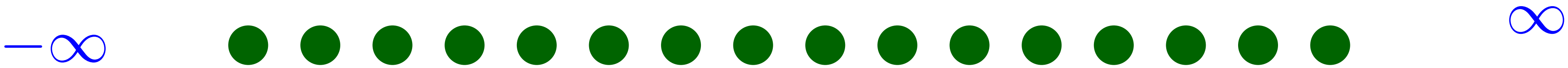
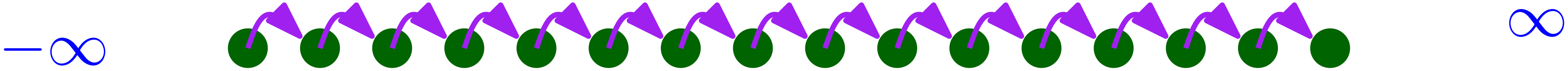
Randomized algorithms for Delaunay triangulations

- Randomized backward analysis of binary trees
- Randomized incremental construction of Delaunay
- Jump and walk
- The Delaunay hierarchy
- Biased randomized incremental order
- Chew algorithm for convex polygon

Sorting



Sorting

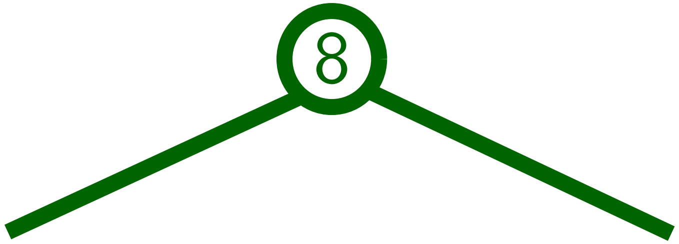


Sorting

Binary tree

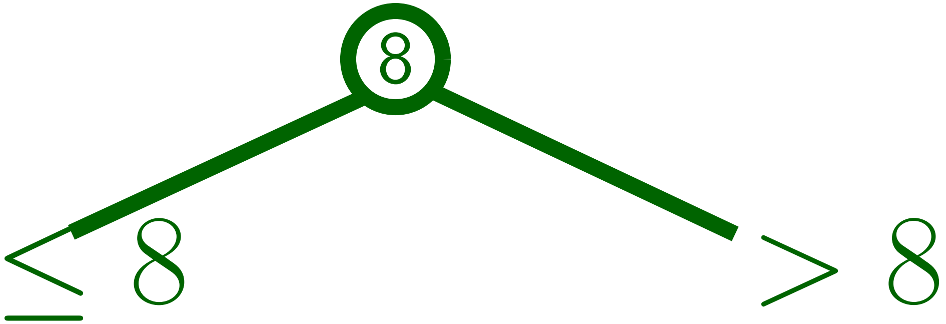
Sorting

Binary tree



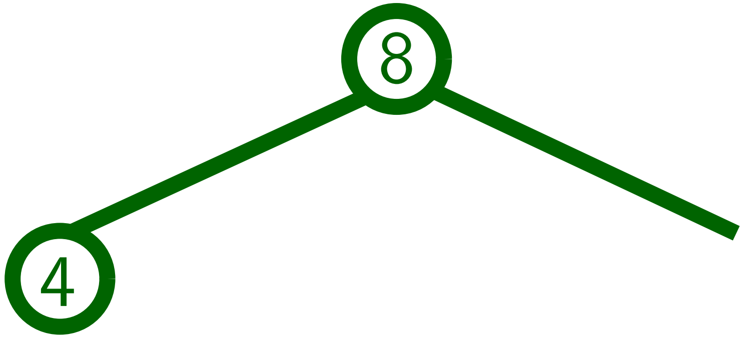
Sorting

Binary tree



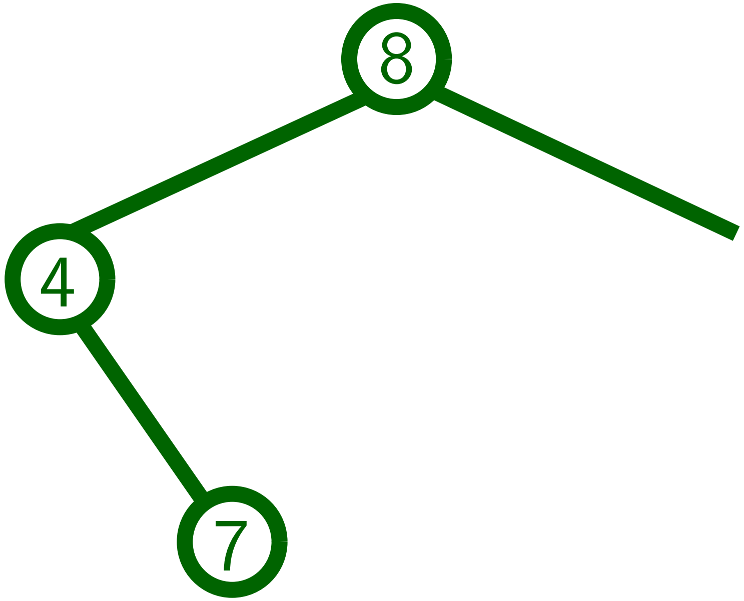
Sorting

Binary tree



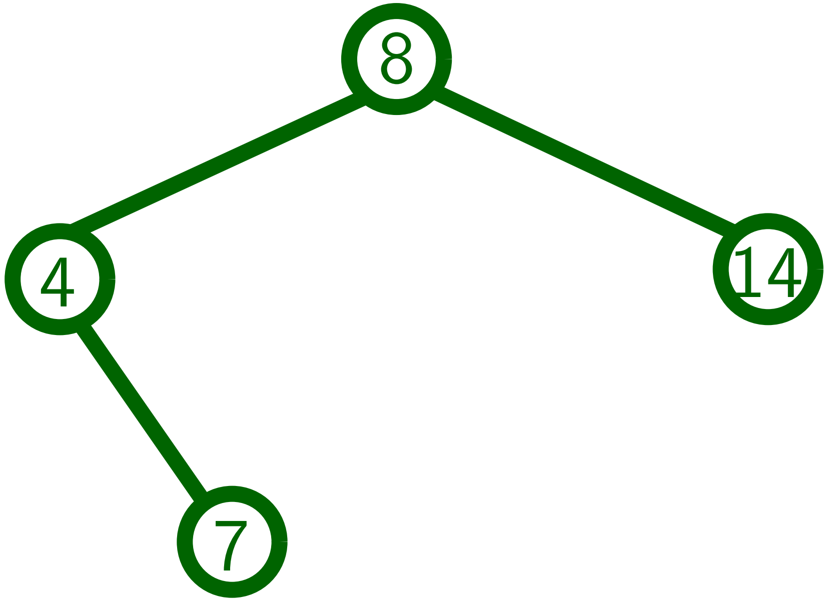
Sorting

Binary tree



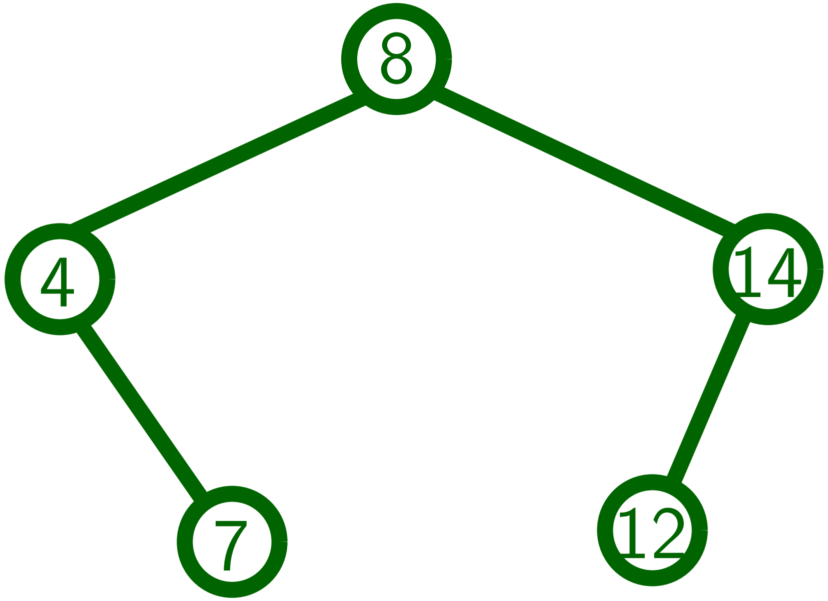
Sorting

Binary tree



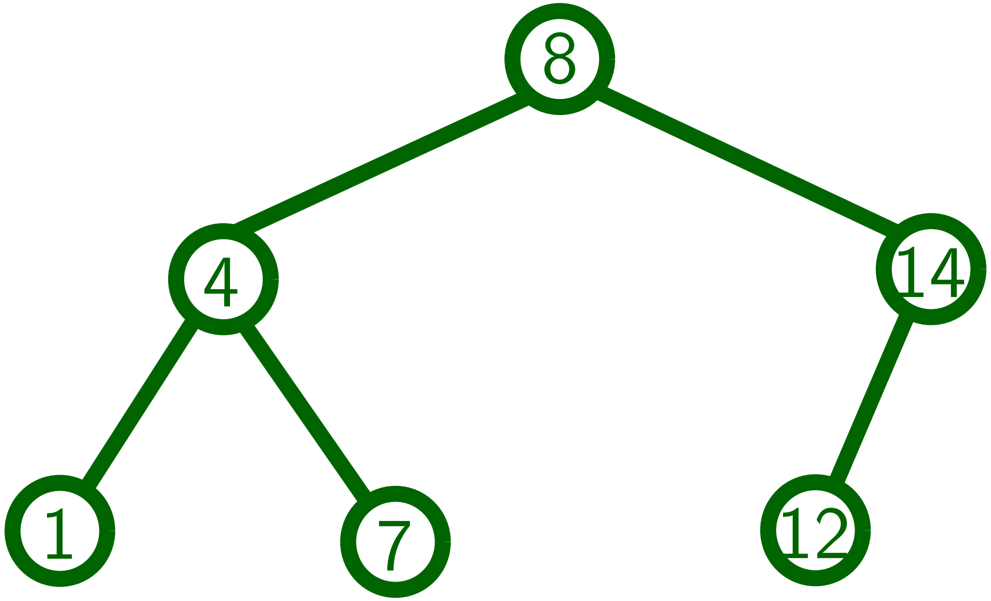
Sorting

Binary tree



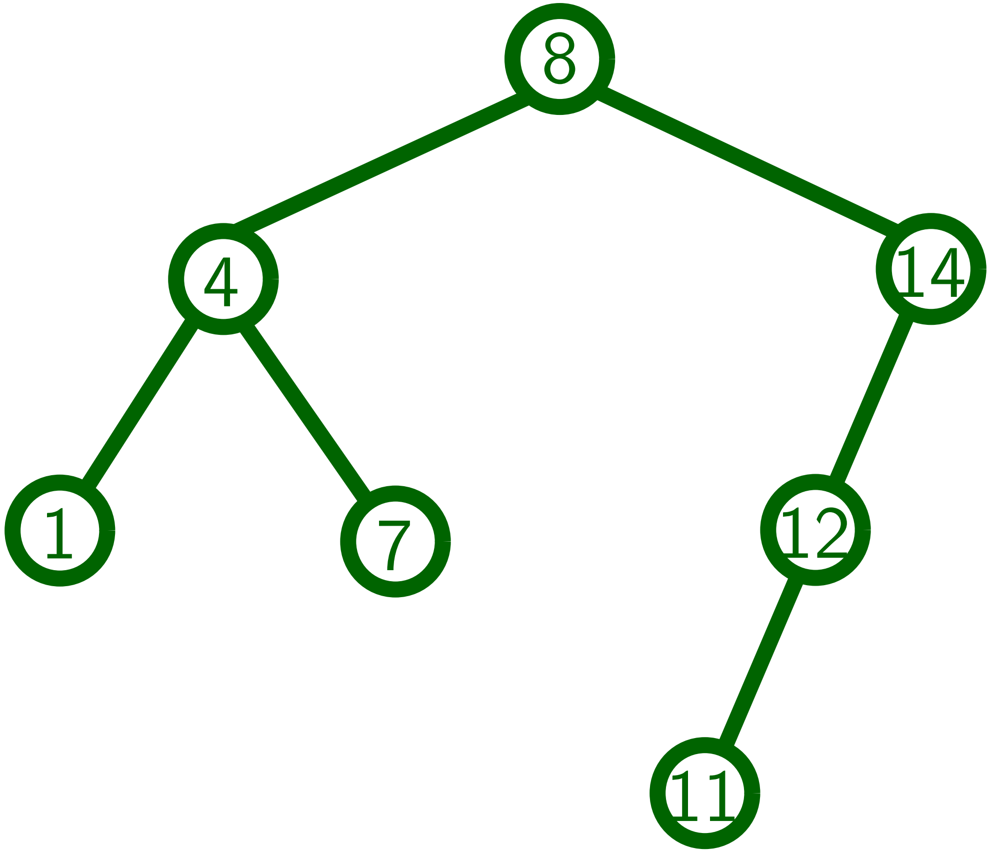
Sorting

Binary tree



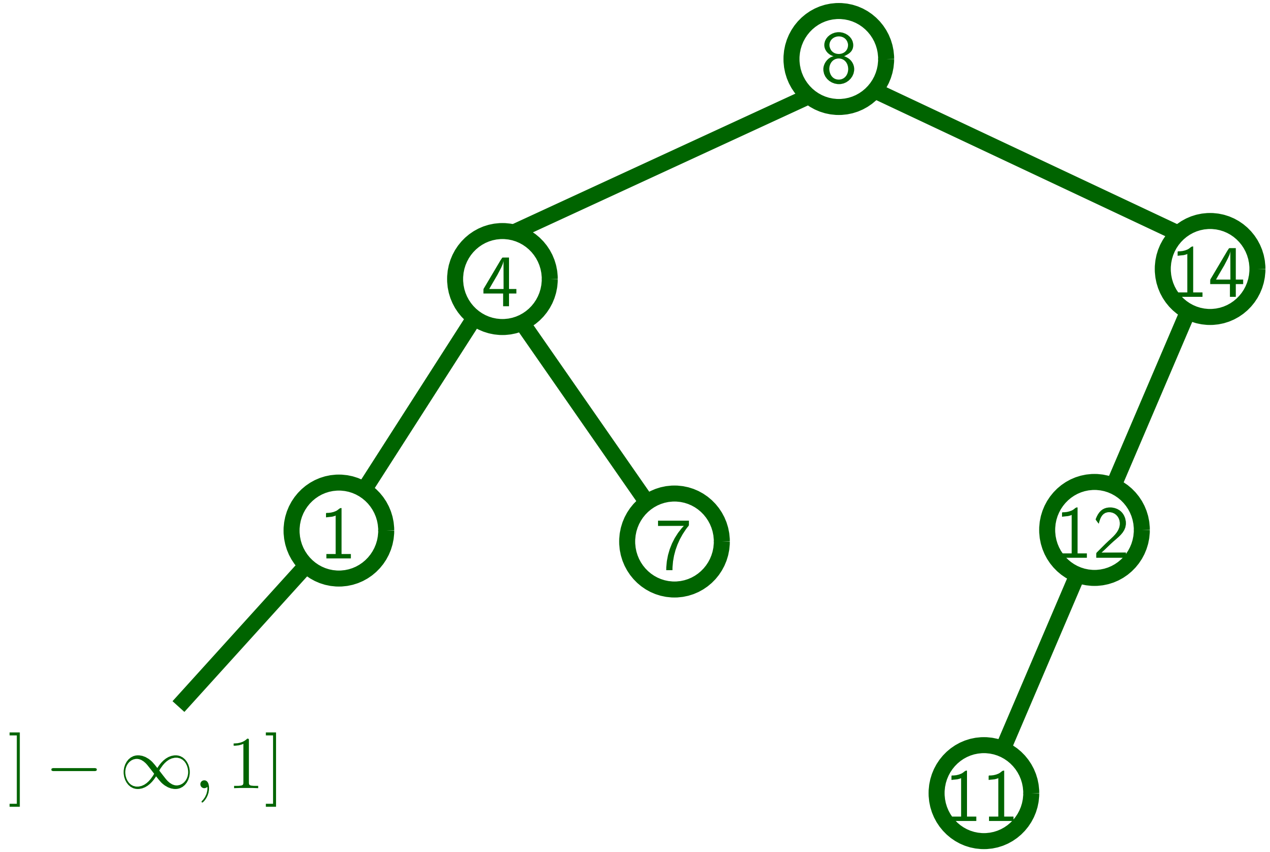
Sorting

Binary tree



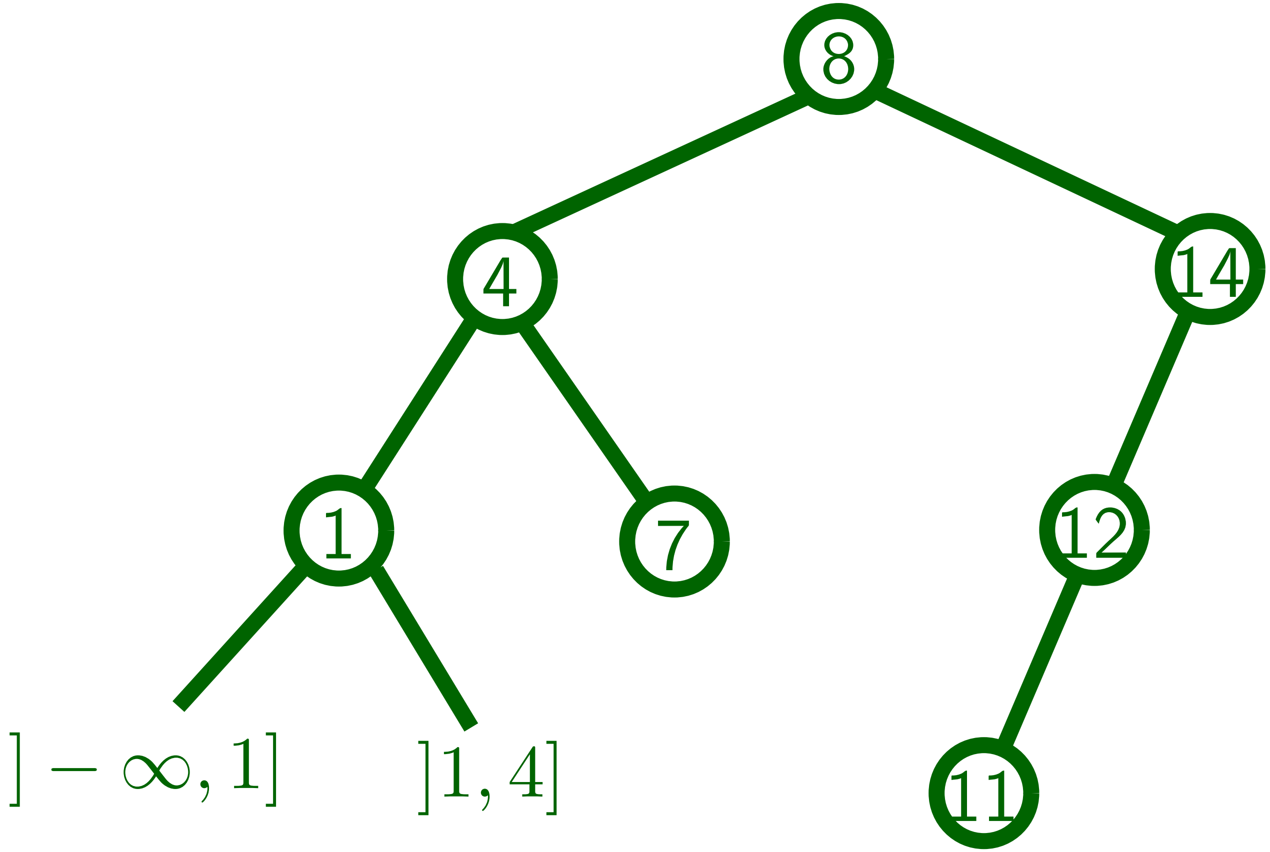
Sorting

Binary tree



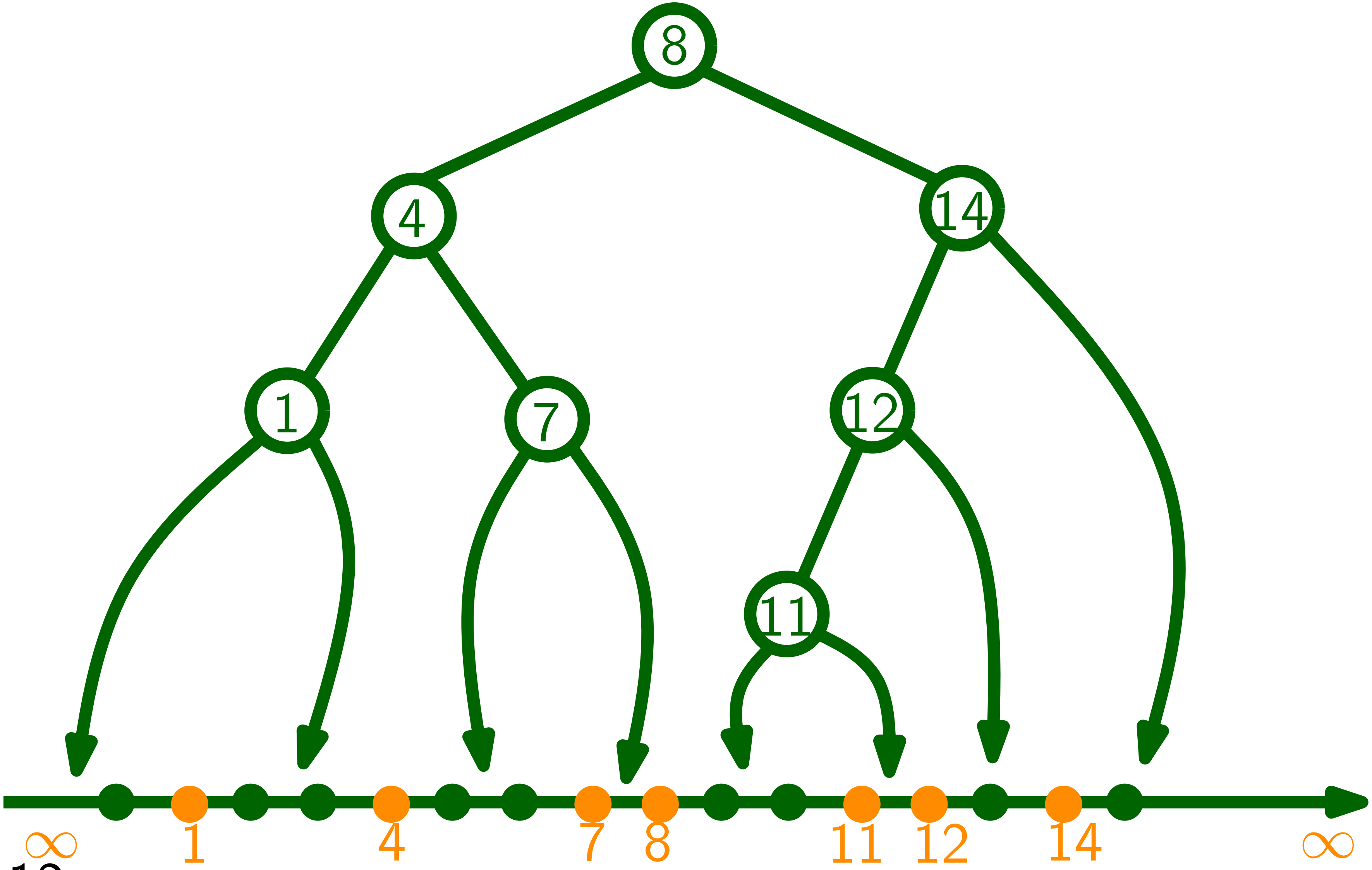
Sorting

Binary tree

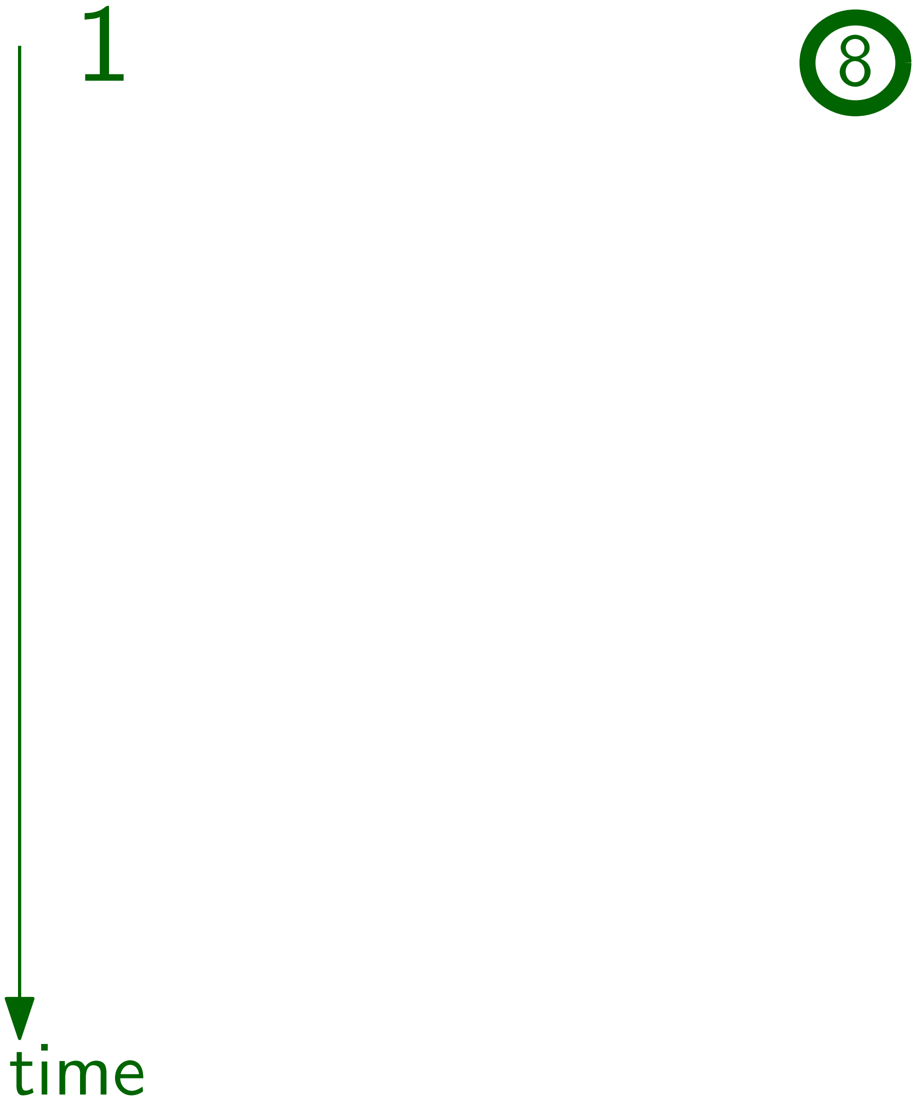


Sorting

Binary tree

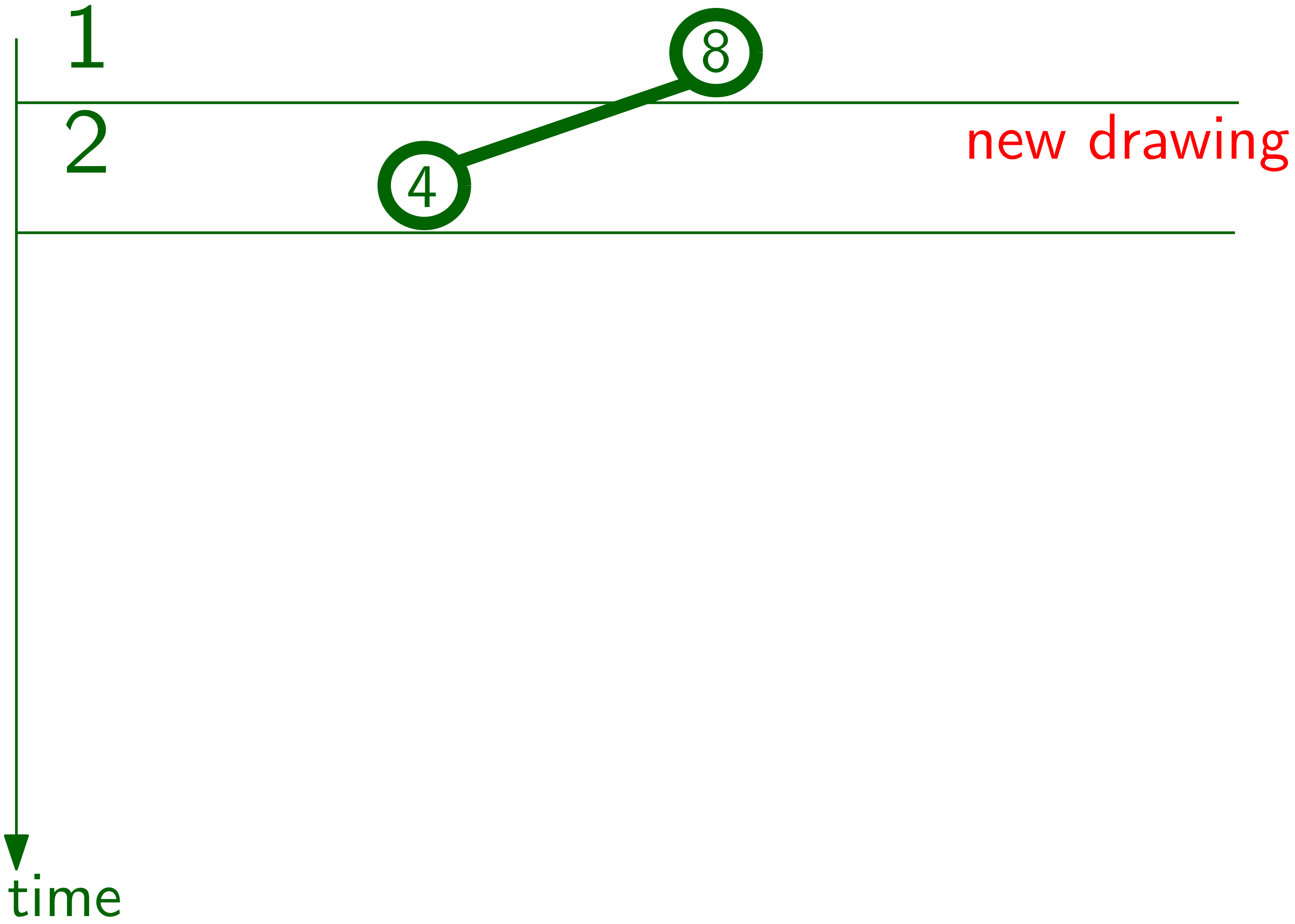


Sorting

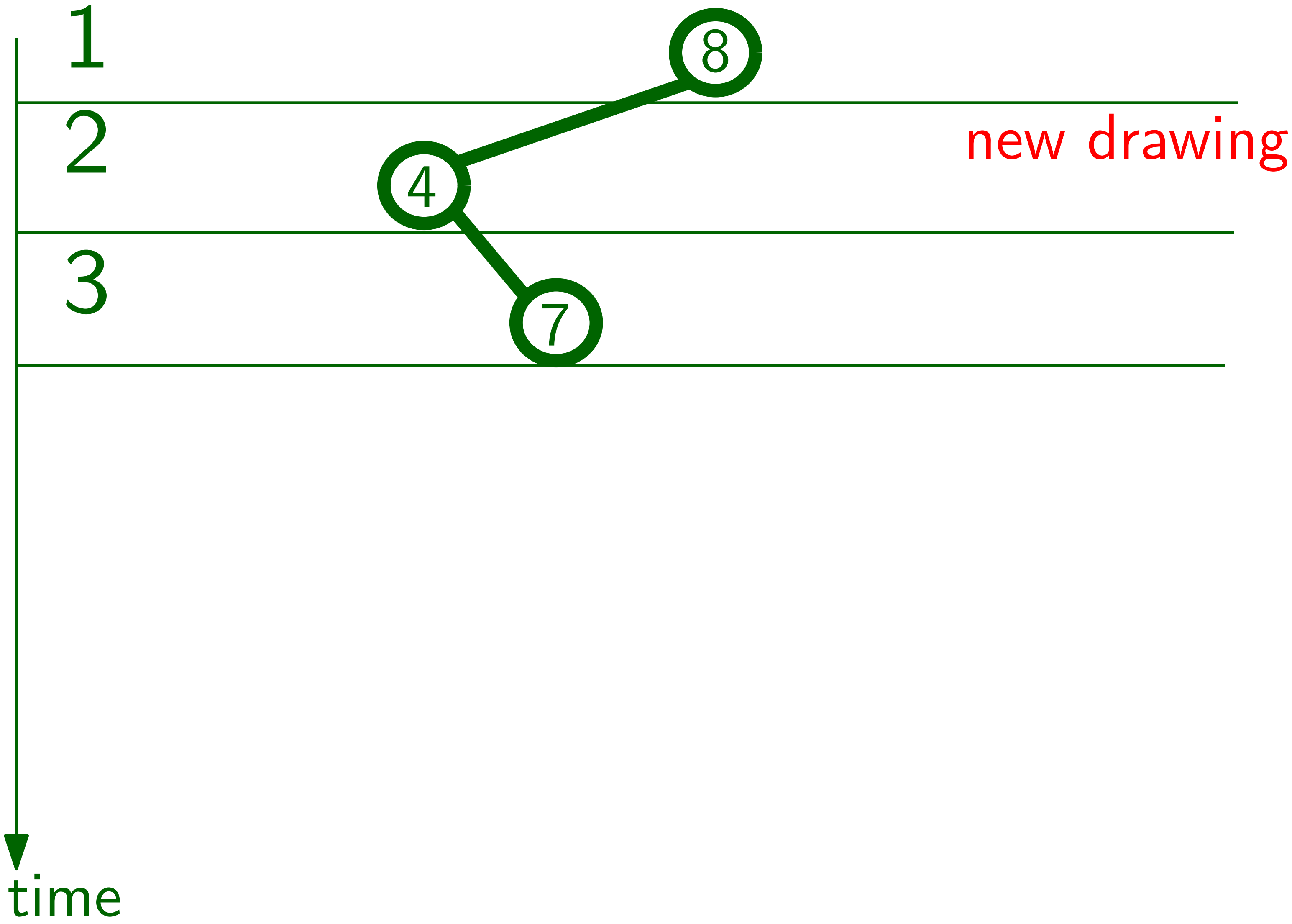


new drawing

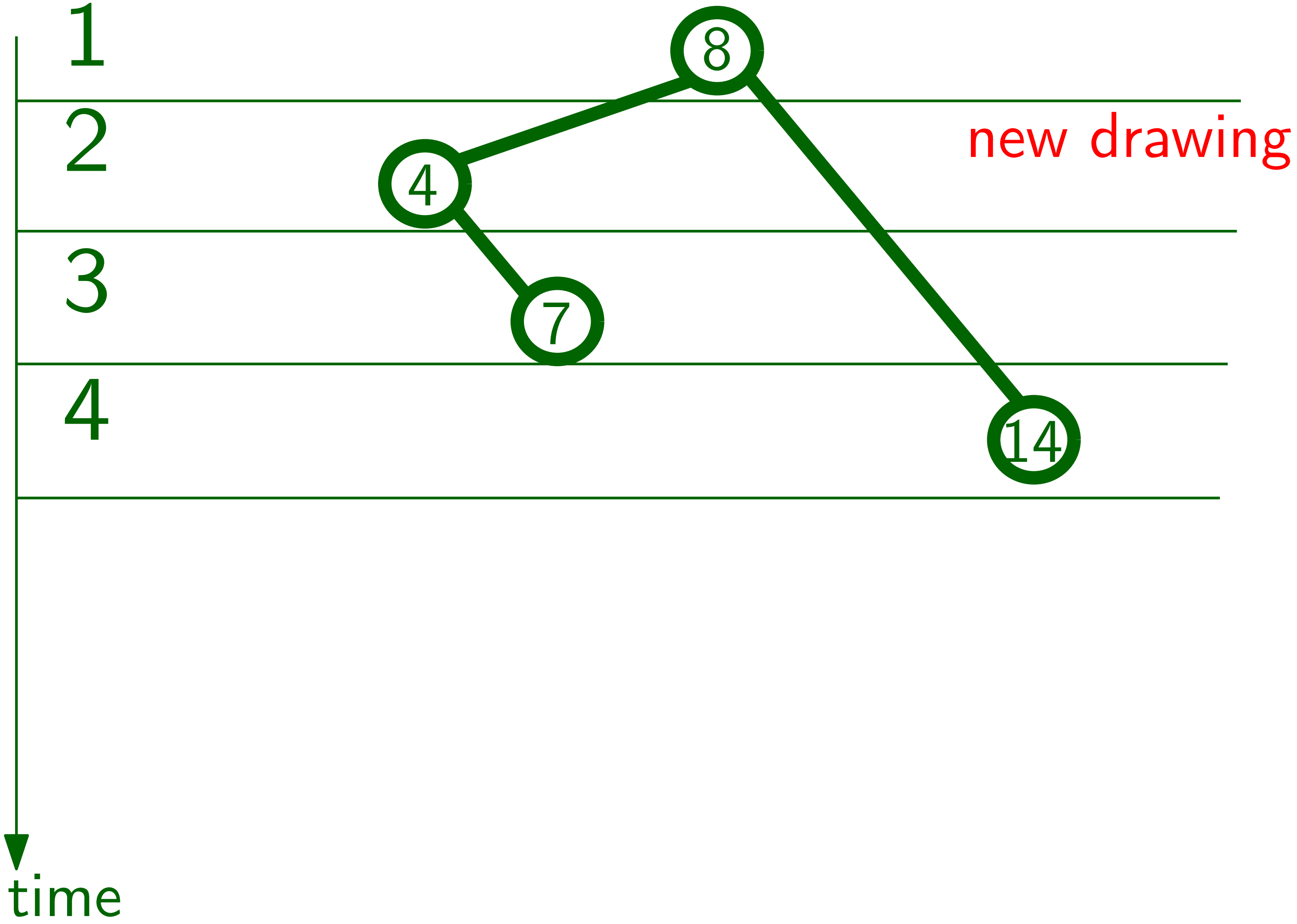
Sorting



Sorting

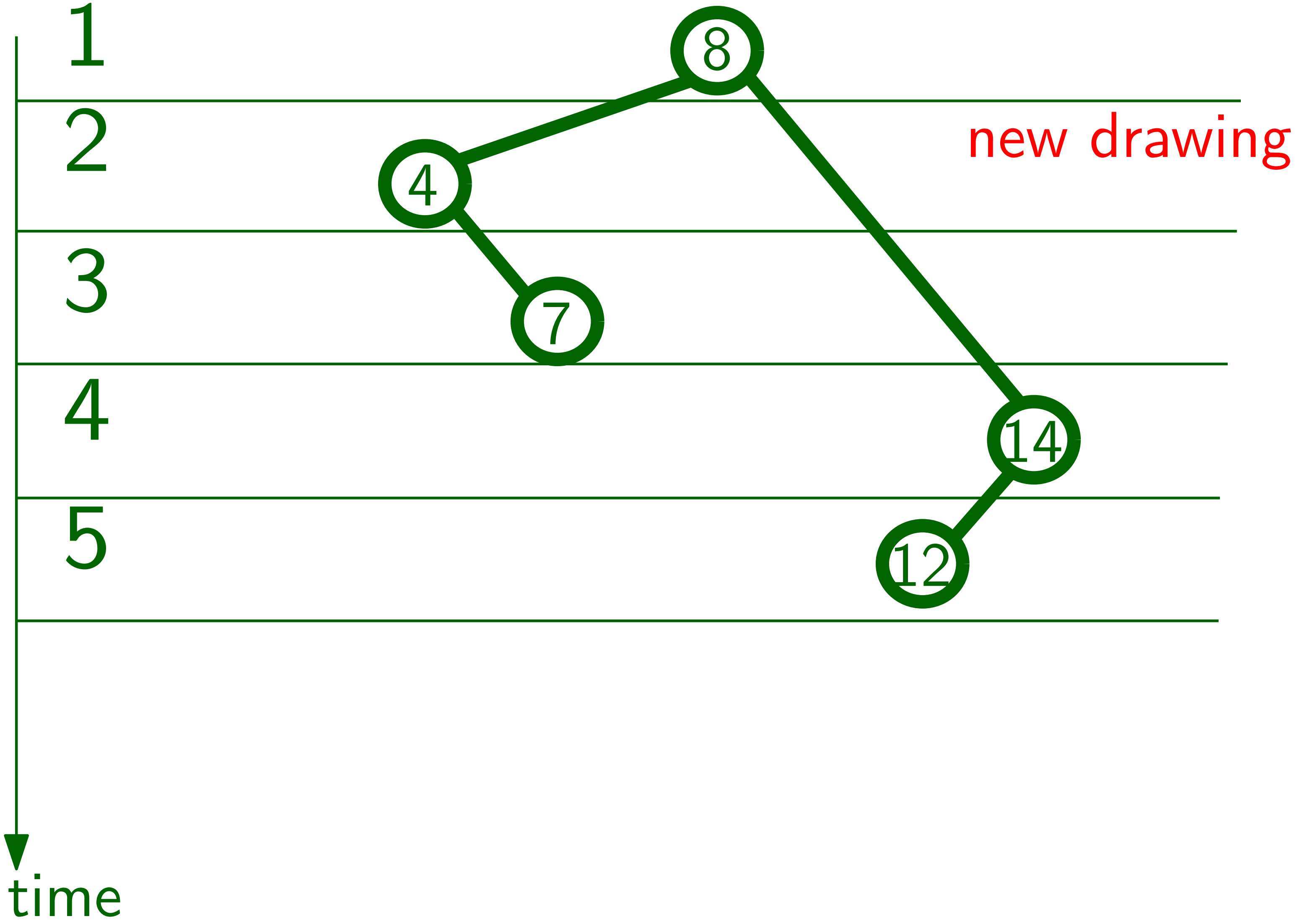


Sorting

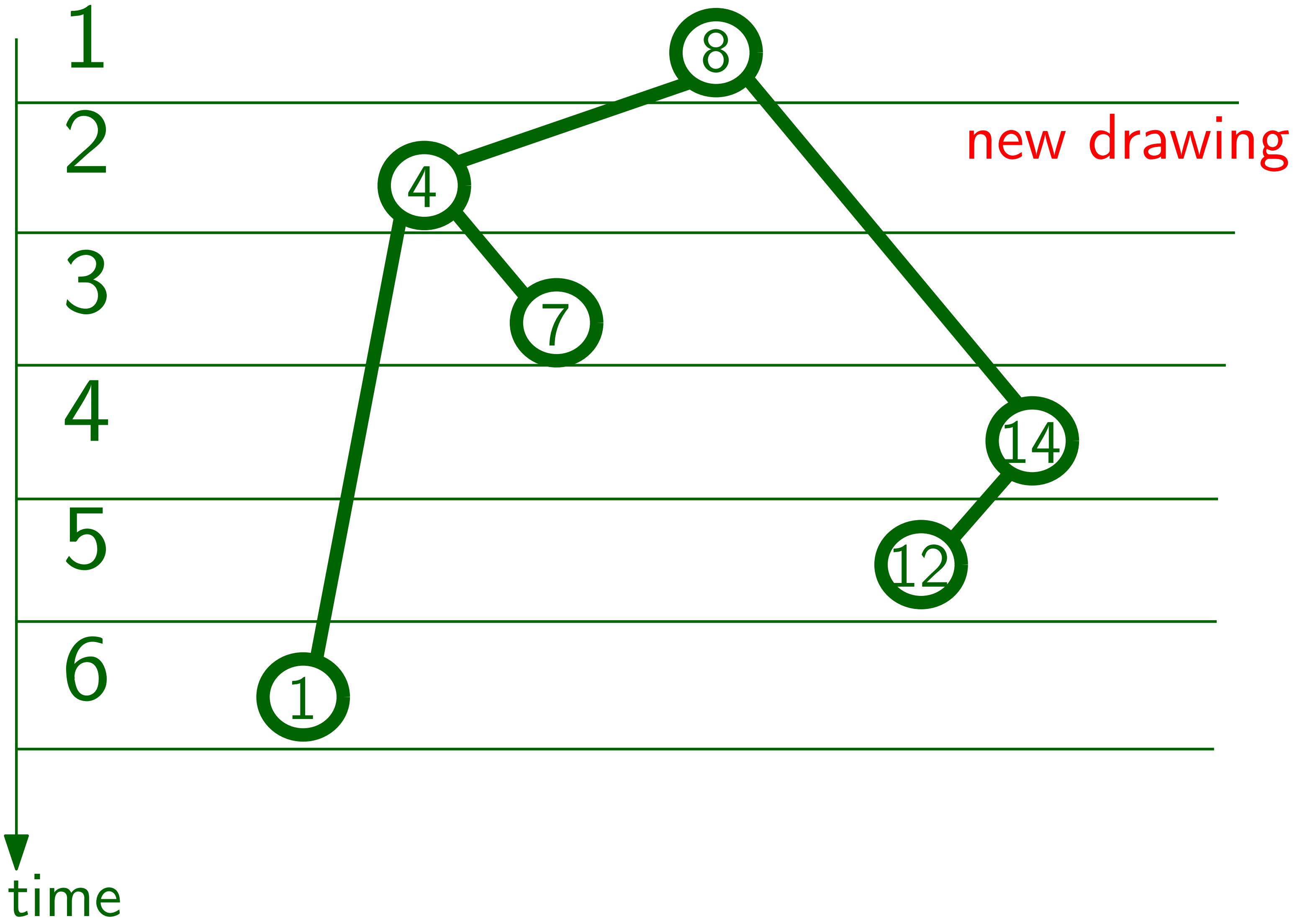


new drawing

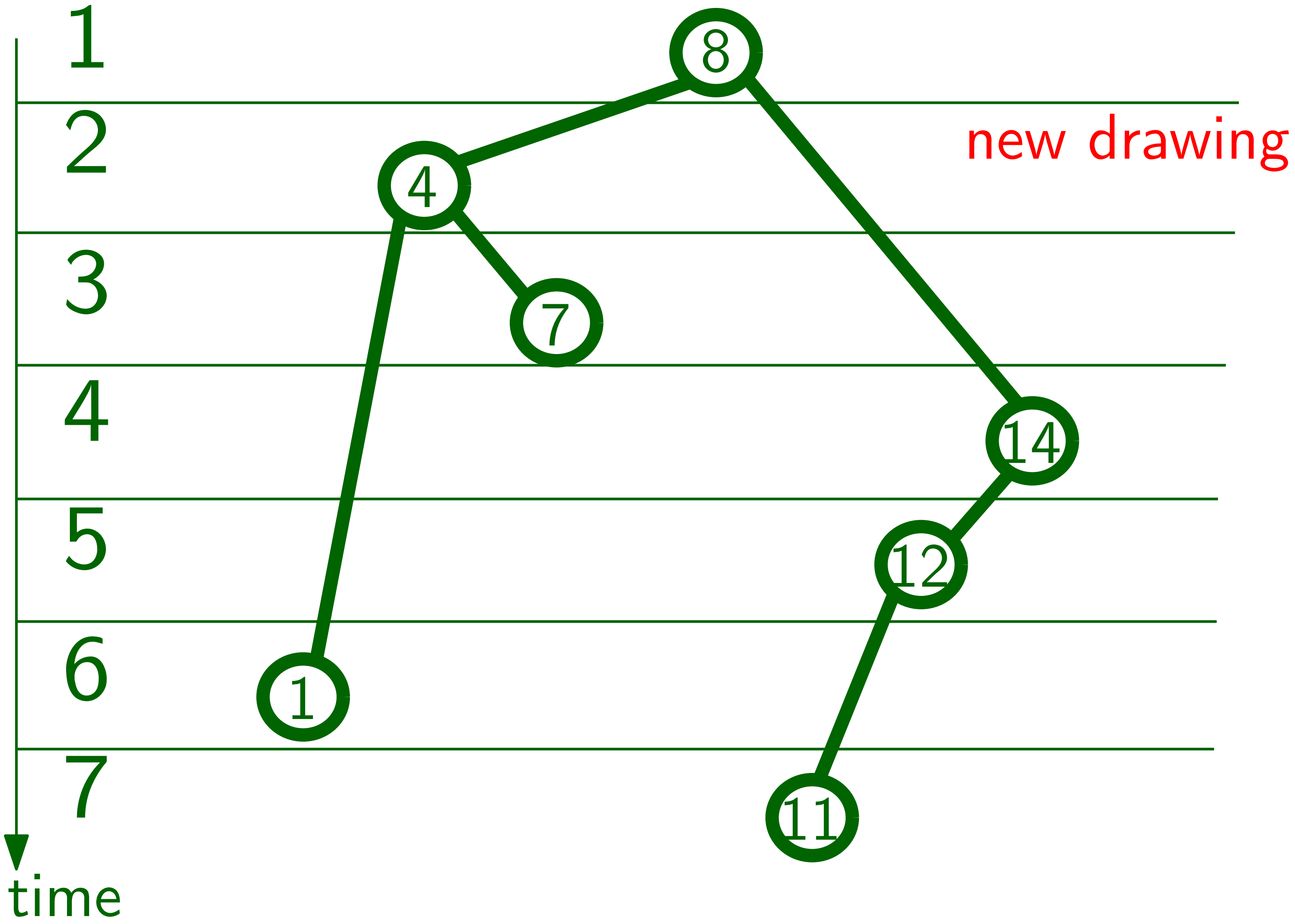
Sorting



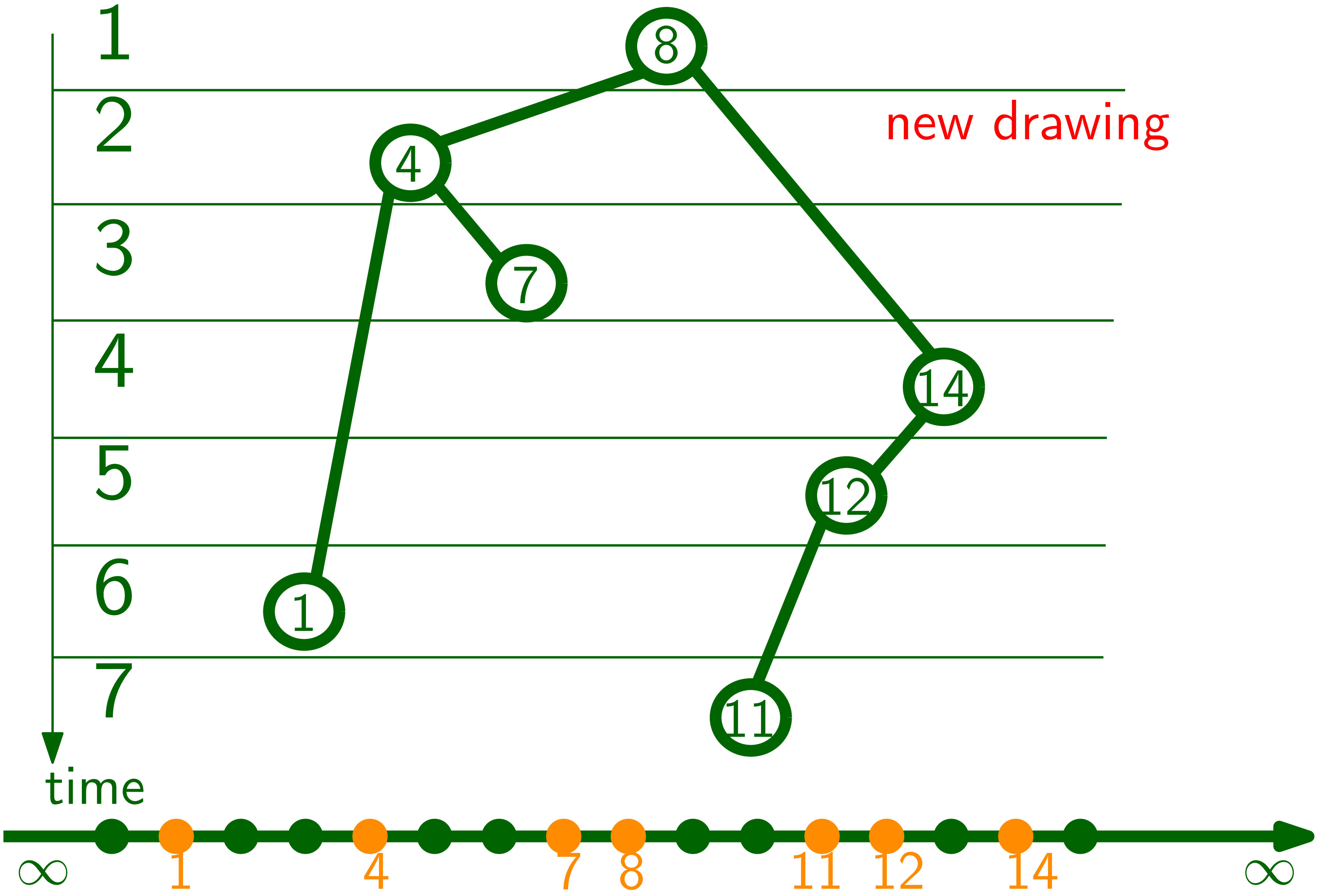
Sorting



Sorting

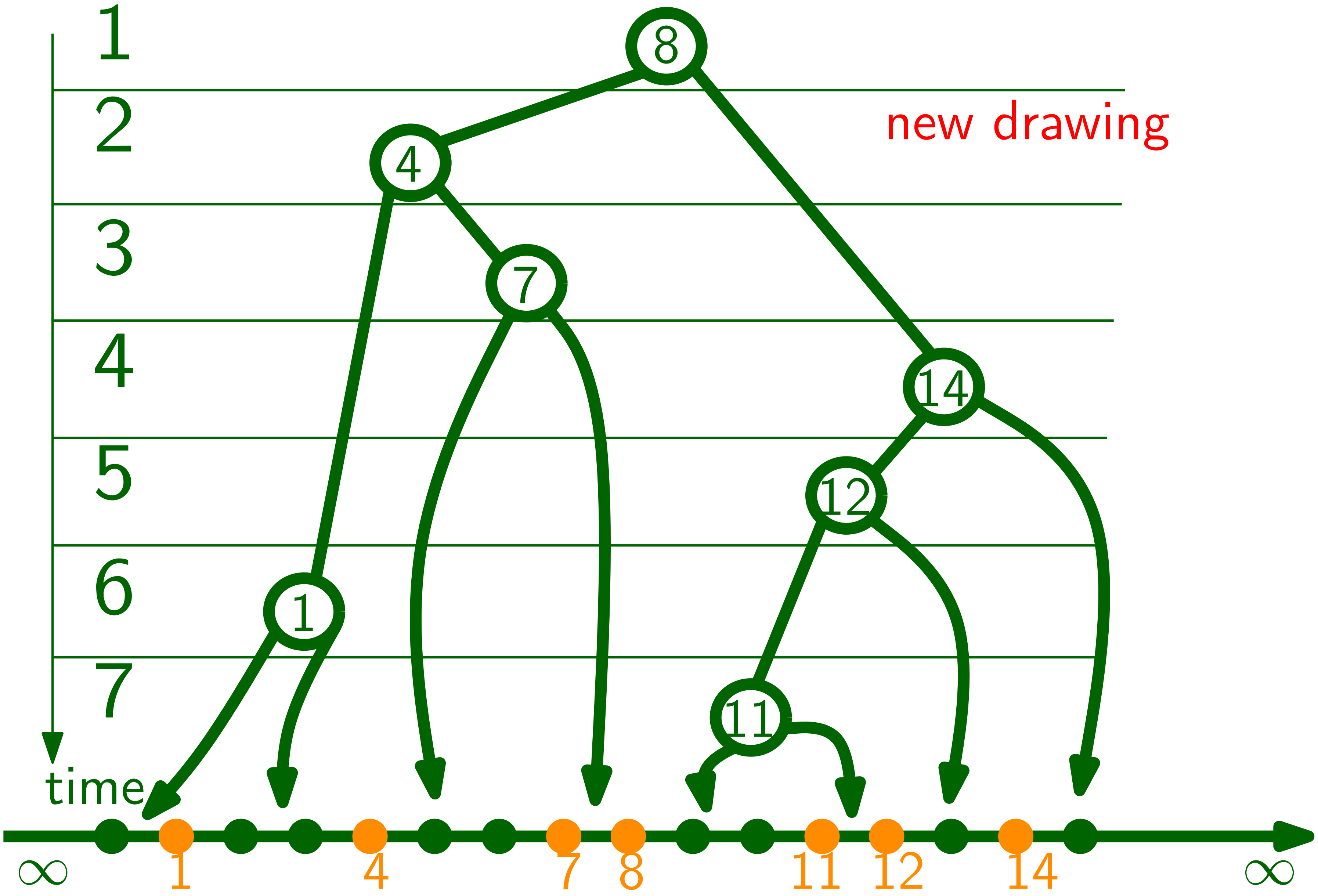


Sorting

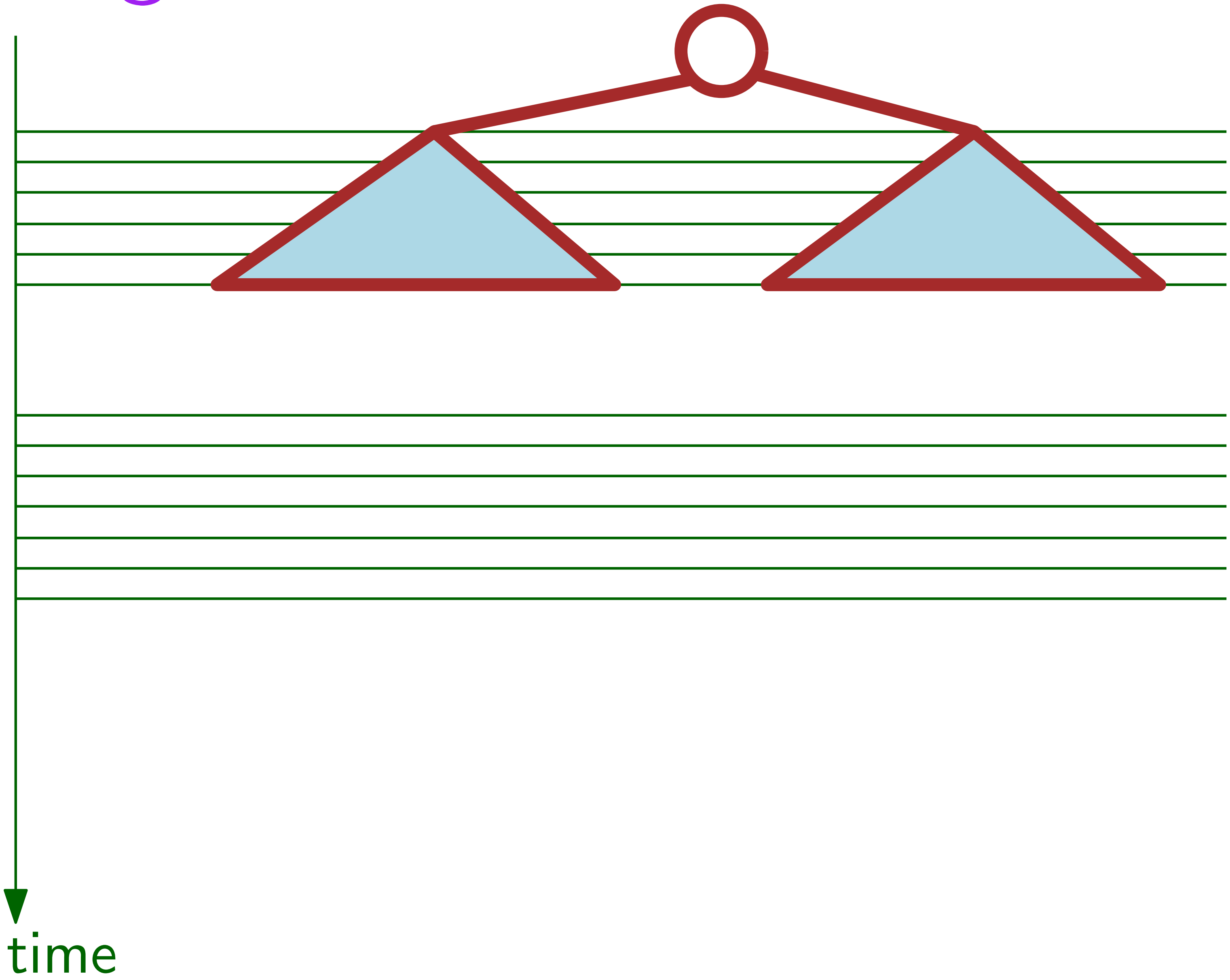


new drawing

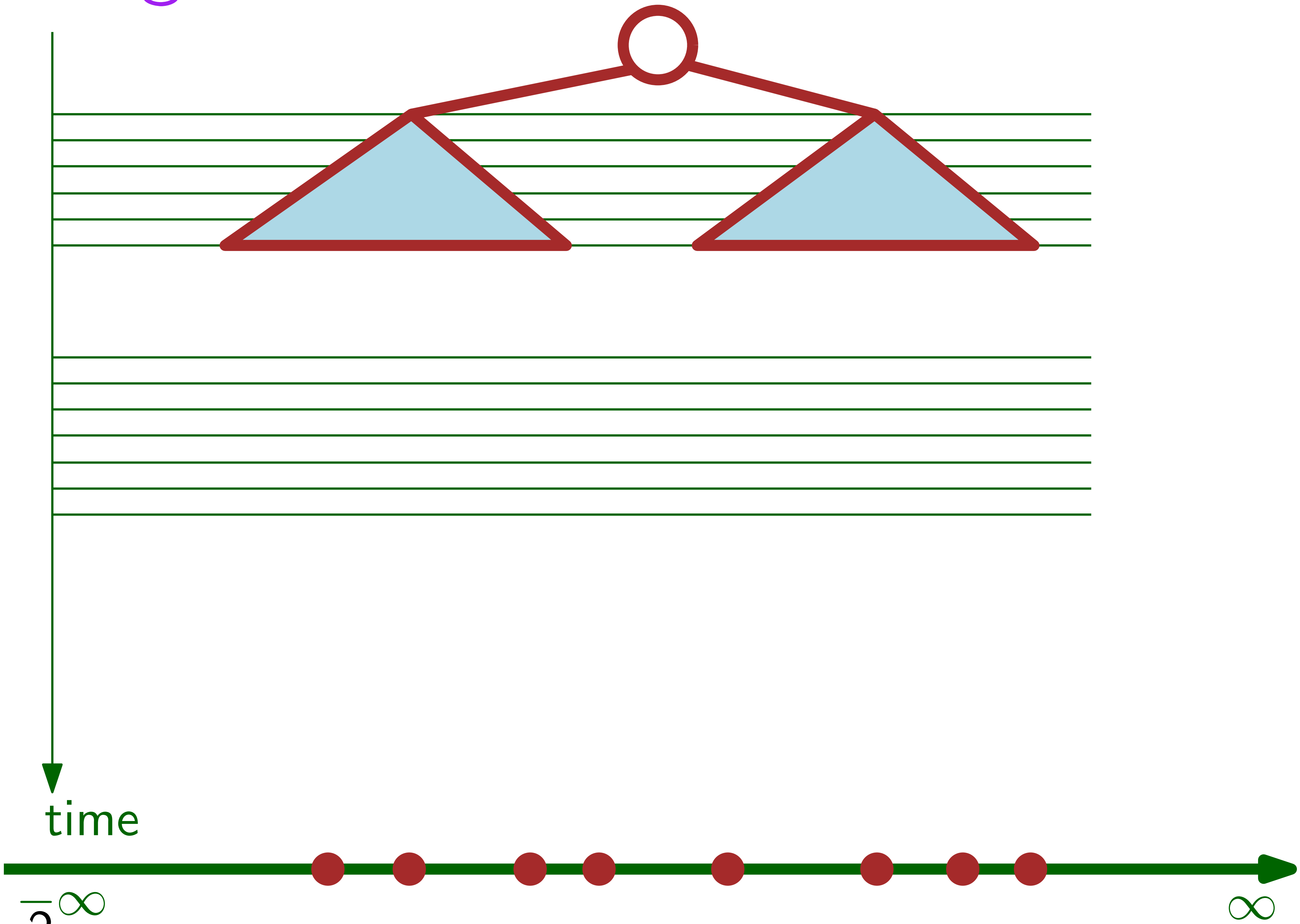
Sorting



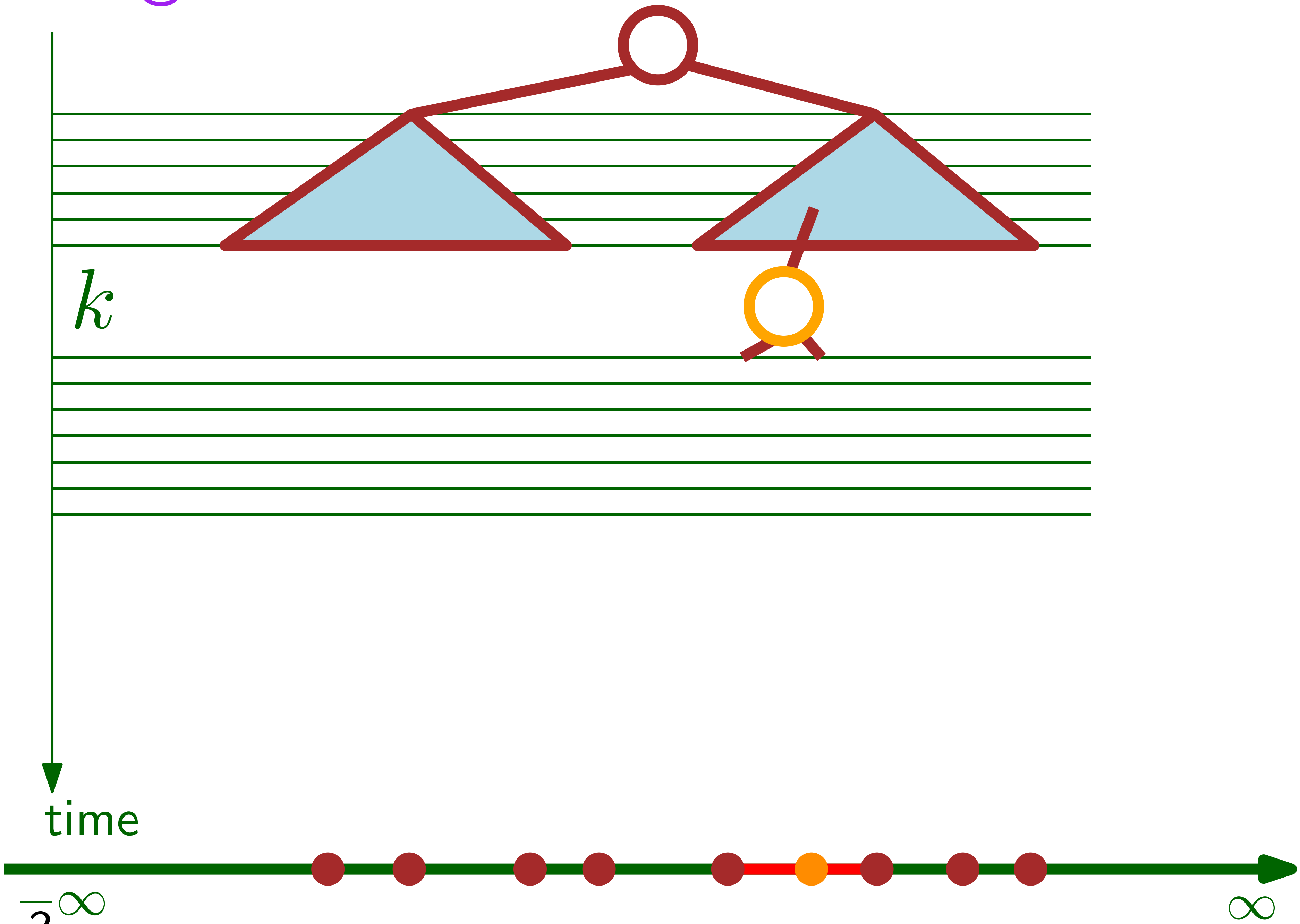
Sorting



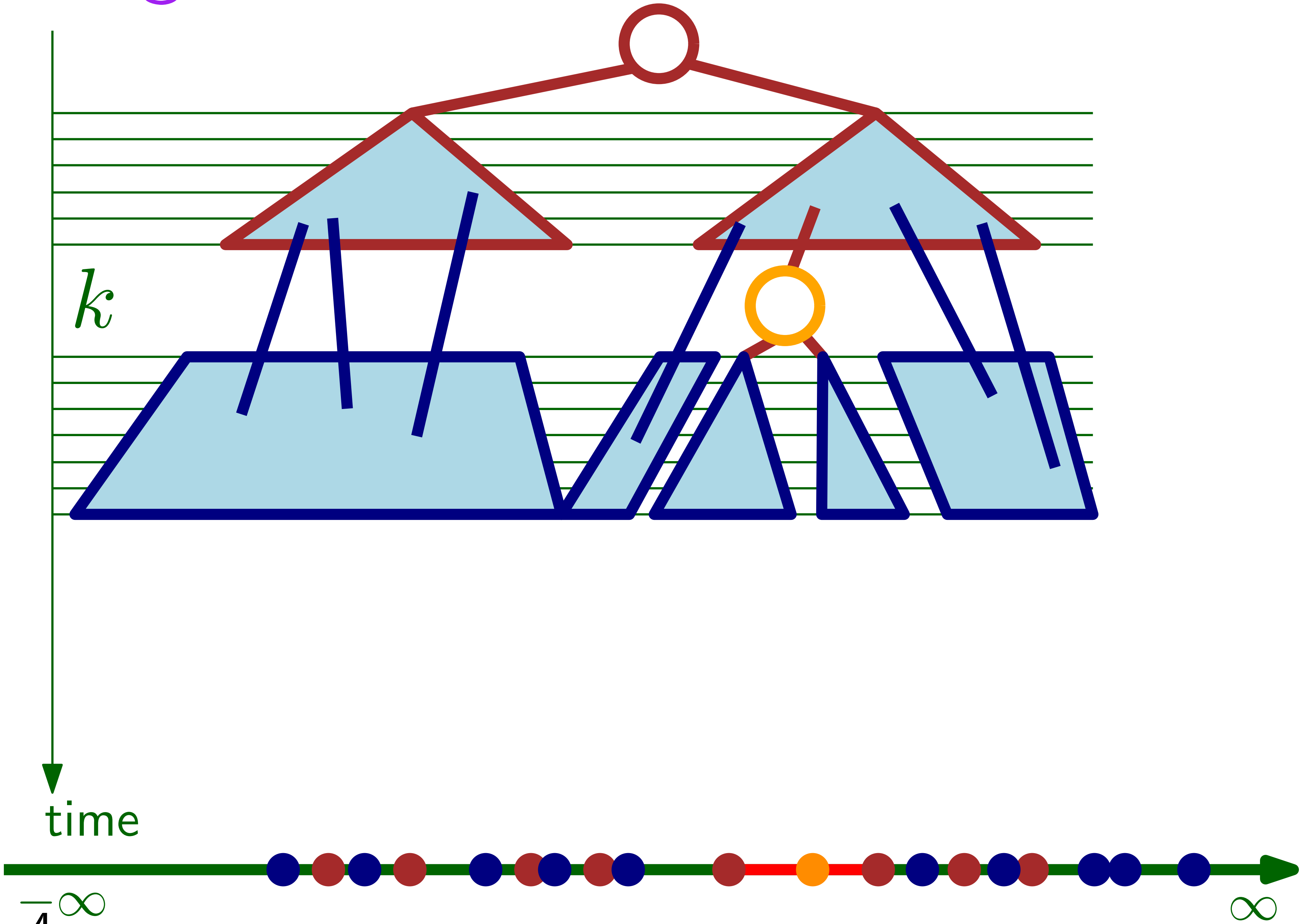
Sorting



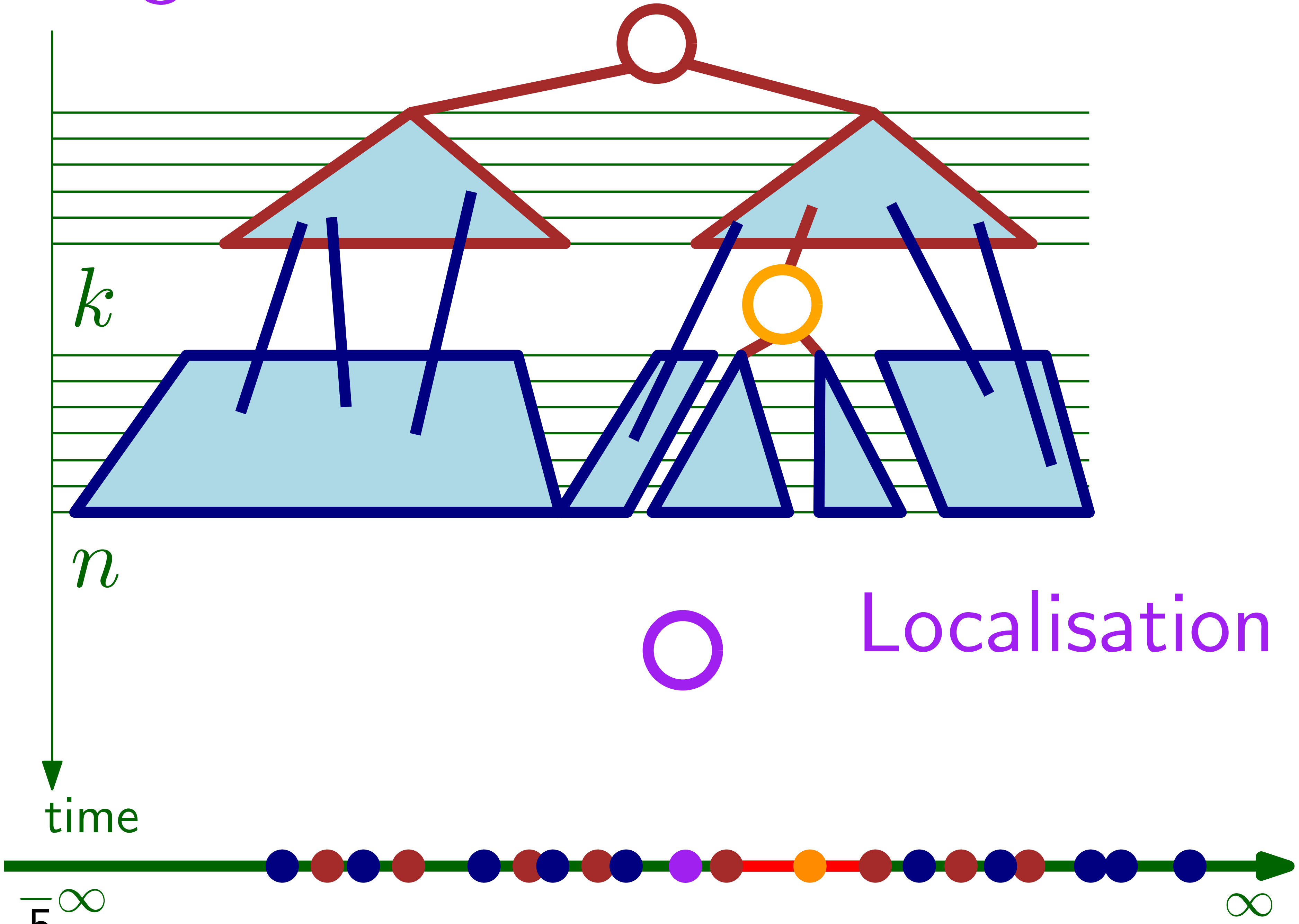
Sorting



Sorting

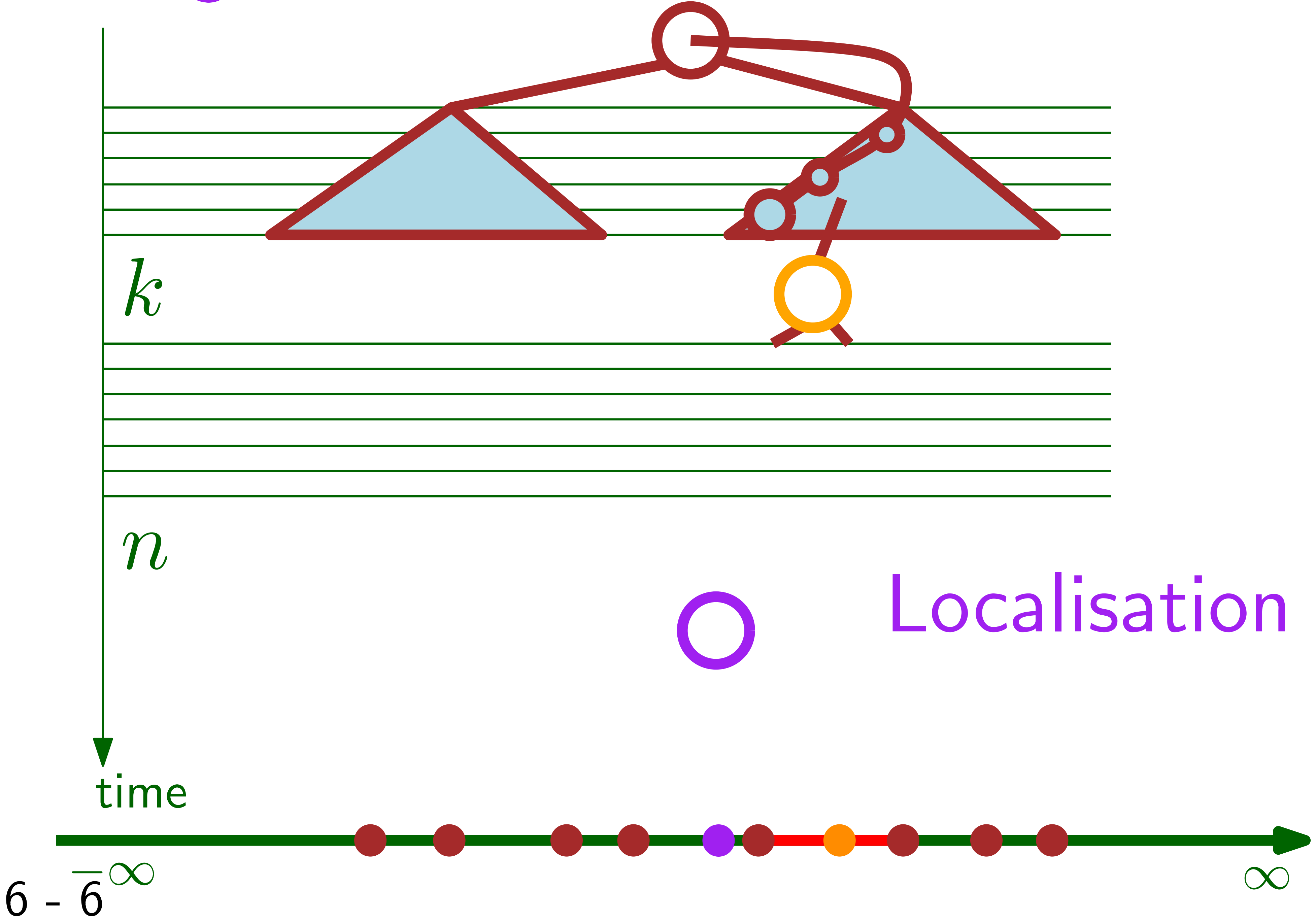


Sorting



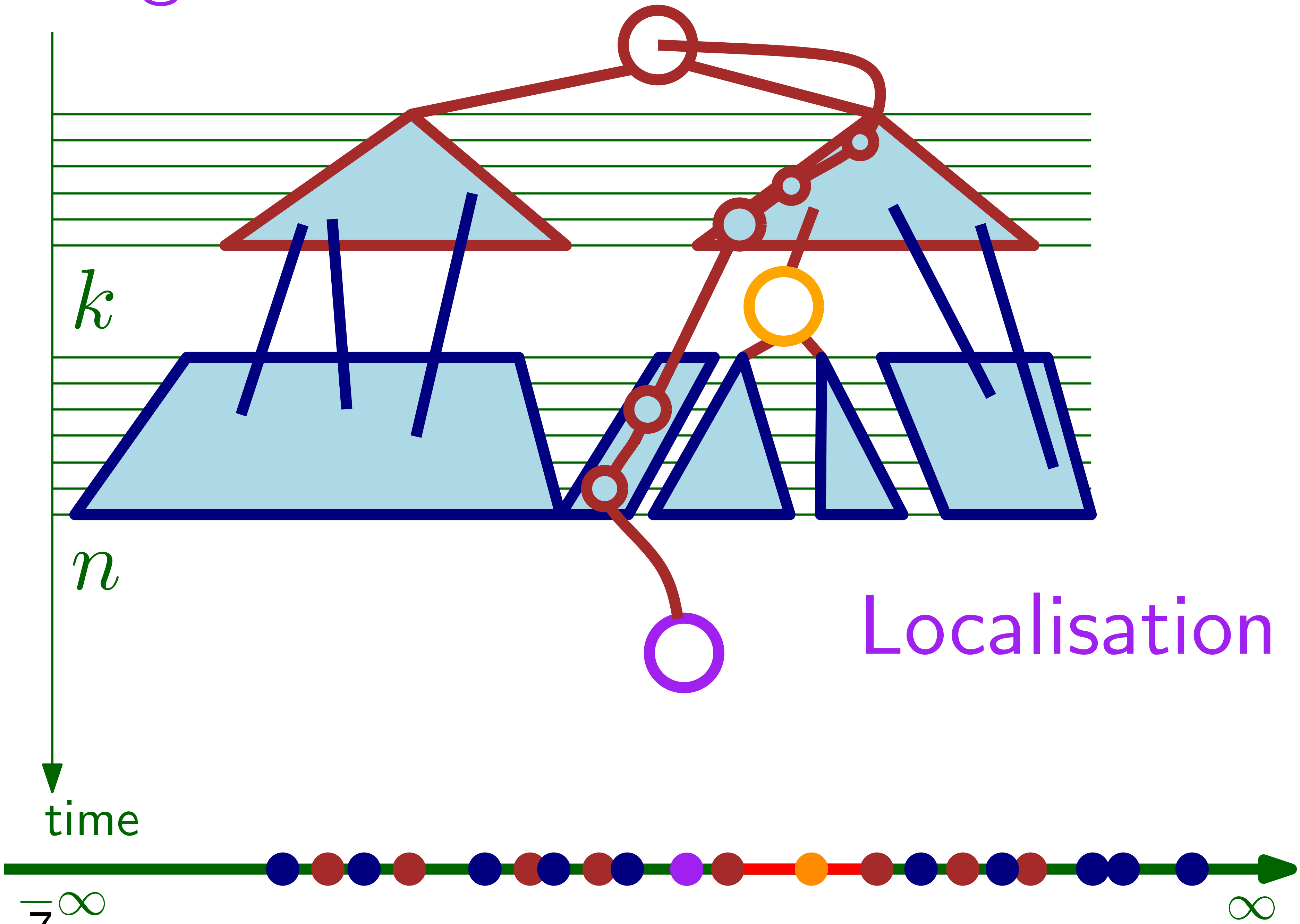
Localisation

Sorting



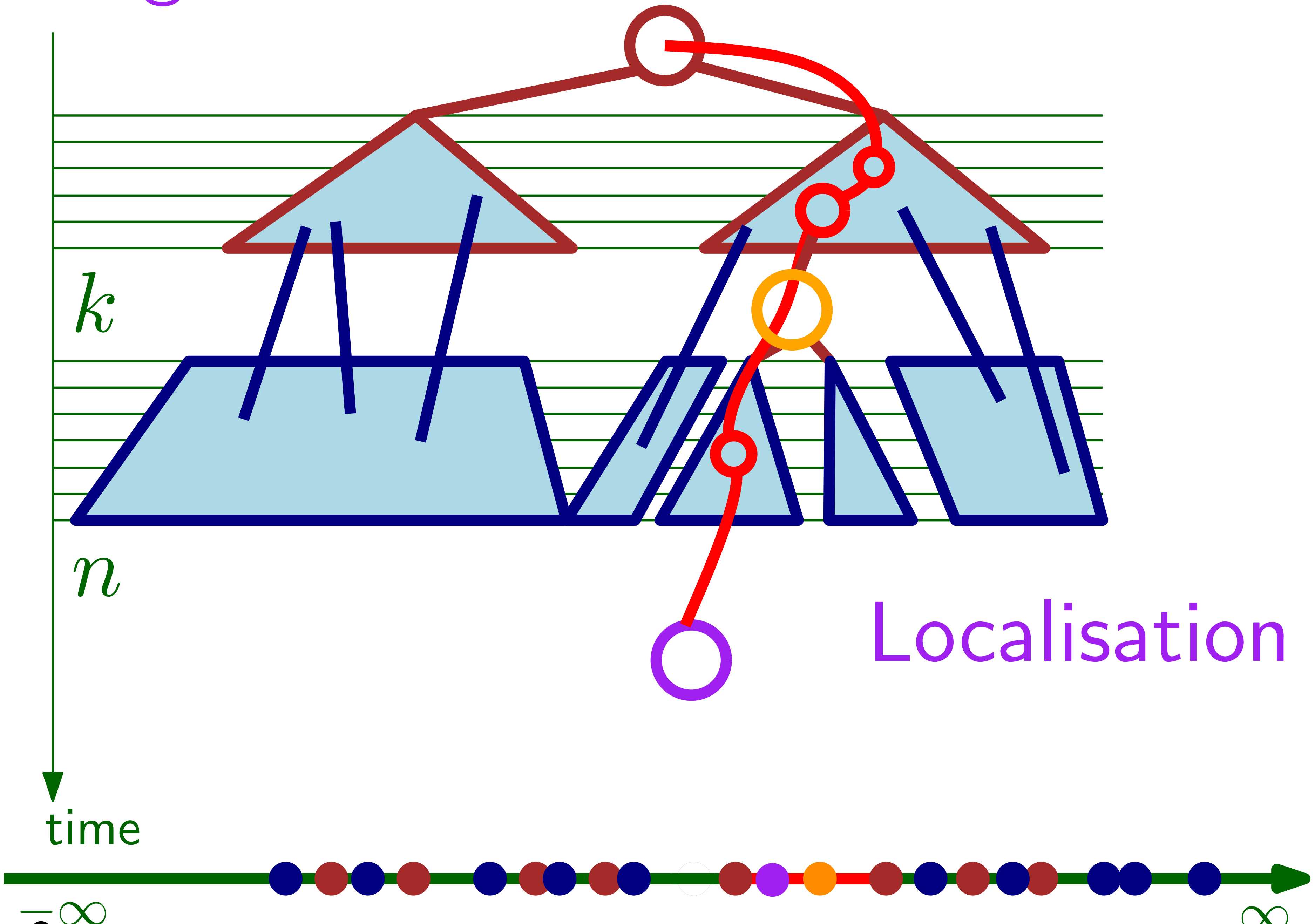
Localisation

Sorting



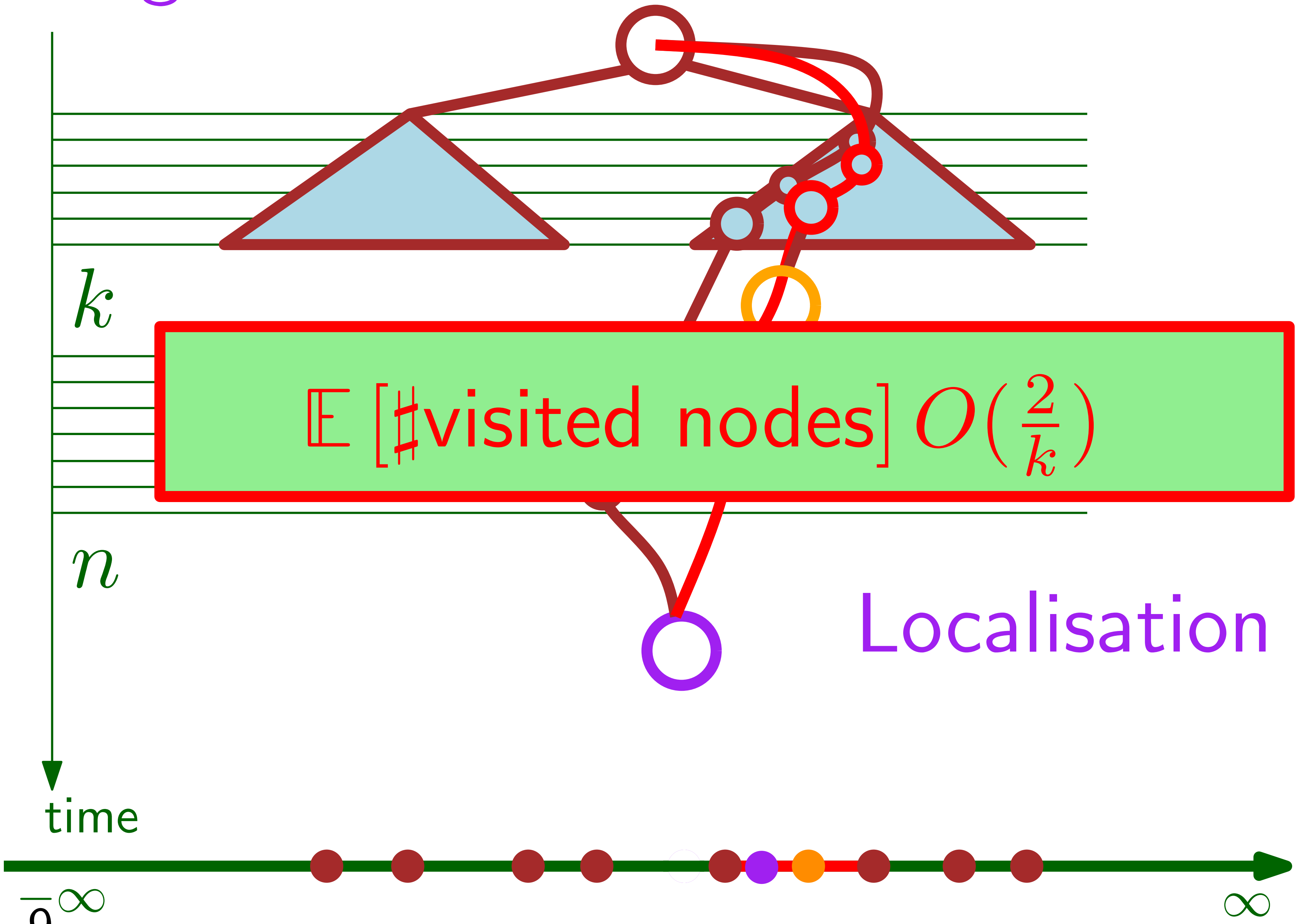
Localisation

Sorting



Localisation

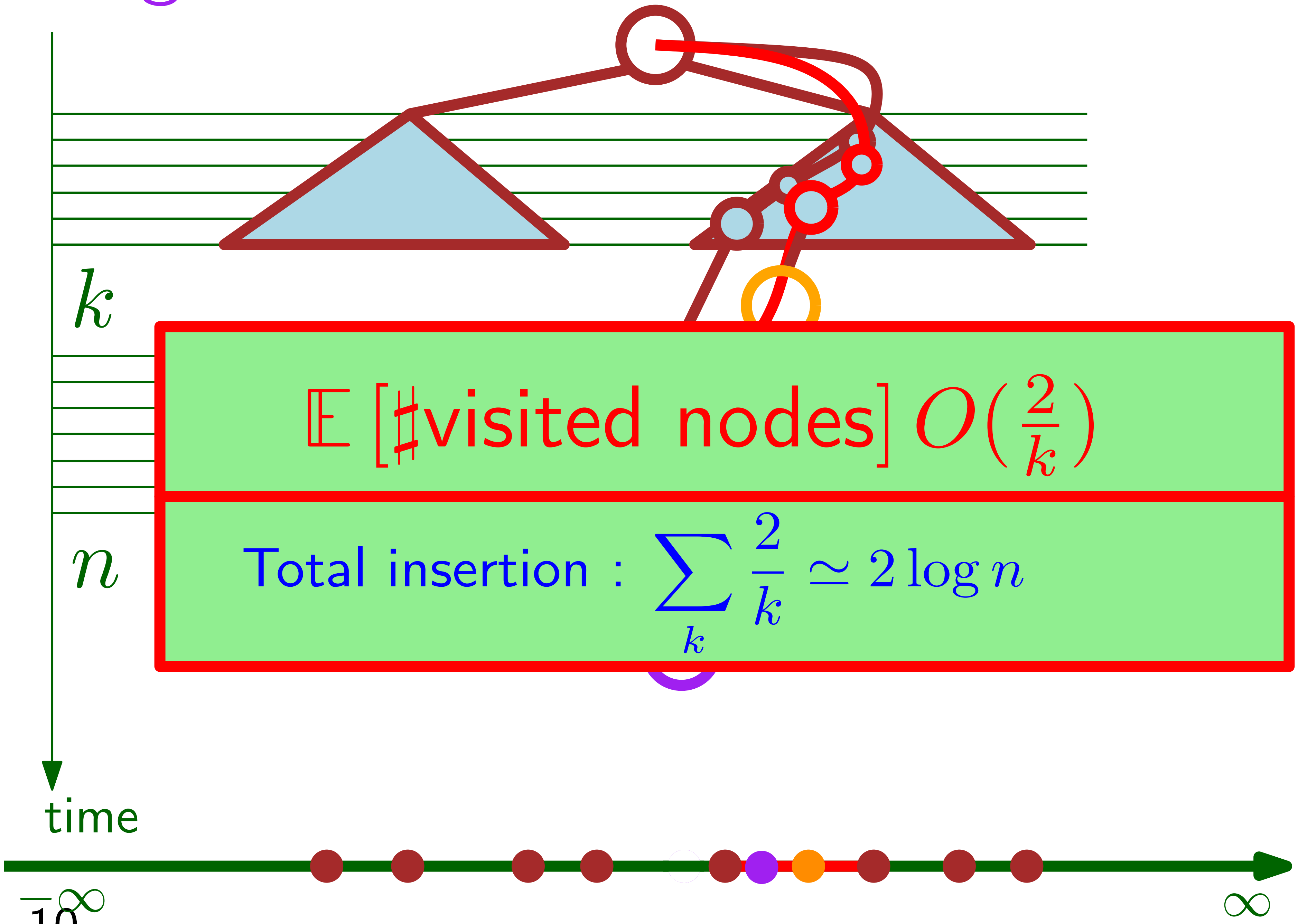
Sorting



$\mathbb{E}\left[\sharp\text{visited nodes}\right] O(\frac{2}{k})$

Localisation

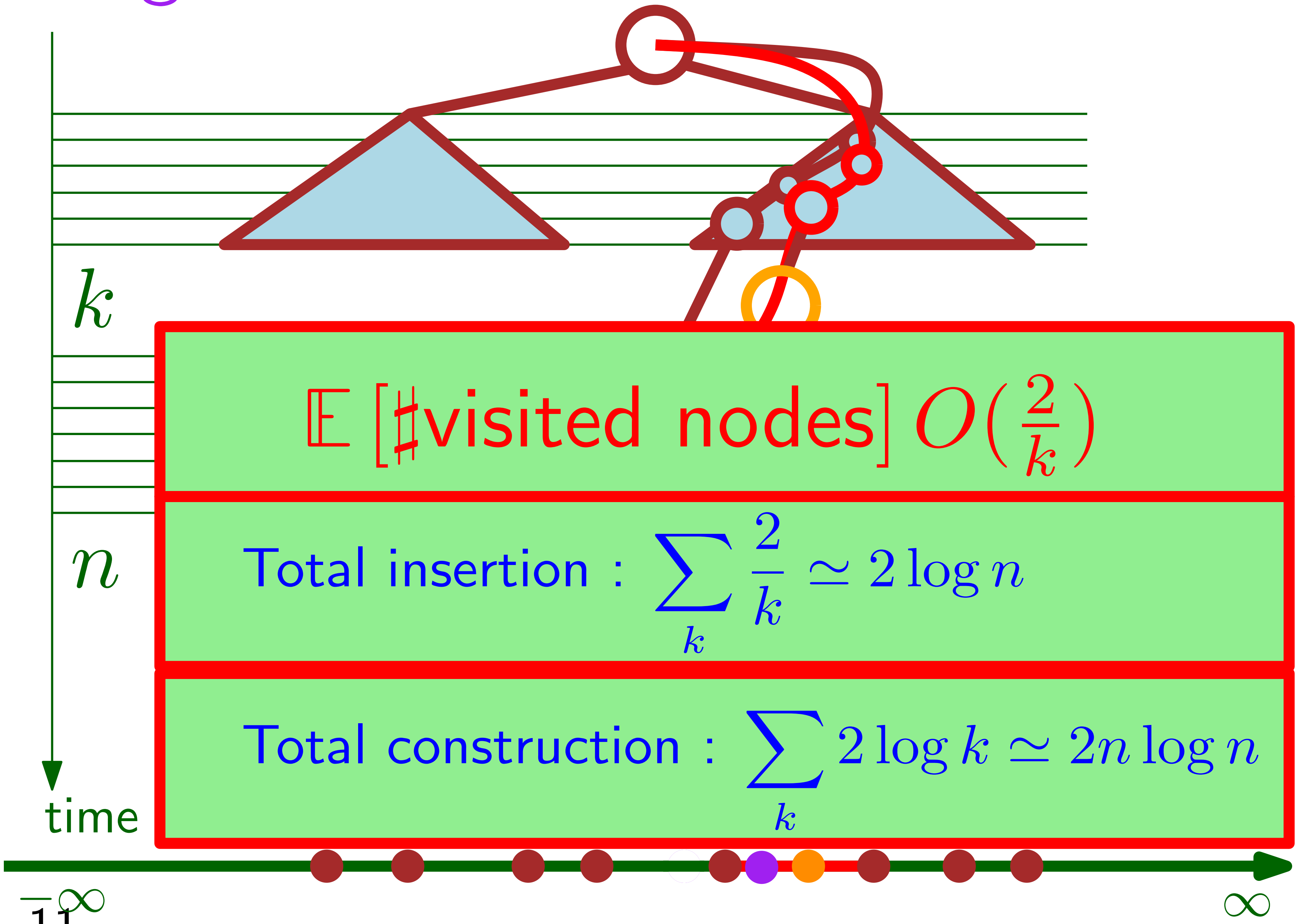
Sorting



$\mathbb{E}\left[\sharp\text{visited nodes}\right] O(\frac{2}{k})$

Total insertion : $\sum_k \frac{2}{k} \simeq 2\log n$

Sorting

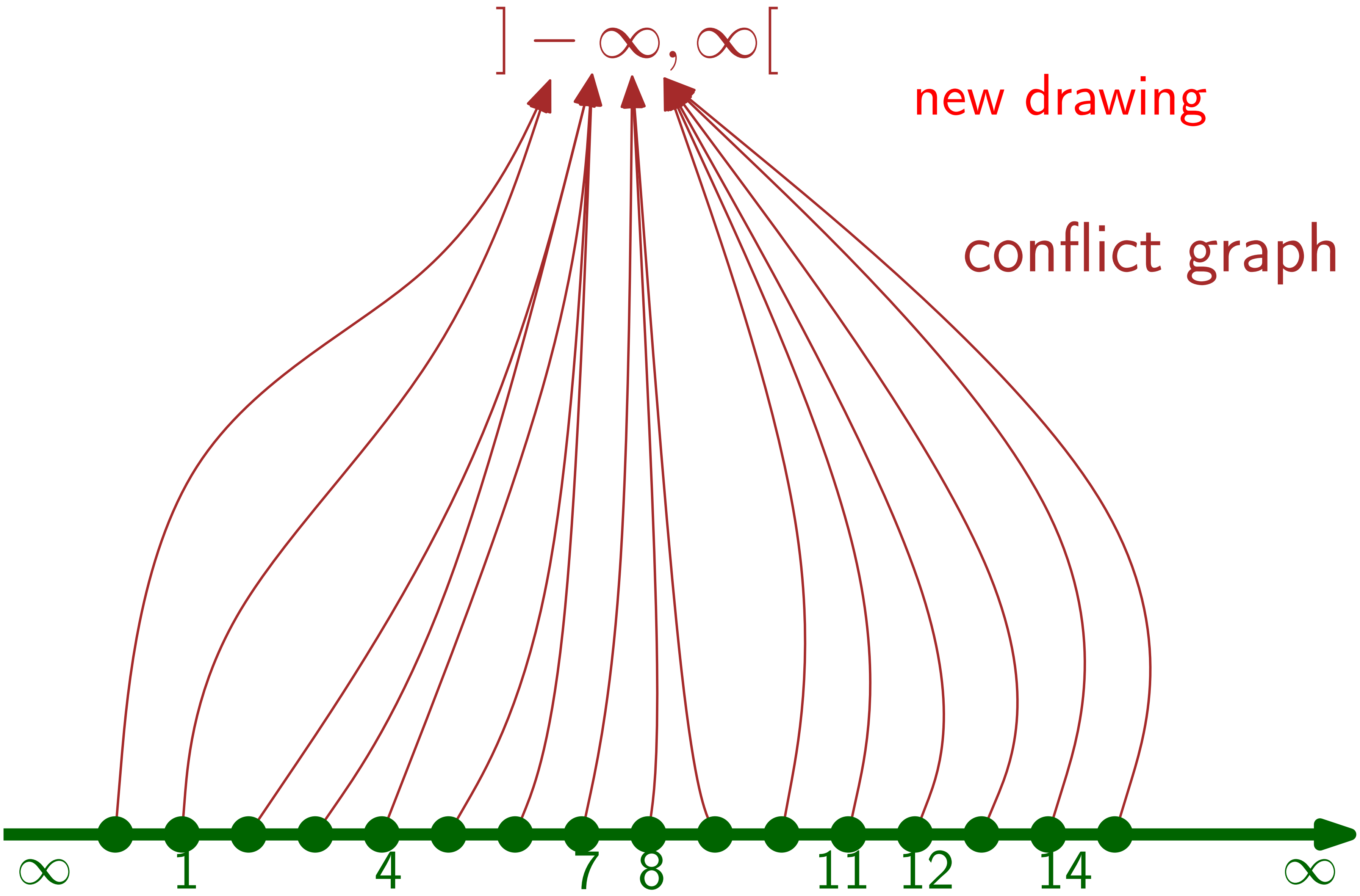


$\mathbb{E}\left[\sharp\text{visited nodes}\right] O(\frac{2}{k})$

Total insertion : $\sum_k \frac{2}{k} \simeq 2\log n$

Total construction : $\sum_k 2\log k \simeq 2n\log n$

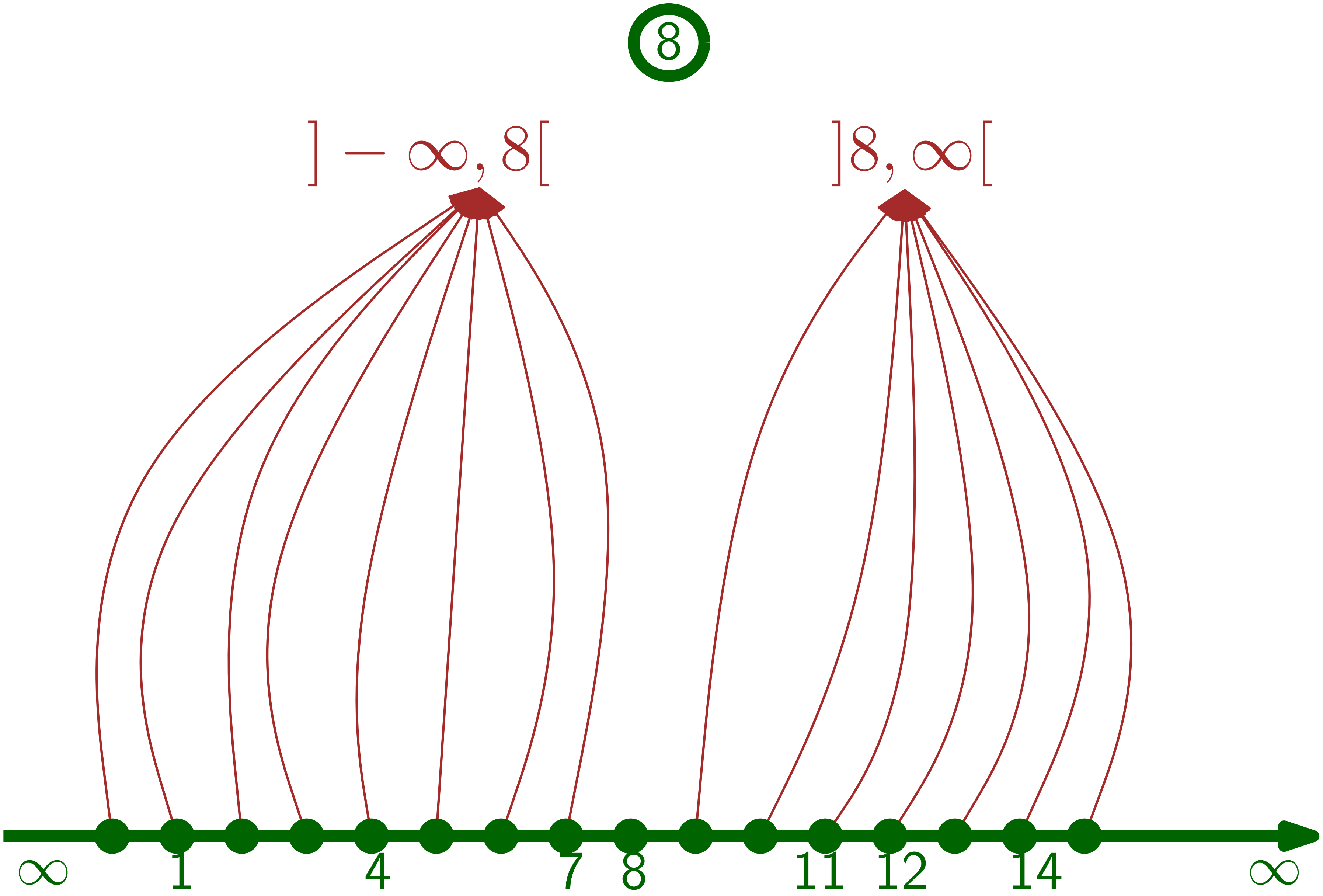
Sorting



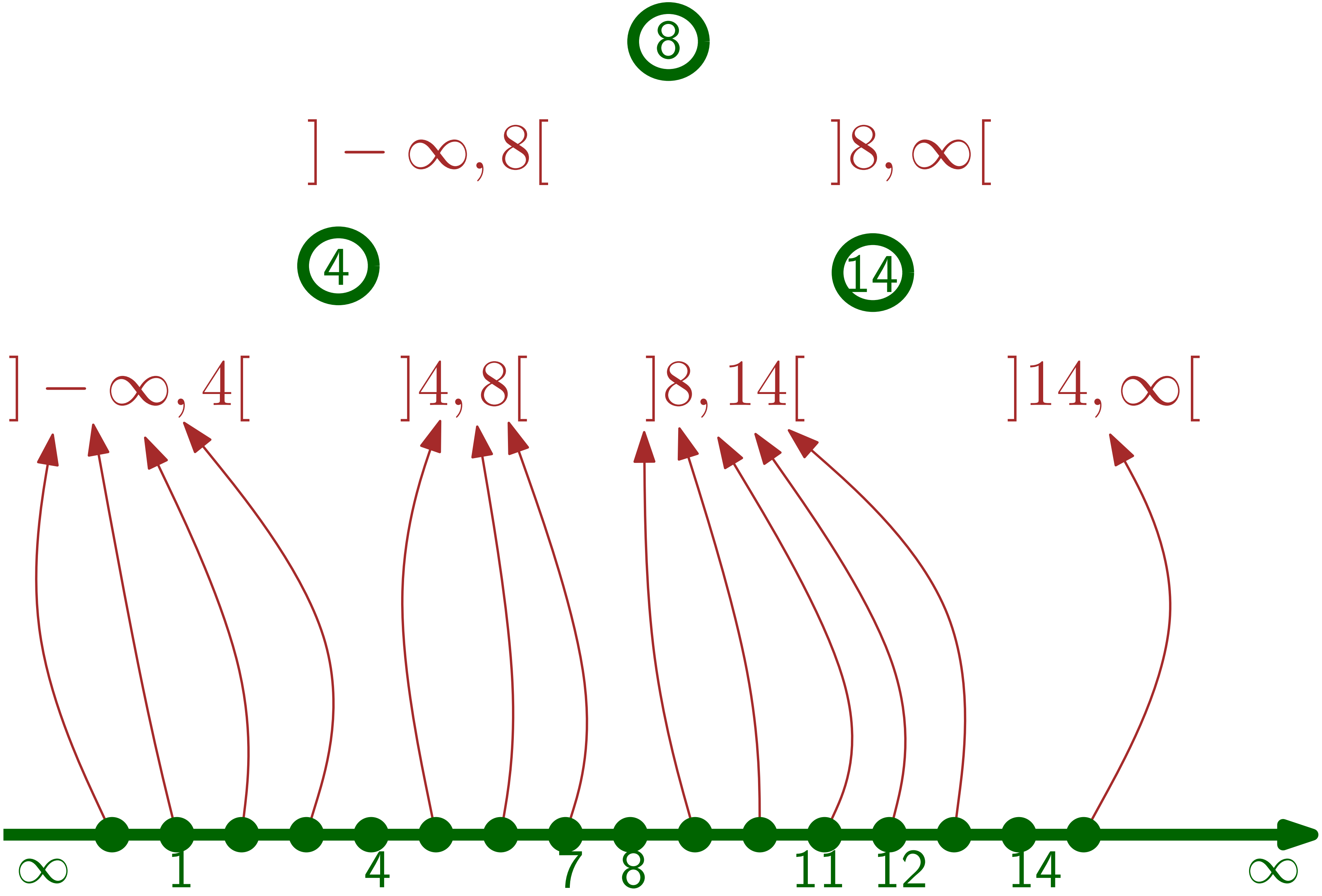
new drawing

conflict graph

Sorting



Sorting



Sorting

Unbalanced binary tree

Quicksort

History graph

Conflict graph

$O(n \log n)$

Same analysis

Backwards analysis

Analyse last insertion and sum

Last object is a random object

Randomization

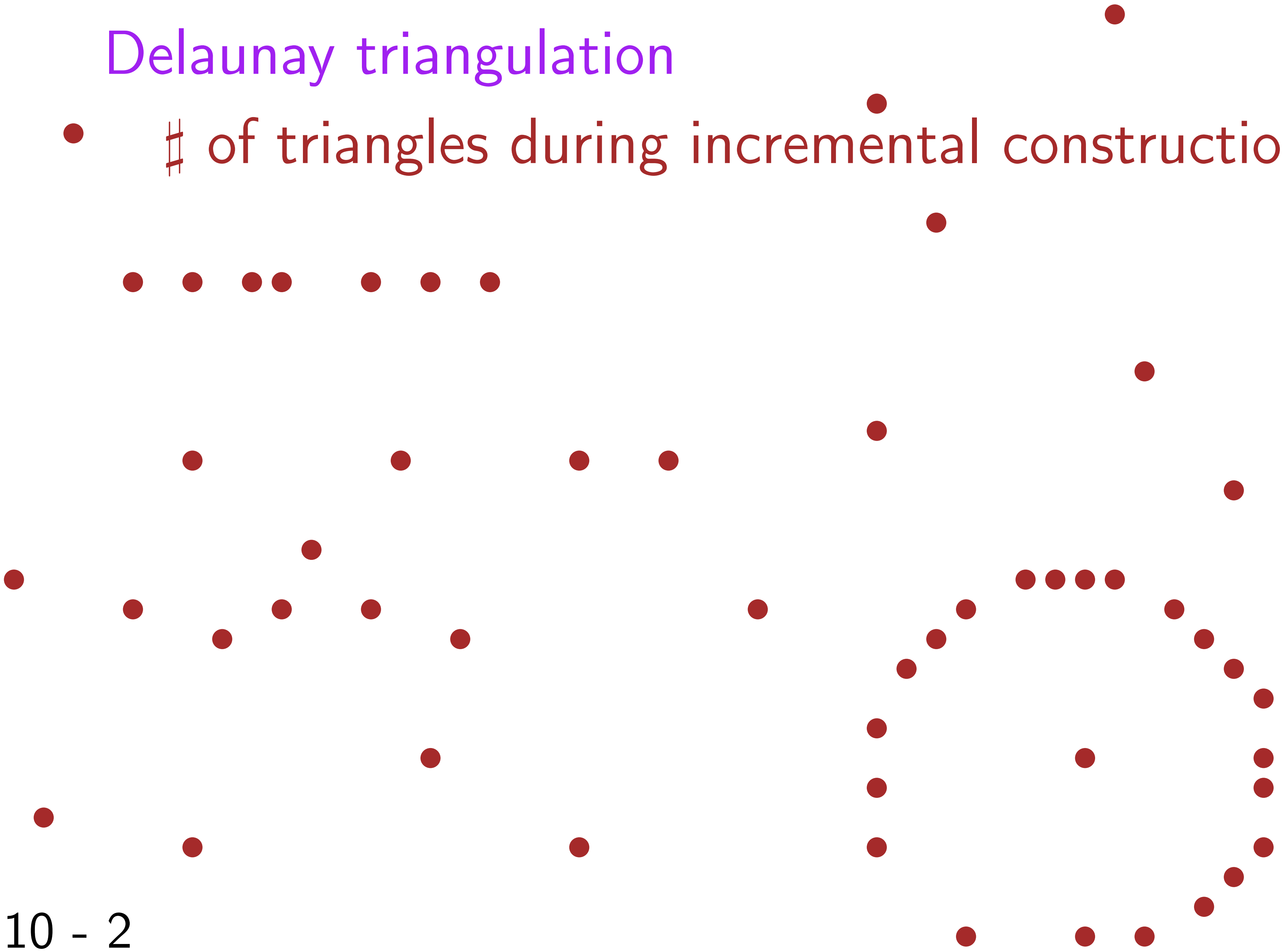
Backwards analysis for Delaunay triangulation

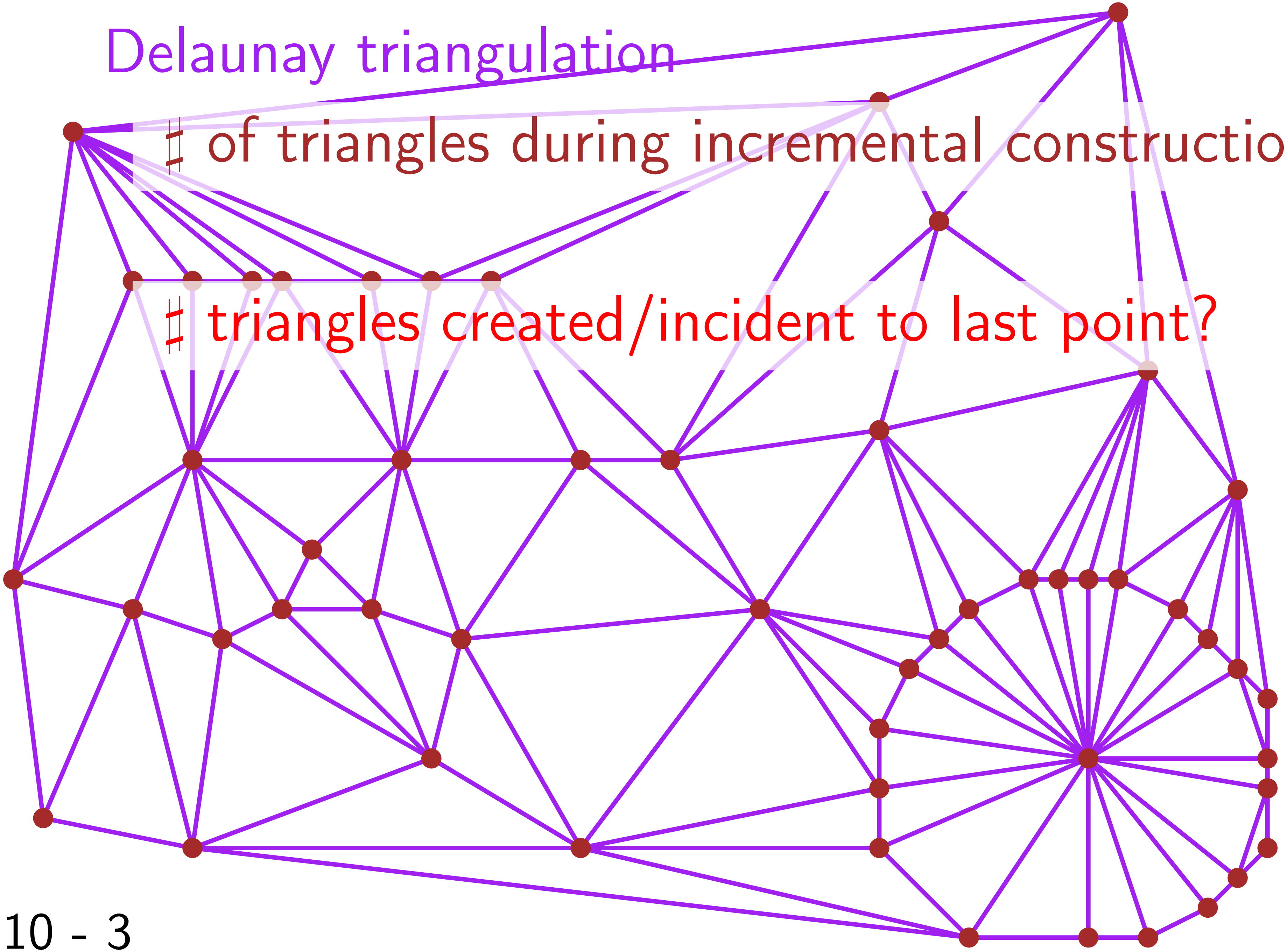
Delaunay triangulation

$\sharp$ of triangles during incremental construction?

Delaunay triangulation

- $\sharp$ of triangles during incremental construction?

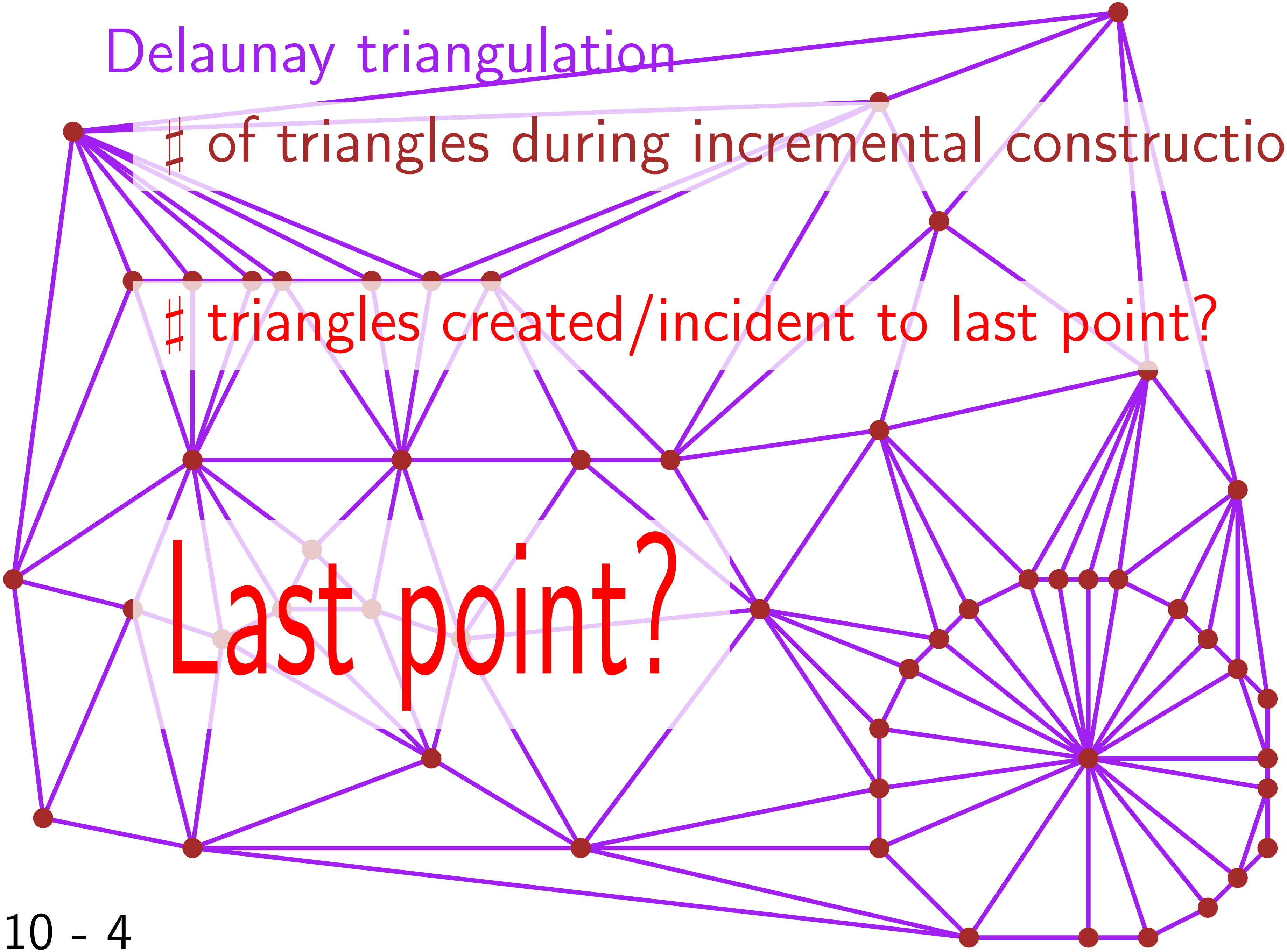




Delaunay triangulation

♯ of triangles during incremental construction?

♯ triangles created/incident to last point?

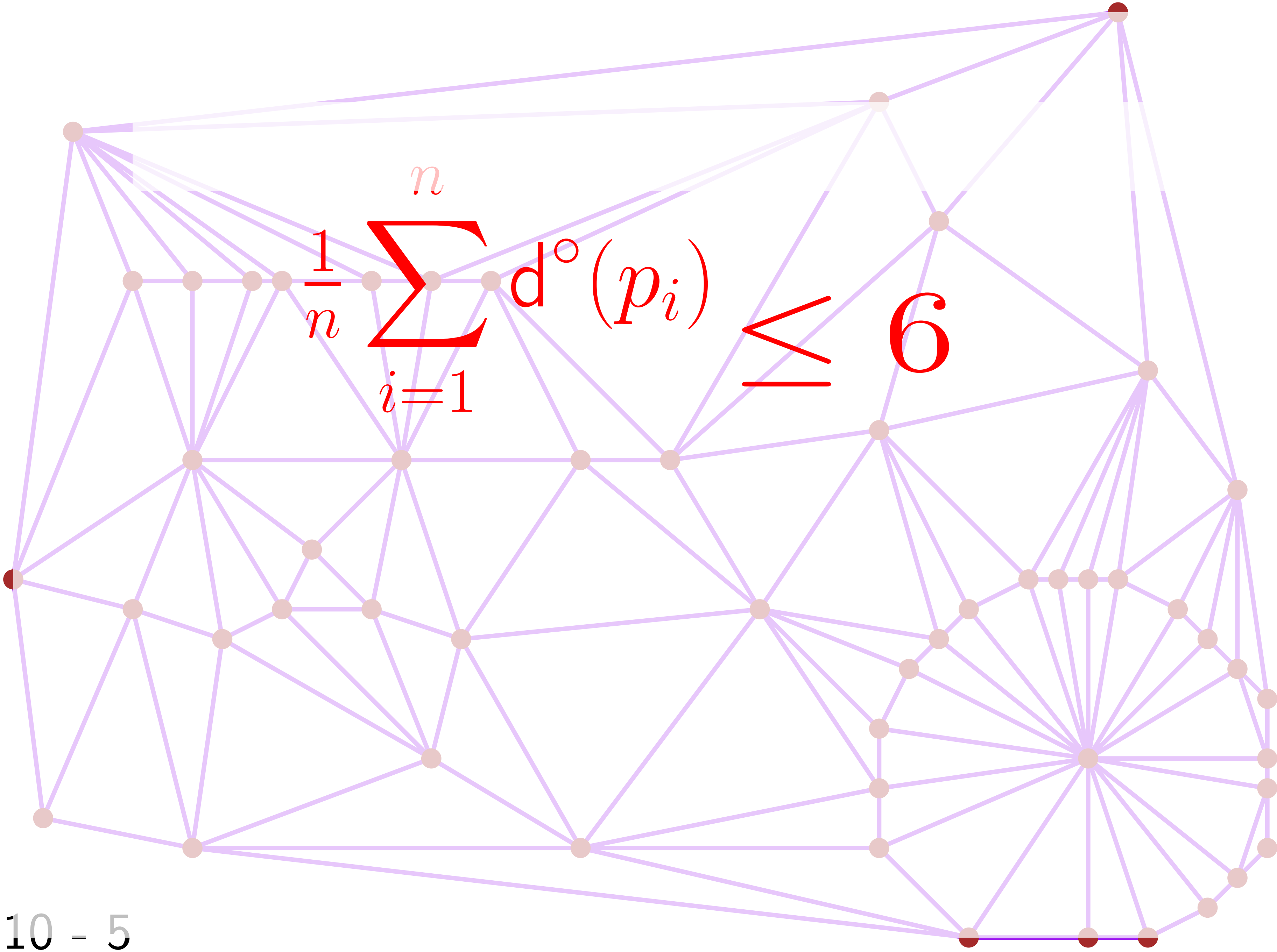


Delaunay triangulation

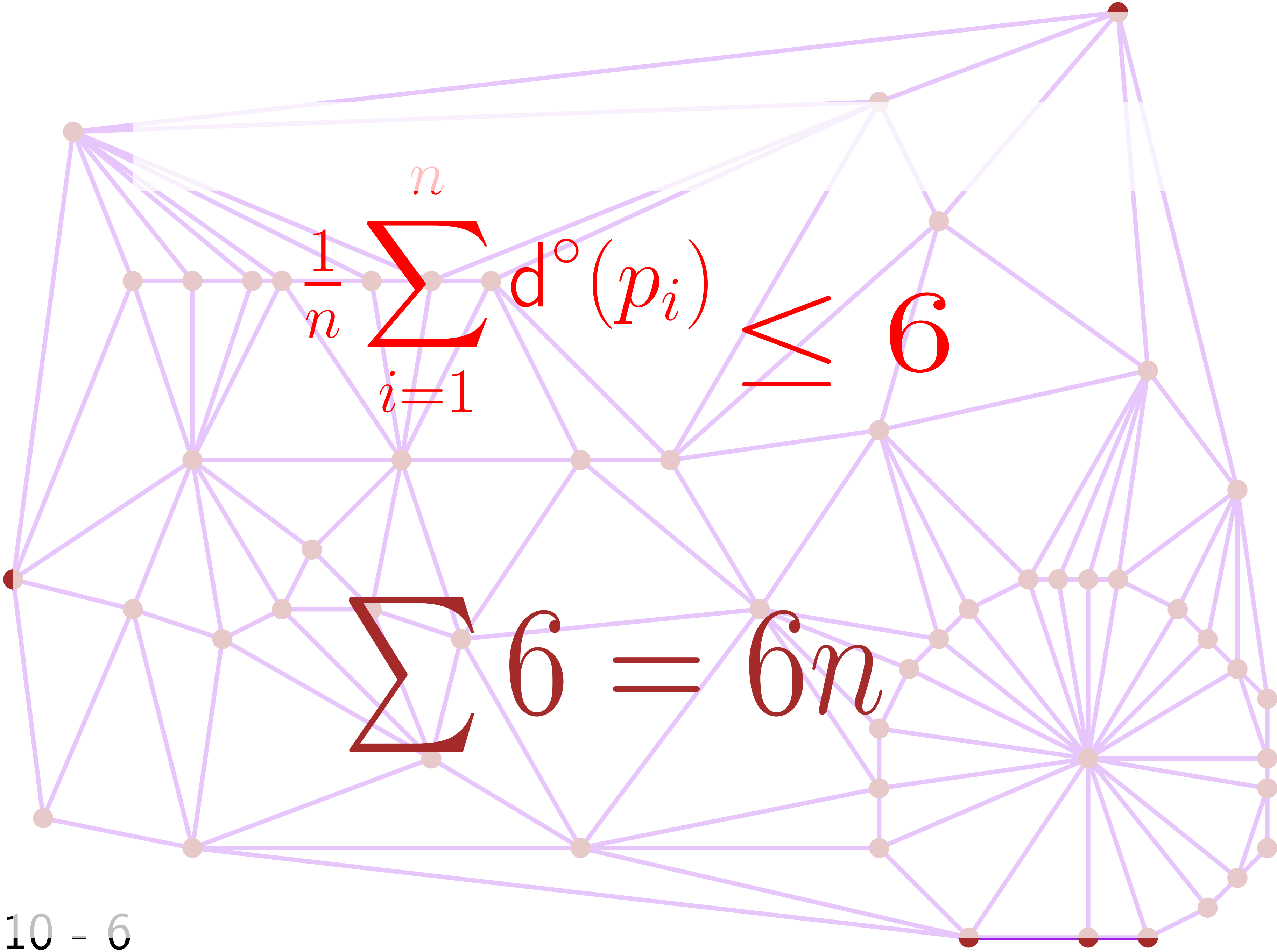
♯ of triangles during incremental construction?

♯ triangles created/incident to last point?

Last point?



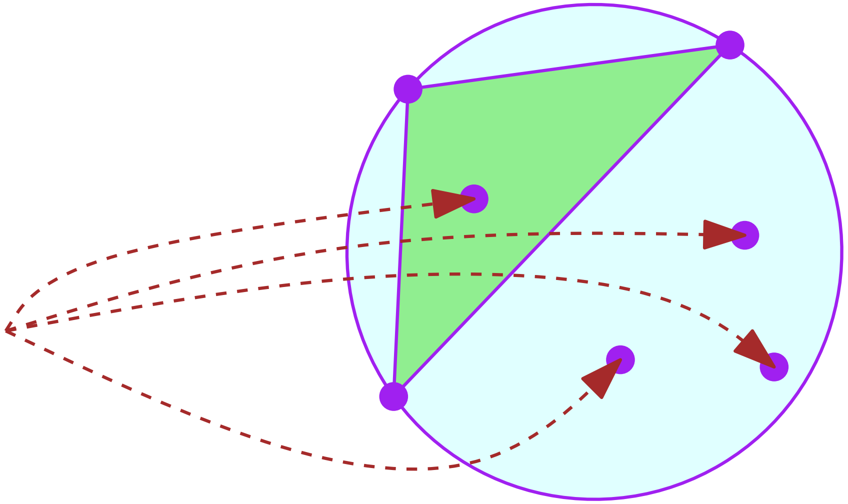
$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{d}^\circ(p_i) \leq 6$$



$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{d}^\circ(p_i) \leq 6$$

$$\sum 6 = 6n$$

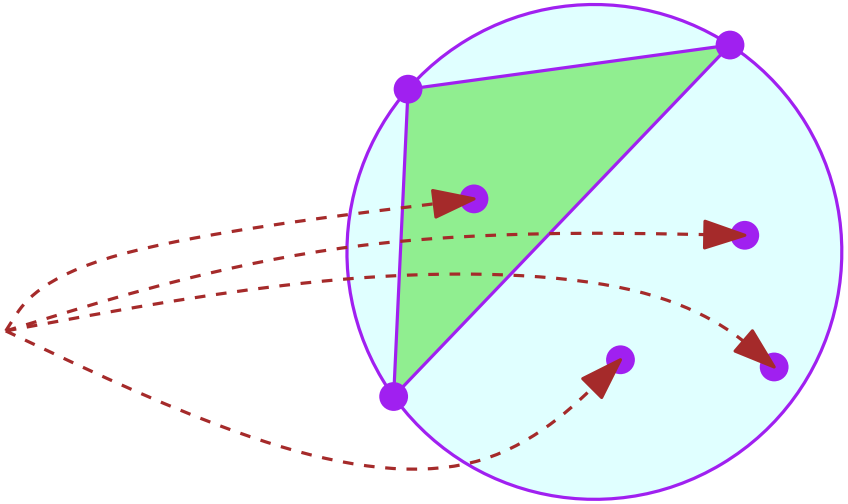
Alternative analysis



Triangle Δ with j stoppers

Alternative analysis

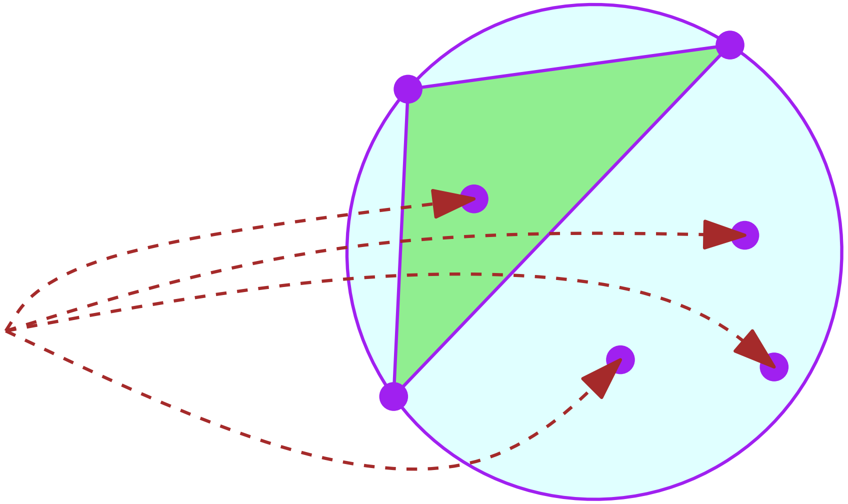
Triangle Δ with j stoppers



Probability that it exists in the triangulation of a sample of size αn

$$\simeq \alpha^3(1-\alpha)^j \geq \alpha^3(1-\alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4}\alpha^3 \qquad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Alternative analysis



Triangle Δ with j stoppers

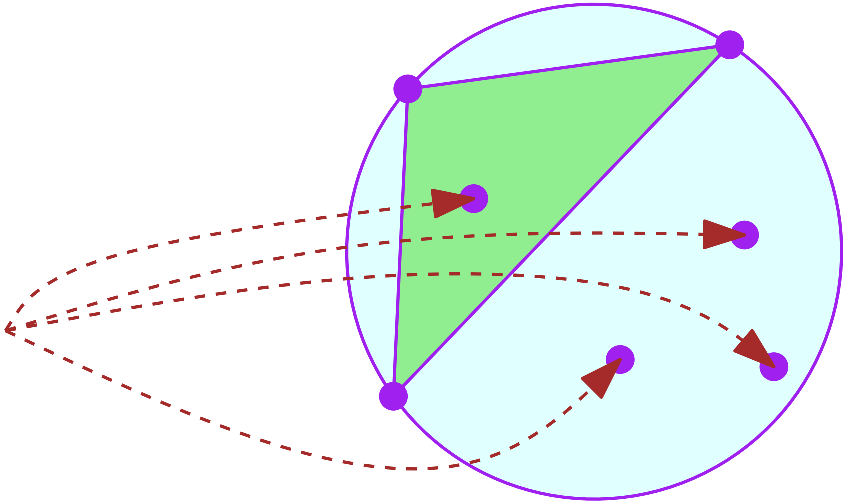
Probability that it exists in the triangulation of a sample of size αn

$$\simeq \alpha^3(1-\alpha)^j \geq \alpha^3(1-\alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4}\alpha^3 \qquad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Size of the triangulation of the sample

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \,\text{with}\, j \,\text{stoppers is there}\right] \times \sharp\Delta \,\text{with}\, j \,\text{stoppers}$$

Alternative analysis



Triangle Δ with j stoppers

Probability that it exists in the triangulation of a sample of size αn

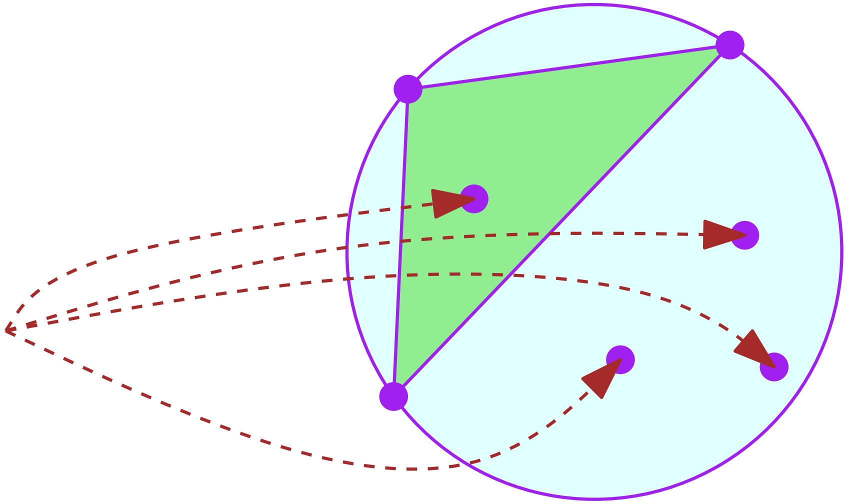
$$\simeq \alpha^3(1-\alpha)^j \geq \alpha^3(1-\alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4}\alpha^3 \qquad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Size of the triangulation of the sample

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers is there}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \sharp\Delta \text{ with } j \text{ stoppers}$$

Alternative analysis



Triangle Δ with j stoppers

Probability that it exists in the triangulation of a sample of size αn

$$\simeq \alpha^3(1-\alpha)^j \geq \alpha^3(1-\alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4}\alpha^3 \qquad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

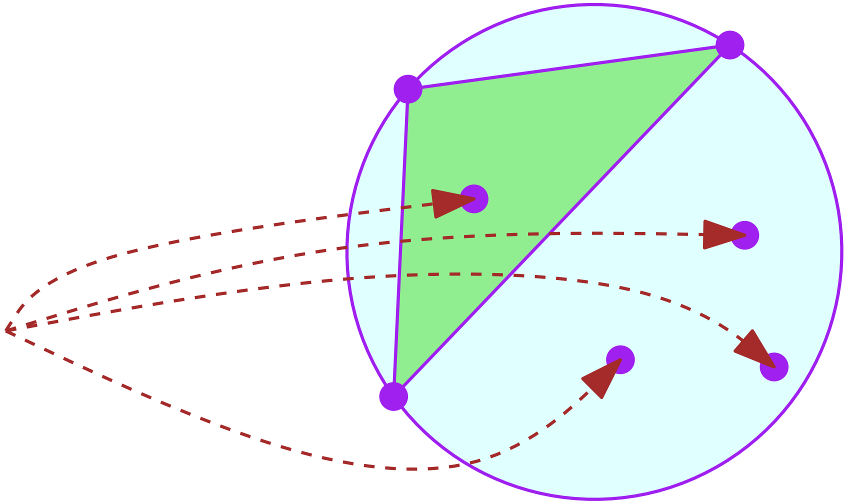
Size of the triangulation of the sample

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers is there}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \sharp\Delta \text{ with } j \text{ stoppers} = \alpha^3 \sharp\Delta \text{ with } \leq \frac{1}{\alpha} \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



Probability that it exists in the triangulation of a sample of size αn

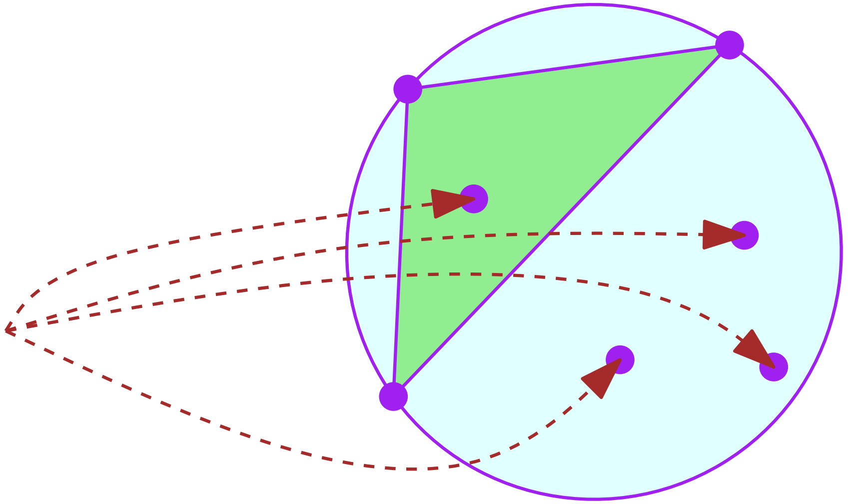
$$\simeq \alpha^3(1-\alpha)^j \geq \alpha^3(1-\alpha)^{\frac{1}{\alpha}} \geq \frac{1}{4}\alpha^3 \qquad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$

Size of the triangulation of the sample

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers is there}\right] \times \sharp\Delta \text{ with } j \text{ stoppers} \qquad = O(\alpha n)$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \sharp\Delta \text{ with } j \text{ stoppers} = \alpha^3 \sharp\Delta \text{ with } \leq \frac{1}{\alpha} \text{ stoppers}$$

Alternative analysis



Triangle Δ with j stoppers

Probability that it exists in the triangulation of a sample of size αn

$$\simeq \alpha^3(1-\alpha)^j \geq \alpha^3(1-\alpha)^{\frac{1}{\alpha}} \geq \tfrac{1}{4}\alpha^3 \qquad \text{if } 2 \leq j \leq \tfrac{1}{\alpha}$$

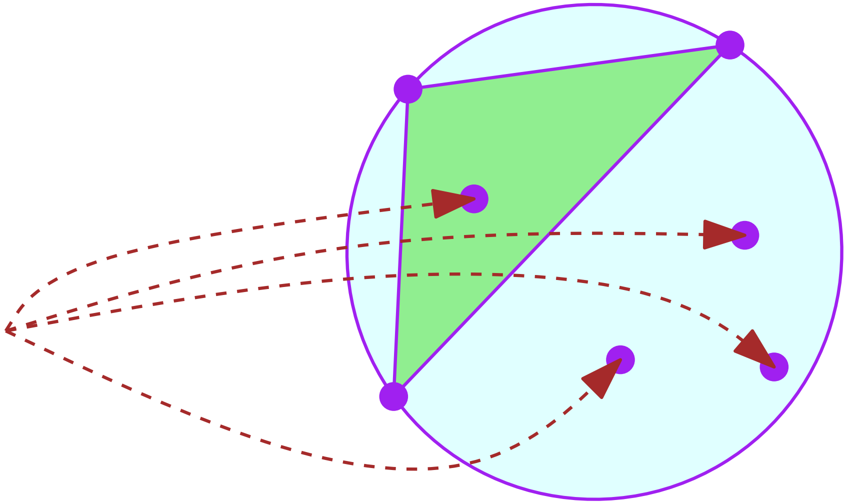
Size of the triangulation of the sample

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers is there}\right] \times \sharp\Delta \text{ with } j \text{ stoppers} \qquad = O(\alpha n)$$

$$\geq \sum_{j=0}^{1/\alpha} \frac{\alpha^3}{4} \times \sharp\Delta \text{ with } j \text{ stoppers} = \alpha^3 \sharp\Delta \text{ with} \leq \frac{1}{\alpha} \text{ stoppers}$$

Size (order $\leq k$ Voronoi) $\leq \frac{\alpha n}{\alpha^3} = nk^2$

Alternative analysis

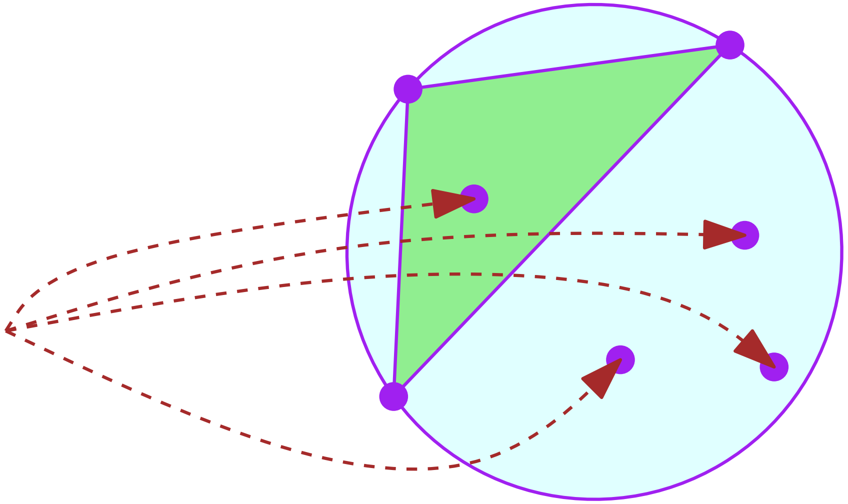


Triangle Δ with j stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

Alternative analysis



Triangle Δ with j stoppers

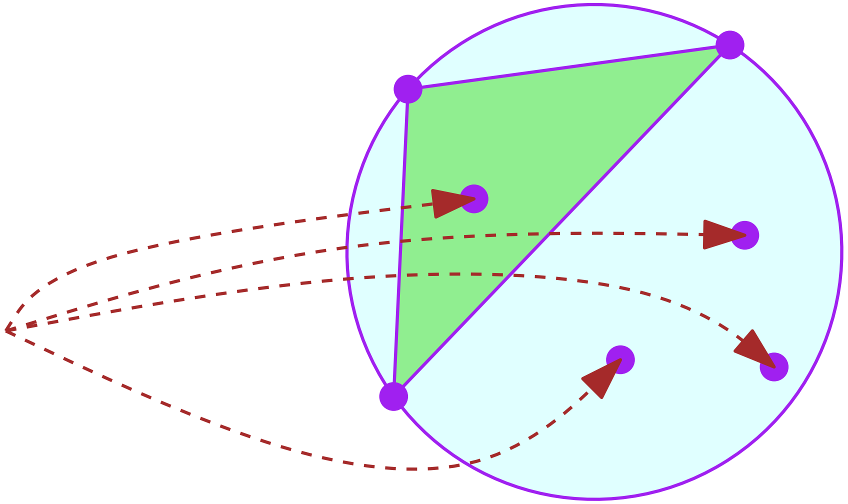
Probability that it exists during the construction

$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

$\sharp$ of created triangles

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

Alternative analysis



Triangle Δ with j stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

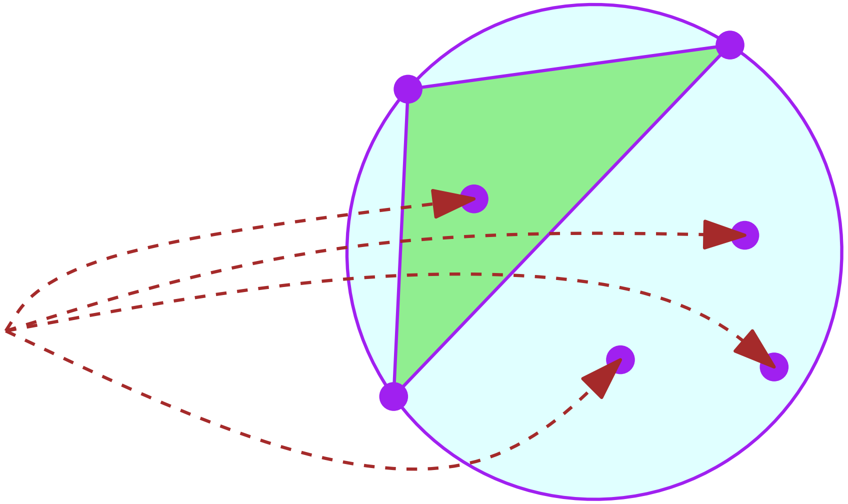
$\sharp$ of created triangles

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with } \leq j \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



Probability that it exists during the construction

$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

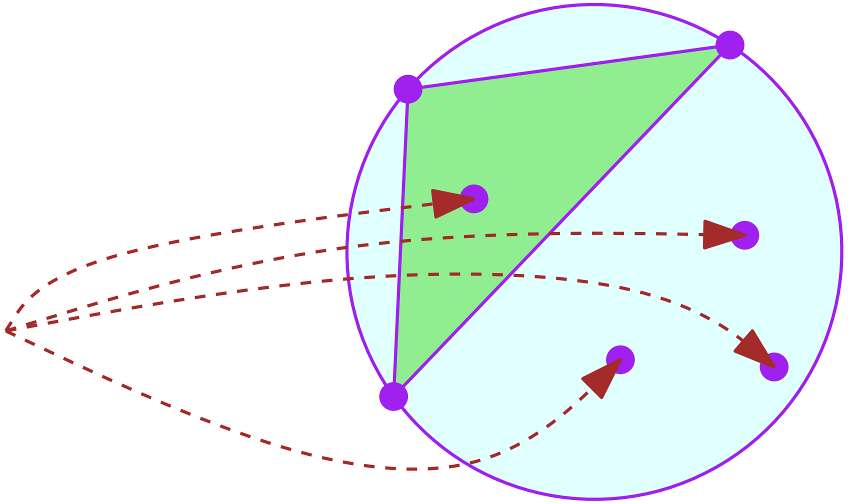
♯ of created triangles

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with } \leq j \text{ stoppers}$$

$$\simeq \sum_{j=0}^{n} \frac{18}{j^4} \times nj^2 = O(n \sum \frac{1}{j^2}) = O(n)$$

Alternative analysis

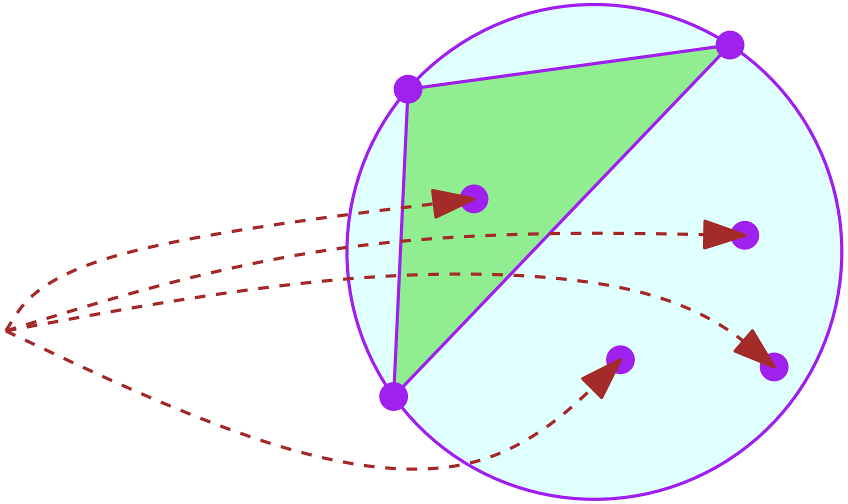


Triangle Δ with j stoppers

Conflict graph / History graph

It remains to analyze conflict location

Alternative analysis



Triangle Δ with j stoppers

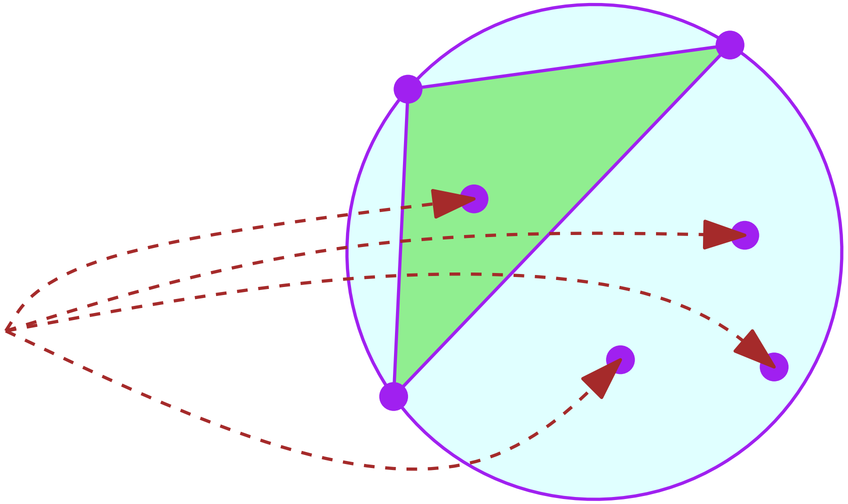
Probability that it exists during the construction

$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

$\sharp$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

Alternative analysis



Triangle Δ with j stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

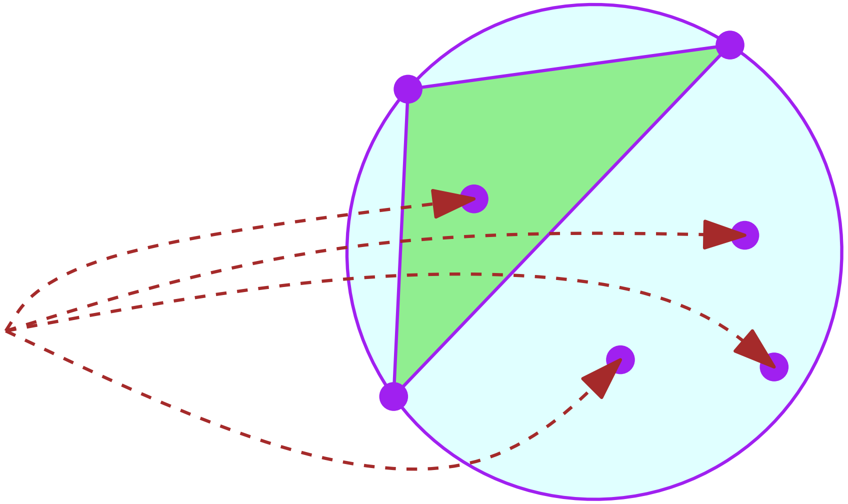
$\sharp$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} j \times \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with } \leq j \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



Probability that it exists during the construction

$$= \frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}$$

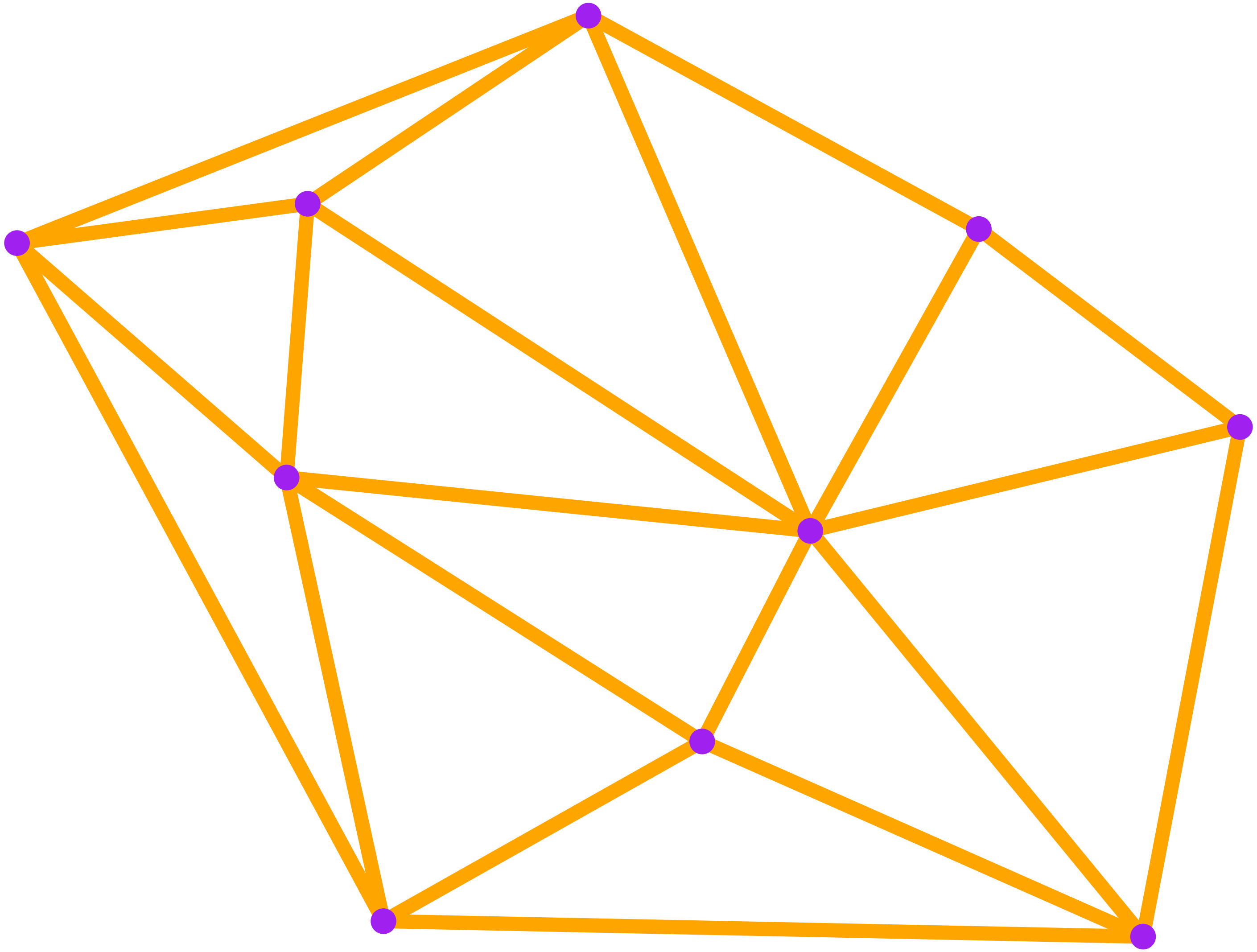
$\sharp$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

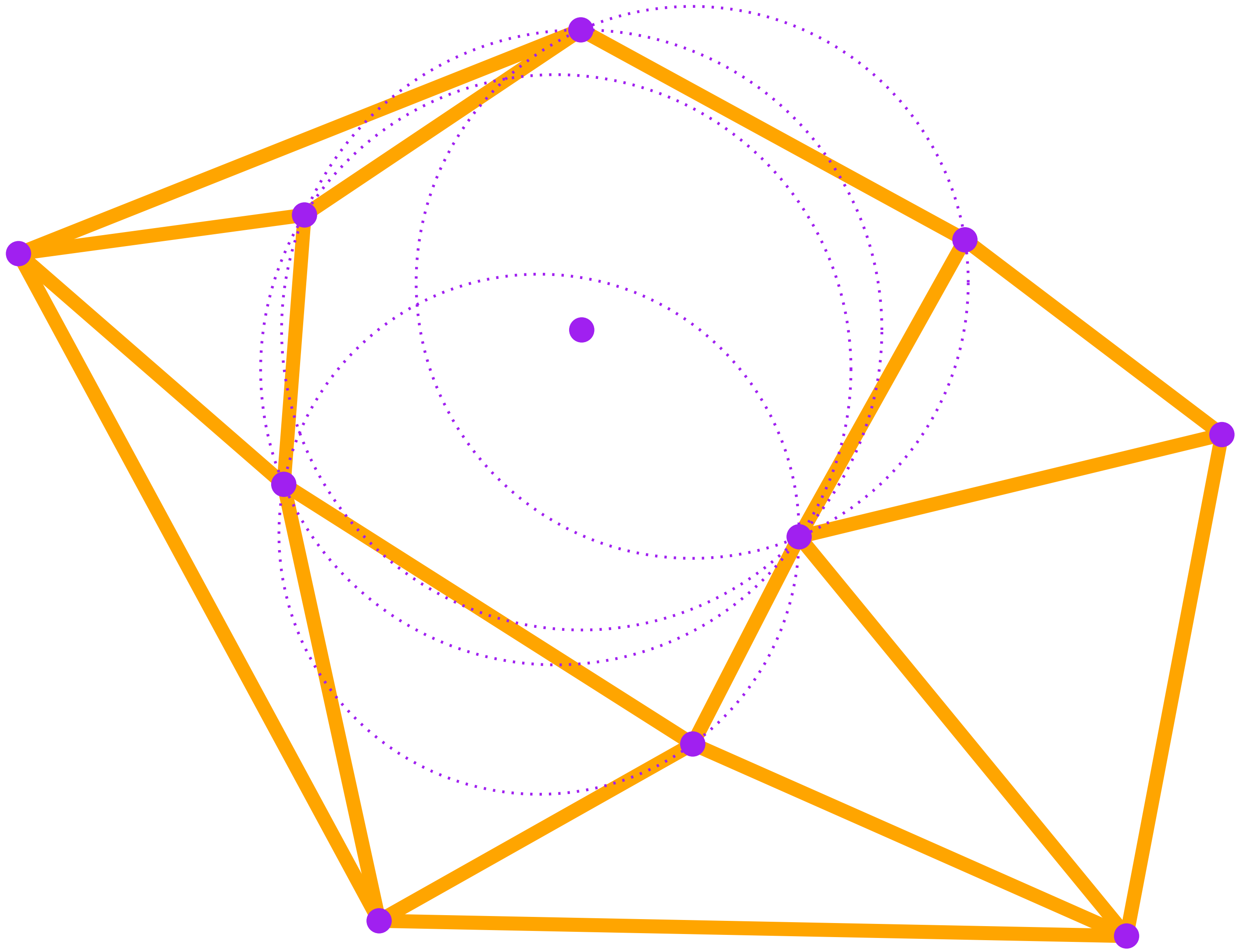
$$= \sum_{j=0}^{n} j \times \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with} \leq j \text{ stoppers}$$

$$\simeq \sum_{j=0}^{n} j \times \frac{18}{j^4} \times nj^2 = O(n\sum\frac{1}{j}) = O(n\log n)$$

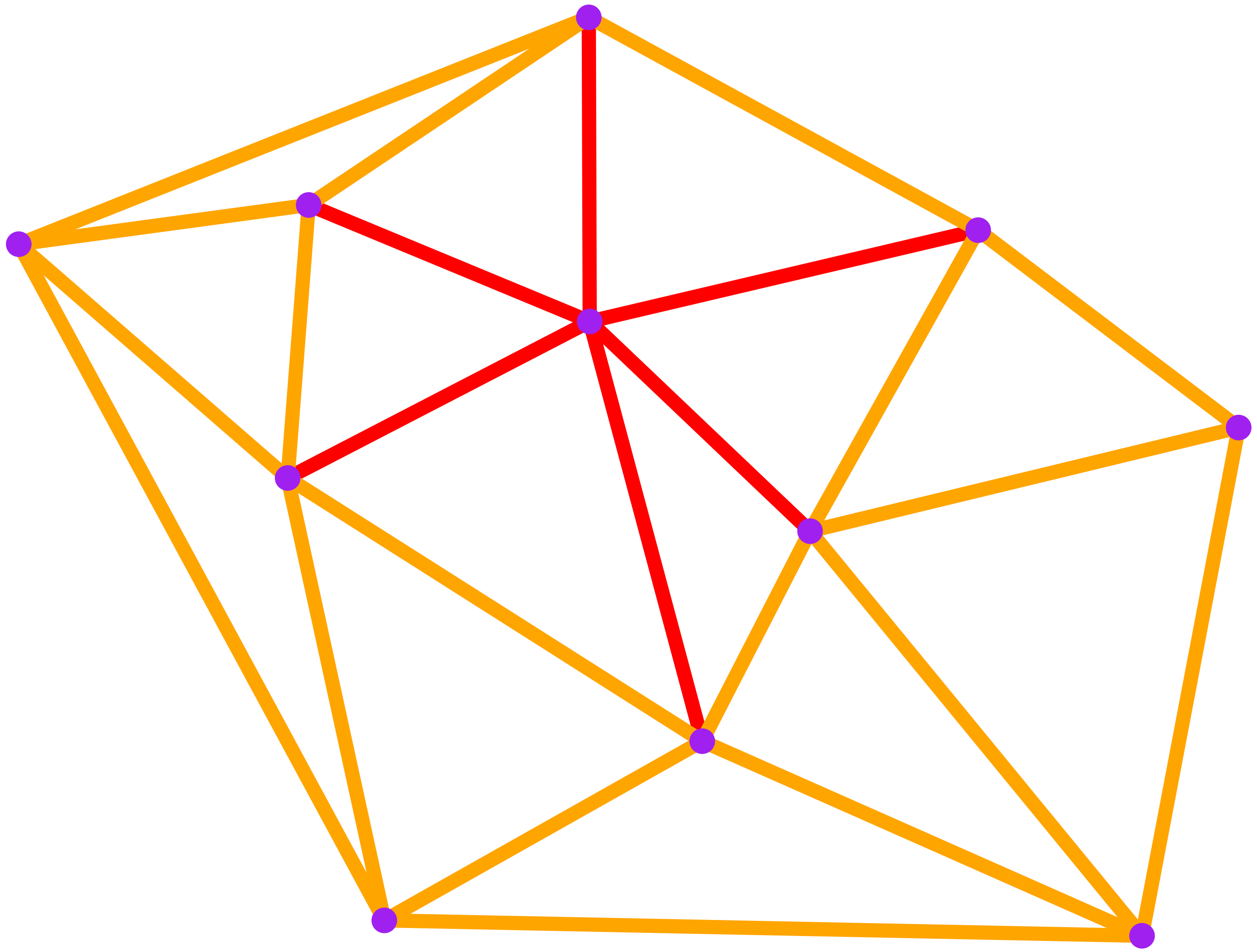
History graph



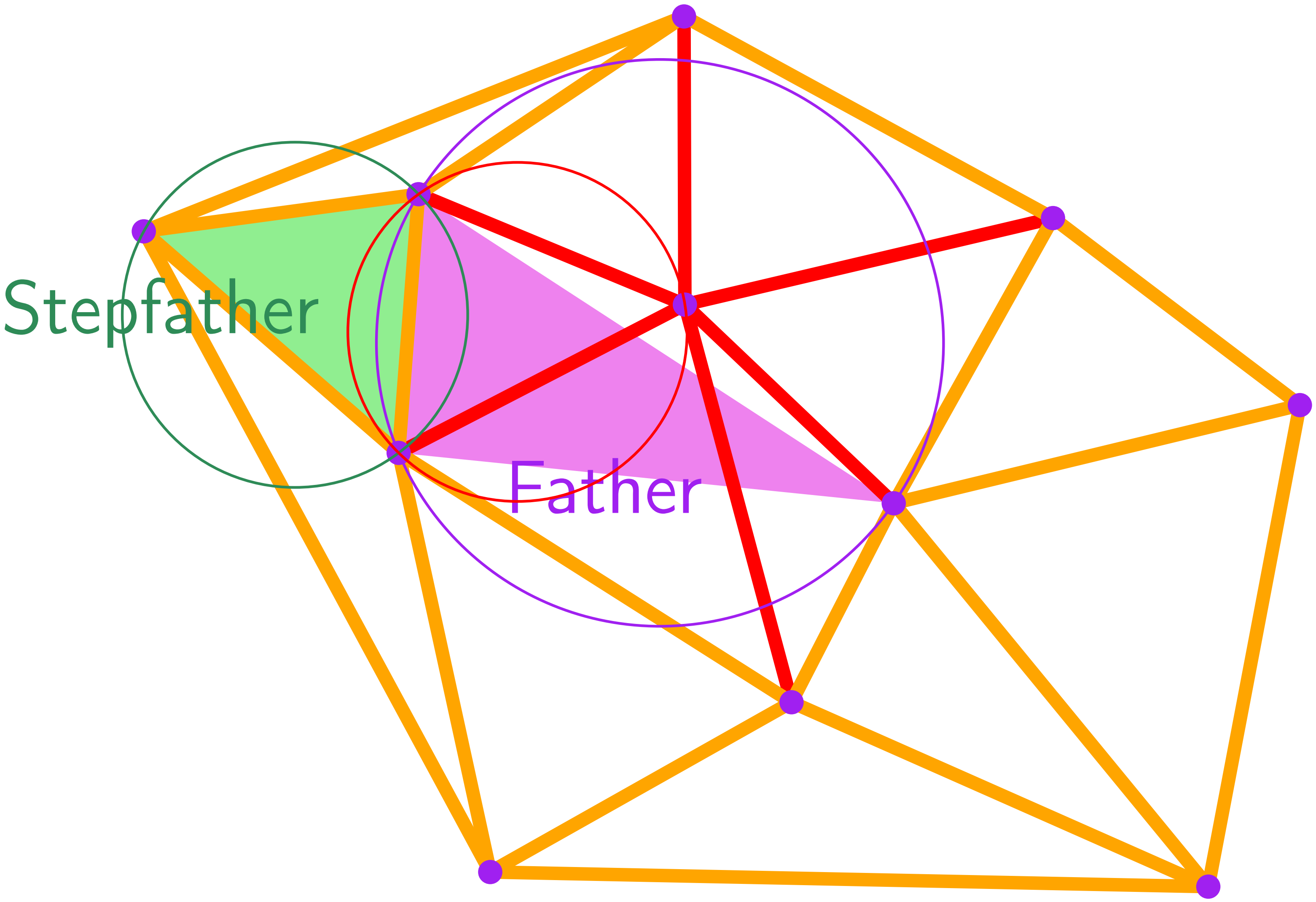
History graph



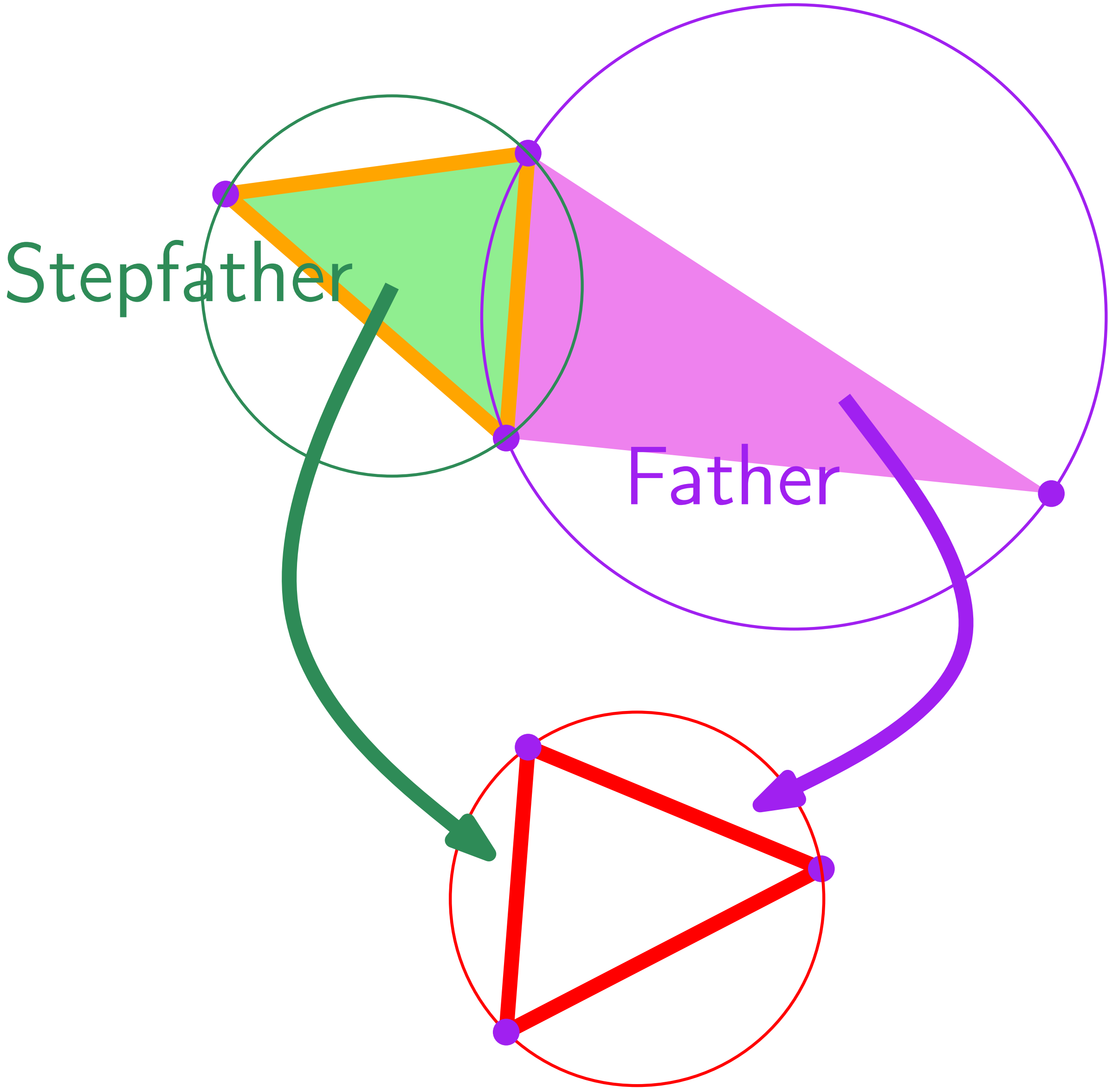
History graph

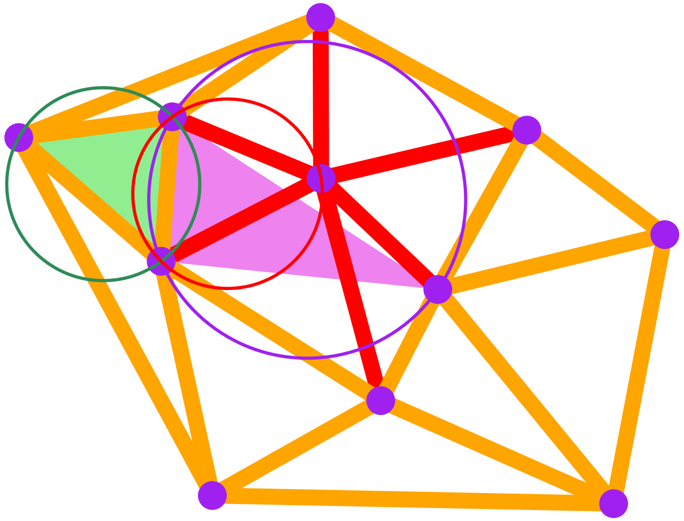


History graph

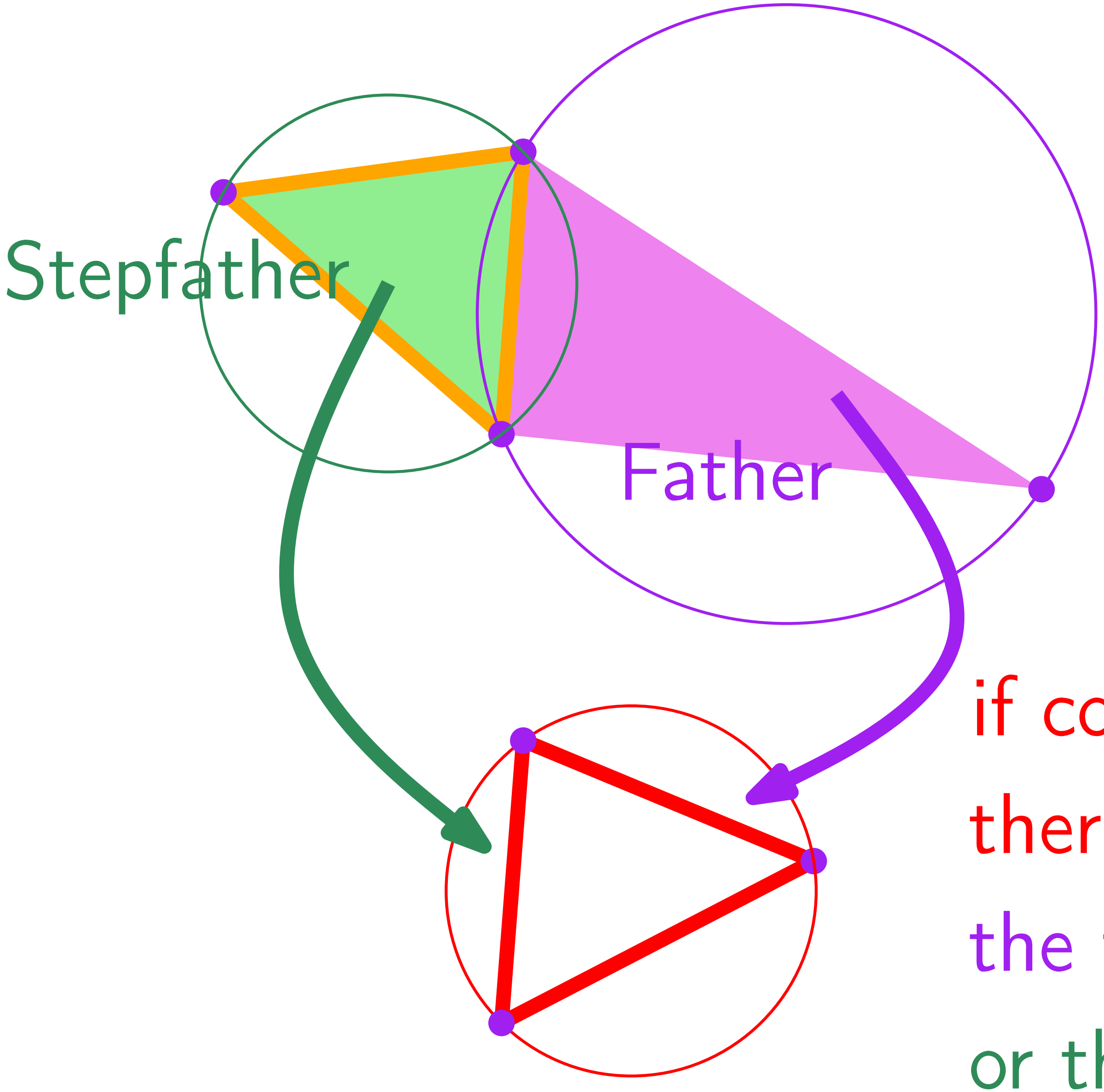


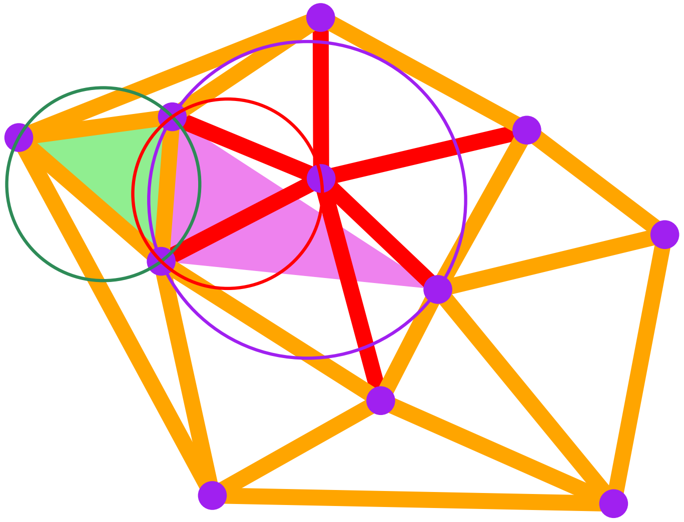
History graph (Delaunay tree)





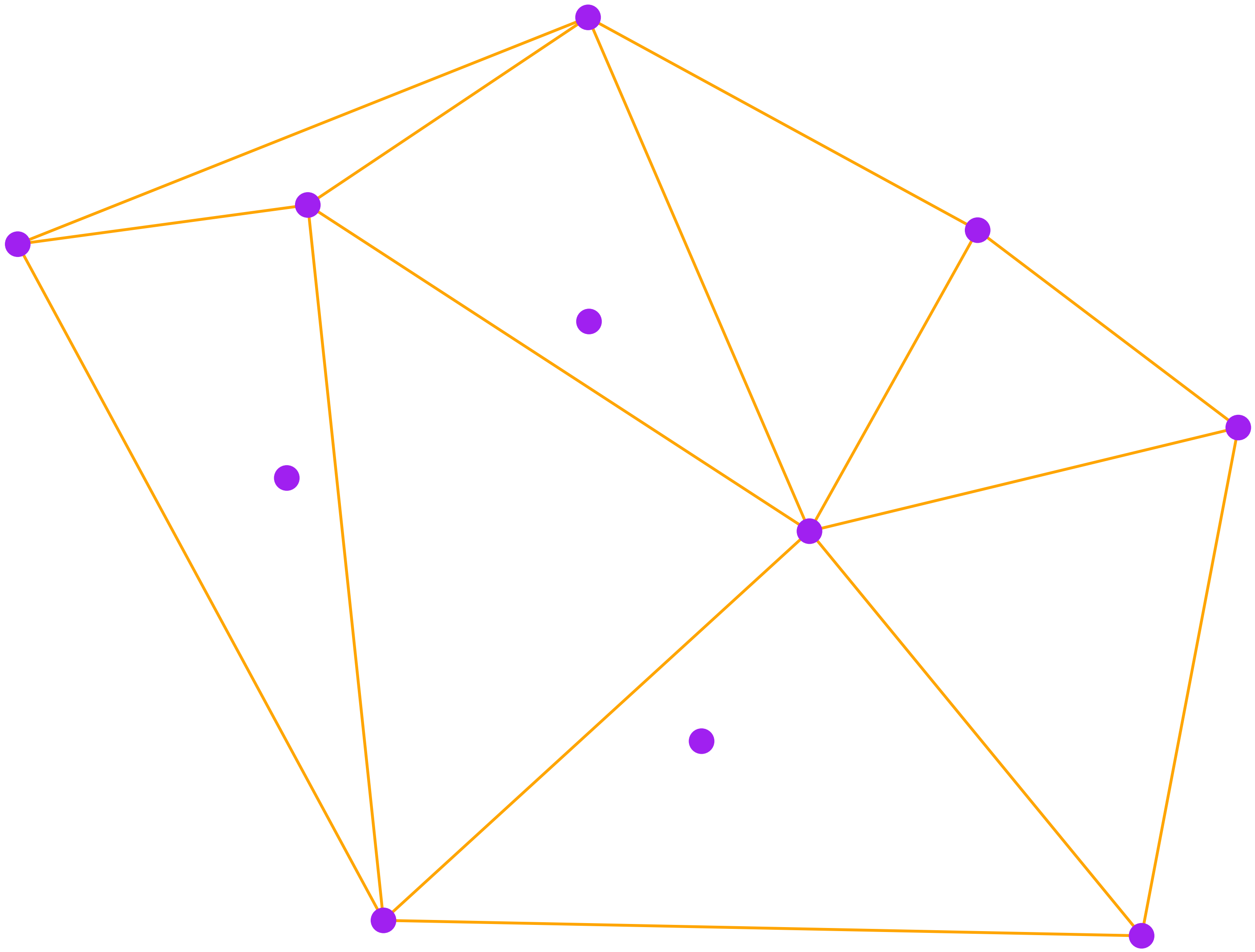
History graph (Delaunay tree)



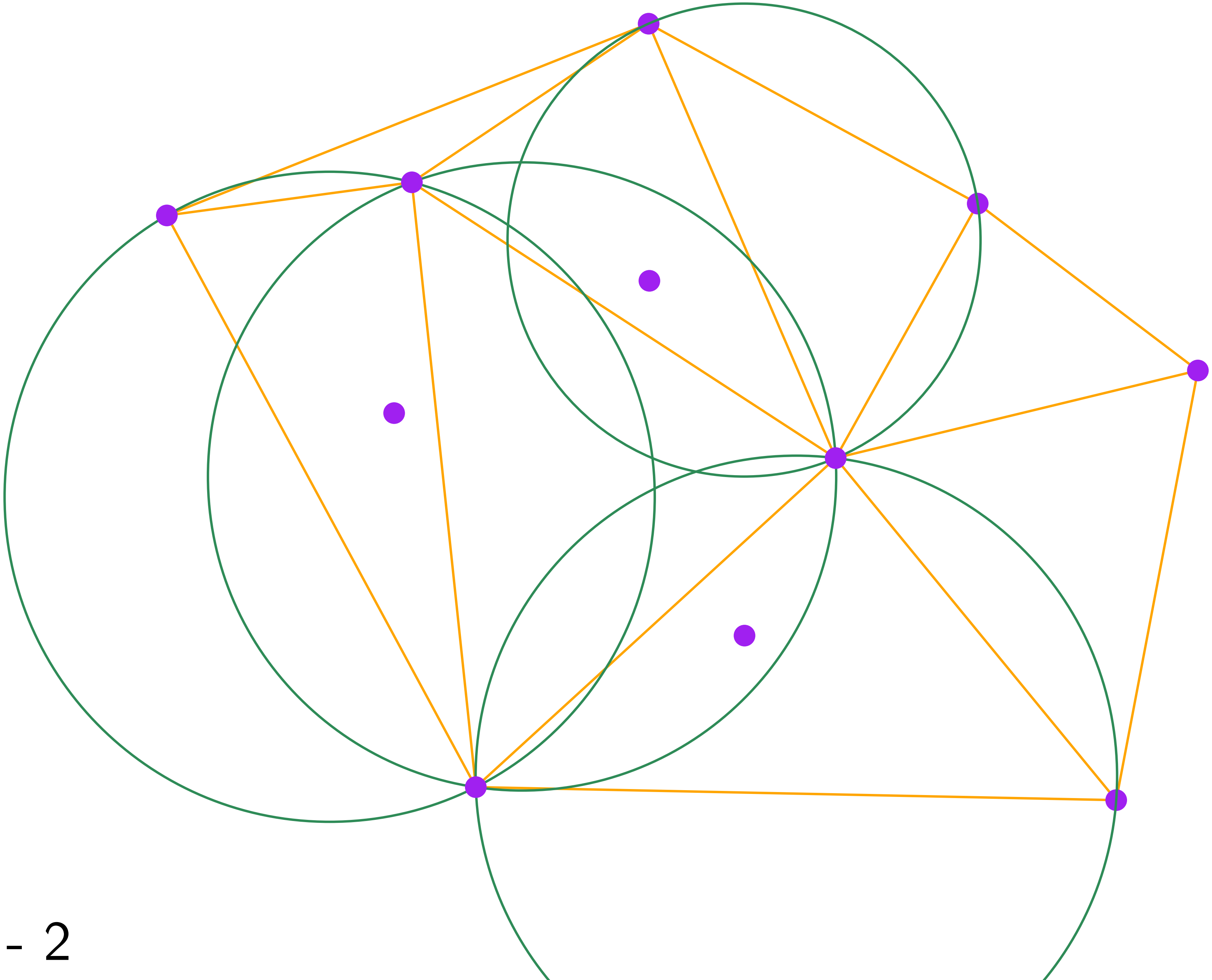


if conflict
there was a conflict with
the father
or the stepfather
or both

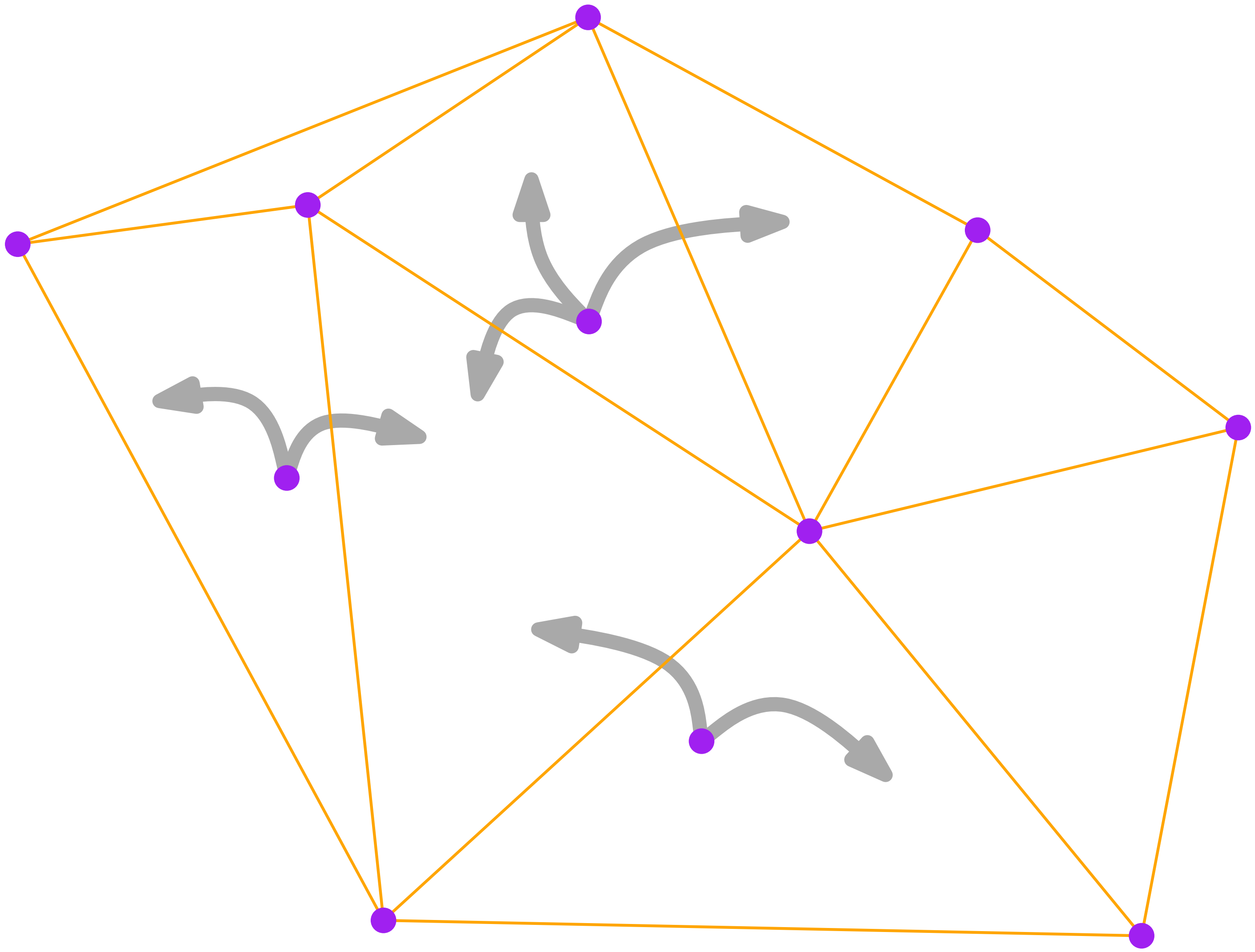
Conflict graph



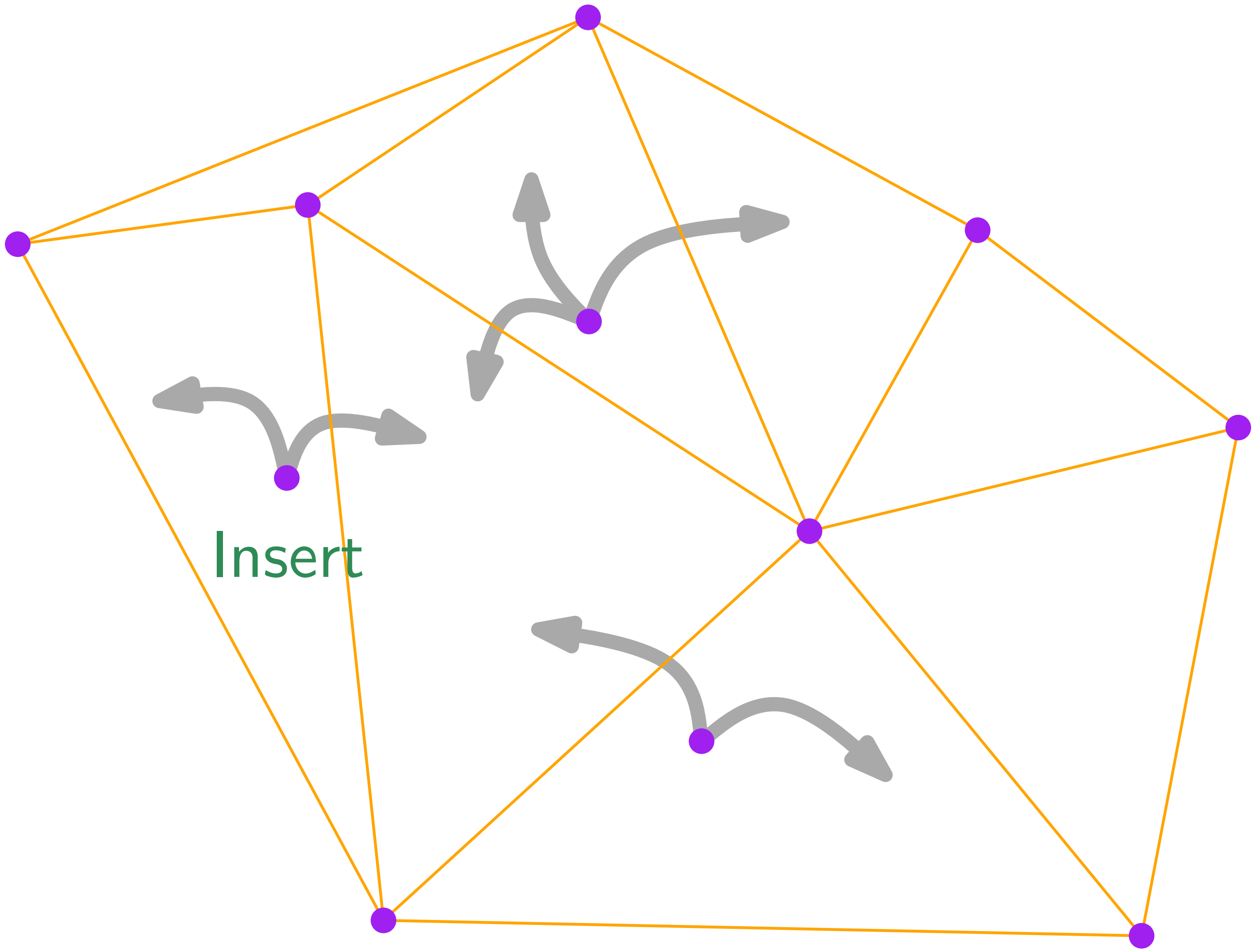
Conflict graph



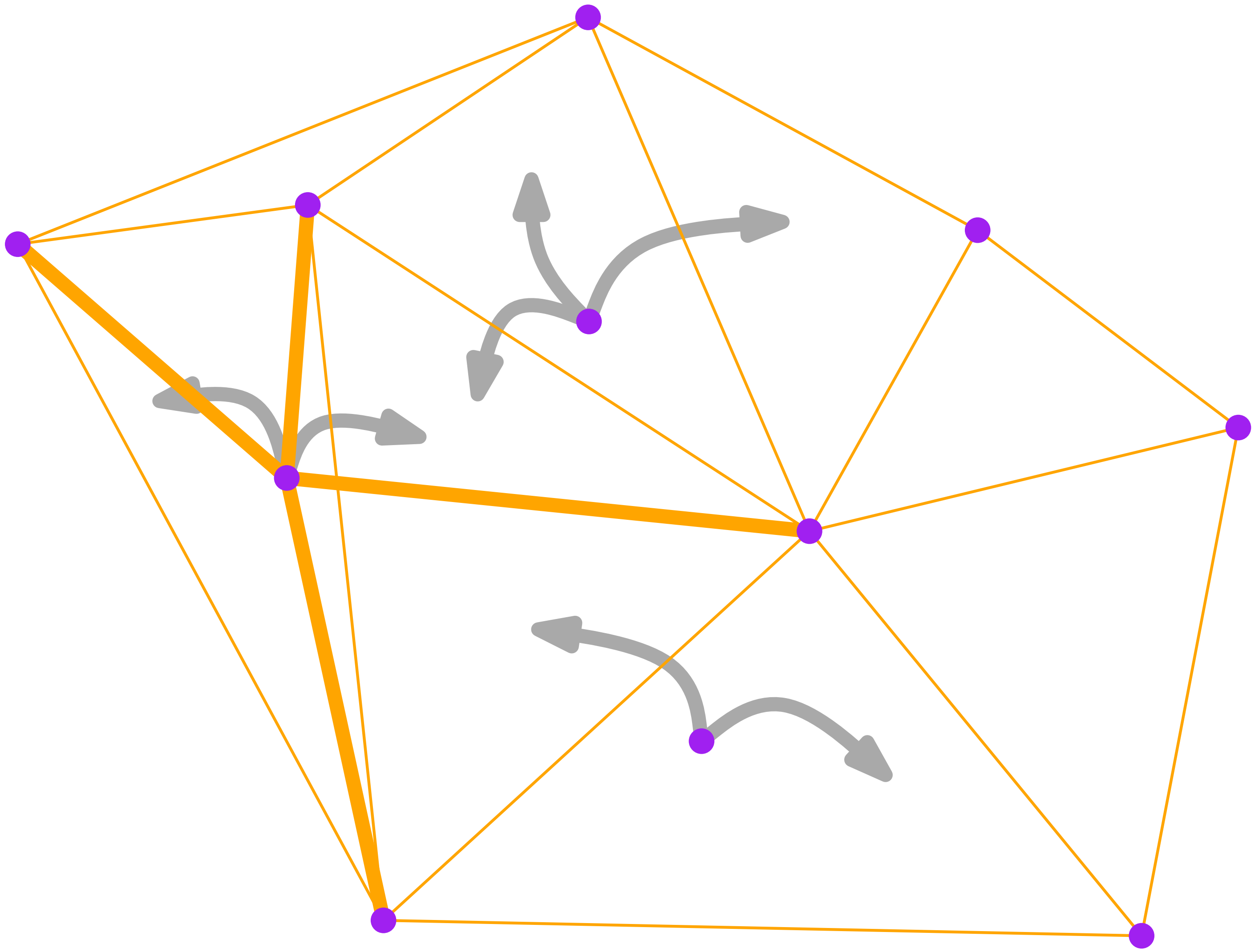
Conflict graph



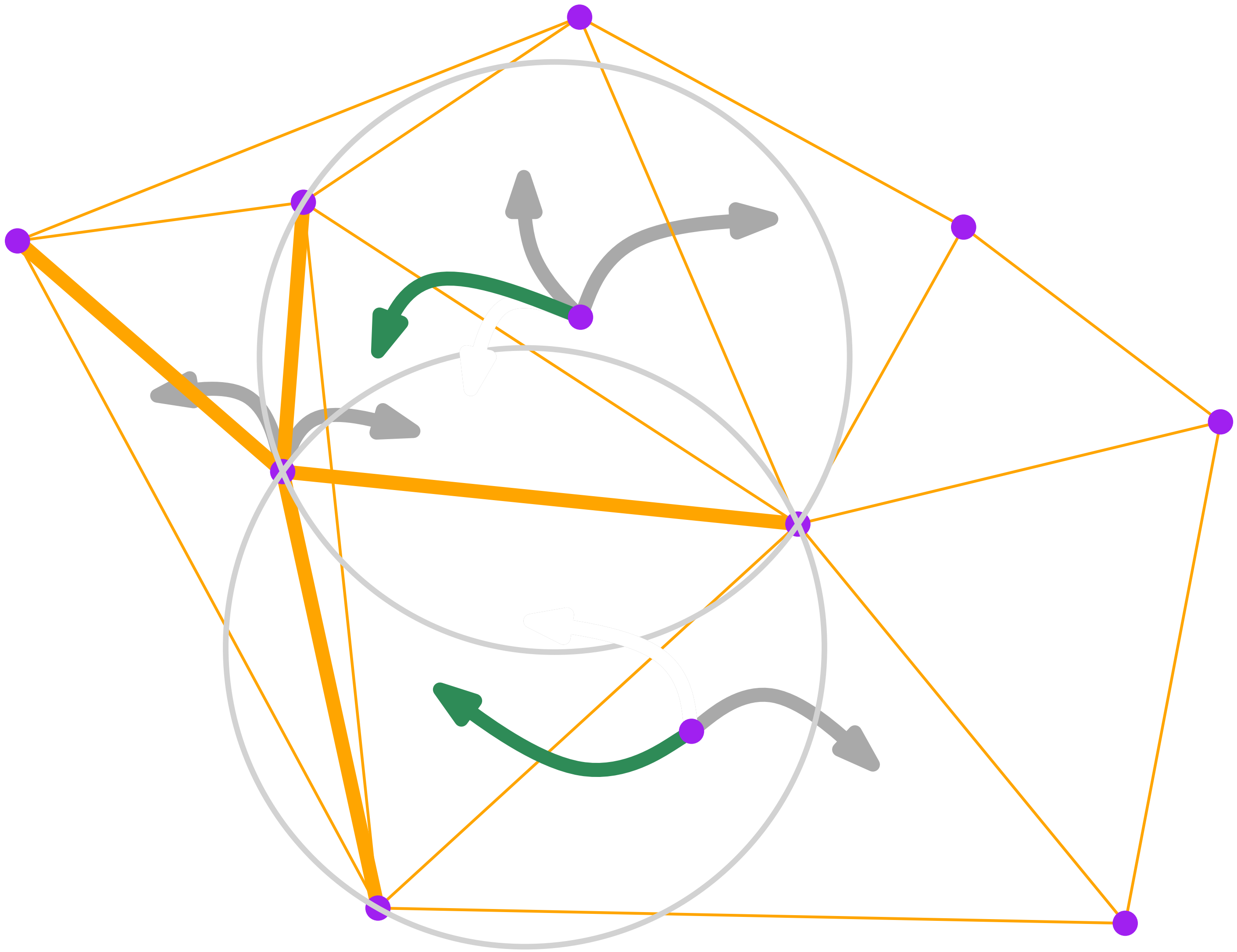
Conflict graph



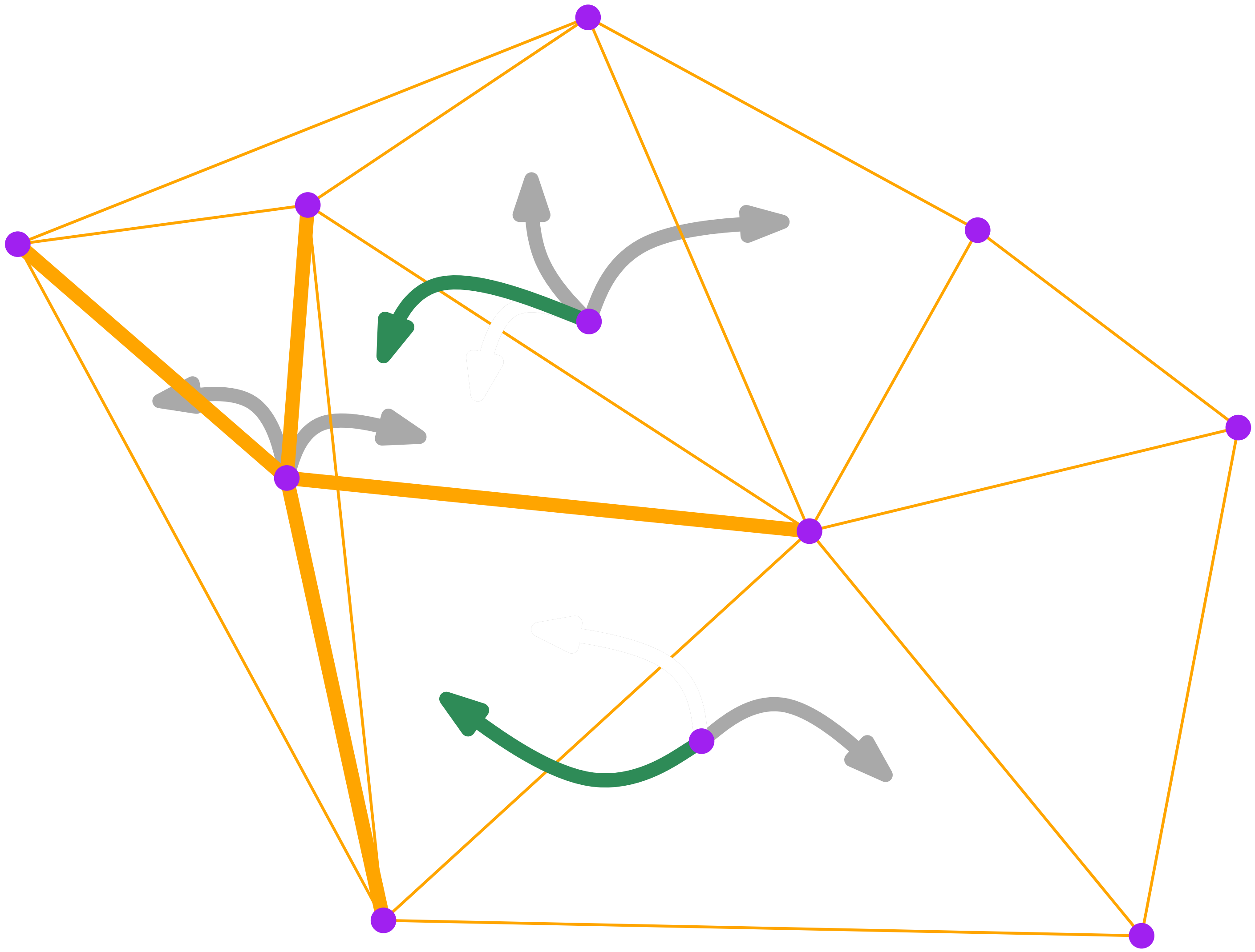
Conflict graph



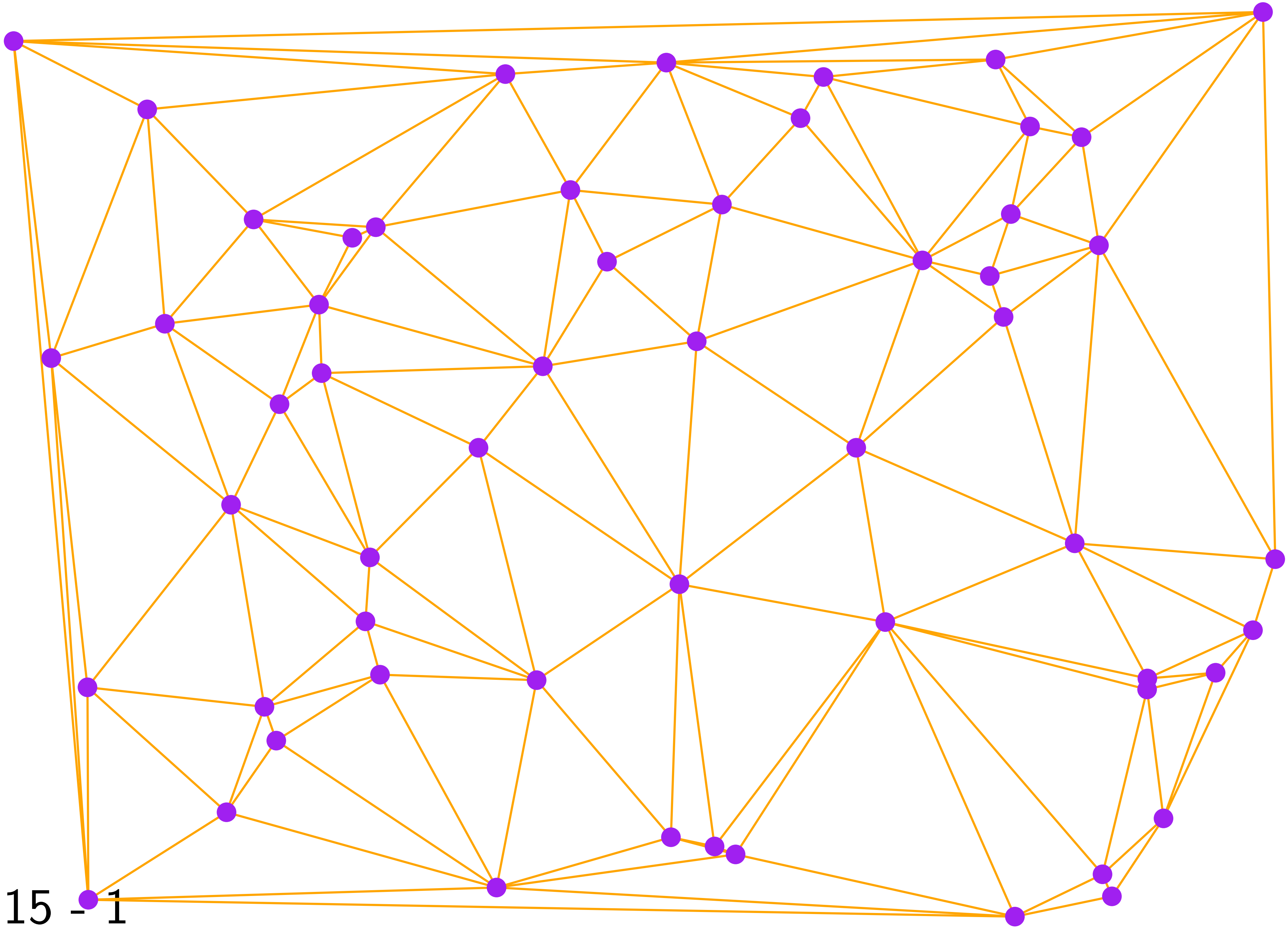
Conflict graph



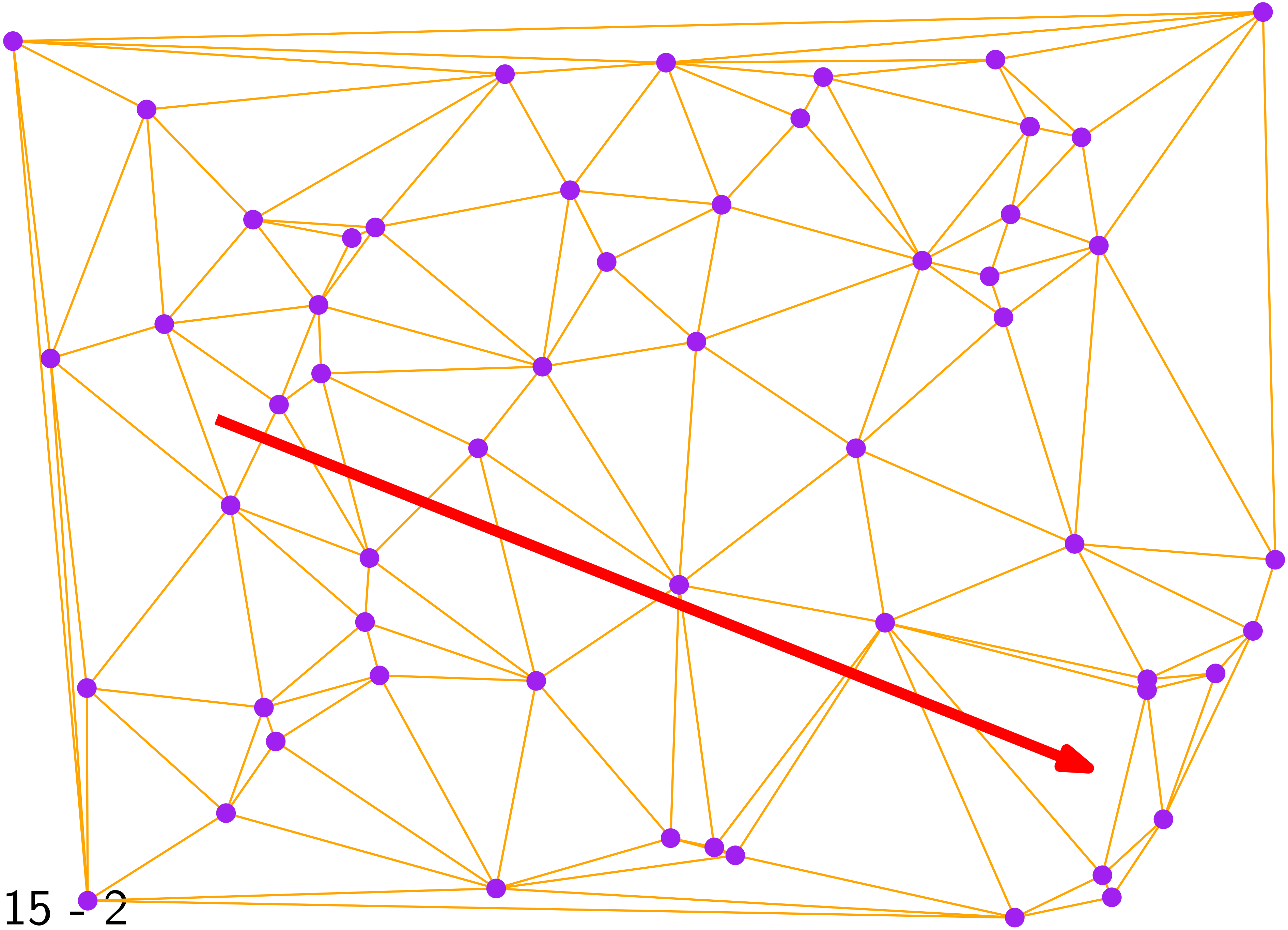
Conflict graph



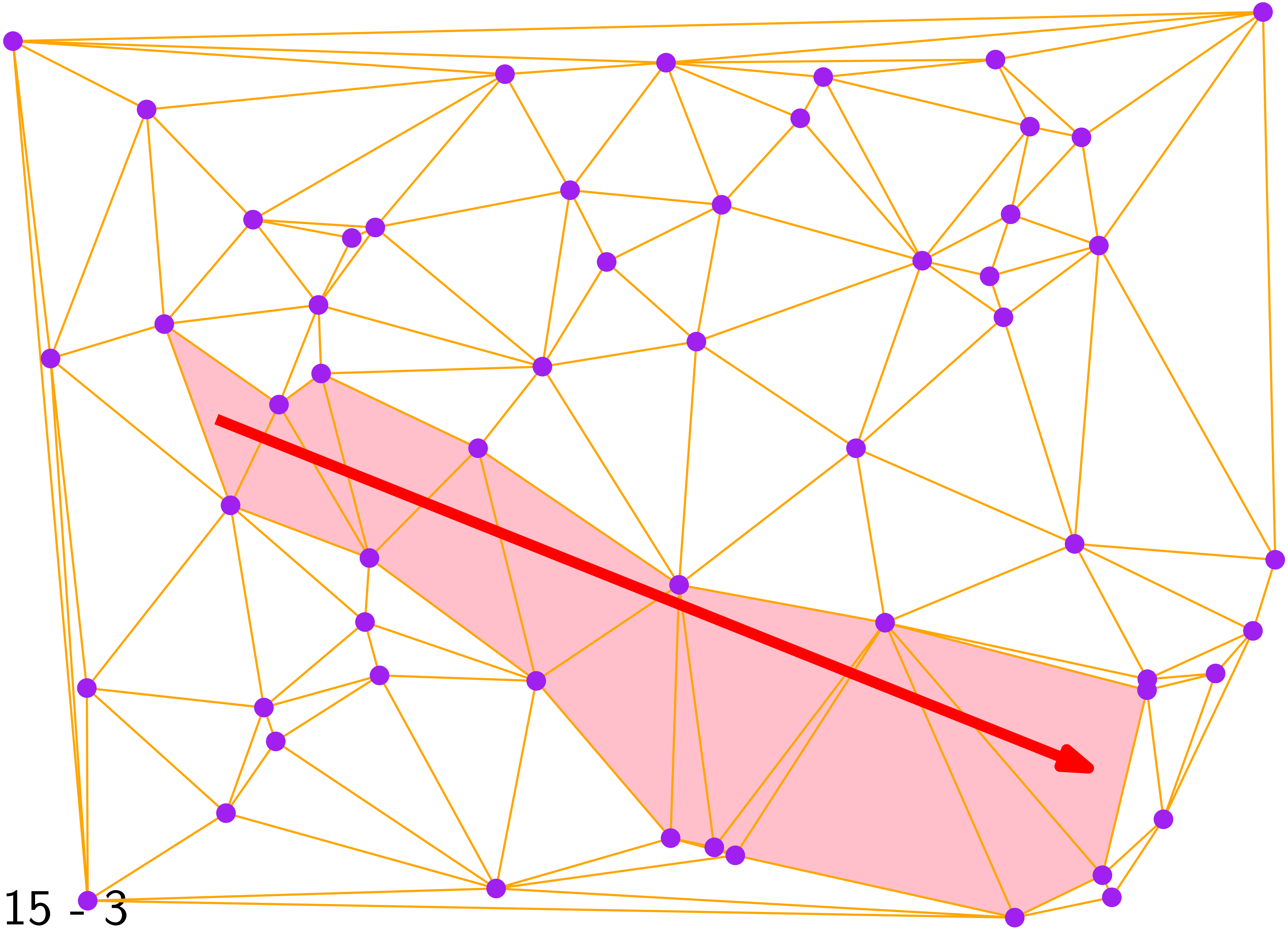
Walk



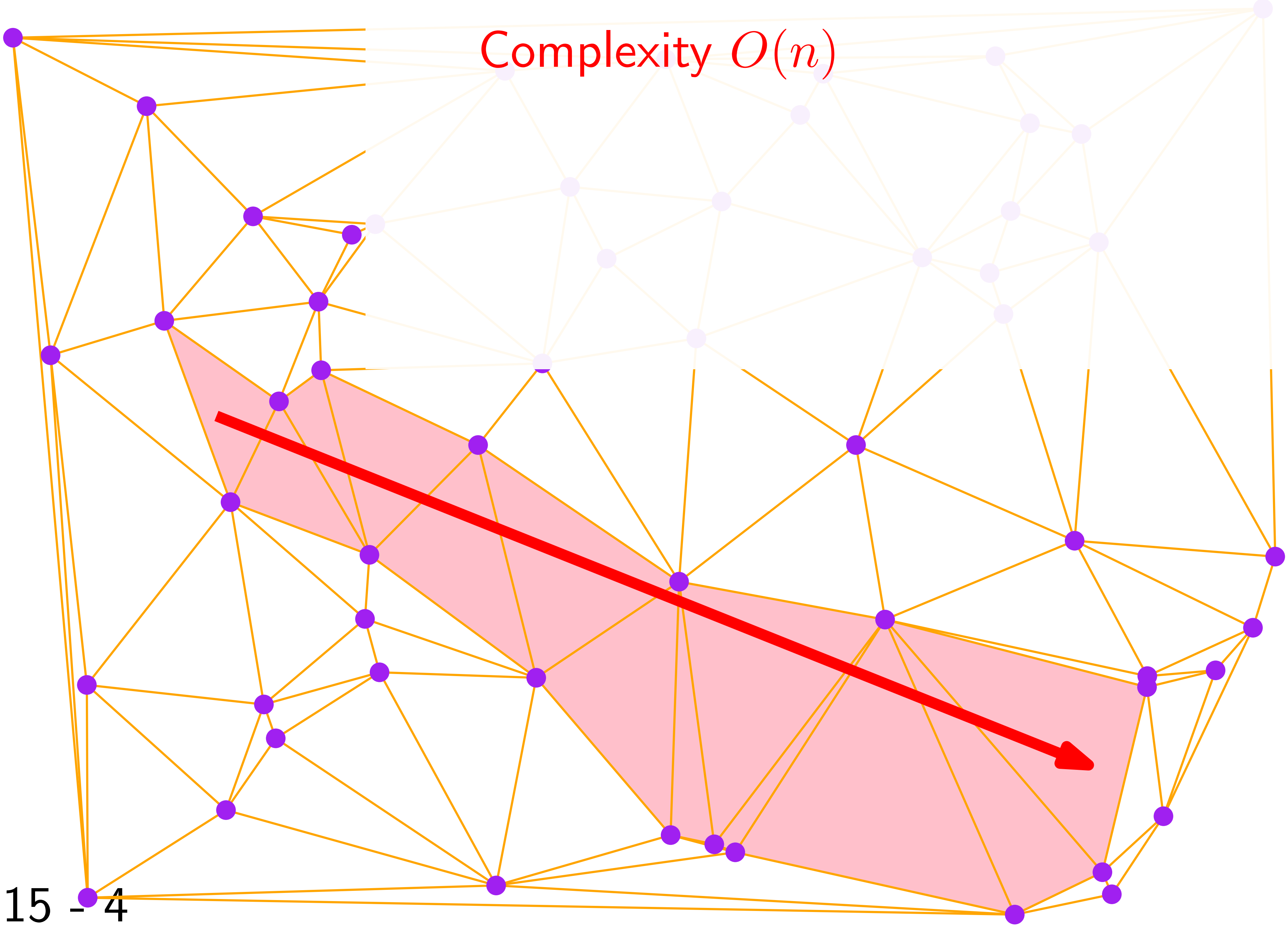
Walk



Walk

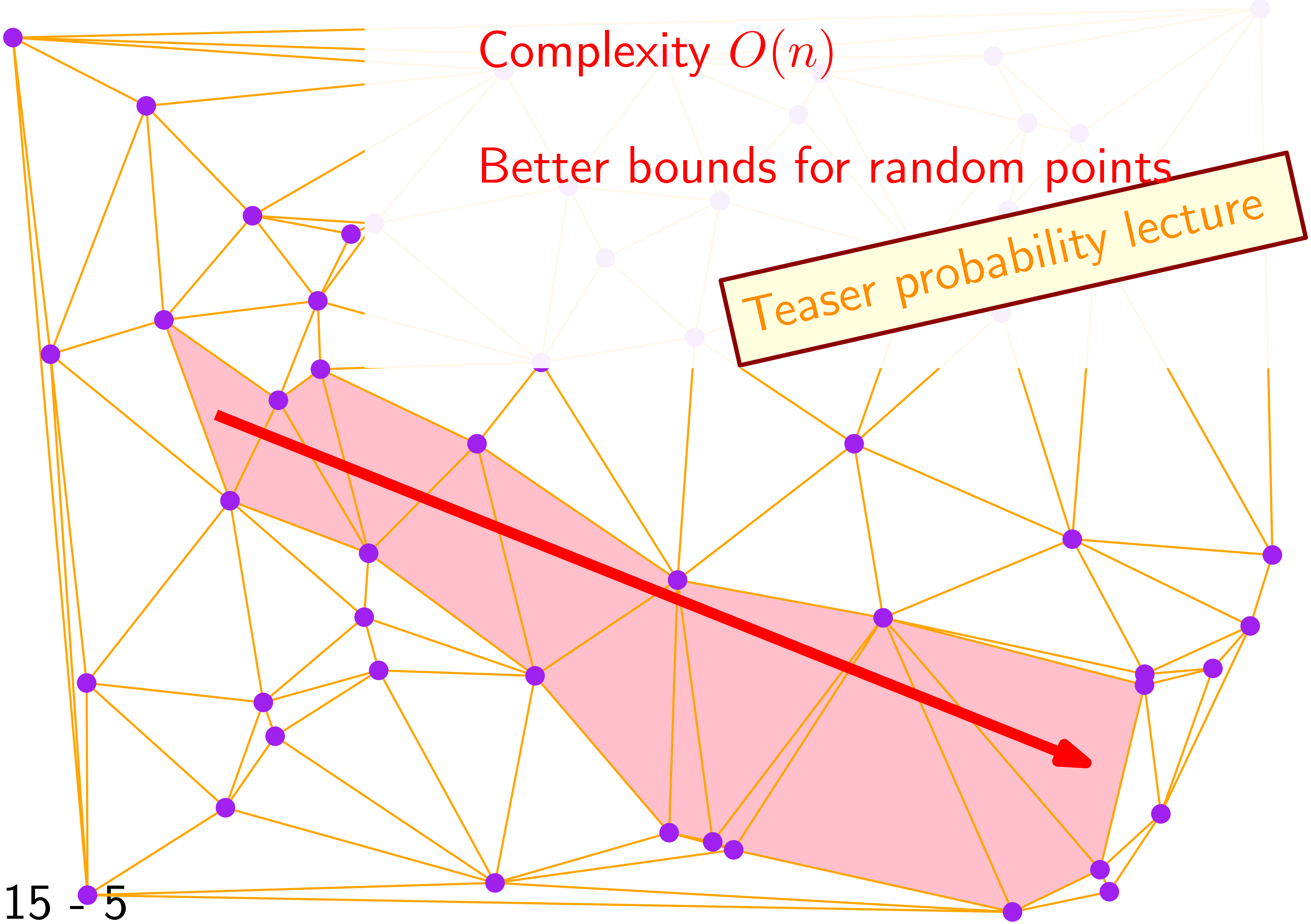


Walk



Complexity $O(n)$

Walk

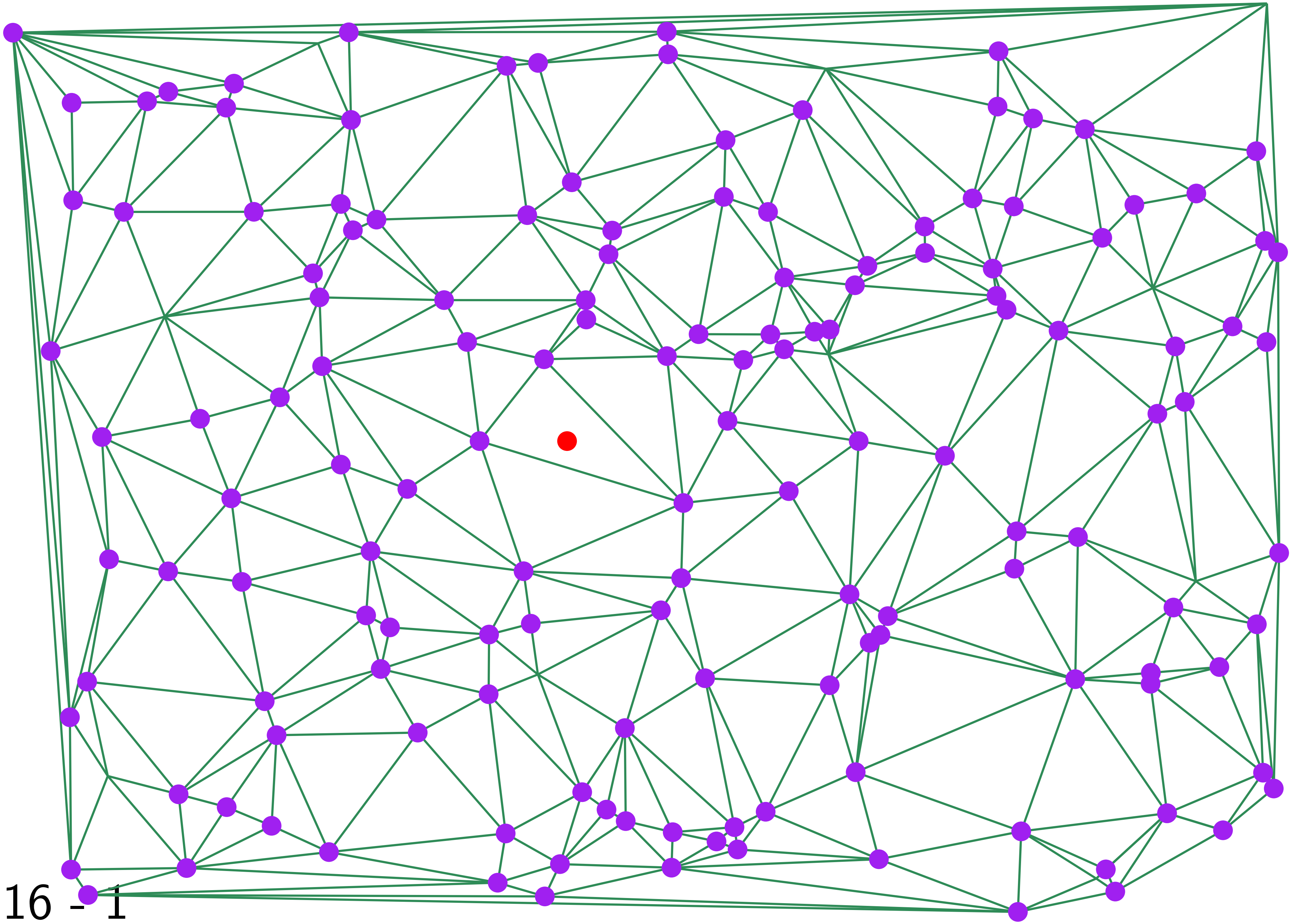


Complexity $O(n)$

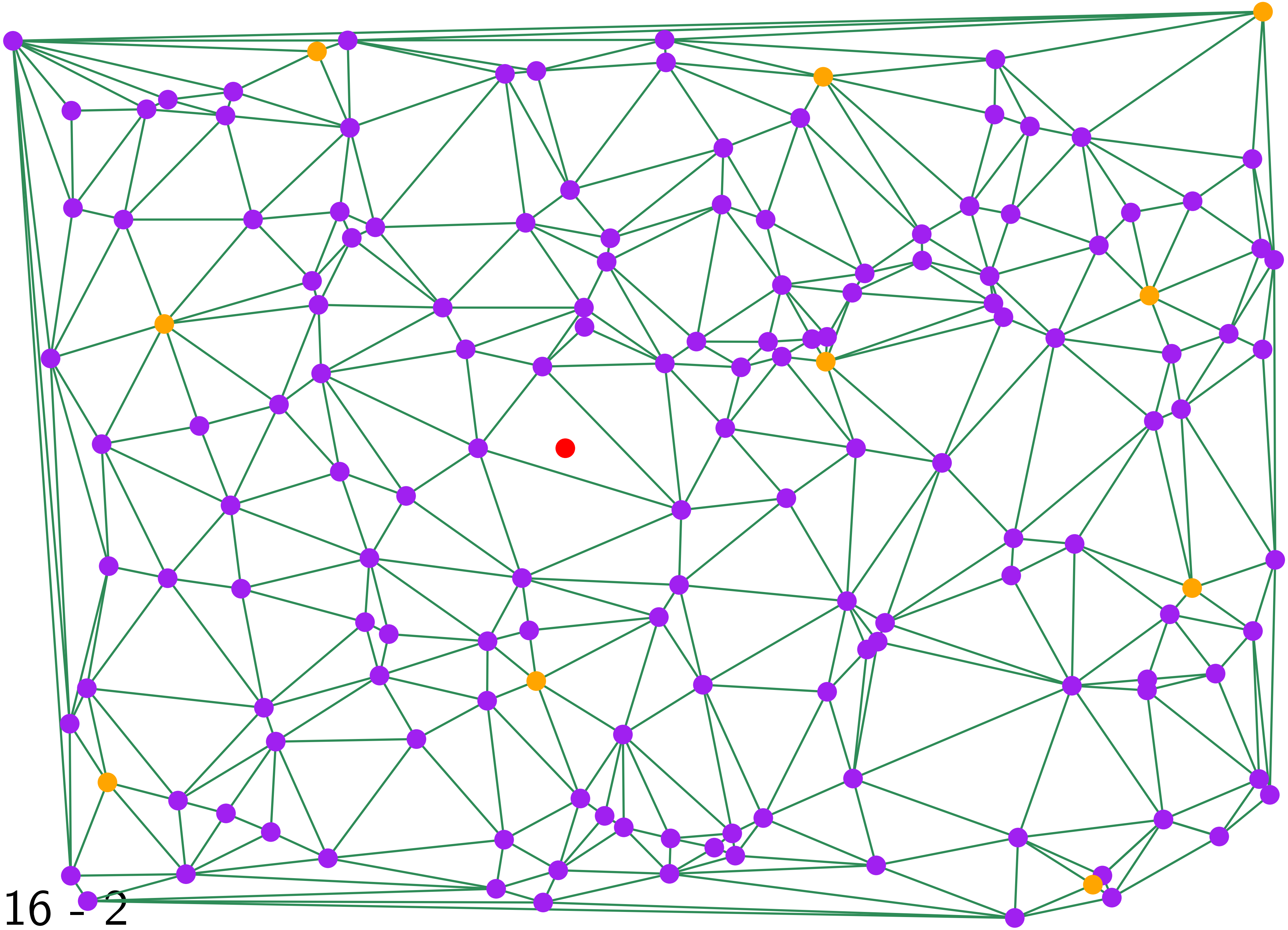
Better bounds for random points

Teaser probability lecture

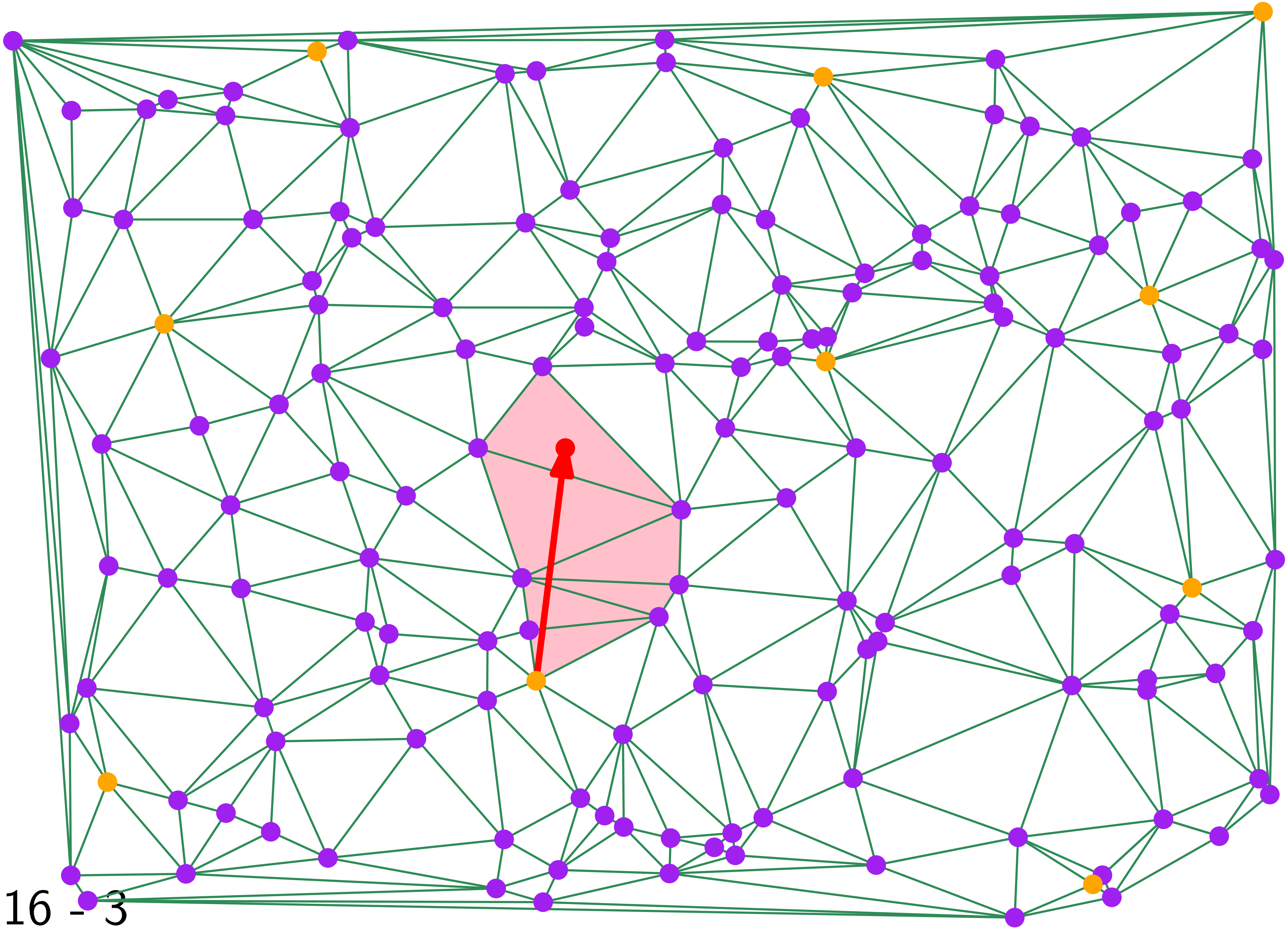
Jump and walk



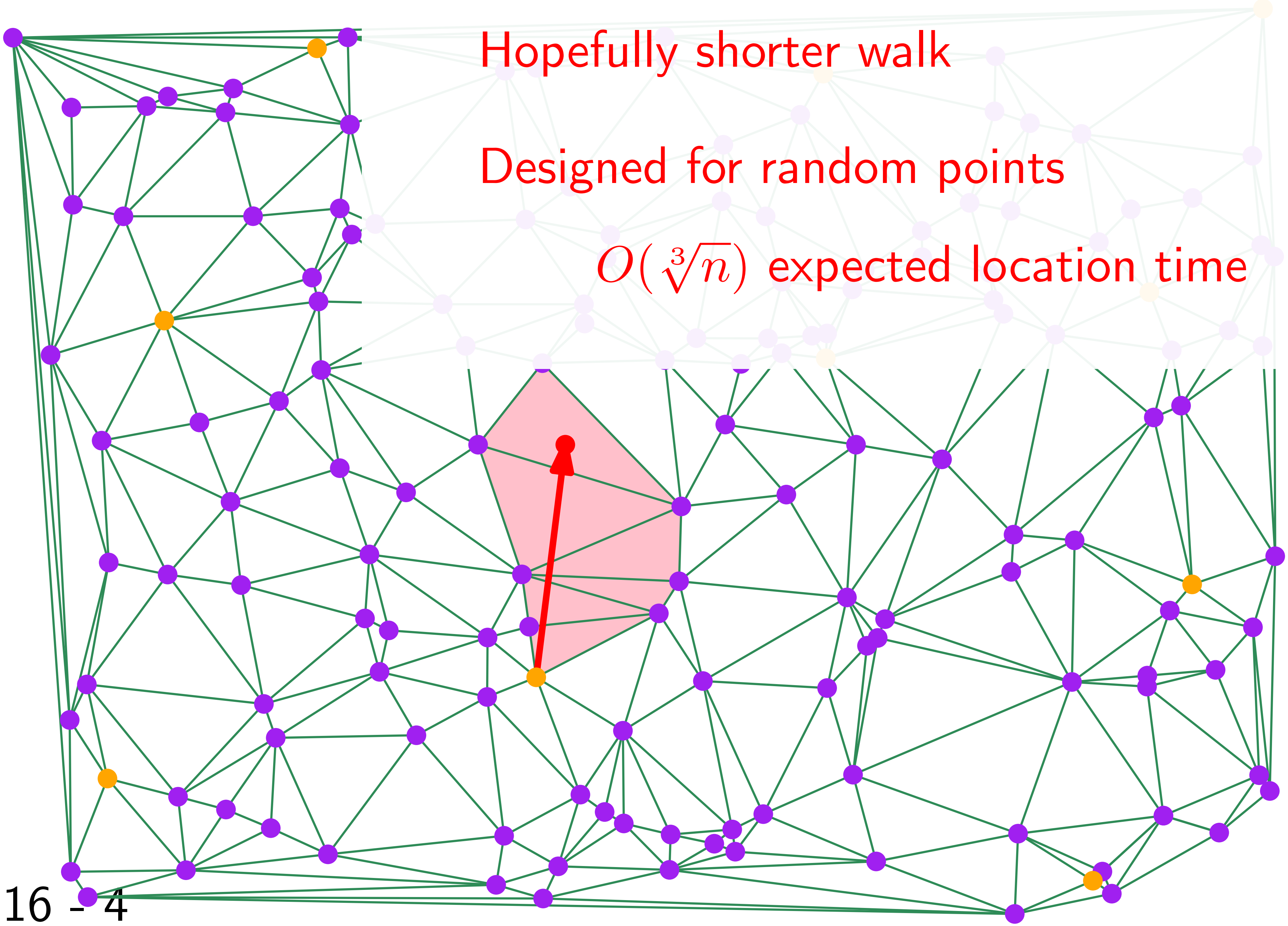
Jump and walk



Jump and walk



Jump and walk

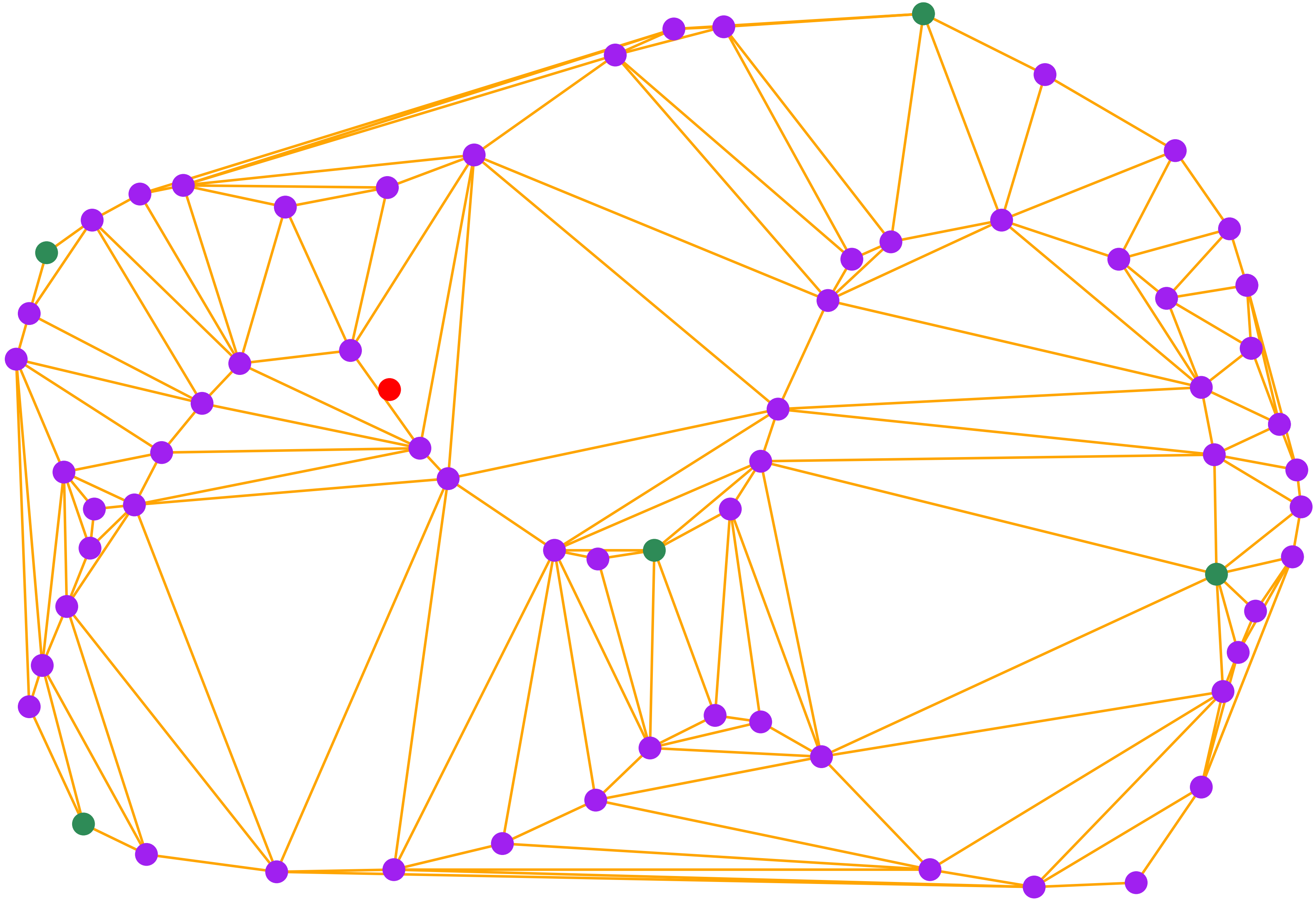


Hopefully shorter walk

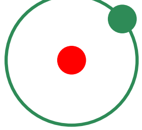
Designed for random points

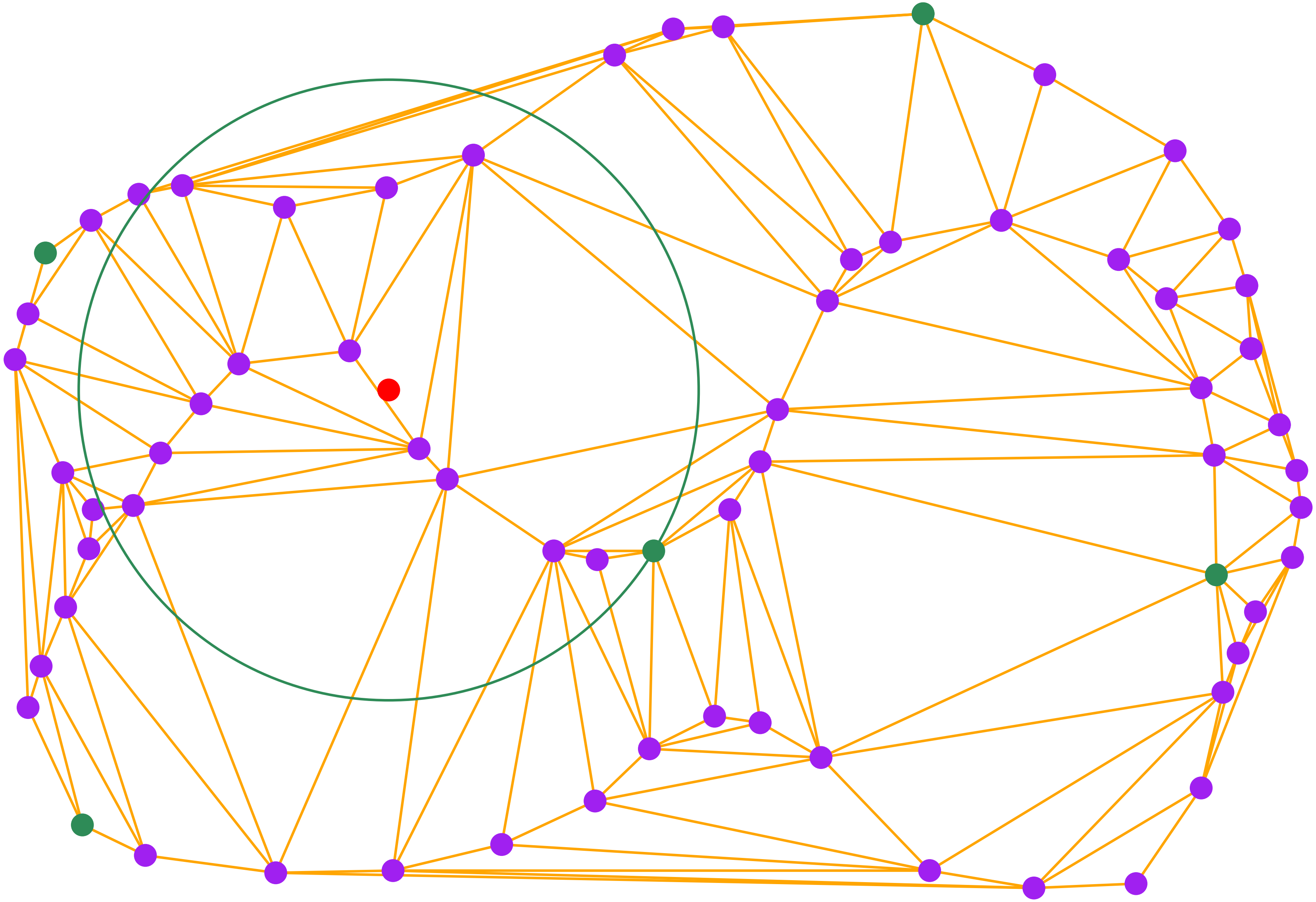
$O(\sqrt[3]{n})$ expected location time

Jump and walk (no distribution hypothesis)



Jump and walk (no distribution hypothesis)

$\mathbb{E}\,[\sharp$ of ● in  $] = \frac{n}{k}$

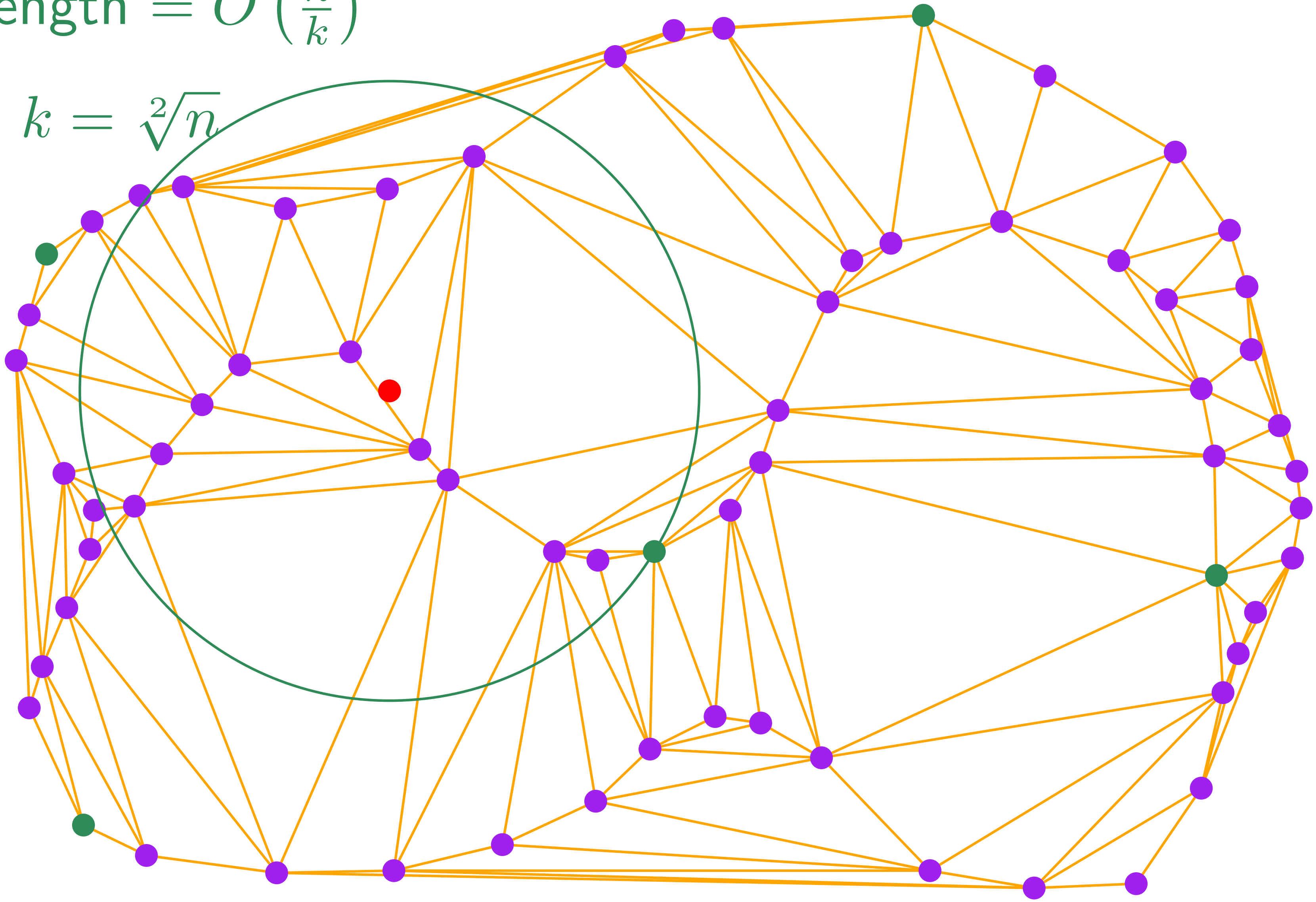


Jump and walk (no distribution hypothesis)

$\mathbb{E}\left[\sharp \text{ of }\right.$ ● in ⊙ $\left.\right] = \frac{n}{k}$

Walk length $= O\left(\frac{n}{k}\right)$

choose $k = \sqrt[2]{n}$

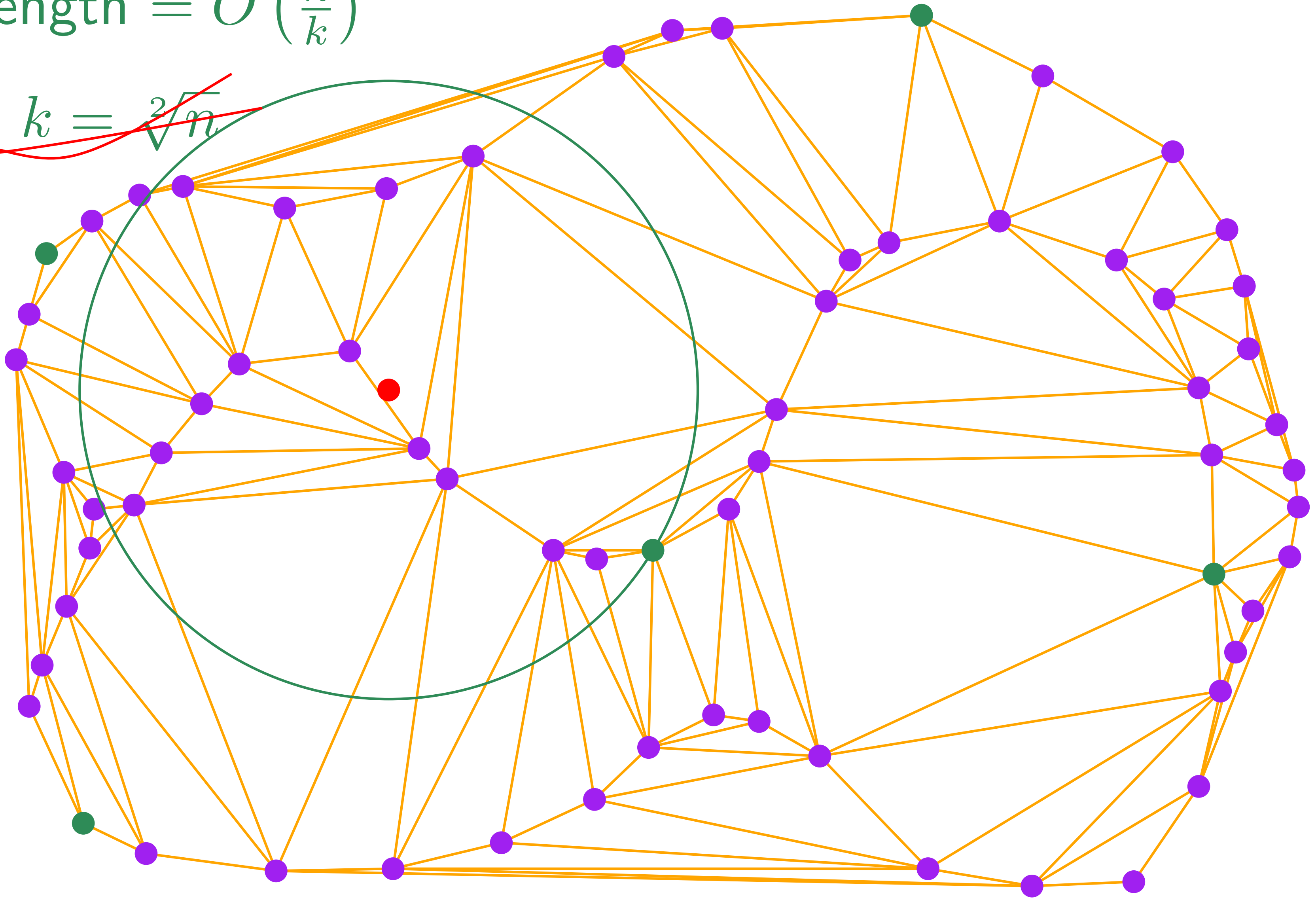


Jump and walk (no distribution hypothesis)Delaunay hierarchy

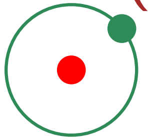
$\mathbb{E}\left[\sharp\right.$ of ● in ◯ $\left.\right] = \frac{n}{k}$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~



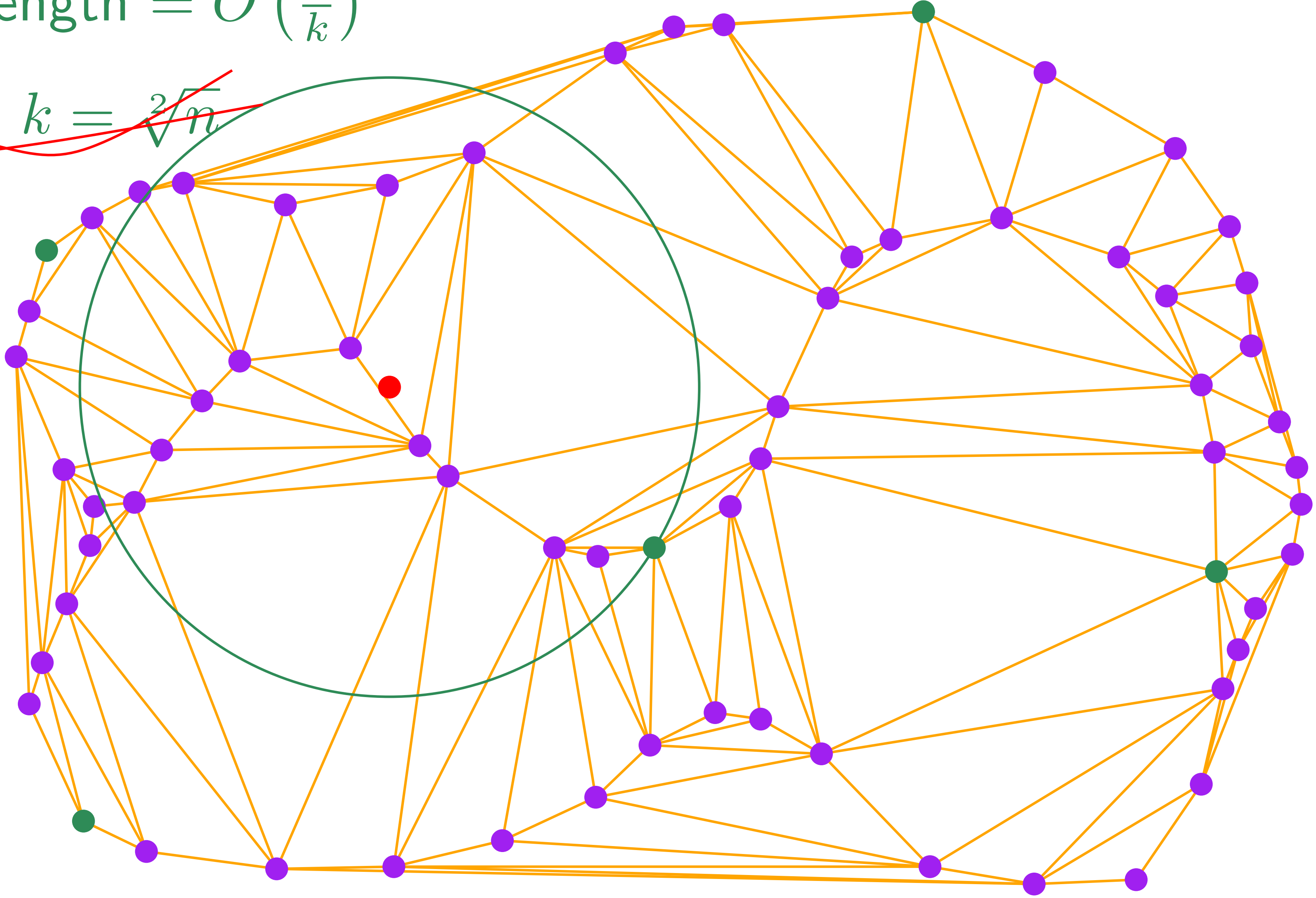
Jump and walk (no distribution hypothesis)Delaunay hierarchy

$\mathbb{E}\left[\sharp\right.$ of ● in  $\left.\right] = \frac{n}{k}$

$\frac{n}{k_1}$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~



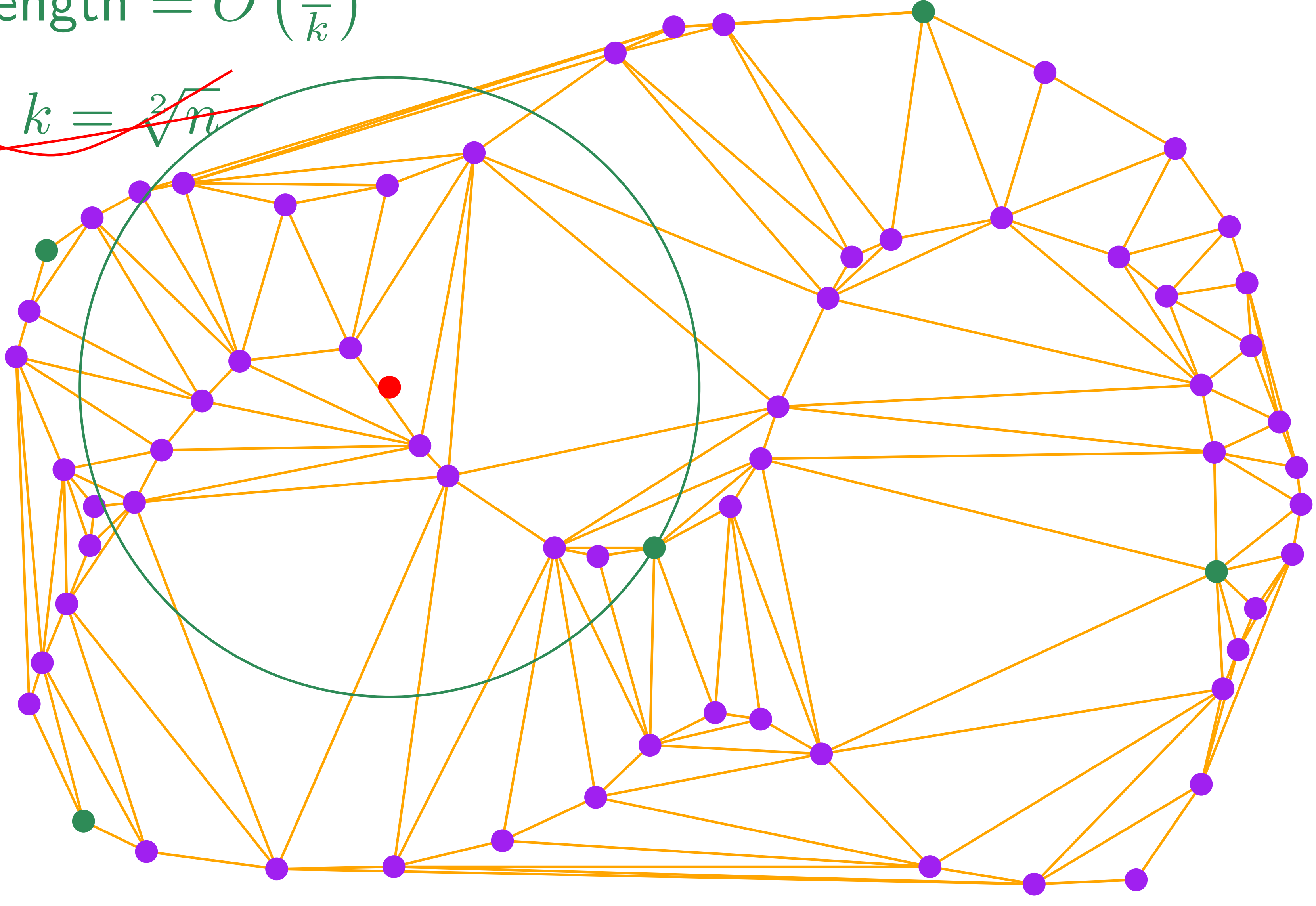
Jump and walk (no distribution hypothesis)Delaunay hierarchy

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \bigcirc\right] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2}$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~



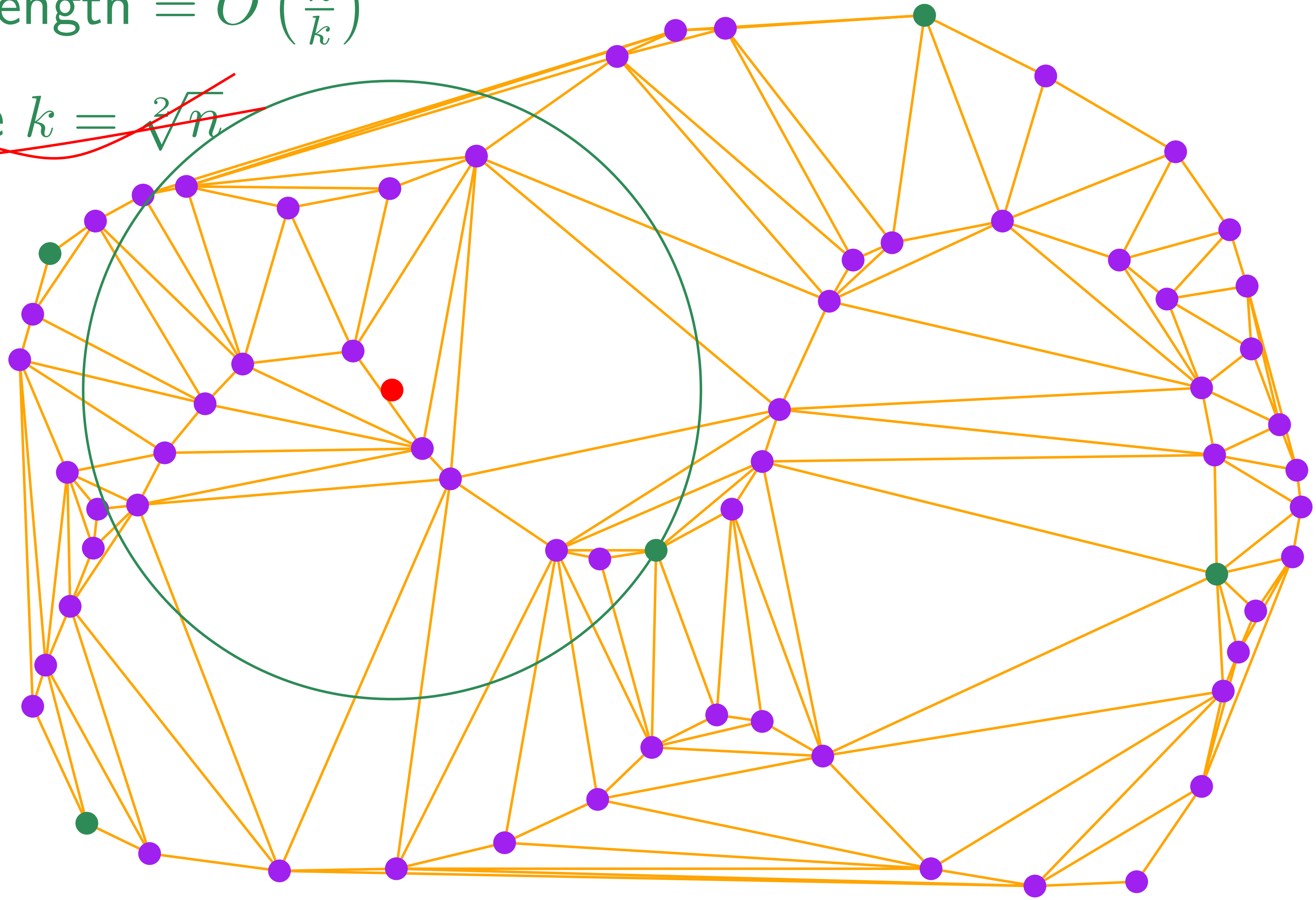
Jump and walk (no distribution hypothesis)Delaunay hierarchy

$\mathbb{E}\left[\sharp\right.$ of ● in ◯$\left.\right] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \dots$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~



Jump and walk (no distribution hypothesis)Delaunay hierarchy

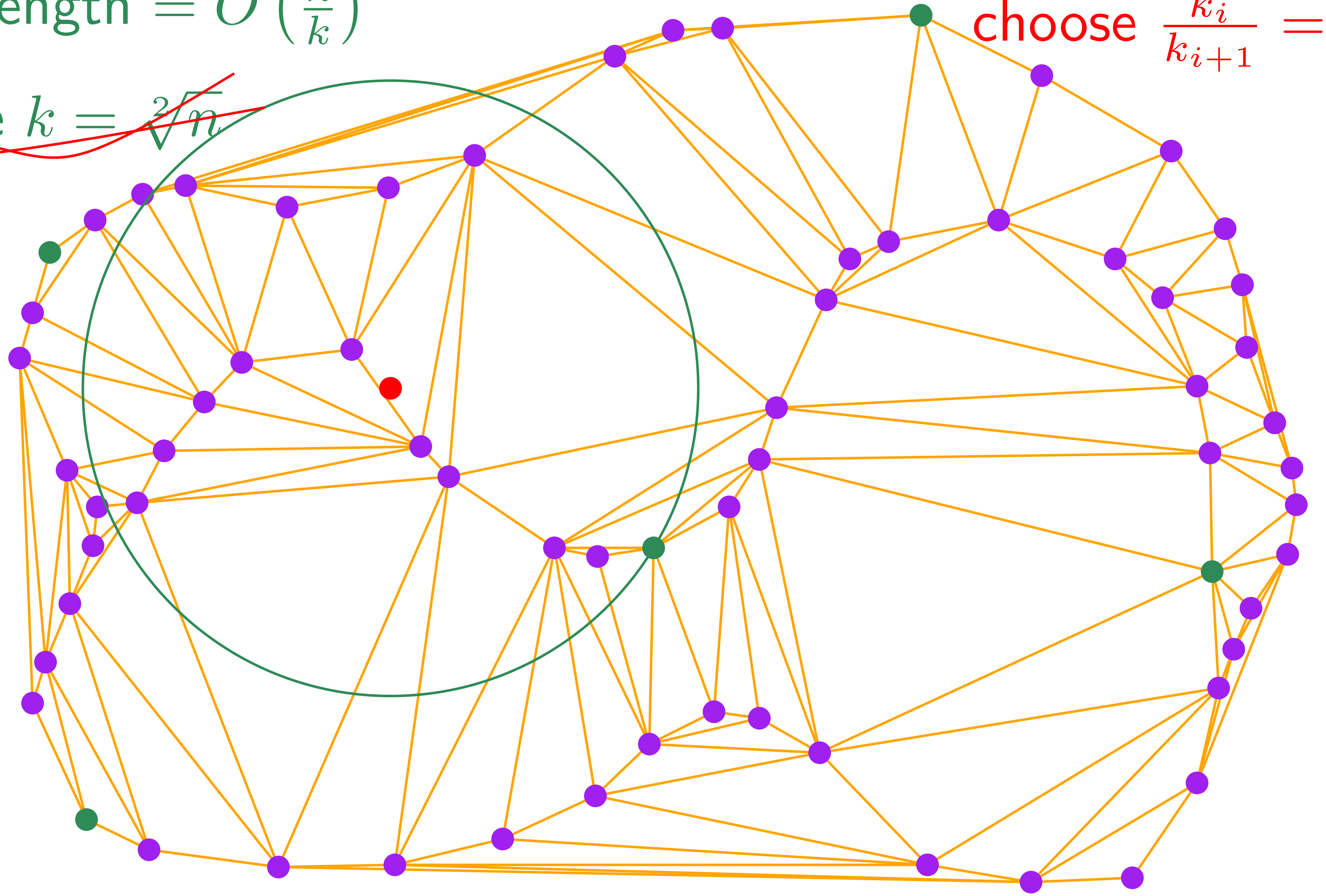
$\mathbb{E}[\sharp$ of ● in ⊙ $] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots$

Walk length $= O\left(\frac{n}{k}\right)$

choose $\frac{k_i}{k_{i+1}} = \alpha$

~~choose $k = \sqrt[2]{n}$~~



Jump and walk (no distribution hypothesis)Delaunay hierarchy

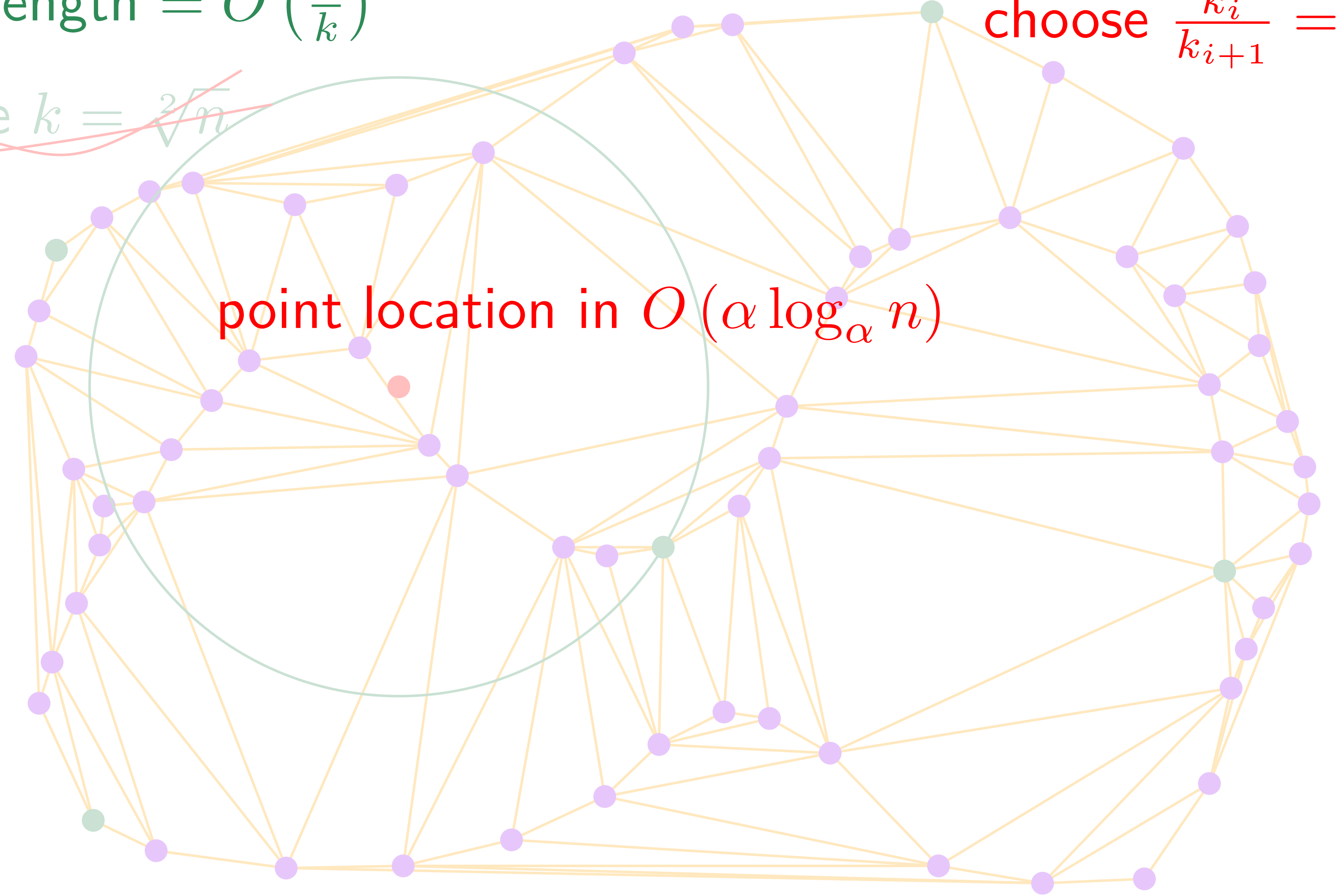
$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \odot \right] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \dots$

Walk length $= O\left(\frac{n}{k}\right)$

choose $\frac{k_i}{k_{i+1}} = \alpha$

~~choose $k = \sqrt[2]{n}$~~



point location in $O\left(\alpha \log_\alpha n\right)$

Jump and walk (no distribution hypothesis)Delaunay hierarchy

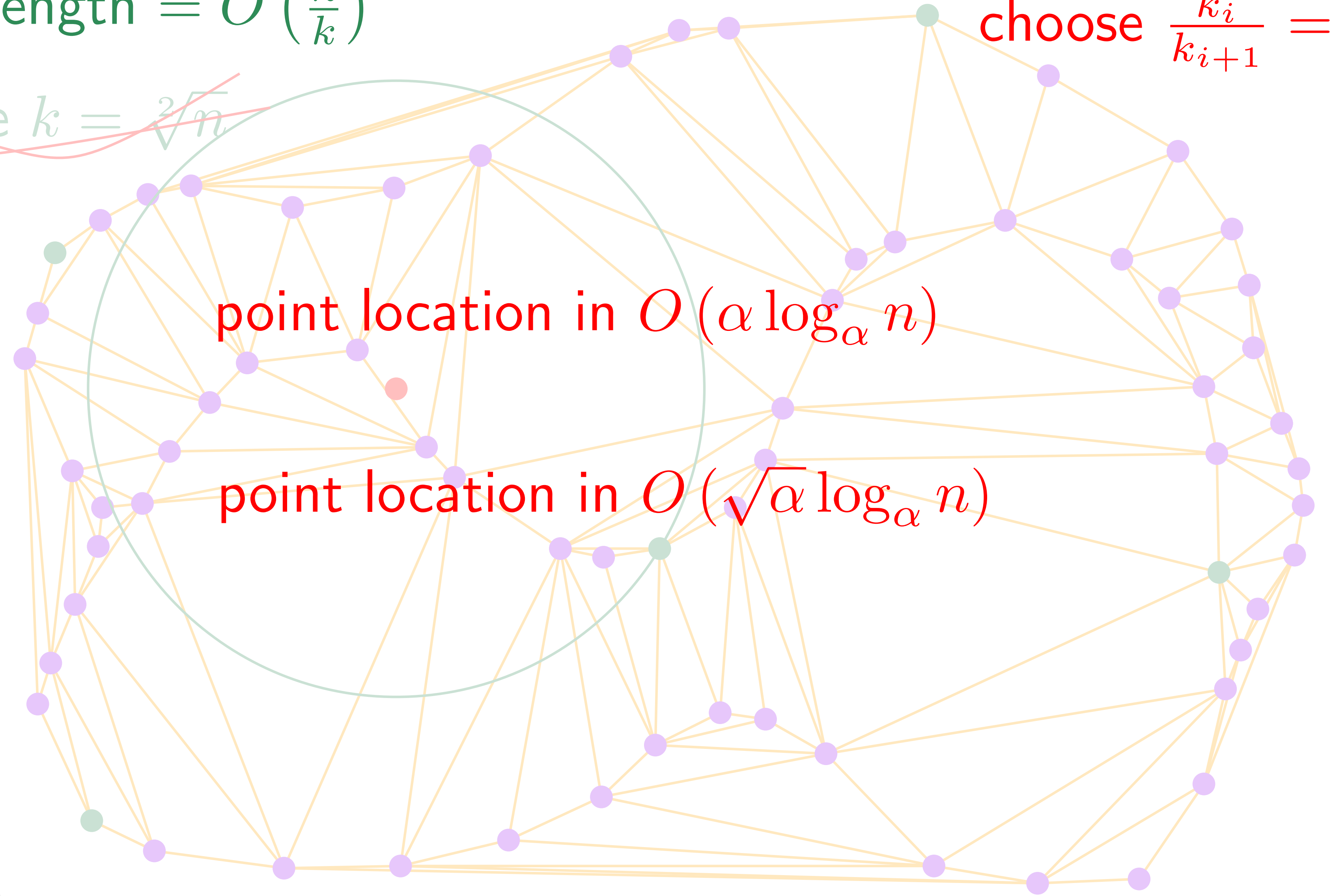
$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \odot\right] = \frac{n}{k}$

$$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots$$

Walk length $= O\left(\frac{n}{k}\right)$

choose $\frac{k_i}{k_{i+1}} = \alpha$

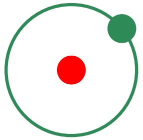
~~choose $k = \sqrt[2]{n}$~~

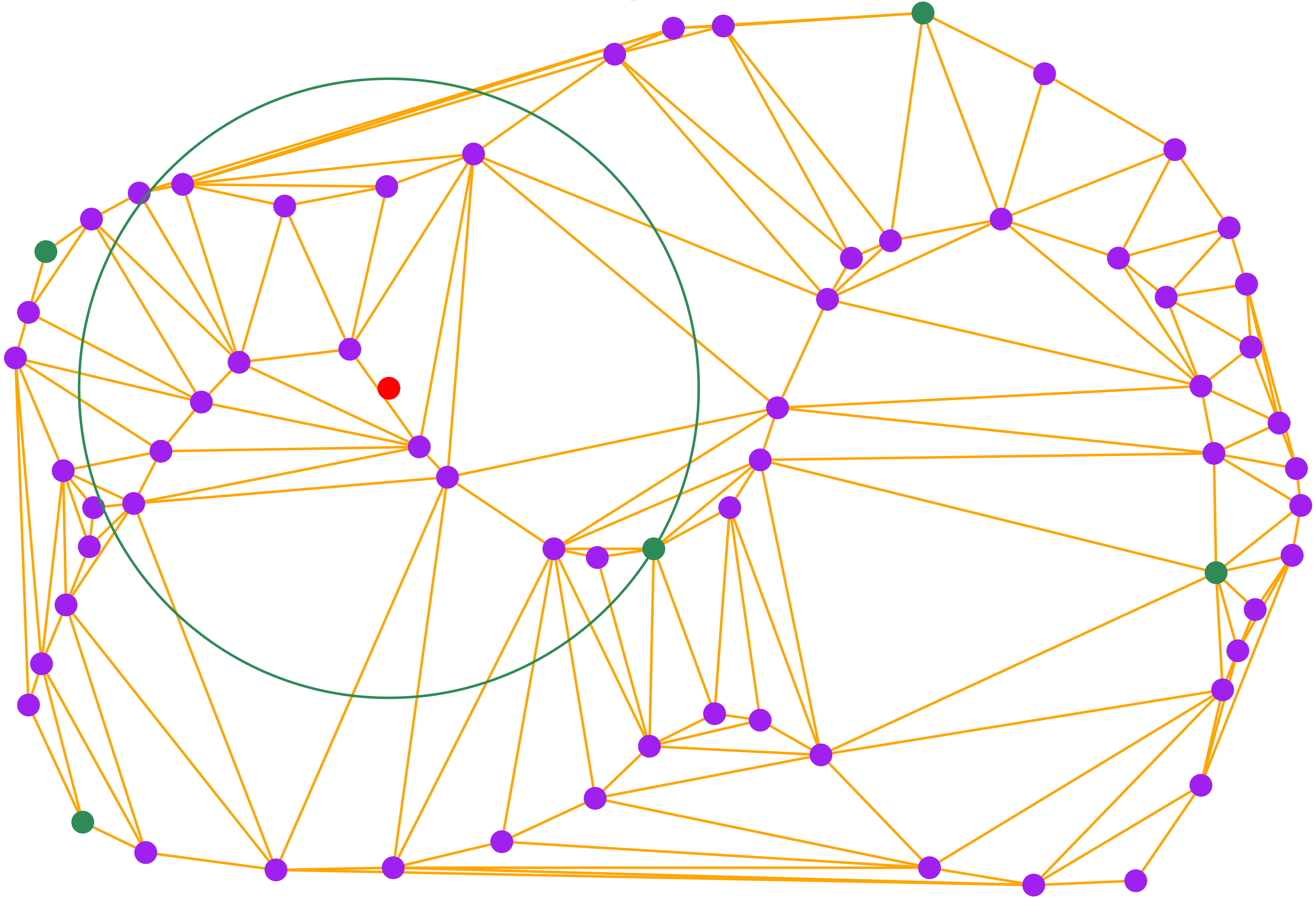


point location in $O\left(\alpha \log_\alpha n\right)$

point location in $O\left(\sqrt{\alpha} \log_\alpha n\right)$

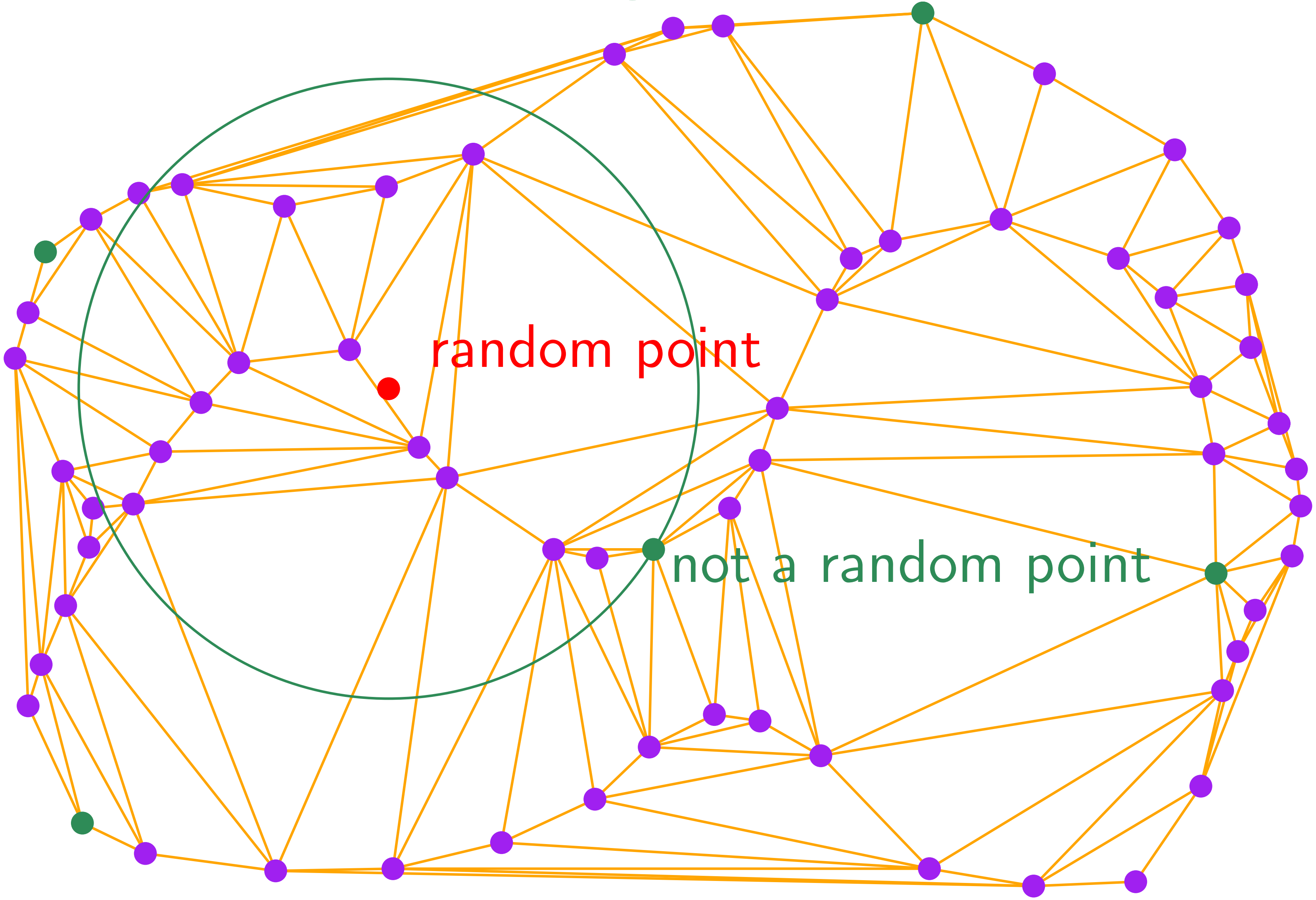
Technical detail

Walk length $= O\Big(\sharp$ of ● in  $\Big) = O\left(\frac{n}{k}\right)$

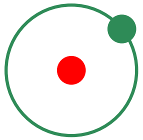


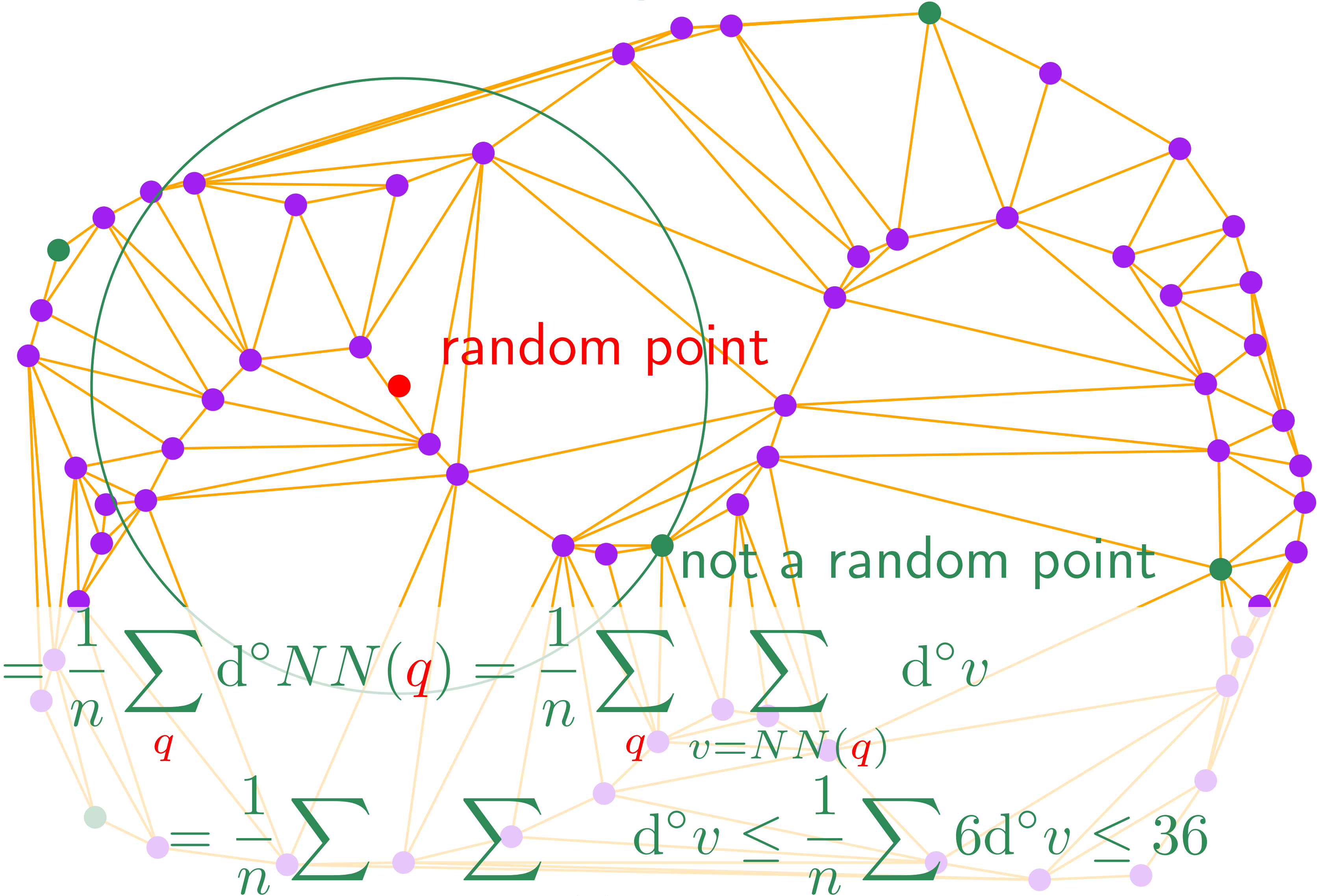
Technical detail

Walk length $= O\Big(\sharp$ of ● in ◉ $\Big) = O\left(\frac{n}{k}\right)$



Technical detail

Walk length $= O\Big(\sharp$ of ● in  $\Big) = O\left(\frac{n}{k}\right)$

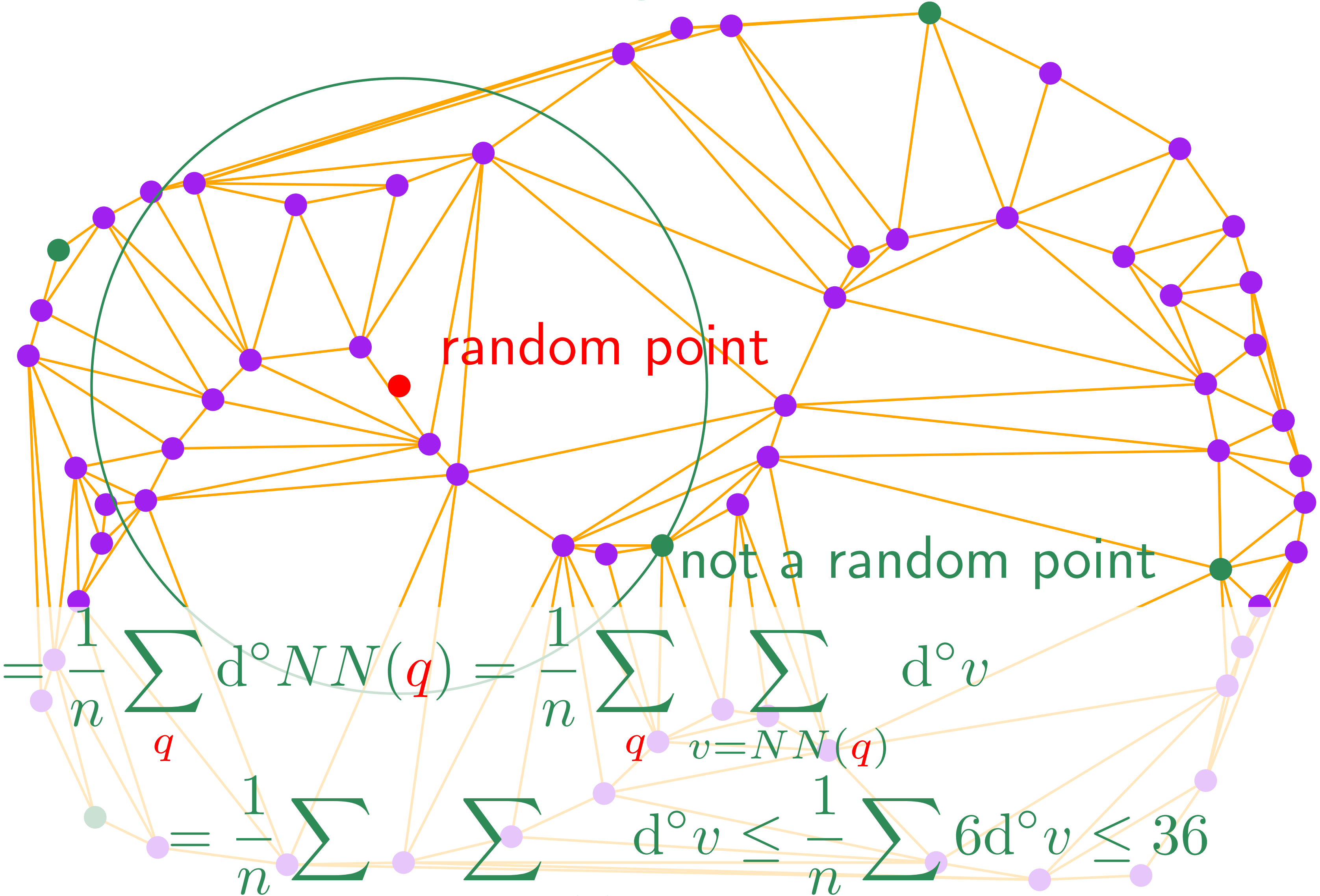


$$\mathbb{E}\left[\mathrm{d}^\circ \bullet\right] = \frac{1}{n}\sum_{q} \mathrm{d}^\circ NN(q) = \frac{1}{n}\sum_{q} \sum_{v=NN(q)} \mathrm{d}^\circ v$$

$$= \frac{1}{n}\sum_{v} \sum_{q;v=NN(q)} \mathrm{d}^\circ v \leq \frac{1}{n}\sum_{v} 6\mathrm{d}^\circ v \leq 36$$

Technical detail

$$\text{Walk length} = O\left(\overset{\sum d^\circ}{\sharp} \text{ of } \bullet \text{ in } \odot \right) = O\left(\tfrac{n}{k}\right)$$



$$\mathbb{E}\left[\mathrm{d}^\circ \bullet\right] = \frac{1}{n}\sum_{q} \mathrm{d}^\circ NN(q) = \frac{1}{n}\sum_{q}\sum_{v=NN(q)} \mathrm{d}^\circ v$$

$$= \frac{1}{n}\sum_{v}\sum_{q;v=NN(q)} \mathrm{d}^\circ v \leq \frac{1}{n}\sum_{v} 6\mathrm{d}^\circ v \leq 36$$

Randomization

How many randomness is necessary?

If the data are not known in advance

shuffle locally

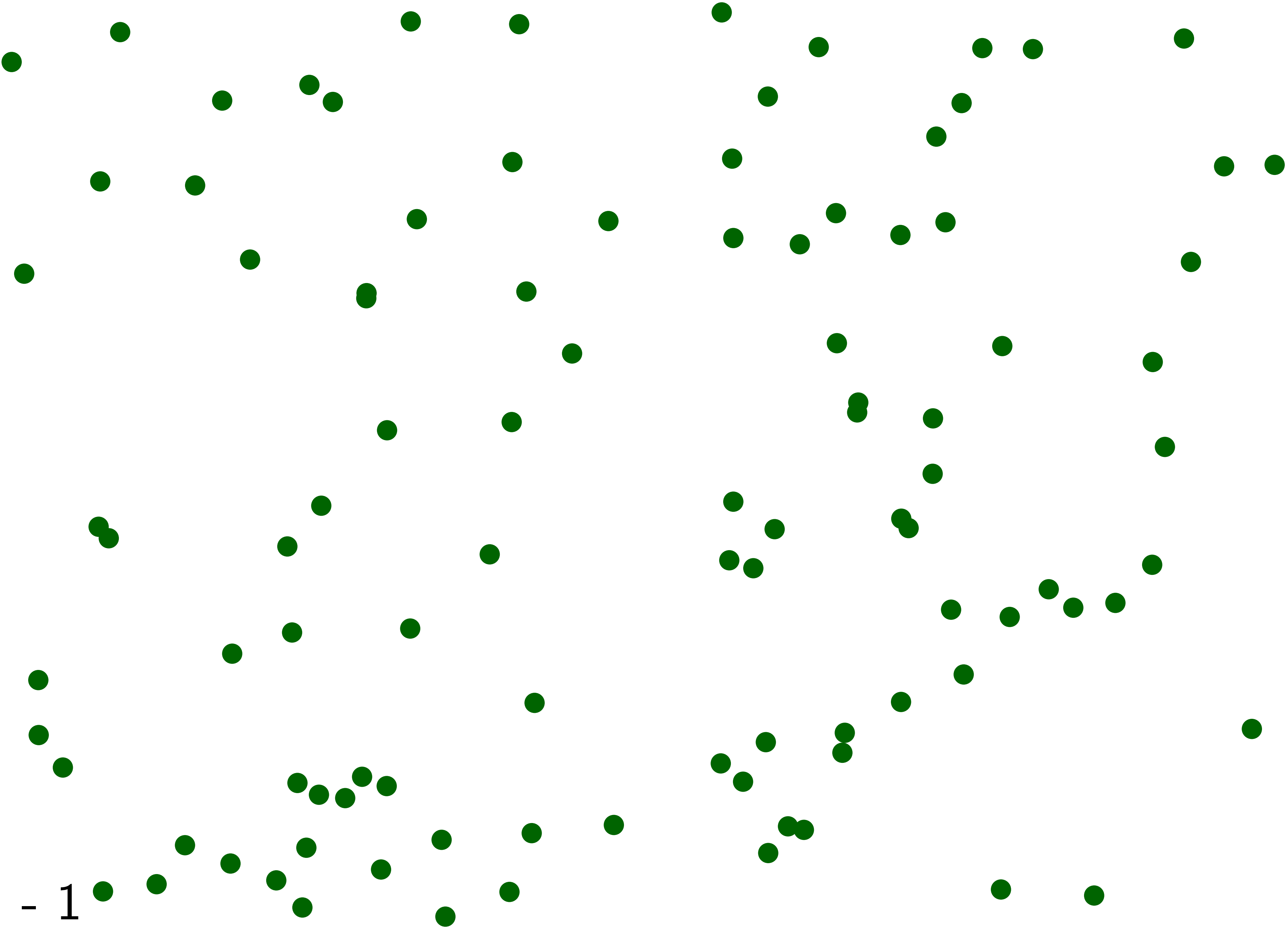
Randomization

Drawbacks of random order

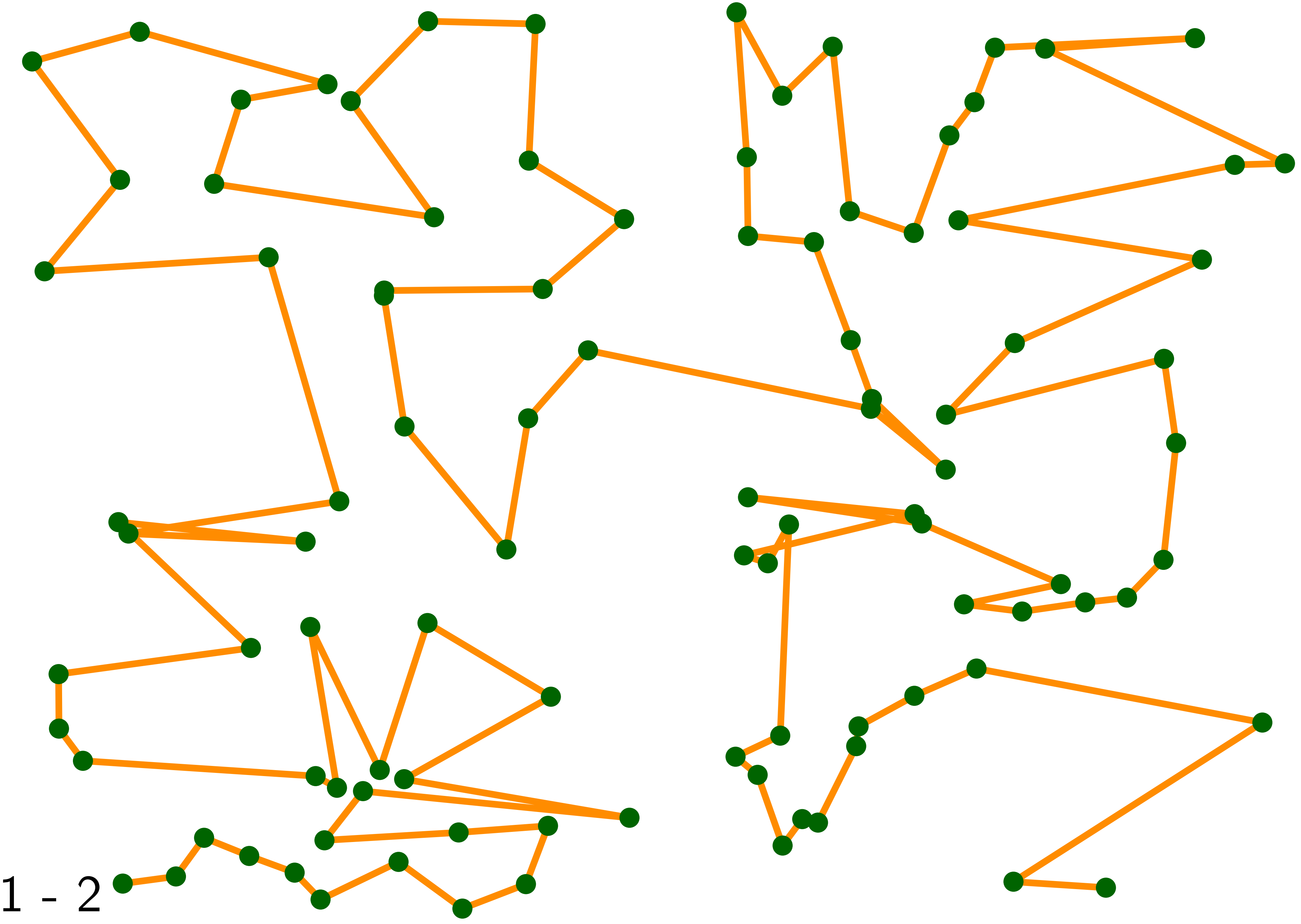
non locality of memory access

data structure for point location

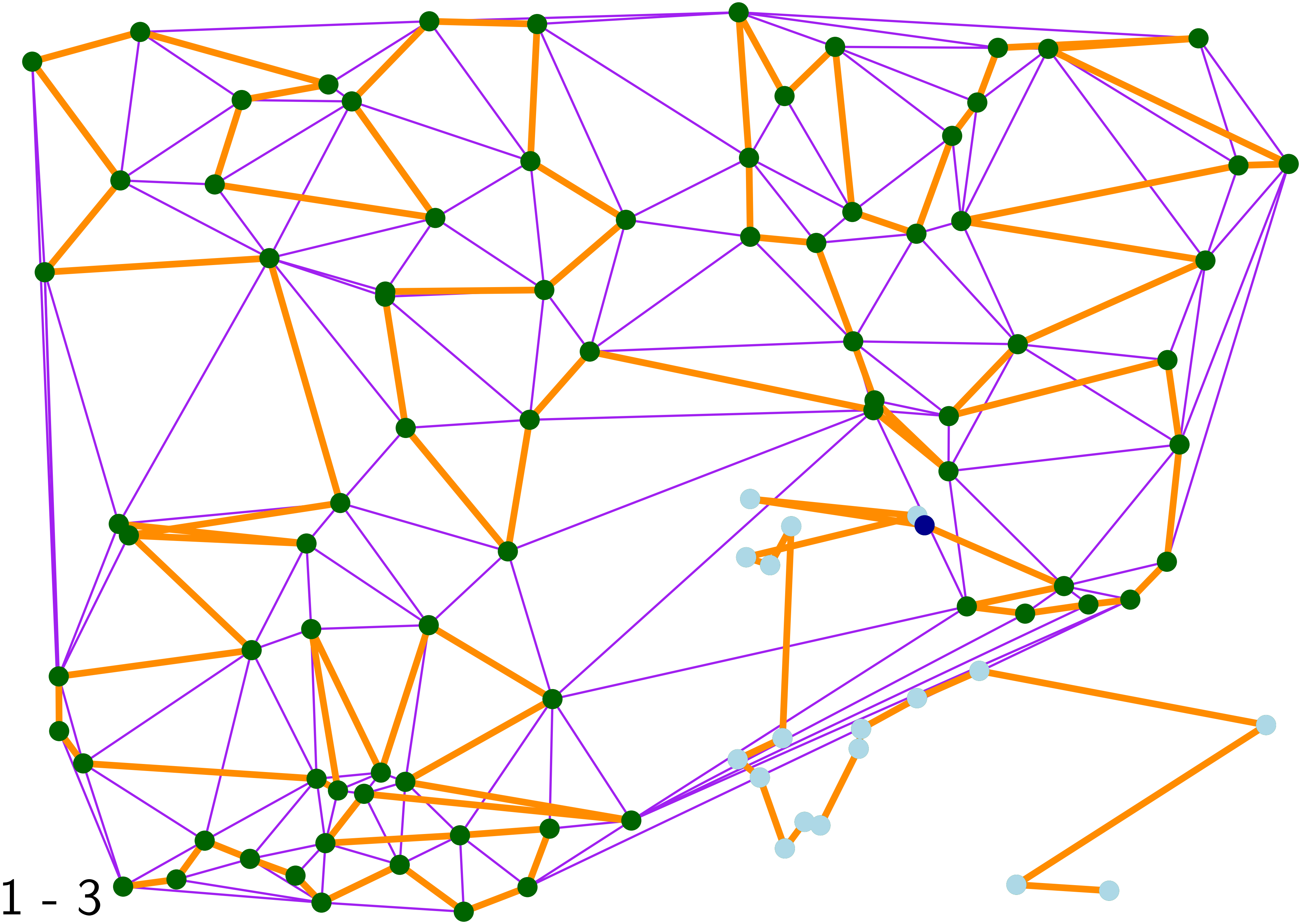
→ Hilbert sort



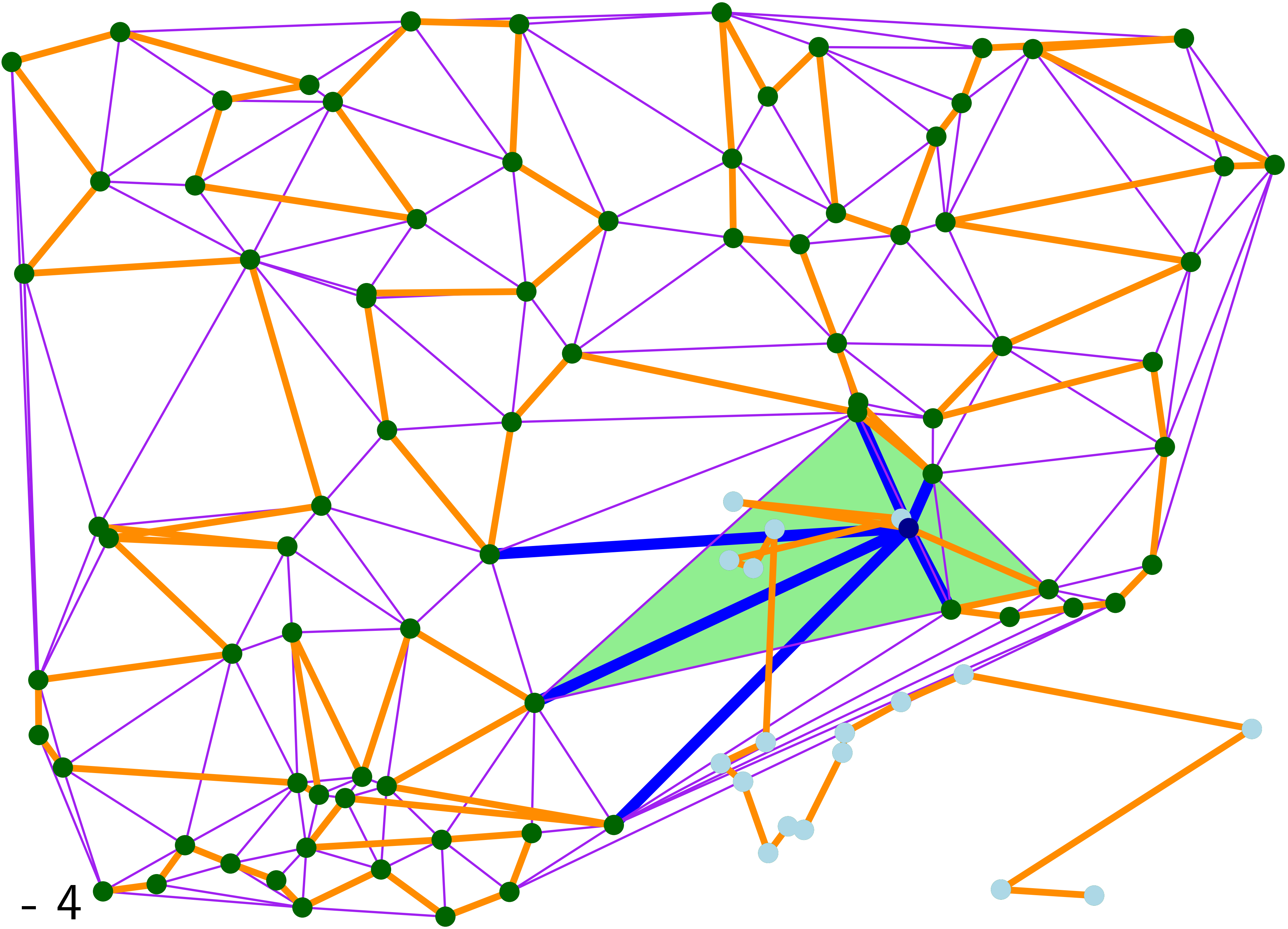
21 - 1



21 - 2



21 - 3


21 - 4

Drawbacks of random order

non locality of memory access

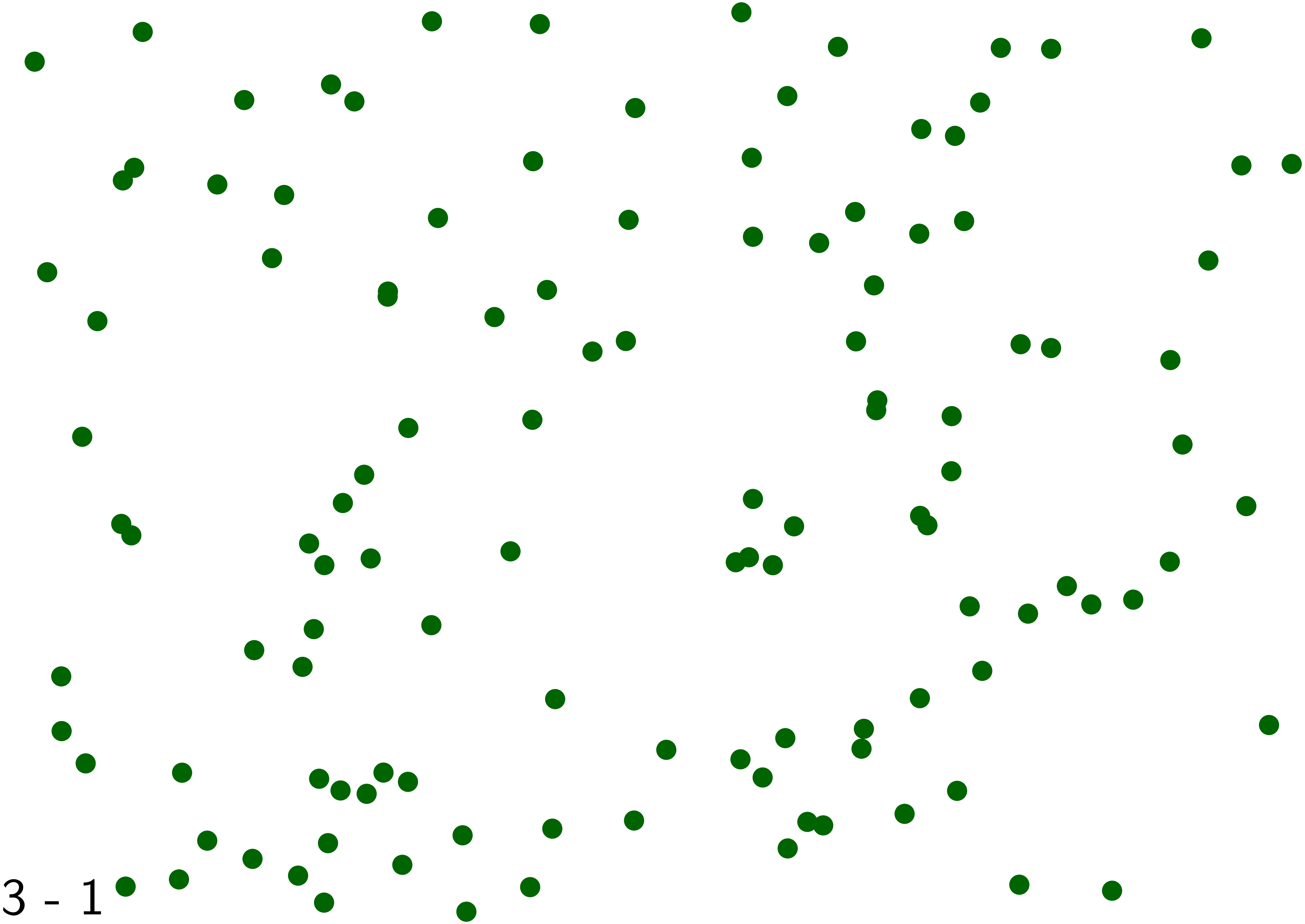
data structure for point location

$\longrightarrow$ Hilbert sort

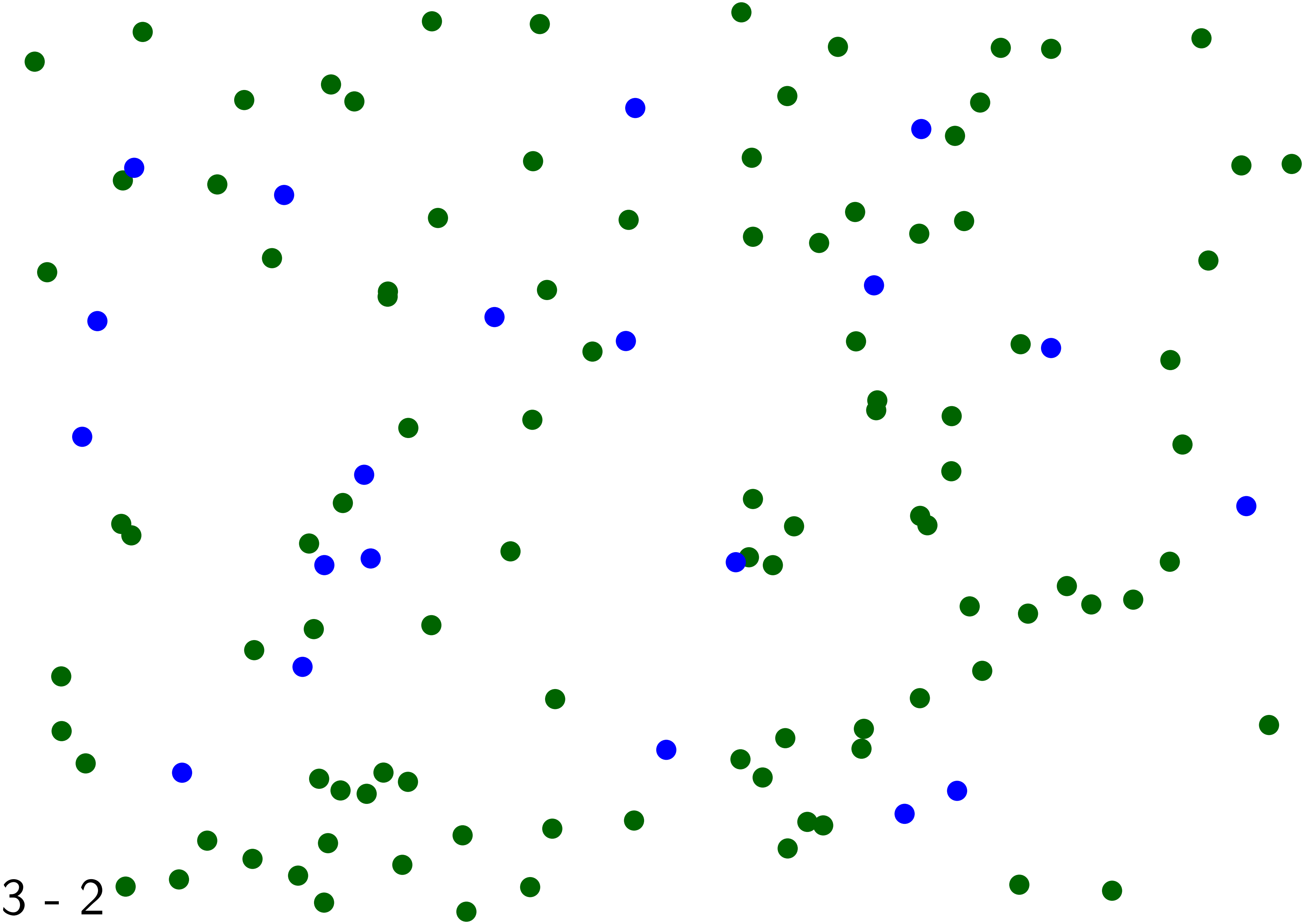
Walk should be fast

Last point is not at all a random point

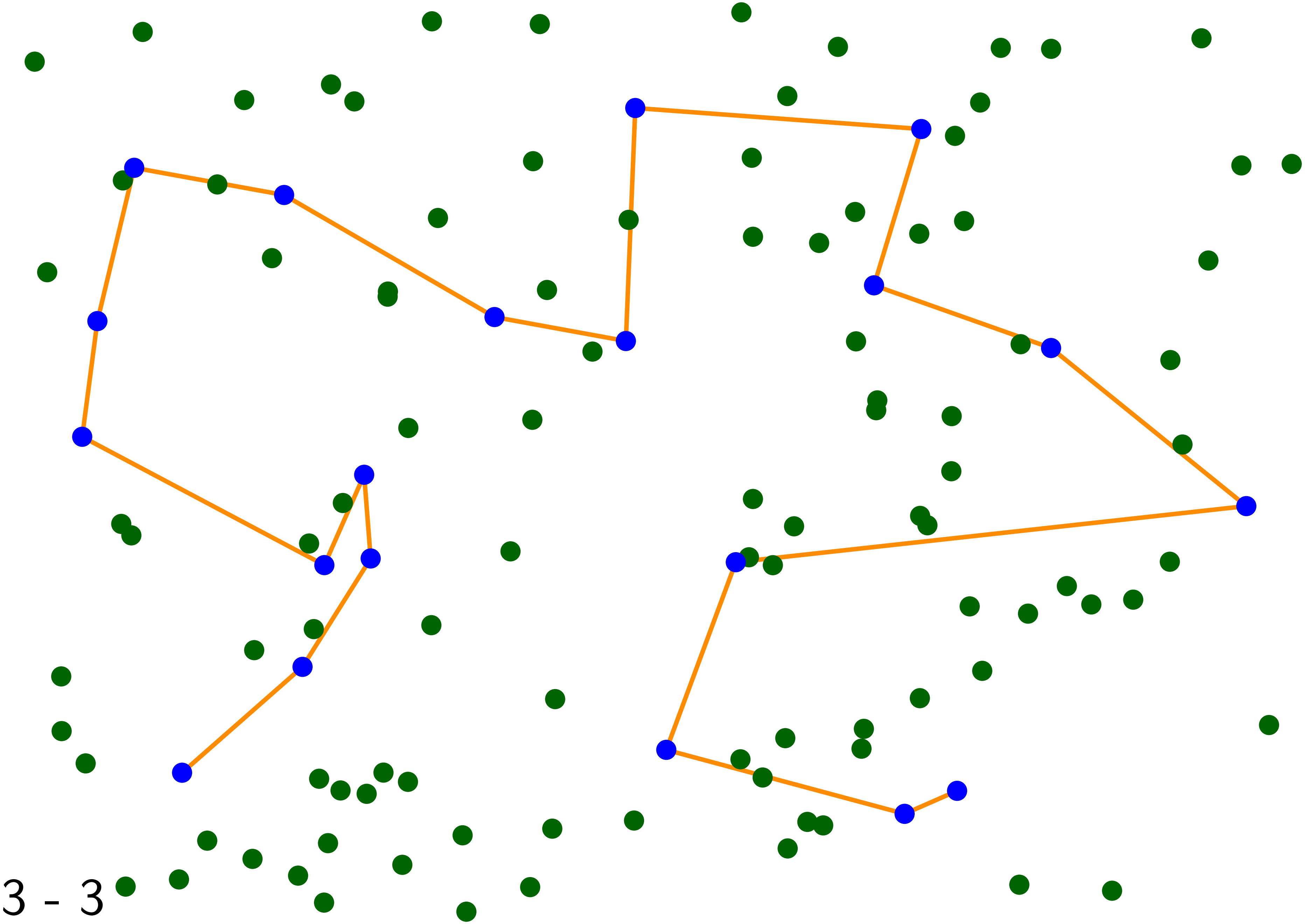
$\longrightarrow$ no control of degree of last point



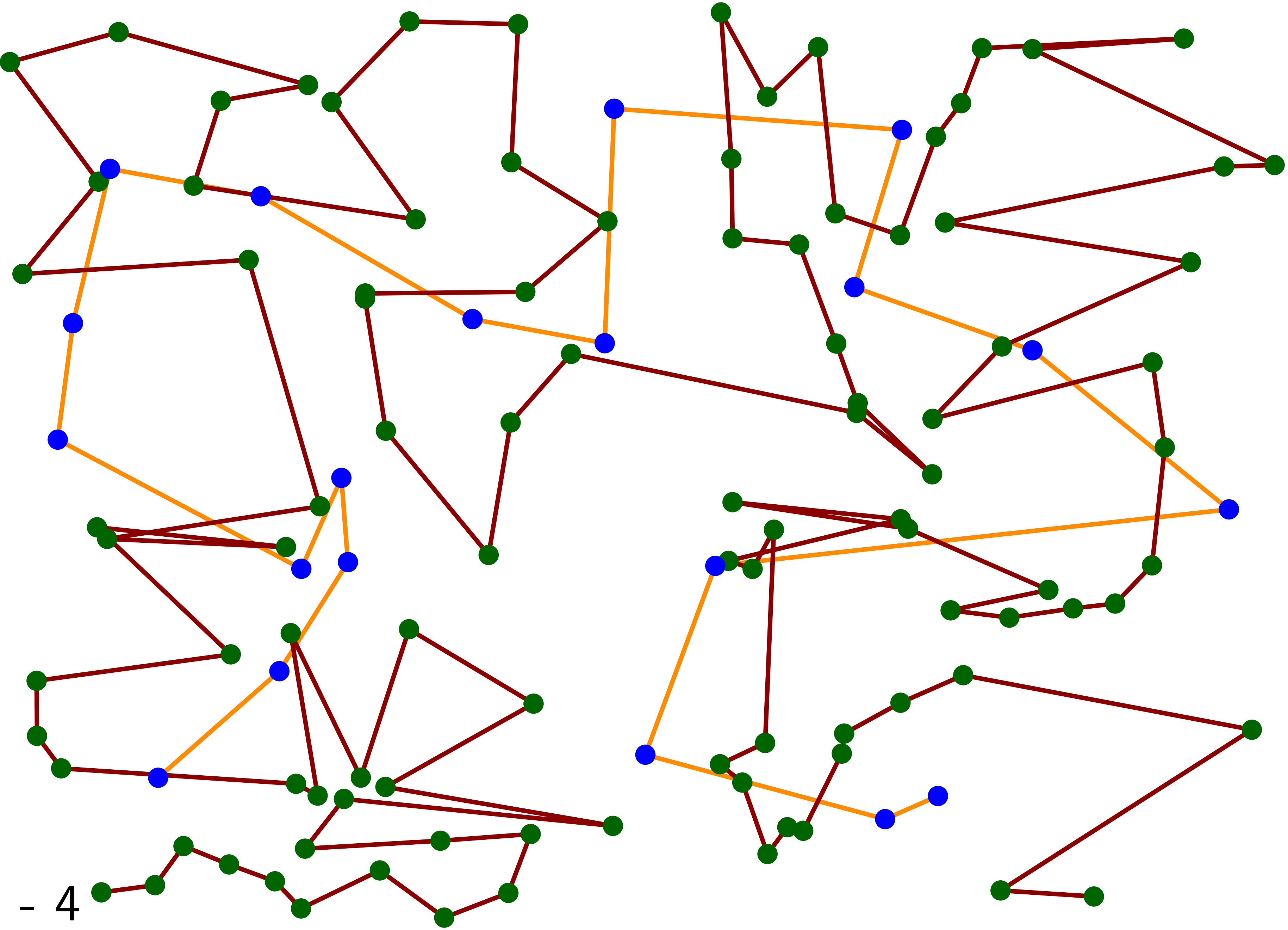
23 - 1



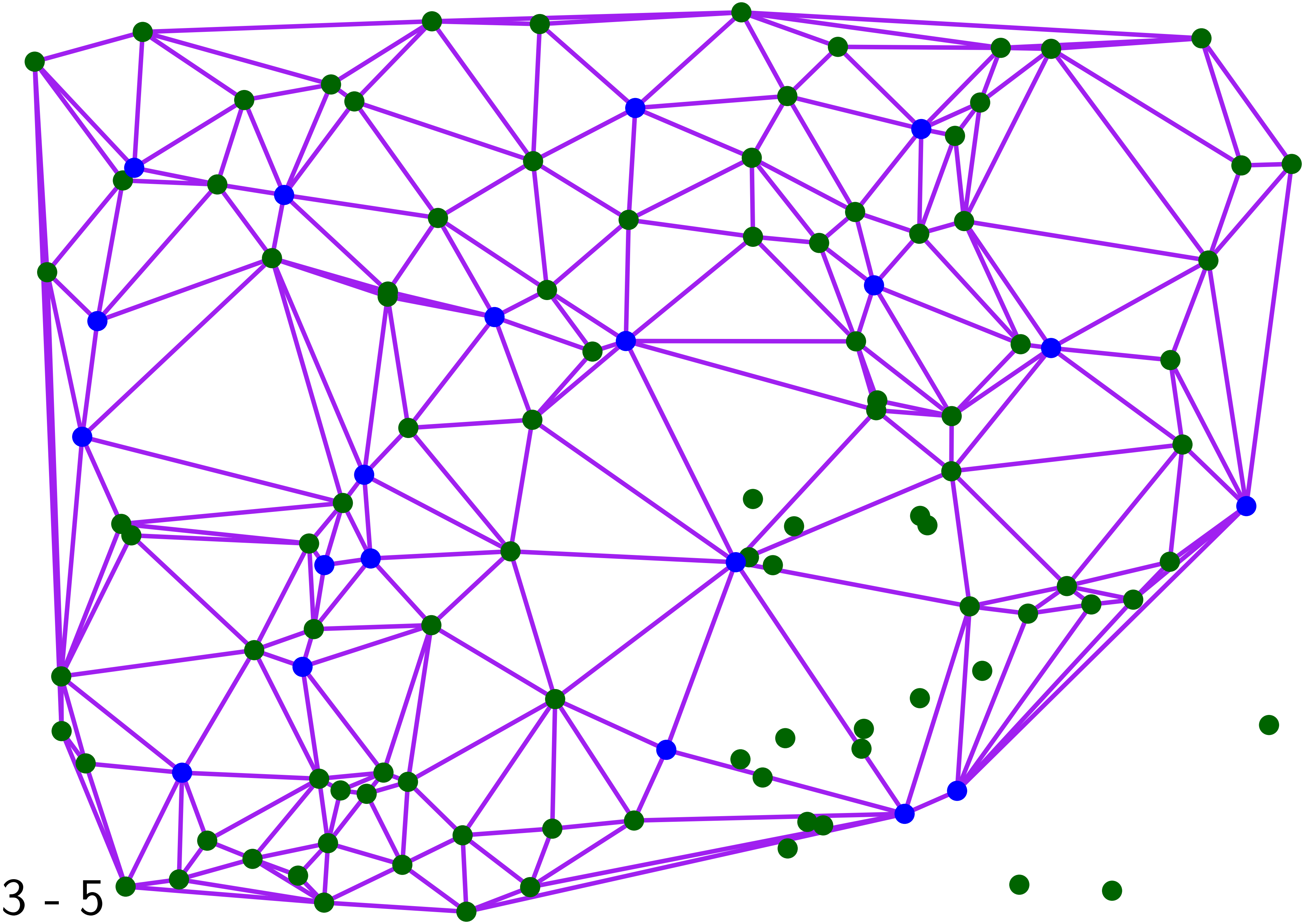
23 - 2



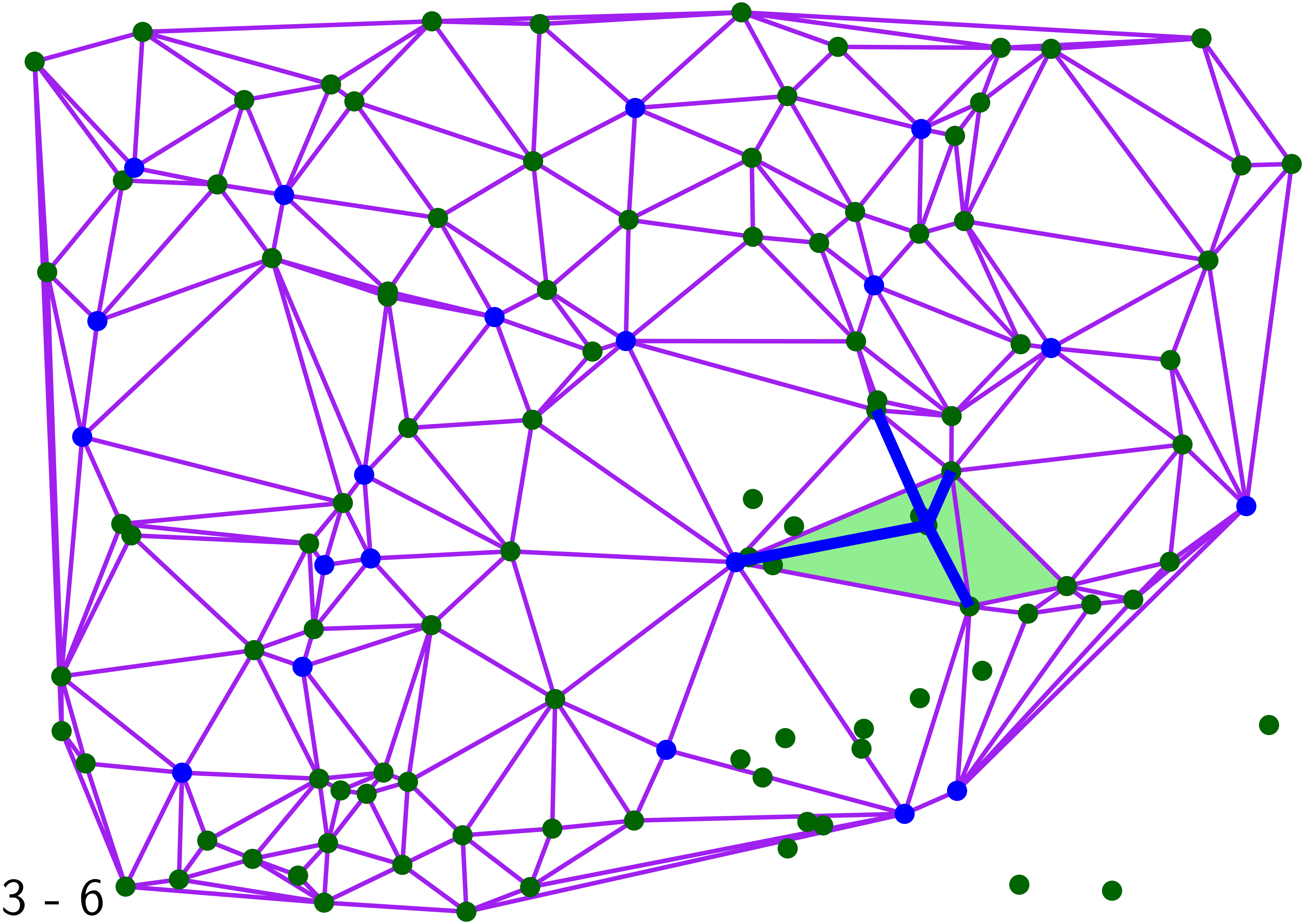
23 - 3



23 - 4

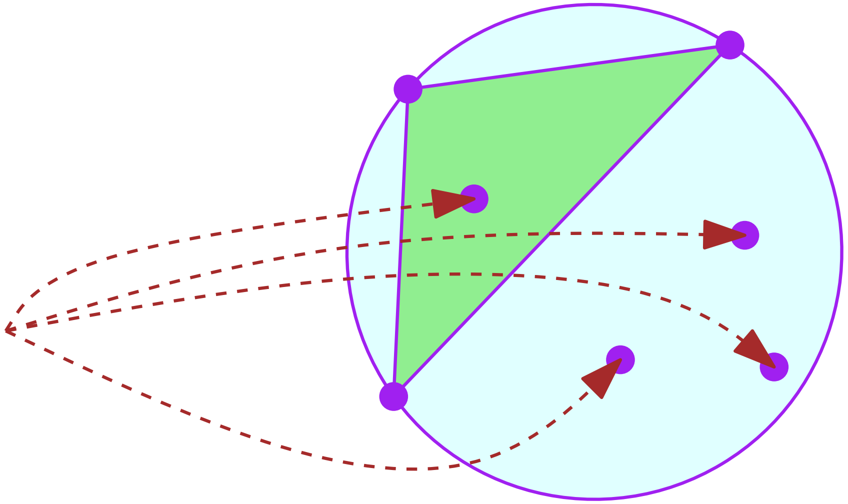


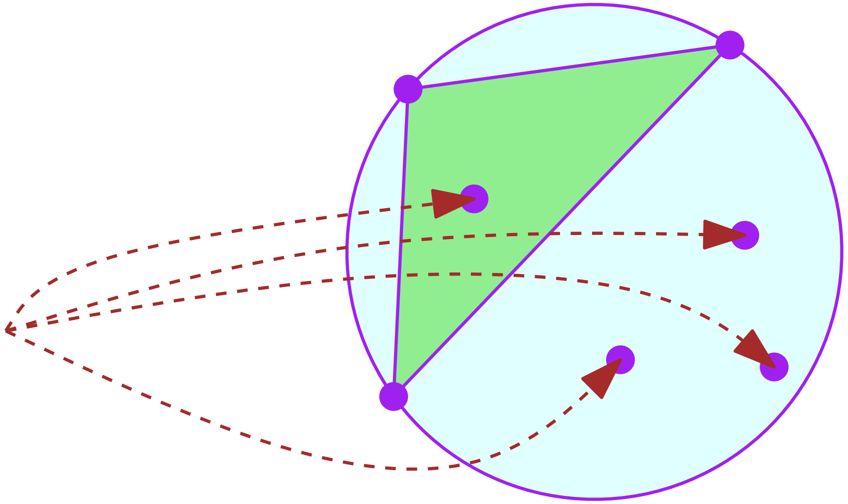
23 - 5



23 - 6

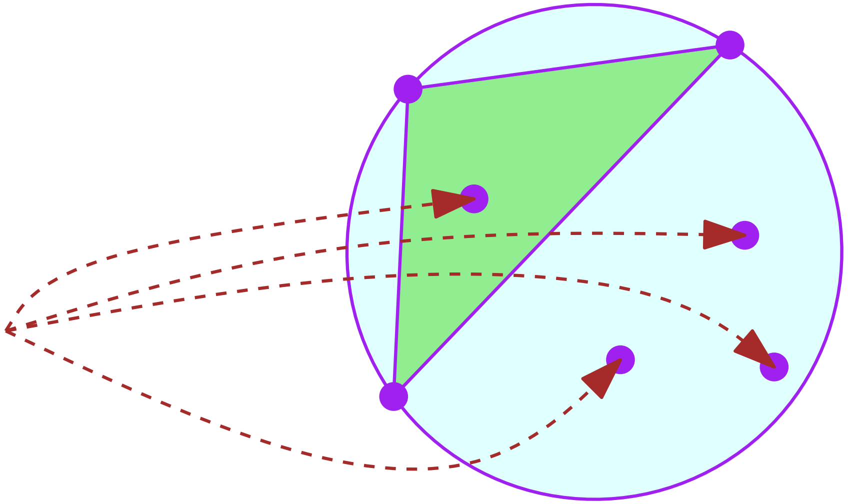
Triangle Δ with j stoppers





Triangle Δ with j stoppers

Size (order $\leq k$ Voronoi) $\leq \frac{\alpha n}{\alpha^3} = nk^2$

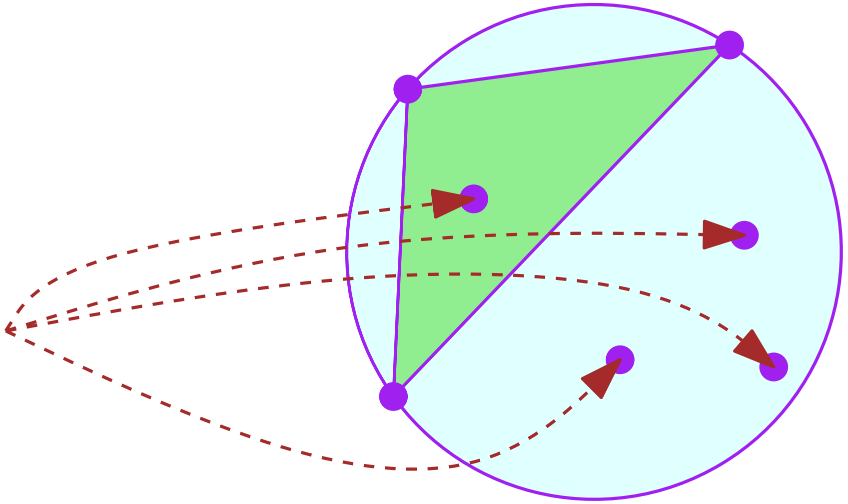


Triangle Δ with j stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3} \frac{2}{j+2} \frac{1}{j+1}$$

Triangle Δ with j stoppers

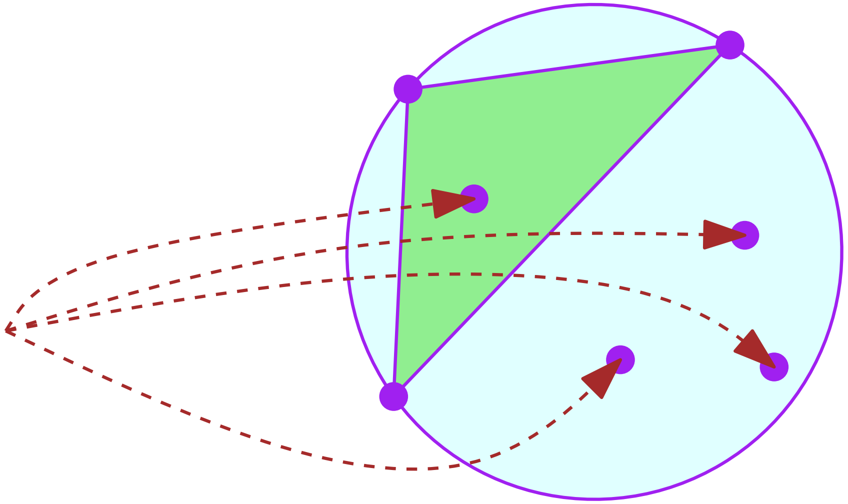


Probability that it exists during the construction

$$=\frac{3}{j+3}\cancel{\frac{2}{j+2}\frac{1}{j+1}}$$

remains $\Theta(j^{-3})$

Triangle Δ with j stoppers



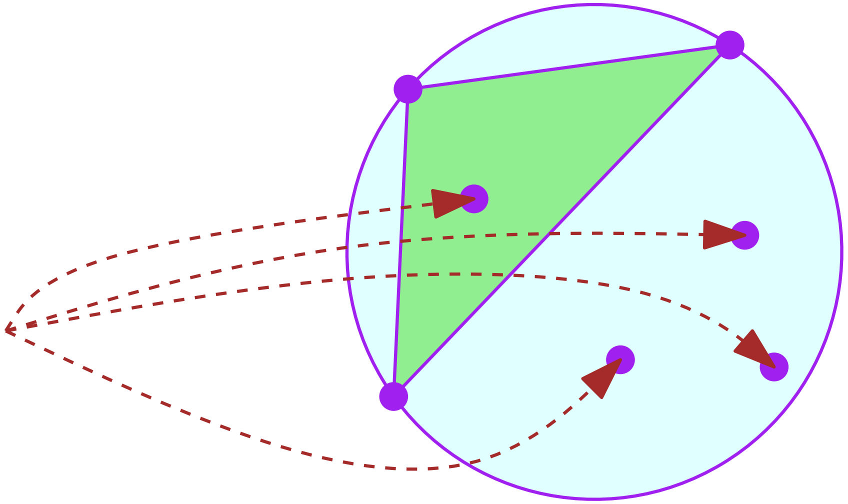
Probability that it exists during the construction

$$= \cancel{\frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}} \quad \text{remains } \Theta(j^{-3})$$

$\sharp$ of created triangles

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$\simeq O(\sum \frac{nj^2}{j^4}) = O(n)$$



Triangle Δ with j stoppers

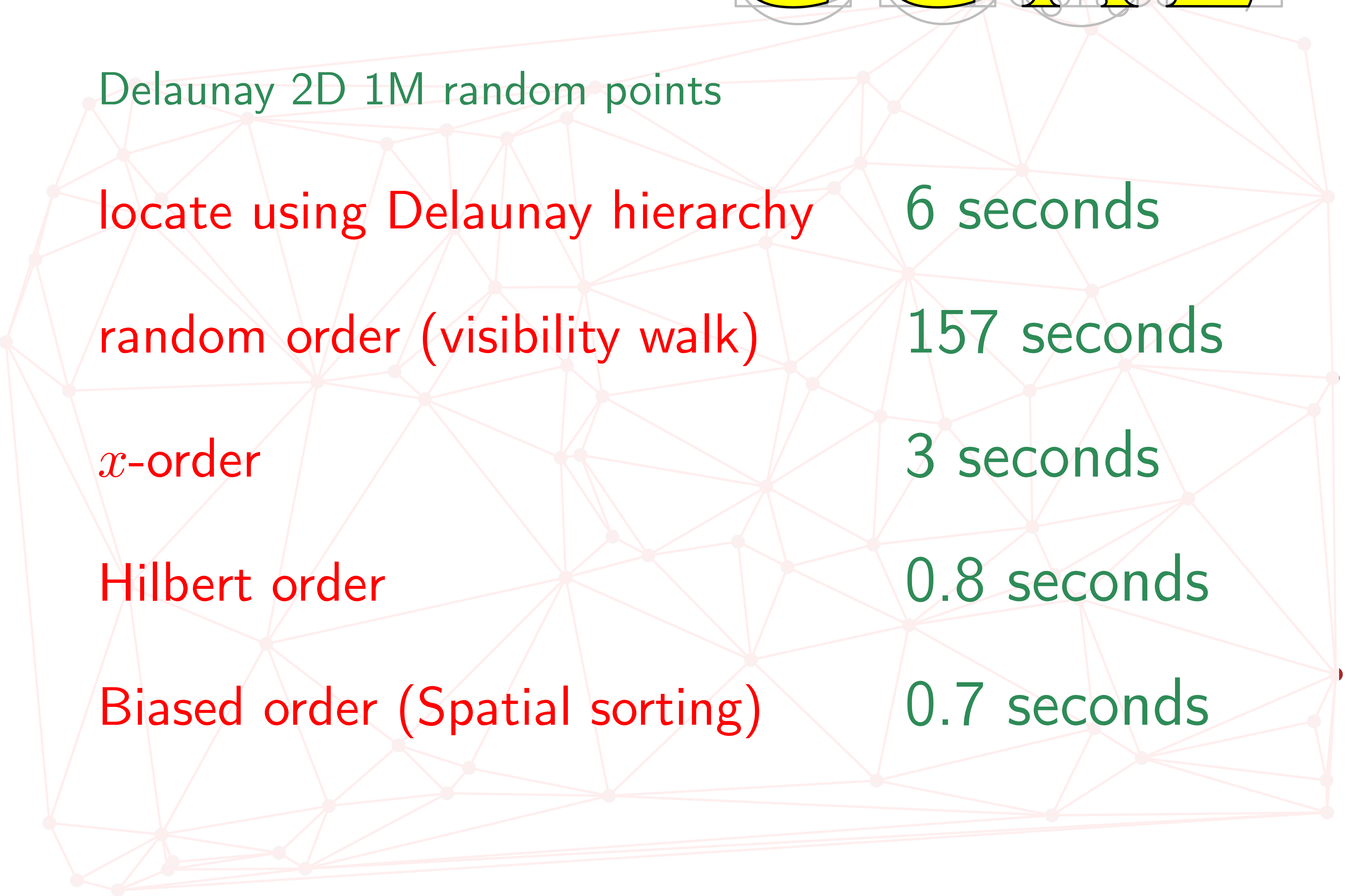
Probability that it exists during the construction

$$= \xcancel{\frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}} \quad \text{remains } \Theta(j^{-3})$$

$\sharp$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

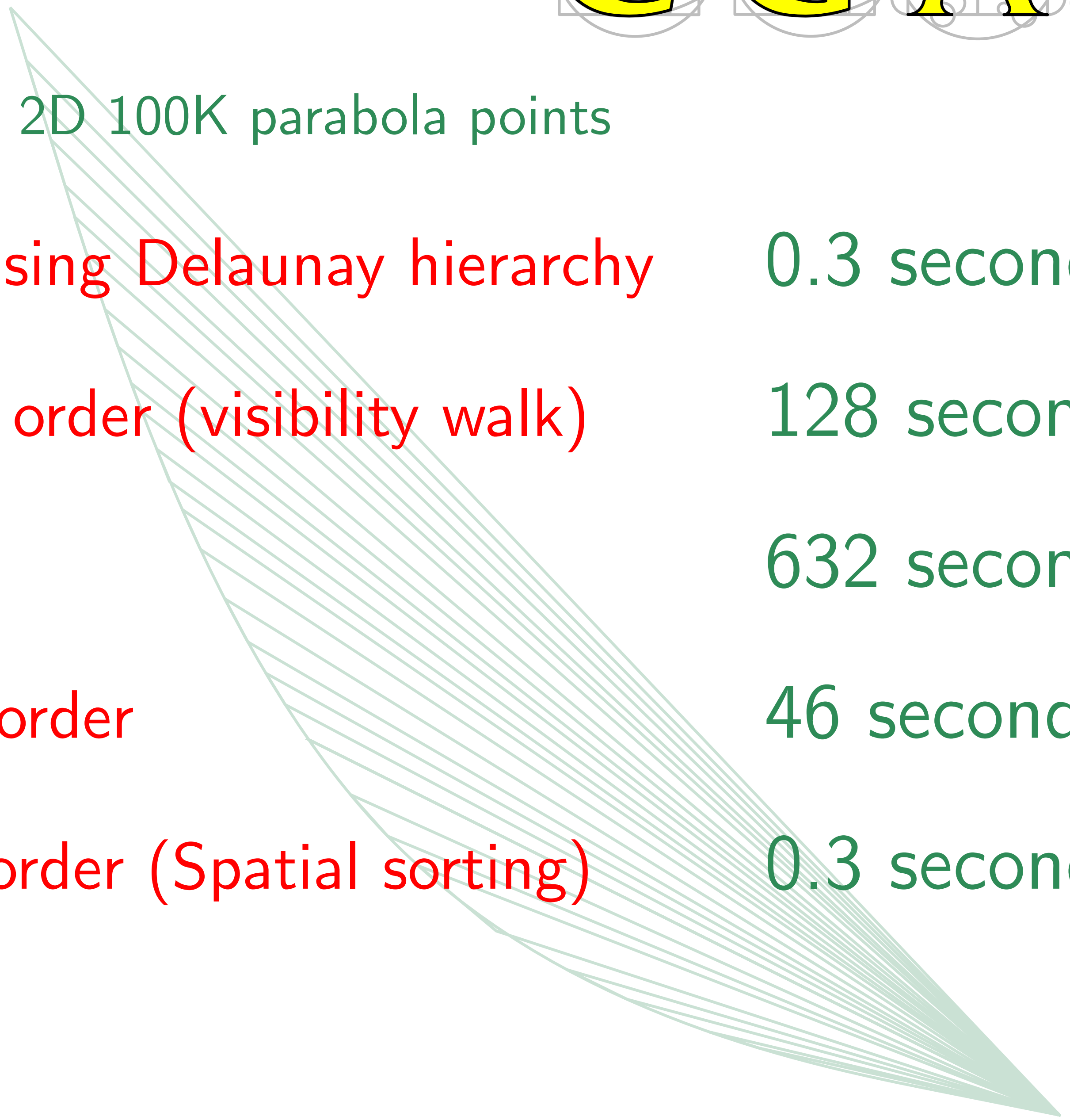
$$\simeq O(\sum j \frac{nj^2}{j^4}) = O(n \log n)$$



Delaunay 2D 1M random points

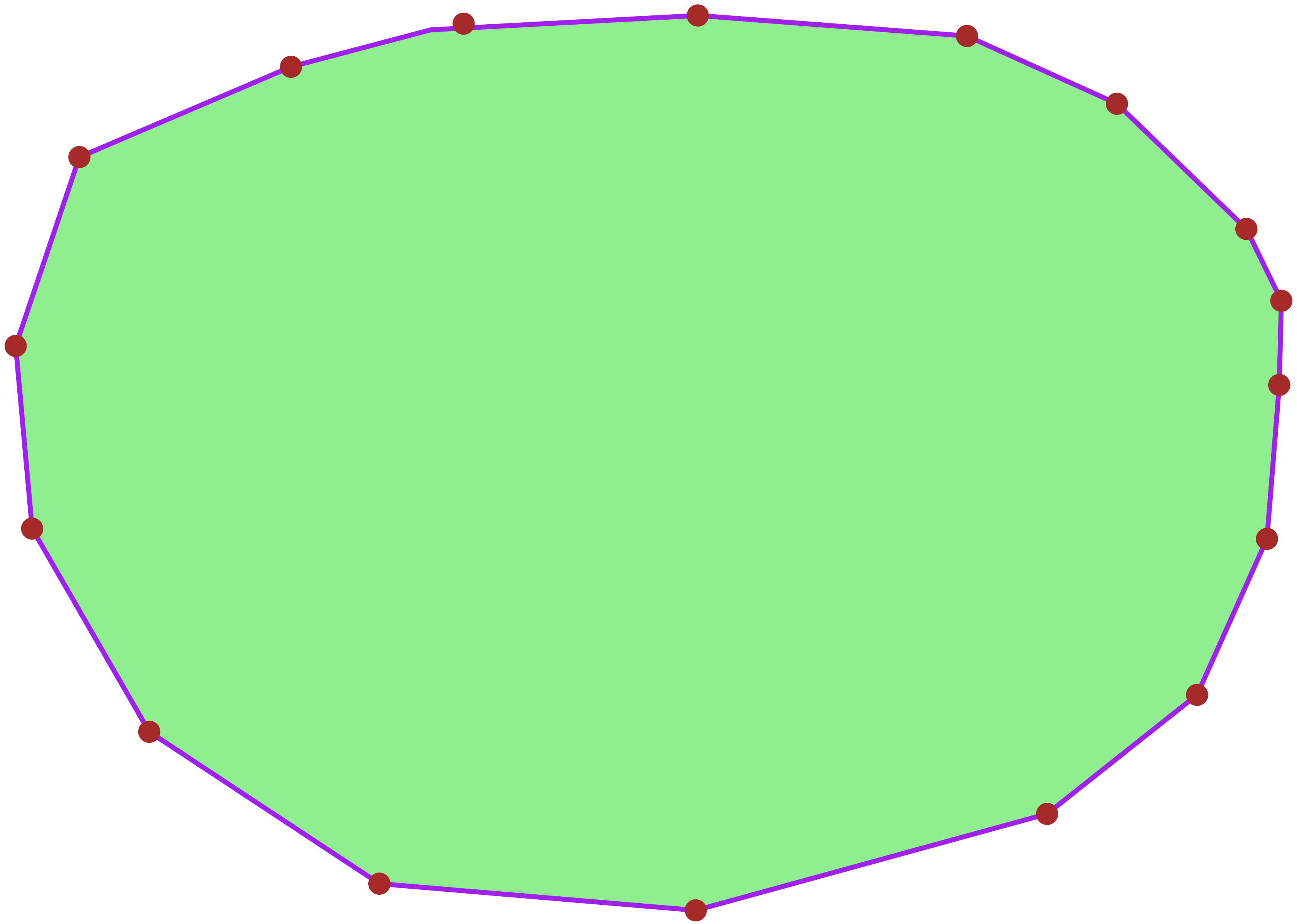
locate using Delaunay hierarchy	6 seconds
random order (visibility walk)	157 seconds
x-order	3 seconds
Hilbert order	0.8 seconds
Biased order (Spatial sorting)	0.7 seconds

Delaunay 2D 100K parabola points

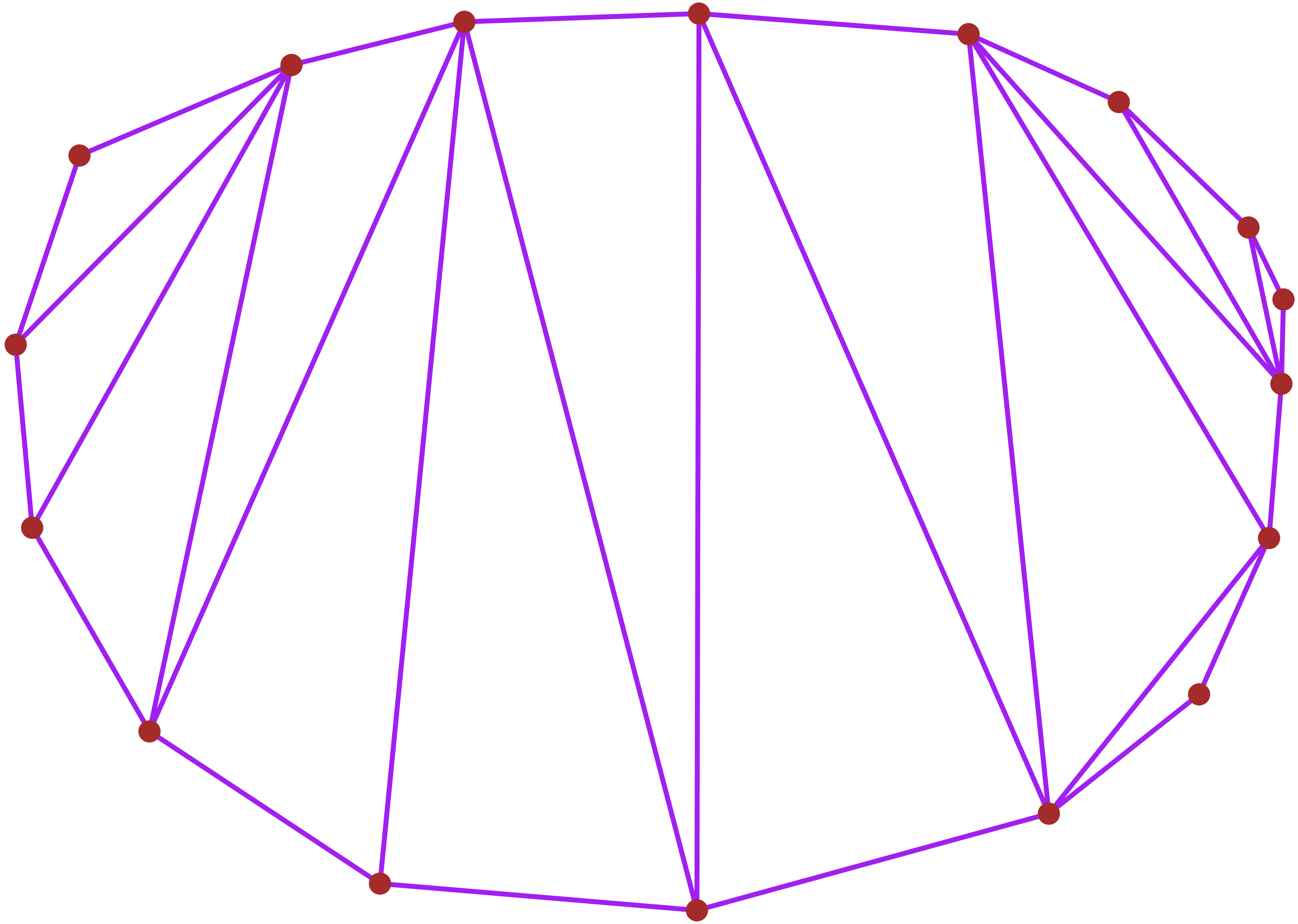


locate using Delaunay hierarchy	0.3 seconds
random order (visibility walk)	128 seconds
x-order	632 seconds
Hilbert order	46 seconds
Biased order (Spatial sorting)	0.3 seconds

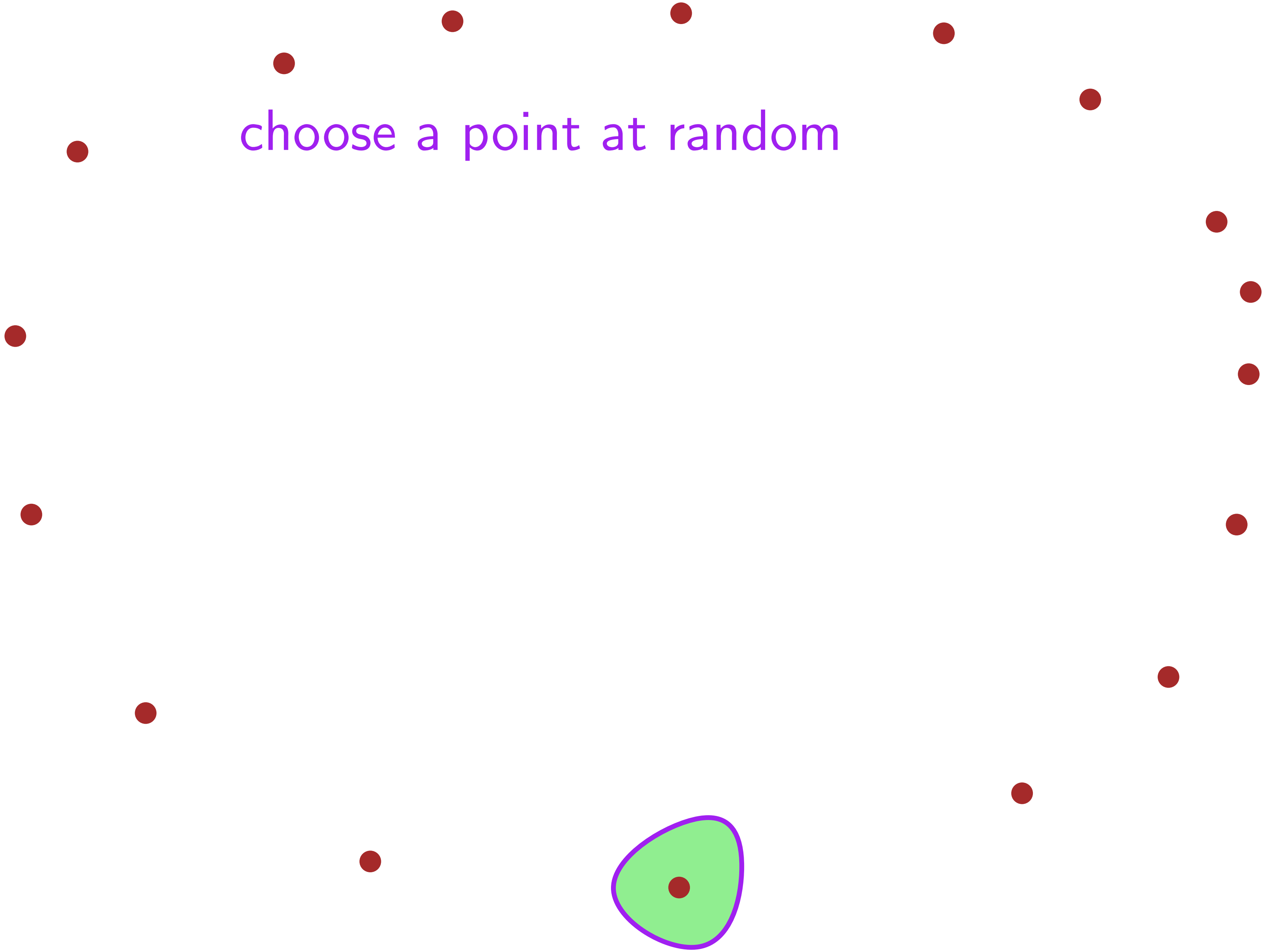
Delaunay of points in convex position



Delaunay of points in convex position

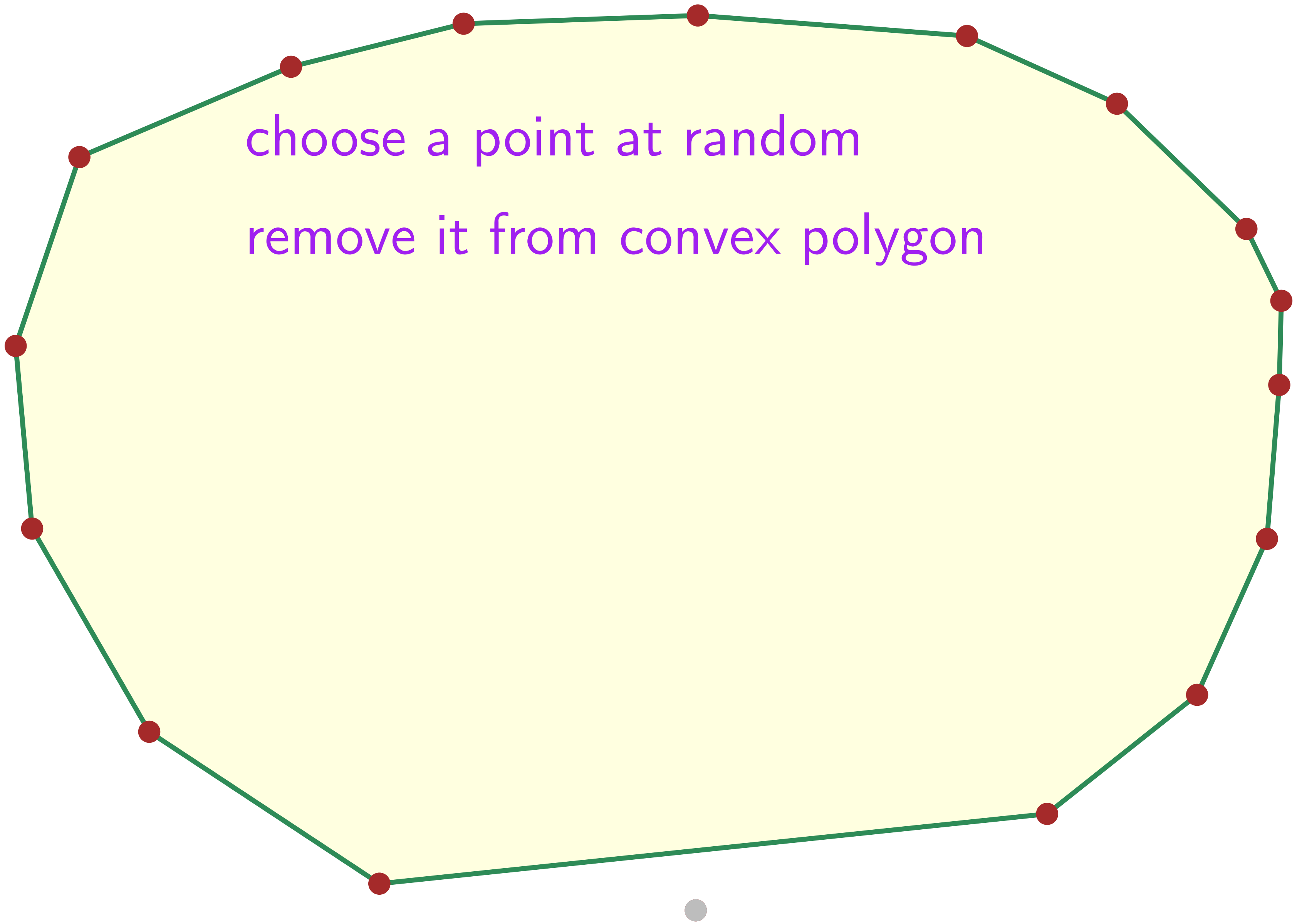


Delaunay of points in convex position



choose a point at random

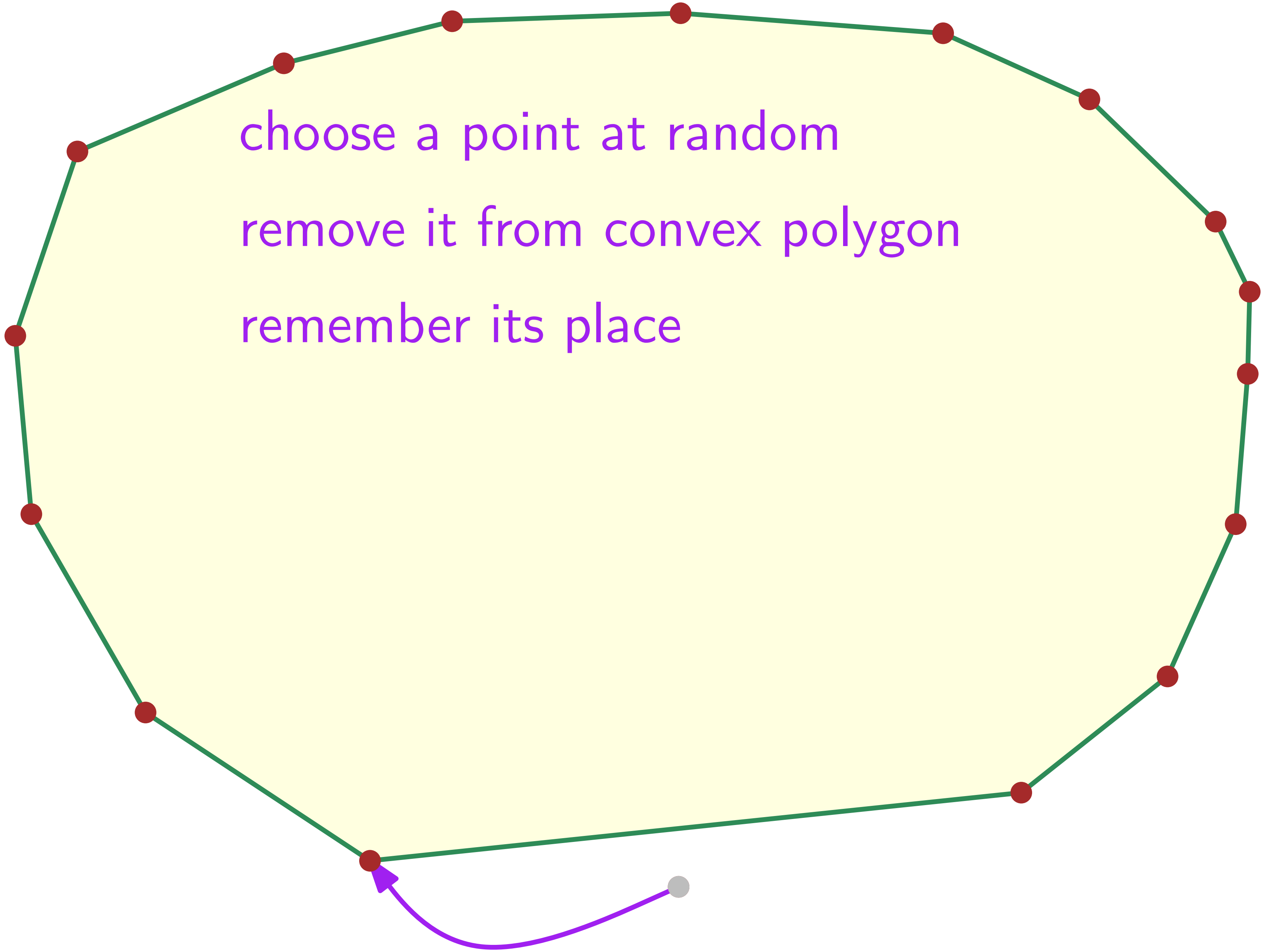
Delaunay of points in convex position



choose a point at random

remove it from convex polygon

Delaunay of points in convex position

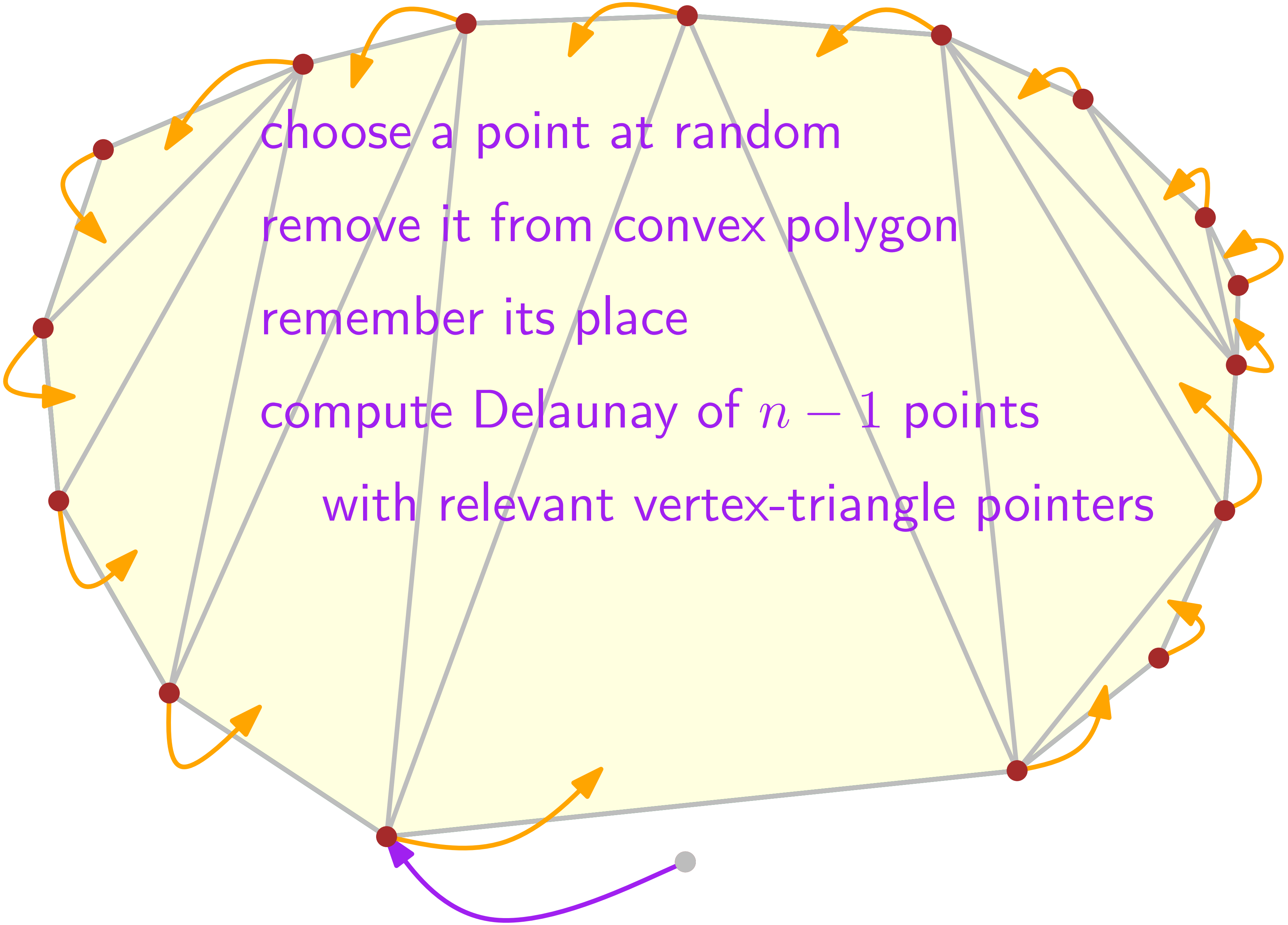


choose a point at random

remove it from convex polygon

remember its place

Delaunay of points in convex position



choose a point at random

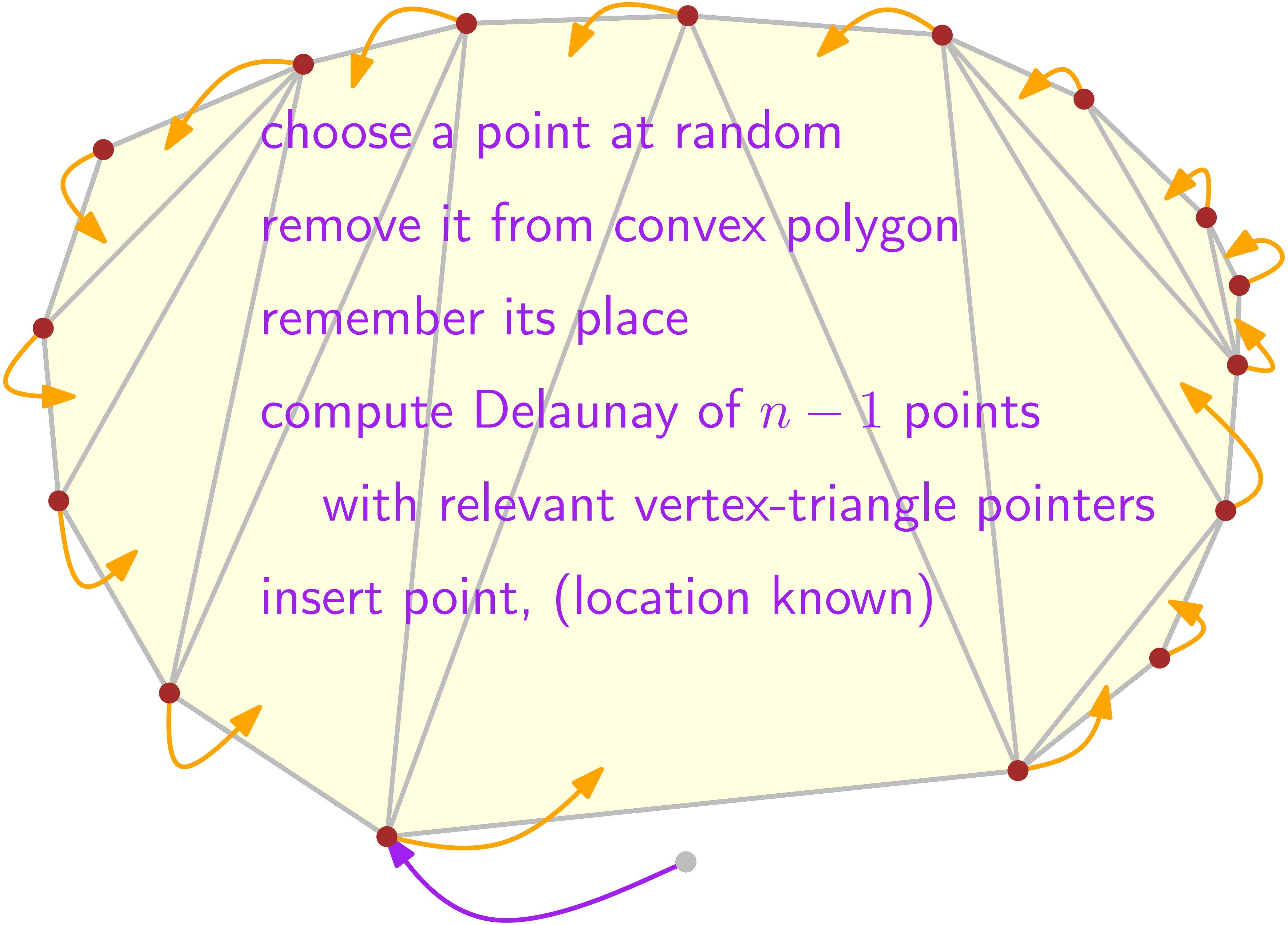
remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

Delaunay of points in convex position



choose a point at random

remove it from convex polygon

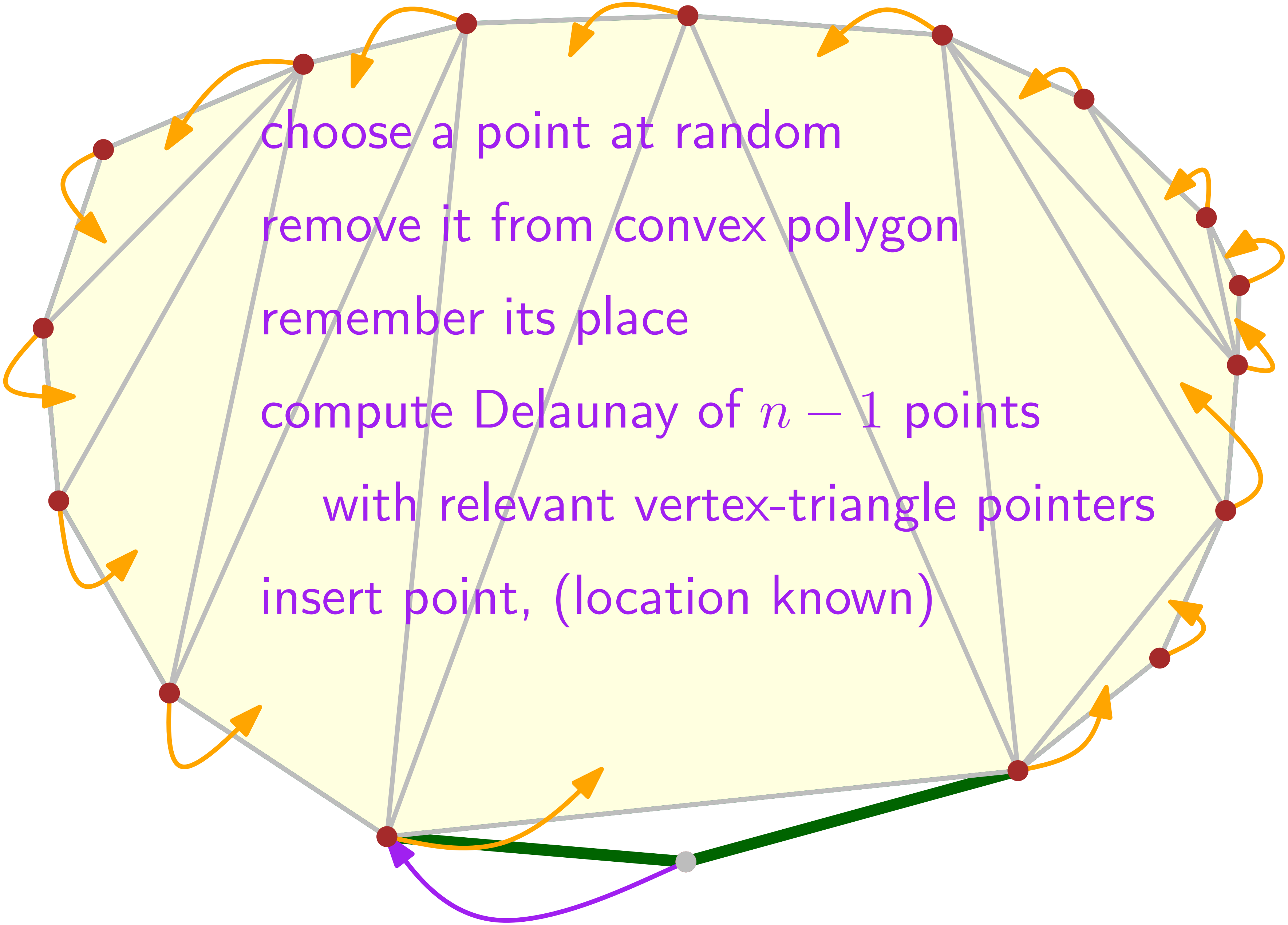
remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position



choose a point at random

remove it from convex polygon

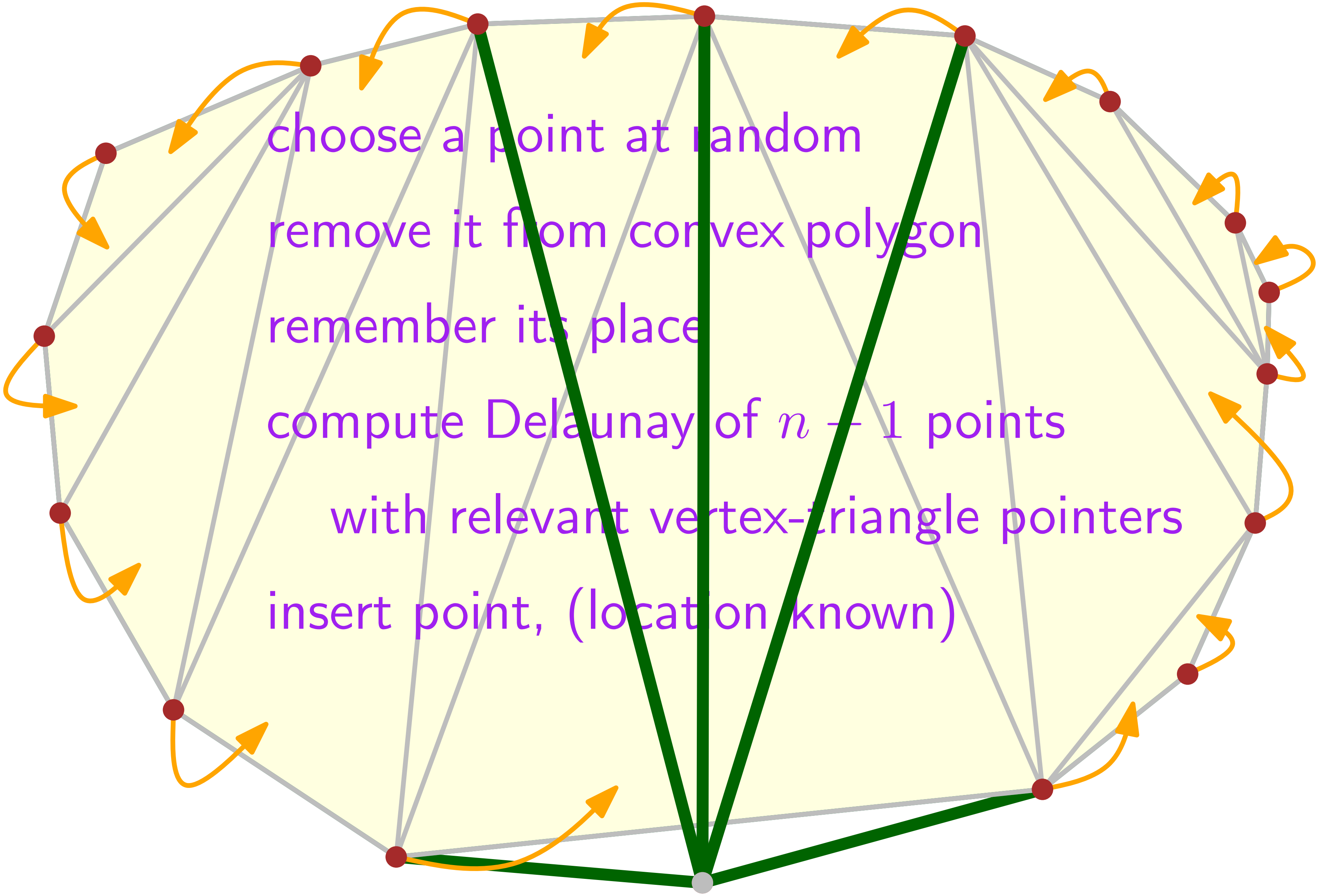
remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position



choose a point at random

remove it from convex polygon

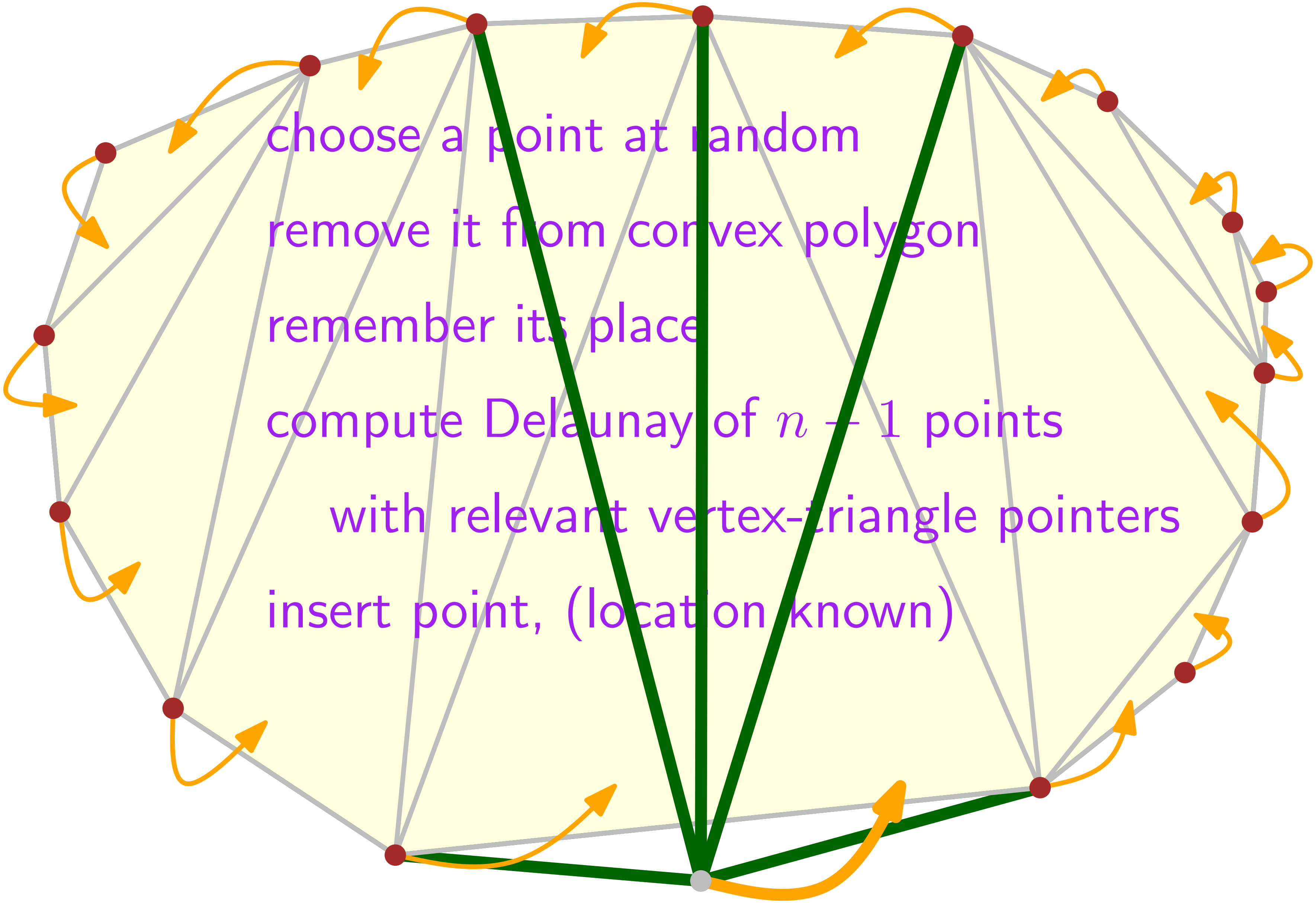
remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position



choose a point at random

remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

choose a point at random

remove it from convex polygon

remember its place

compute Delaunay of $n - 1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

choose a point at random $O(1)$ [model]

remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

  with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n - 1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known) $O(d^\circ p)$

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\}O(1)$$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known) $O(d^\circ p) = O(1)$

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers $f(n-1)$

insert point, (location known) $O(d^\circ p) = O(1)$

Delaunay of points in convex position

Analysis

choose a point at random

remove it from convex polygon

remember its place

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points
with relevant vertex-triangle pointers $f(n-1)$

insert point, (location known) $O(d^\circ p) = O(1)$

$$f(n) = f(n-1) + O(1)$$

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points
with relevant vertex-triangle pointers $f(n-1)$

insert point, (location known) $O(d^\circ p) = O(1)$

$$f(n) = f(n-1) + O(1) \; = O(n)$$

Delaunay of points in convex position

Analysis

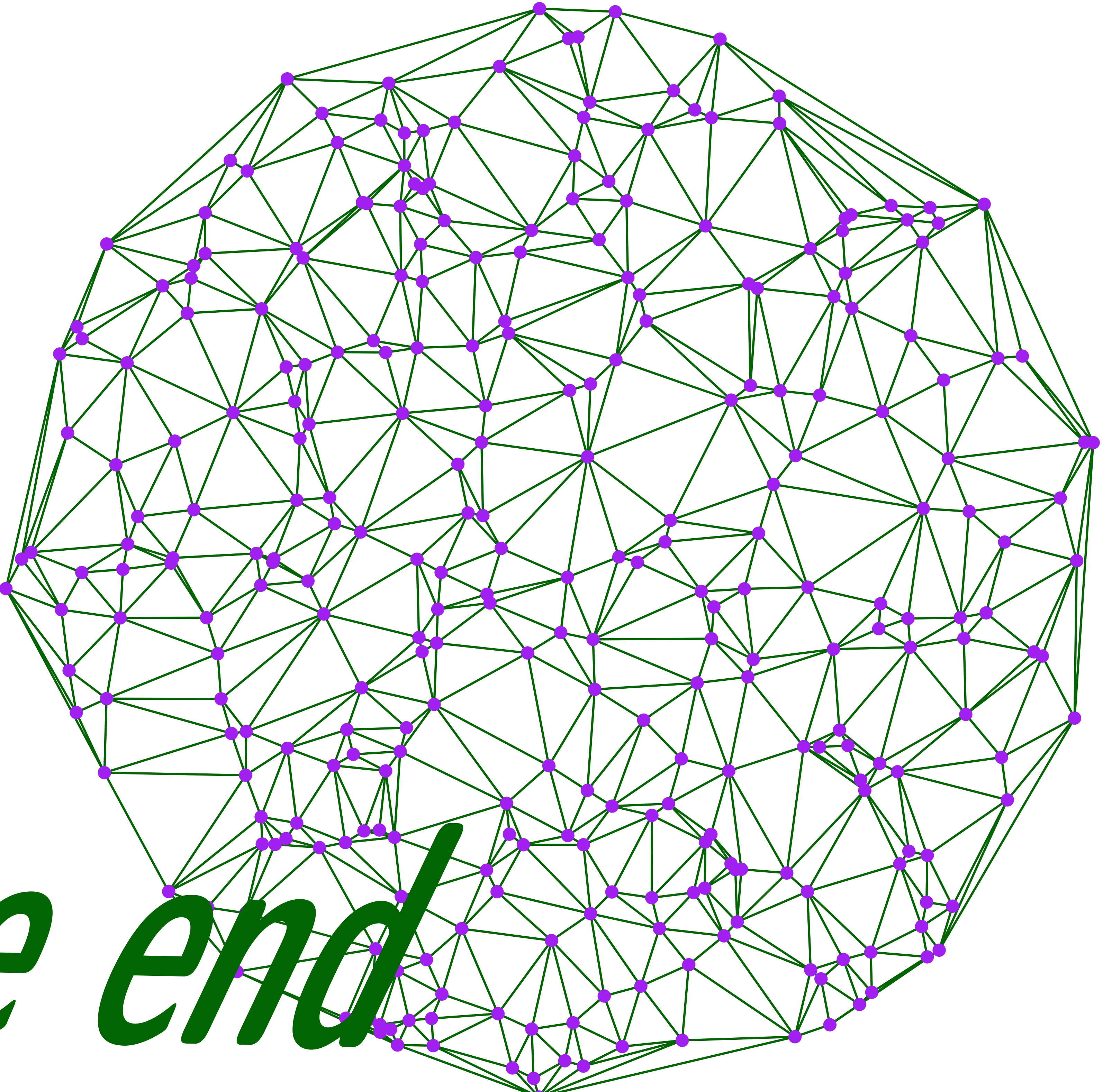
$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points
with relevant vertex-triangle pointers $f(n-1)$

insert point, (location known) $O(d^\circ p) = O(1)$

$$f(n) = f(n-1) + O(1) \;= O(n)$$

[Chew 86]



The end