

Size (order $\leq k$ Voronoi) $\leq \frac{\alpha n}{\alpha^3} = nk^2$
Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}$$
Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

\[ \frac{3}{j+3} \cdot \frac{2}{j+2} \cdot \frac{1}{j+1} \]
remains $\Theta(j^{-3})$
Probability that it exists during the construction remains \( \Theta(j^{-3}) \)

\[
\sum_{j=0}^{n} P[\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers} \\
\approx O\left(\sum_{j=0}^{n} \frac{n j^2}{j^4}\right) = O(n)
\]
Triangle $\triangle$ with $j$ stoppers

Probability that it exists during the construction

\[ \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1} \]

remains $\Theta(j^{-3})$

$\#$ of conflicts occurring

\[ \sum_{j=0}^{n} j \times \mathbb{P}[\triangle \text{ with } j \text{ stoppers appears}] \times \#\triangle \text{ with } j \text{ stoppers} \]

\[ \simeq O\left(\sum_{j} j \frac{n j^2}{j^4}\right) = O(n \log n) \]
Delaunay 2D 1M random points

locate using Delaunay hierarchy 6 seconds
random order (visibility walk) 157 seconds
$x$-order 3 seconds
Hilbert order 0.8 seconds
Biased order (Spatial sorting) 0.7 seconds
<table>
<thead>
<tr>
<th>Method</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delaunay 2D 100K parabola points</td>
<td>0.3</td>
</tr>
<tr>
<td>locate using Delaunay hierarchy</td>
<td></td>
</tr>
<tr>
<td>random order (visibility walk)</td>
<td>128</td>
</tr>
<tr>
<td>$x$-order</td>
<td>632</td>
</tr>
<tr>
<td>Hilbert order</td>
<td>46</td>
</tr>
<tr>
<td>Biased order (Spatial sorting)</td>
<td>0.3</td>
</tr>
</tbody>
</table>
3D
3D

Degree of a random point?

$O(n)$ worst case.

$O(1)$ in practical cases?

$O(\log n)$ for random points on a cylinder.

$O(\sqrt{n})$ for “good” samples.

Final size of the triangulation is not enough.
Randomization

Avoiding point location
Delaunay randomized construction

$O(n)$
Delaunay randomized construction

$O(n) + \text{point location}$
Delaunay randomized construction

$O(n) + \text{point location}$

Use additional information to save on point location
Delaunay randomized construction

\[ O(n) \ + \ \text{point location} \]

Use additional information to save on point location

e.g. points are sorted by spatial sort
Delaunay randomized construction

\[ O(n) + \text{point location} \]

Use additional information to save on point location

e.g. points are sorted by spatial sort

Delaunay of points in convex position

Splitting Delaunay
Delaunay of points in convex position
Delaunay of points in convex position
Delaunay of points in convex position

choose a point at random
Delaunay of points in convex position

choose a point at random
remove it from convex polygon
Delaunay of points in convex position

choose a point at random
remove it from convex polygon
remember its place
Delaunay of points in convex position

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
Delaunay of points in convex position

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points
with relevant vertex-triangle pointers
insert point, (location known)
choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points
with relevant vertex-triangle pointers
insert point, (location known)
choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points
with relevant vertex-triangle pointers
insert point, (location known)
choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n-1 \) points
with relevant vertex-triangle pointers
insert point, (location known)
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points
    with relevant vertex-triangle pointers
insert point, (location known)
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points
with relevant vertex-triangle pointers
insert point, (location known)
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
    with relevant vertex-triangle pointers
insert point, (location known)

\[ O(1) \]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points

with relevant vertex-triangle pointers

insert point, (location known)

\[
\begin{align*}
\{ & O(1) \\
O(d^o p) & \}
\end{align*}
\]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
insert point, (location known)

\[
\begin{align*}
\{ & O(1) \\
O(d^\circ p) &= O(1)
\end{align*}
\]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
    with relevant vertex-triangle pointers
insert point, (location known)

\[
\begin{align*}
\{ \text{compute } & f(n - 1) \} \\
\text{compute } & O(d^o p) = O(1)
\end{align*}
\]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points

\[
\text{with relevant vertex-triangle pointers}
\]
insert point, (location known)

\[
f(n) = f(n - 1) + O(1)
\]

\[
O(d^\circ p) = O(1)
\]

\[
f(n - 1) = O(1)
\]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
insert point, (location known)

\[
f(n) = f(n - 1) + O(1) = O(n)
\]

\[
O(d^p) = O(1)
\]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points
with relevant vertex-triangle pointers
insert point, (location known)

$$f(n) = f(n - 1) + O(1) = O(n)$$

$$O(d^o p) = O(1)$$

$$f(n - 1) \leq O(1)$$

[Chew 86]
Randomization
Randomization

Randomized incremental constructions

Simple algorithms
non trivial analysis
good complexities
efficient in practice
Randomization

Randomized incremental constructions

Simple algorithms
non trivial analysis
good complexities
efficient in practice

Delaunay hierarchy
Spatial sorting
Randomization

Randomized incremental constructions

Simple algorithms
non trivial analysis
good complexities
efficient in practice

Other tools
divide and conquer
\( \epsilon \) nets
Good sample with high probability

Delaunay hierarchy
Spatial sorting

CGAL
Poisson Delaunay triangulation
Poisson Delaunay triangulation

- Poisson distribution
- Slivnyak-Mecke formula
- Blaschke-Petkanschin variables substitution
- Stupid analysis of the expected degree
- Straight walk expected analysis
- Catalog of properties
Poisson distribution

\[ X \] a Poisson point process

Distribution in \( A \) independent from distribution in \( B \).

when \( A \cap B = \emptyset \)

Unit uniform rate

\[
P \left[ |X \cap A| = k \right] = \frac{\text{vol}(A)^k}{k!} e^{-\text{vol}(A)}
\]
Poisson distribution

\[ X \text{ a Poisson point process} \]

Distribution in \( A \) independent from distribution in \( B \).

when \( A \cap B = \emptyset \)

Unit uniform rate

\[
\mathbb{P} [|X \cap A| = k] = \frac{\text{vol}(A)^k}{k!} e^{-\text{vol}(A)}
\]
Poisson distribution

\( X \) a Poisson point process

Distribution in \( A \) independent from distribution in \( B \).

when \( A \cap B = \emptyset \)

Unit uniform rate

\[
P[|X \cap A| = k] = \frac{\text{vol}(A)^k}{k!} e^{-\text{vol}(A)}
\]

\[
P[|X \cap A| = 0] = e^{-\text{vol}(A)}
\]

\[
\mathbb{E}[|X \cap A|] = \sum_{k=0}^{\infty} k \frac{\text{vol}(A)^k}{k!} e^{-\text{vol}(A)} = \text{vol}(A)
\]
Slivnyak-Mecke formula

$X$ a Poisson point process of density $n$

Sum $\rightarrow$ Integral
Slivnyak-Mecke formula

$X$ a Poisson point process of density $n$

$\mathbb{E} \left[ \sum_{q \in X} 1_{P(X,q)} \right]$
Slivnyak-Mecke formula

$X$ a Poisson point process of density $n$

\[
\sum_{q \in X} \mathbb{1}_{P(X,q)} = n \int_{\mathbb{R}^2} \mathbb{P}(P(X \cup \{q\}, q)) \, dq
\]
Slivnyak-Mecke formula

\[ X \text{ a Poisson point process of density } n \]

\[ \sum_{q \in X} \mathbb{1}[P(X,q)] = n \int_{\mathbb{R}^2} \mathbb{P}[P(X \cup \{q\}, q)] \, dq \]

e.g.,

\[ \mathbb{E} \left[ \sum_{q \in X} \mathbb{1}[NN_{X}(0)=q] \right] \]
Slivnyak-Mecke formula

$X$ a Poisson point process of density $n$

Sum $\rightarrow$ Integral

$$
\mathbb{E} \left[ \sum_{q \in X} 1_{P(X,q)} \right] = n \int_{\mathbb{R}^2} \mathbb{P} [P(X \cup \{q\}, q)] \, dq
$$

e.g.,

$$
\mathbb{E} \left[ \sum_{q \in X} 1_{NN_X(0)=q} \right] = n \int_{\mathbb{R}^2} \mathbb{P} [D(0, ||q||) \cap X = \emptyset] \, dq
$$
Slivnyak-Mecke formula

Let $X$ be a Poisson point process of density $n$.

The Slivnyak-Mecke formula is given by:

$$
\mathbb{E} \left[ \sum_{q \in X} 1_{P(X,q)} \right] = n \int_{\mathbb{R}^2} \mathbb{P} \left[ P(X \cup \{q\}, q) \right] \, dq
$$

e.g.,

$$
\mathbb{E} \left[ \sum_{q \in X} 1_{NN_X(0)=q} \right] = n \int_{\mathbb{R}^2} \mathbb{P} \left[ D(0, ||q||) \cap X = \emptyset \right] \, dq
$$

$$
= n \int_{\mathbb{R}^2} e^{-n\pi ||q||^2} \, dq
$$
Slivnyak-Mecke formula

$X$ a Poisson point process of density $n$

**Sum** $\rightarrow$ **Integral**

$$
\mathbb{E} \left[ \sum_{q \in X} 1_{P(X,q)} \right] = n \int_{\mathbb{R}^2} \mathbb{P} \left[ P(X \cup \{q\}, q) \right] \, dq
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$$

$$
= n \int_{\mathbb{R}^2} e^{-n\pi ||q||^2} \, dq
$$

$$
= n \int_{0}^{2\pi} \int_{0}^{\infty} e^{-n\pi r^2} \, r \, d\theta \, dr = n \times 2\pi \times \frac{1}{2n\pi} = 1
$$
The end