Probability and Delaunay triangulations
Randomized algorithms for Delaunay triangulations

Poisson Delaunay triangulation
Randomized algorithms for Delaunay triangulations

- Randomized backward analysis of binary trees
- Randomized incremental construction of Delaunay
- Jump and walk
- The Delaunay hierarchy
- Biased randomized incremental order
- Chew algorithm for convex polygon
Sorting

\[ -\infty \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \infty \]

3 - 1
Sorting
Sorting

Binary tree

4 - 1
Sorting
Binary tree
Sorting

Binary tree

8

\leq 8 \quad 8 \quad > 8
Sorting
Binary tree
Sorting
Binary tree
Sorting
Binary tree

4 - 6
Sorting

Binary tree

4 - 7
Sorting

Binary tree
Binary tree

Sorting
Sorting

Binary tree

\[ [\pi, 1] \]

4 - 10
Sorting

Binary tree

\[
\begin{align*}
8 & \rightarrow 4 \rightarrow 1 \\
4 & \rightarrow 7 \\
8 & \rightarrow 14 \rightarrow 12 \\
14 & \rightarrow 11
\end{align*}
\]

\[
\begin{align*}
] - \infty, 1] & \rightarrow ] 1, 4] \\
1 & \rightarrow 7 \\
12 & \rightarrow 11
\end{align*}
\]
Sorting

Binary tree
Sorting

1

time

5 - 1

new drawing
Sorting

time

new drawing

5 - 2
Sorting

1
2
3

new drawing

time

5 - 3
Sorting

1
2
3
4

new drawing

time

5 - 4
Sorting

time

new drawing
Sorting

new drawing

1
2
3
4
5
6
time

1
4
7
8
14
12
1
5 - 6
Sorting

new drawing

time
Sorting

new drawing

1
2
3
4
5
6
7

time

5 - 8
Sorting

A new drawing.
Sorting

time

6 - 1
Sorting
Sorting

\[ k \]

\[ \text{time} \]

\[ 6 - 3 \]
Sorting

$k$

time

$6 - 4$
Sorting

Localisation

$\mathcal{nn}$

$k$

time

6 - 5 $\infty$
Sorting

Localisation

$k$

$n$

time
Sorting

Localisation

\[ k \]

\[ n \]

\[ \text{time} \]

6 - 7
Sorting

\[ k \]

\[ \mathcal{E} [\# \text{visited nodes}] O\left(\frac{2}{k}\right) \]

Localisation

\[ n \]

time

6 - 9
Sorting

\[ E[\# \text{visited nodes}] \mathcal{O}\left(\frac{2}{k}\right) \]

Total insertion: \[ \sum_k \frac{2}{k} \approx 2 \log n \]
Sorting

Localisation

Total insertion: $\sum_k \frac{2}{k} \approx 2 \log n$

Total construction: $\sum_k 2 \log k \approx 2n \log n$

$\mathbb{E}[\text{\#visited nodes}] \mathcal{O}(\frac{2}{k})$
Sorting

\[ - \infty, \infty \]

new drawing

conflict graph

\[ \infty \quad 1 \quad 4 \quad 7 \quad 8 \quad 11 \quad 12 \quad 14 \quad \infty \]
Sorting

\(-\infty, 8\[ \\
8, \infty[ \\
\]

\(\infty\) 1 4 7 8 11 12 14 \(\infty\)

7 - 2
Sorting

\[-\infty, 8[ \quad ]8, \infty[ \quad ]-\infty, 4[ \quad ]4, 8[ \quad ]8, 14[ \quad ]14, \infty[\]
Sorting

Unbalanced binary tree
Quicksort

$O(n \log n)$

History graph
Conflict graph

Same analysis

Backwards analysis
Analyse last insertion and sum
Last object is a random object
Randomization

Backwards analysis for Delaunay triangulation
Delaunay triangulation

# of triangles during incremental construction?
Delaunay triangulation of triangles during incremental construction?
Delaunay triangulation

# of triangles during incremental construction?

# triangles created/incident to last point?
Delaunay triangulation

Number of triangles during incremental construction?

Number of triangles created/incident to last point?

Last point?
\( \frac{1}{n} \sum_{i=1}^{n} d^\circ(p_i) \leq 6 \)
\[
\frac{1}{n} \sum_{i=1}^{n} d^\circ(p_i) \leq 6
\]

\[
\sum 6 = 6n
\]
Alternative analysis

Triangle $\Delta$ with $j$ stoppers
Alternative analysis

Triangle \( \triangle \) with \( j \) stoppers

Probability that it exists in the triangulation of a sample of size \( \alpha n \)

\[
\preceq \alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4} \alpha^3 \quad \text{if} \quad 2 \leq j \leq \frac{1}{\alpha}
\]
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists in the triangulation of a sample of size $\alpha n$

$$\approx \alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4} \alpha^3 \quad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Size of the triangulation of the sample

$$= \sum_{j=0}^{n} P[\Delta \text{ with } j \text{ stoppers is there}] \times \#\Delta \text{ with } j \text{ stoppers}$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists in the triangulation of a sample of size $\alpha n$

$$\alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha)^{1/\alpha} \geq \frac{1}{4} \alpha^3 \quad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Size of the triangulation of the sample

$$= \sum_{j=0}^{n} P[\Delta \text{ with } j \text{ stoppers is there}] \times \#\Delta \text{ with } j \text{ stoppers}$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \#\Delta \text{ with } j \text{ stoppers}$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists in the triangulation of a sample of size $\alpha n$

$$\simeq \alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4} \alpha^3 \quad \text{if} \quad 2 \leq j \leq \frac{1}{\alpha}$$

Size of the triangulation of the sample

$$= \sum_{j=0}^{1/\alpha} \mathbb{P} \left[ \Delta \text{ with } j \text{ stoppers is there} \right] \times \#\Delta \text{ with } j \text{ stoppers}$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \#\Delta \text{ with } j \text{ stoppers} = \alpha^3 \#\Delta \text{ with } \leq \frac{1}{\alpha} \text{ stoppers}$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists in the triangulation of a sample of size $\alpha n$

$$\simeq \alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4} \alpha^3 \quad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Size of the triangulation of the sample

$$= \sum_{j=0}^{\frac{1}{\alpha}} P[\Delta \text{ with } j \text{ stoppers is there}] \times \# \Delta \text{ with } j \text{ stoppers} = O(\alpha n)$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \# \Delta \text{ with } j \text{ stoppers} = \alpha^3 \# \Delta \text{ with } \leq \frac{1}{\alpha} \text{ stoppers}$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists in the triangulation of a sample of size $\alpha n$

$$\approx \alpha^3 (1 - \alpha)^j \geq \alpha^3 (1 - \alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4} \alpha^3 \quad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Size of the triangulation of the sample

$$= \sum_{j=0}^{n} P[\Delta \text{ with } j \text{ stoppers is there}] \times \#\Delta \text{ with } j \text{ stoppers} = O(\alpha n)$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \#\Delta \text{ with } j \text{ stoppers} = \alpha^3 \#\Delta \text{ with } \leq \frac{1}{\alpha} \text{ stoppers}$$

Size (order $\leq k$ Voronoi) $\leq \frac{\alpha n}{\alpha^3} = nk^2$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \cdot \frac{2}{j+2} \cdot \frac{1}{j+1}$$

$\#$ of created triangles

$$= \sum_{j=0}^{n} P[\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers}$$
Alternative analysis

Triangle \( \Delta \) with \( j \) stoppers

Probability that it exists during the construction

\[
= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}
\]

\# of created triangles

\[
= \sum_{j=0}^{n} P[\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers}
\]

\[
= \sum_{j=0}^{n} (P[\Delta \text{ with } j] - P[\Delta \text{ with } j+1]) \times \#\Delta \text{ with } \leq j \text{ stoppers}
\]
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}$$

$\#$ of created triangles

$$= \sum_{j=0}^{n} \mathbb{P} [\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} (\mathbb{P} [\Delta \text{ with } j] - \mathbb{P} [\Delta \text{ with } j + 1]) \times \#\Delta \text{ with } \leq j \text{ stoppers}$$

$$\approx \sum_{j=0}^{n} \frac{18}{j^4} \times n j^2 = O(n \sum \frac{1}{j^2}) = O(n)$$
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Conflict graph / History graph

It remains to analyze conflict location
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \cdot \frac{2}{j+2} \cdot \frac{1}{j+1}$$

$\#$ of conflicts occurring

$$= \sum_{j=0}^{n} j \times \mathbb{P} [\Delta \text{ with } j \text{ stoppers appears}] \times \# \Delta \text{ with } j \text{ stoppers}$$
Alternative analysis

Triangle \( \Delta \) with \( j \) stoppers

Probability that it exists during the construction

\[
= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}
\]

\( \# \) of conflicts occurring

\[
= \sum_{j=0}^{n} j \times \mathbb{P} [\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers}
\]

\[
= \sum_{j=0}^{n} j \times (\mathbb{P} [\Delta \text{ with } j] - \mathbb{P} [\Delta \text{ with } j + 1]) \times \#\Delta \text{ with } \leq j \text{ stoppers}
\]
Alternative analysis

Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}$$

$\#$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P} [\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} j \times (\mathbb{P} [\Delta \text{ with } j] - \mathbb{P} [\Delta \text{ with } j+1]) \times \#\Delta \text{ with } \leq j \text{ stoppers}$$

$$\approx \sum_{j=0}^{n} j \times \frac{18}{j^4} \times nj^2 = O(n \sum \frac{1}{j}) = O(n \log n)$$
History graph
History graph
History graph
History graph
History graph  
(Delaunay tree)

Stepfather  
Father
if conflict there was a conflict with the father or the stepfather or both
Conflict graph
Conflict graph
Conflict graph

Insert
Conflict graph
Conflict graph
Conflict graph
Walk

15 - 1
Walk

15 - 2
Walk

Complexity $O(n)$
Walk

Complexity $O(n)$

Better bounds for random points

Teaser probability lecture
Jump and walk
Jump and walk
Jump and walk
Jump and walk

Hopefully shorter walk

Designed for random points

$O\left(\frac{3}{\sqrt{n}}\right)$ expected location time
Jump and walk (no distribution hypothesis)
Jump and walk (no distribution hypothesis)

\[ \mathbb{E} [\text{# of in } \bigcirc] = \frac{n}{k} \]
Jump and walk (no distribution hypothesis)

\[ \mathbb{E} \left[ \text{# of } \bullet \text{ in } \circ \right] = \frac{n}{k} \]

Walk length = \( O \left( \frac{n}{k} \right) \)

choose \( k = \sqrt[2]{n} \)
Jump and walk (no distribution hypothesis) Delaunay hierarchy

$$\mathbb{E} \left[ \text{\# of } \bullet \text{ in } \bigcirc \right] = \frac{n}{k}$$

Walk length $= O\left( \frac{n}{k} \right)$

choose $k = 2\sqrt{n}$
Jump and walk (no distribution hypothesis) Delaunay hierarchy

\[
\mathbb{E} \left[ \text{# of } \bullet \text{ in } \bigcirc \right] = \frac{n}{k}
\]

Walk length = \( O \left( \frac{n}{k} \right) \)

choose \( k = \sqrt{n} \)
Jump and walk (no distribution hypothesis)

\[ \mathbb{E} [\# \text{ of } \bullet \text{ in } \bigcirc] = \frac{n}{k} \]

\[ \frac{n}{k_1} + \frac{k_1}{k_2} \]

Walk length = \( O \left( \frac{n}{k} \right) \)

choose \( k = \sqrt[4]{n} \)
Jump and walk (no distribution hypothesis) Delaunay hierarchy

\[ \mathbb{E} \left[ \# \text{ of } \bullet \text{ in } \bigcirc \right] = \frac{n}{k} \]

Walk length = \( O \left( \frac{n}{k} \right) \)

Choose \( k = \sqrt[2]{n} \)

\[ \frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots \]
Jump and walk (no distribution hypothesis)

\[ \mathbb{E} [\text{\# of } \bullet \text{ in } \bigcirc] = \frac{n}{k} \]

Delaunay hierarchy

\[ \frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots \]

Walk length = \( O \left( \frac{n}{k} \right) \)

choose \( k = \sqrt{n} \)

choose \( \frac{k_i}{k_{i+1}} = \alpha \)
Jump and walk (no distribution hypothesis) Delaunay hierarchy

\[ \mathbb{E} [\text{\# of \ in } \bigcirc] = \frac{n}{k} \]

Walk length = \( O \left( \frac{n}{k} \right) \)

Choose \( k = \sqrt{n} \)

Choose \( \frac{k_i}{k_{i+1}} = \alpha \)

Point location in \( O(\alpha \log_\alpha n) \)
Jump and walk (no distribution hypothesis) Delaunay hierarchy

\[ \mathbb{E} [\# \text{ of } \bullet \text{ in } \bigcirc] = \frac{n}{k} \]

Walk length = \( O \left( \frac{n}{k} \right) \)

Choose \( k = \sqrt{n} \)

Choose \( \frac{k_i}{k_{i+1}} = \alpha \)

Point location in \( O \left( \alpha \log_{\alpha} n \right) \)

Point location in \( O \left( \sqrt{\alpha} \log_{\alpha} n \right) \)
Technical detail

Walk length = \( O\left( \#\ of\ in\ \bigcirc \right) = O\left( \frac{n}{k} \right) \)
Technical detail

Walk length $= O\left(\text{# of } \bullet \text{ in } \bigcirc \right) = O\left(\frac{n}{k}\right)$
Walk length $= O\left(\# \text{ of } \bullet \text{ in } \circ \right) = O\left(\frac{n}{k}\right)$

$\mathbb{E}[d^\circ \bullet] = \frac{1}{n} \sum_q d^\circ NN(q) = \frac{1}{n} \sum_q \sum_{v=NN(q)} d^\circ v$

$\leq \frac{1}{n} \sum_{v} 6d^\circ v \leq 36$
Technical detail

Walk length = $O\left(\frac{n}{k}\right)$

\[
\mathbb{E}[d^\circ \bullet] = \frac{1}{n} \sum_q d^\circ NN(q) = \frac{1}{n} \sum_q \sum_{v=NN(q)} d^\circ v \\
\leq \frac{1}{n} \sum_q \sum_{v;v=NN(q)} d^\circ v \leq 36
\]
Randomization

How many randomness is necessary?

If the data are not known in advance

shuffle locally
Randomization

Drawbacks of random order

non locality of memory access

data structure for point location

Hilbert sort
Drawbacks of random order

- non locality of memory access
- data structure for point location

→ Hilbert sort

Walk should be fast

Last point is not at all a random point

→ no control of degree of last point
23 - 2
23 - 4
Triangle $\triangle$ with $j$ stoppers
Triangle $\Delta$ with $j$ stoppers

Size (order $\leq k$ Voronoi) $\leq \frac{\alpha n}{\alpha^3} = nk^2$
Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}$$
Triangle $\triangle$ with $j$ stoppers

Probability that it exists during the construction remains $\Theta(j^{-3})$
Triangle $\Delta$ with $j$ stoppers

Probability that it exists during the construction

\[
\frac{3}{j+3} \times \frac{2}{j+2} \times \frac{1}{j+1} \quad \text{remains } \Theta(j^{-3})
\]

\# of created triangles

\[
= \sum_{j=0}^{n} P[\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers}
\]

\[
\approx O\left(\sum \frac{n j^2}{j^4}\right) = O(n)
\]
Probability that it exists during the construction remains $\Theta(j^{-3})$

$$\sum_{j=0}^{n} j \times \mathbb{P}[\Delta \text{ with } j \text{ stoppers appears}] \times \#\Delta \text{ with } j \text{ stoppers}$$

$$\simeq O\left(\sum j \frac{n^2 j^2}{j^4}\right) = O(n \log n)$$
Delaunay 2D 1M random points

locate using Delaunay hierarchy 6 seconds
random order (visibility walk) 157 seconds
$x$-order 3 seconds
Hilbert order 0.8 seconds
Biased order (Spatial sorting) 0.7 seconds
Delaunay 2D 100K parabola points

locate using Delaunay hierarchy 0.3 seconds

random order (visibility walk) 128 seconds

$x$-order 632 seconds

Hilbert order 46 seconds

Biased order (Spatial sorting) 0.3 seconds
Delaunay of points in convex position
Delaunay of points in convex position
Delaunay of points in convex position

choose a point at random
Delaunay of points in convex position

choose a point at random
remove it from convex polygon
Delaunay of points in convex position

choose a point at random
remove it from convex polygon
remember its place
Delaunay of points in convex position

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
Delaunay of points in convex position

- choose a point at random
- remove it from convex polygon
- remember its place
- compute Delaunay of $n - 1$ points
- with relevant vertex-triangle pointers
- insert point, (location known)
Delaunay of points in convex position

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
insert point, (location known)
choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
insert point, (location known)
choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n-1$ points
with relevant vertex-triangle pointers
insert point, (location known)
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points
with relevant vertex-triangle pointers
insert point, (location known)
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
insert point, (location known)

\( O(1) \) [model]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
insert point, (location known)

\[ O(1) \]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points
with relevant vertex-triangle pointers
insert point, (location known)
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points
with relevant vertex-triangle pointers
insert point, (location known)

\[ O(d^0 p) = O(1) \]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
insert point, (location known)

\[
O(1)
\]

\[
f(n - 1) = O(d^p) = O(1)
\]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of $n - 1$ points
with relevant vertex-triangle pointers
insert point, (location known)

\[
O(1)
\]

\[
f(n - 1) = O(d^o p) = O(1)
\]

\[
f(n) = f(n - 1) + O(1)
\]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
insert point, (location known)

\[
f(n) = f(n - 1) + O(1) = O(n)
\]

\[
O(d^o_p) = O(1)
\]
Delaunay of points in convex position

Analysis

choose a point at random
remove it from convex polygon
remember its place
compute Delaunay of \( n - 1 \) points
with relevant vertex-triangle pointers
insert point, (location known)

\[
\begin{align*}
O(1) \\
O(d^\circ p) &= O(1) \\
f(n - 1) \\
f(n) &= f(n - 1) + O(1) = O(n)
\end{align*}
\]

[Chew 86]
The end