Walking in Poisson Delaunay triangulations

Olivier Devillers

[Chenavier & D., 2018]
[de Castro & D., 2018]
[D. & Noizet, 2018]
Walking in Delaunay triangulations

Straight walk
Walking in Delaunay triangulations

Straight walk

2 - 2
Walking in Delaunay triangulations

Straight walk
Walking in Delaunay triangulations

Straight walk
Walking in Delaunay triangulations

Straight walk
Walking in Delaunay triangulations

Straight walk
Walking in Delaunay triangulations

Straight walk

Exit edge?

One orientation predicate
Walking in Delaunay triangulations

Straight walk

End of walk?

A second orientation predicate
Walking in Delaunay triangulations

Straight walk

Two orientation predicates per edge
Walking in Delaunay triangulations
Walking in Delaunay triangulations

Visibility walk
Walking in Delaunay triangulations

Visibility walk
Walking in Delaunay triangulations

Visibility walk

3 - 4
Walking in Delaunay triangulations

Visibility walk
Walking in Delaunay triangulations

Visibility walk

Triangle with two exits

One orientation predicate
Walking in Delaunay triangulations

Visibility walk

Triangle with one exit

1.5 orientation predicate

One predicate if this neighbor tried first

Two predicates if this neighbor tried first
Walking in Delaunay triangulations

Visibility walk

1.25 orientation predicate per edge?
Walking in Delaunay triangulations

How many edges crossed?

Straight walk

Visibility walk
Walking in Delaunay triangulations

How many edges crossed?

Straight walk

\[2n\]

Visibility walk

Worst case in a triangulation (non Delaunay)
Walking in Delaunay triangulations

How many edges crossed?

Straight walk

$2n$

Visibility walk

$\infty$

Worst case in a triangulation (non Delaunay)
Walking in Delaunay triangulations

How many edges crossed?

Straight walk

$$2n$$

Visibility walk

$$\infty \geq 2^{\frac{3}{n}}$$ randomized

Worst case in a triangulation (non Delaunay)
random choice

[D., Pion, & Teillaud 2002]
Walking in Delaunay triangulations

How many edges crossed?

Straight walk

$2n$

Visibility walk

$2n$

Worst case in a Delaunay triangulation
Not Delaunay

May loop
Green power < Red power

Power decreases
Walking in Delaunay triangulations

How many edges crossed?

Straight walk

\[ 2n \quad O(\sqrt{n}) \]

\[
\frac{64}{3\pi^2} \sqrt{n} + O\left(\frac{1}{\sqrt{n}}\right) \approx 2.16\sqrt{n}
\]

Visibility walk

\[ 2n \quad O(\sqrt{n}) \]

random

Worst case in a Delaunay triangulation

uniform in domain

Stretch in infinite Poisson distribution

[D. & Hemsley, 2016]

[D. & Hemsley, 2016]
Walking in Delaunay triangulations

Walk between vertices
Walking in Delaunay triangulations

Walk between vertices
Walking in Delaunay triangulations

Walk between vertices

Shortest path
Walking in Delaunay triangulations

Walk between vertices

Upper path

s

Upper path

5 - 4

5 - 4
Walking in Delaunay triangulations

Walk between vertices

Compass walk
Walking in Delaunay triangulations

Walk between vertices

Voronoï path
Walking in Delaunay triangulations

Walk between vertices

Voronoi path with shortcuts
Walking in Delaunay triangulations

Walk between vertices

Shortest path
Upper path
Compass walk
Voronoi path with shortcuts
Walking in Delaunay triangulations

Walk between vertices, worst case

Shortest path
Walking in Delaunay triangulations

Walk between vertices, worst case

Shortest path

Search this "subgraph"
Walking in Delaunay triangulations

Walk between vertices, worst case

Shortest path

Search this "subgraph"
Walking in Delaunay triangulations

Walk between vertices, worst case

Shortest path

Search this "subgraph"
Walking in Delaunay triangulations

Walk between vertices, worst case

Shortest path

Search this "subgraph"

Upper bound

Dobkin, Friedman, and Supowit 1987

Keil and Gutwin 1989

Xia 2011

5.08 \frac{1 + \sqrt{5}}{2} \pi

2.42 \frac{2\pi}{3 \cos(\pi/6)}

1.998
Walking in Delaunay triangulations

Walk between vertices, worst case

Shortest path

Search this "subgraph"

Lower bound

- Chew 1989: $1.5708 \frac{\pi}{2}$
- Bose, Devroye, Löffler, Snoeyink, & Verma 2011: 1.5846
- Xia & Zhang 2011: 1.5932
Walking in Delaunay triangulations

Walk between vertices, worst case

Voronoi path
Unbounded

\[ 7 - 1 \]
Walking in Delaunay triangulations

Walk between vertices, worst case

Voronoï path

Unbounded
Walking in Delaunay triangulations

Walk between vertices, worst case

Voronoi path Unbounded
Walking in Delaunay triangulations

Walk between vertices, worst case

Upper path

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Walking in Delaunay triangulations

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Walking in Delaunay triangulations

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Upper path Unbounded
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Walk between vertices, worst case

Compass walk

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[Bose & Morin 2004]
Walking in Delaunay triangulations

Walk between vertices, worst case

Compass walk

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[Bose & Morin 2004]
Walking in Delaunay triangulations

Walk between vertices, worst case

Compass walk

Unbounded

$\alpha - \epsilon$

$\alpha - 2\epsilon$

[Bose & Morin 2004]
Walking in Delaunay triangulations

Walk between vertices, worst case

Compass walk
Unbounded

\[ \alpha - \epsilon \]
\[ \alpha - 2\epsilon \]

[Bose & Morin 2004]
Walking in Delaunay triangulations

Walk between vertices
Walking in Delaunay triangulations

Expected length (experiments)

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# Walking in Delaunay triangulations

## Expected length (experiments)

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References:

- [Chenavier & D., 2018]
- [D. & Noizet, 2018]
- [Chenavier & D., 2018]
- [Baccelli et al., 2000]
Walking in Delaunay triangulations

Expected length (experiments)             theory

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[Baccelli et al., 2000]
Expected length of upper path

Poisson Delaunay triangulation, rate $n$

$$\mathbb{E}[\text{length}] = \mathbb{E}\left[ \sum_{\text{triangle} \in X_n^3} 1_{\text{triangle is Delaunay}} 1_{\text{first edge above } st} \text{length(first edge)} \right]$$
\[ \mathbb{E}[\text{length}] = \mathbb{E} \left[ \sum_{\text{triangle} \in X_n^3} \mathbb{1}_{[\text{triangle is Delaunay}]} \mathbb{1}_{[\text{first edge above } st]} \text{length(first edge)} \right] \]

\[ = n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P}[\text{triangle is Delaunay}] \mathbb{1}_{[\text{first edge above } st]} \text{length(first edge)} \, d\text{triangle} \]

\text{Slivnyak-Mecke}
\[ E[\text{length}] = \mathbb{E} \left[ \sum_{\text{triangle} \in X_n^3} \mathbb{1}_{[\text{triangle is Delaunay}] \mathbb{1}_{[\text{first edge above st}]}} \text{length(} \text{first edge} \text{)} \right] \]

\[ = n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P}[\text{triangle is Delaunay}] \mathbb{1}_{[\text{first edge above st}]} \text{length(} \text{first edge} \text{)} \, dt_{\text{triangle}} \]

\[ = n^3 \int_{r=0}^{\infty} \int_{x_z=0}^{1} \int_{y_z=-r}^{r} \int_{\alpha_3=0}^{\alpha_1} \int_{\alpha_2=0}^{\alpha_2} e^{-n\pi r^2} \mathbb{1}_{[\text{first edge above st}]} 2r \sin \frac{\alpha_1 - \alpha_2}{2} r^3 2A(triangle) \, d\alpha_1 \, d\alpha_2 \, dy_z \, dx_z \, dr \]

Blaschke-Petkantschin
Expected length of upper path

Poisson Delaunay triangulation, rate $n$

\[
\mathbb{E} [\text{length}] = \mathbb{E} \left[ \sum_{\text{triangle} \in \mathcal{X}_n^3} \mathbb{1}_{\text{triangle is Delaunay}} \mathbb{1}_{\text{first edge above } st} \text{length(first edge)} \right]
\]

\[
= n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P} [\text{triangle is Delaunay}] \mathbb{1}_{\text{first edge above } st} \text{length(first edge)} \, dt_{\text{triangle}}
\]

\[
= n^3 \int_{r=0}^{\infty} \int_{x=0}^{1} \int_{y=-r}^{r} \int_{[0,2\pi]} e^{-n\pi r^2} \mathbb{1}_{\text{first edge above } st} 2r \sin \frac{\alpha_1 - \alpha_2}{2} r^3 A(\text{triangle}) d\alpha_1 \, dy \, dx \, dr
\]

\[
= 4n^3 \left( \int_{r=0}^{\infty} e^{-nr^2} r^5 \, dr \right) \cdot \left( \int_{h=\frac{y}{r}=-1}^{1} \int_{[0,2\pi]} \mathbb{1}_{\text{first edge above } st} \sin \frac{\alpha_1 - \alpha_2}{2} A(\text{triangle}) d\alpha_1 \, dh \right)
\]
\[ \mathbb{E} [\text{length}] = \mathbb{E} \left[ \sum_{\text{triangle} \in X_n^3} 1_{[\text{triangle is Delaunay]}} 1_{[\text{first edge above } st]} \text{length(first edge)} \right] \]

\[ = n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P} [\text{triangle is Delaunay}] 1_{[\text{first edge above } st]} \text{length(first edge)} \, dt_{\text{triangle}} \]

\[ = n^3 \int_{r=0}^{\infty} \int_{x_z=0}^{1} \int_{y_z=-r}^{r} \int_{[0,2\pi]^3} e^{-n\pi r^2} 1_{[\text{first edge above } st]} 2r \sin \frac{\alpha_1 - \alpha_2}{2} r^3 2A(\text{triangle}) d\alpha_1 d\alpha_2 dy_z dx_z dr \]

\[ = 4n^3 \left( \int_{r=0}^{\infty} e^{-n\pi r^2} r^5 dr \right) \cdot \left( \int_{h=\frac{\pi}{r}=-1}^{1} \int_{[0,2\pi]^3} 1_{[\text{first edge above } st]} \sin \frac{\alpha_1 - \alpha_2}{2} A(\text{triangle}) d\alpha_1 dh \right) \]

\[ = 4n^3 \cdot \frac{1}{\pi^3 n^3} \cdot \frac{35\pi}{12} = \frac{35}{3\pi^2} \]
## Walking in Delaunay triangulations

**Expected length (experiments)**

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Walking in Delaunay triangulations

Shortest path

$E[\text{length}(\text{shortest path})]$
Walking in Delaunay triangulations

Shortest path

\[
\mathbb{E}[\text{length}(\text{shortest path})] \quad \xrightarrow{\text{density } \to \infty} \quad \text{limit}
\]
Walking in Delaunay triangulations

Shortest path

\[ \mathbb{E}[\text{length}(\text{shortest path})] \quad \text{limit} \]

by subadditivity

remove start and target from point set

look at shortest path between closest neighbor

\[ \neq \text{between path from start to target negligible} \]
Bad edge = almost horizontal edge
Many bad edges $\iff$ length close to 1
$P[bad] = \text{small constant}$
difficult dependencies to handle
Many bad cells ⇐ length close to 1
Still dependencies
Make a grid
Make a grid
Make a grid

If $E[\# \in cell] = \text{constant}$

$\# \text{ possible paths} = 4^n$ (too big)
Make a grid

If $\mathbb{E}[\# \in cell] = \text{constant}$

$\# \text{ possible paths} = 4^n$ (too big)

big cells $\sqrt{n} \times \sqrt{n}$  $4\sqrt{n} \times n$
A good cell?

No Delaunay circles go outside
A good cell?

No Delaunay circles go outside

No short path from left to right
A good cell?

No Delaunay circles go outside

No short path from left to right

Not enough edges to make a short path
A good cell?

No Delaunay circles go outside

No short path from left to right

Not enough edges to make a short path

Choose \# points in cell,

Choose what "short path" means

\[
P \left[ \text{length} \geq 1 + 2.5 \times 10^{-11} \right] \leq O \left( \frac{1}{\sqrt{n}} \right)
\]
## Walking in Delaunay triangulations

### Expected length (experiments) vs. theory

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- **References:**
  - [Baccelli et al., 2000](#)
  - [De Castro & D., 2018](#)
  - [D. & Noizet, 2018](#)
  - [Chenavier & D., 2018](#)
Walking in Delaunay triangulations

Voronoi path 1.27 $\frac{4}{\pi} \simeq 1.27$

[Baccelli et al., 2000]
Walking in Delaunay triangulations

Voronoi path

\[ \frac{4}{\pi} \approx 1.27 \]

Voronoi path in higher dimension

[Baccelli et al., 2000]
Walking in Delaunay triangulations

Voronoi path

$\frac{4}{\pi} \simeq 1.27$

[Baccelli et al., 2000]

$$E[\ell(VPx)] = \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int (S_{d-1})^2 P[B((x,0,...,0), r) \cap X = \emptyset] 1[(x,0,...,0)\in[st]]$$

$$\cdot r||u_1u_2|| \det(J\Phi)|d\alpha_1,1 \ldots d\alpha_1,d-1d\alpha_2,1 \ldots d\alpha_2,d-1drdx$$

Integral form
Walking in Delaunay triangulations

Voronoi path

\[\frac{4}{\pi} \approx 1.27\]

[Baccelli et al., 2000]

\[
\mathbb{E}[\ell(VPX)] = \frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int (S_{d-1})^2 \mathbb{P}[B((x,0,...,0), r) \cap X = \emptyset] \mathbb{1}_{[(x,0,...,0) \in [st]]} \cdot r \|u_1 u_2\| |\text{det}(J\Phi)| \, d\alpha_1,1 ... d\alpha_{1,d-1} d\alpha_{2,1} ... d\alpha_{2,d-1} dr \, dx
\]

Integral form

Use Taylor expansion to be able to integrate
Walking in Delaunay triangulations

Voronoi path

\[ \frac{4}{\pi} \simeq 1.27 \]

[Baccelli et al., 2000]

\[
\frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5} d}{\pi^2 (2d - 2)!} \left(1 - \frac{d - 1}{4d^2 - 1}\right) \sqrt{2} \leq \mathbb{E}[\ell(VP_x)] \leq \frac{\Gamma\left(\frac{d}{2}\right)^4 2^{4d-5} d}{\pi^2 (2d - 2)!} \left(1 + \frac{1}{4d - 2}\right) \sqrt{2}
\]
Walking in Delaunay triangulations

Voronoi path

\[ \frac{4}{\pi} \simeq 1.27 \]

[\text{Baccelli et al., 2000}]

asymptotic behavior between

\[ \sqrt{\frac{2d}{\pi}} \]

\[ - \frac{1}{4\sqrt{2d\pi}} + O(d^{3/2}) \]

\[ + \frac{3}{4\sqrt{2d\pi}} + O(d^{3/2}) \]
Walking in Delaunay triangulations

Voronoi path

\[ \frac{4}{\pi} \approx 1.27 \]  

[Baccelli et al., 2000]

asymptotic behavior

\[ \sqrt{\frac{2d}{\pi}} \]
Walking in Delaunay triangulations

Voronoi path

\[ \frac{4}{\pi} \approx 1.27 \]

[Baccelli et al., 2000]

<table>
<thead>
<tr>
<th>(d)</th>
<th>(k)</th>
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<th>(\approx)</th>
<th>correct value</th>
<th>upper bound</th>
<th>(\approx)</th>
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<tbody>
<tr>
<td>3</td>
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<td>1.49770</td>
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</tr>
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<td>4</td>
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<td>(\sqrt{2} \cdot \frac{121774997}{10270260})</td>
<td>1.6990</td>
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<tr>
<td>5</td>
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<td>(\sqrt{2} \cdot \frac{135}{104})</td>
<td>1.8357</td>
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<td>7</td>
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numerical integration
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**References:**

- Baccelli et al., 2000
- Chenavier & D., 2018
- Ongoing work
Walking in Delaunay triangulations
Compass walk

Complicated dependencies

Stretch factor
Walking in Delaunay triangulations

Compass walk

Complicated dependencies

Stretch factor

$$\frac{\ell_1 + \ell_2 + \ldots \ell_k}{x_1 + x_2 + \ldots x_k}$$

edge lengths

length of horizontal projections

21 - 2
Walking in Delaunay triangulations

Compass walk

Complicated dependencies

\[
\text{Stretch factor} = \frac{\ell_1 + \ell_2 + \ldots + \ell_k}{x_1 + x_2 + \ldots + x_k}
\]

If independent:

\[
\frac{\mathbb{E}[\ell]}{\mathbb{E}[x]} = \frac{175\pi}{512} \approx 1.0738
\]
Walking in Delaunay triangulations

Compass walk

Complicated dependencies

Stretch factor \[ \frac{\ell_1 + \ell_2 + \ldots + \ell_k}{x_1 + x_2 + \ldots + x_k} \]

If independent:

\[ \frac{\mathbb{E} [\ell]}{\mathbb{E} [x]} = \frac{175\pi}{512} \approx 1.0738 \]

Experimental: 1.06777

There is a positive bias
Walking in Delaunay triangulations

Compass walk

Complicated dependencies

Stretch factor

If independent:
\[
\frac{E[\ell]}{E[x]} = \frac{175\pi}{512} \approx 1.0738
\]

Experimental:
\[
\frac{E[\ell]}{E[x]} \approx \frac{1.061}{0.988} \approx 1.0738
\]

Experimental: 1.06777

There is a positive bias
Walking in Delaunay triangulations

Compass walk

Experimental: \[
\frac{E[\ell]}{E[x]} \approx \frac{1.061}{0.988} \approx 1.0738
\]

at depth 1
Experimental: \[
\frac{E[\ell]}{E[x]} \approx \frac{1.116}{1.045} \approx 1.0681
\]

at depth 2
Experimental: \[
\frac{E[\ell]}{E[x]} \approx \frac{1.117}{1.046} \approx 1.0678
\]

Com complicated dependencies

Stretch factor

If independent: \[
\frac{E[\ell]}{E[x]} = \frac{175\pi}{512} \approx 1.0738
\]

Experimental: 1.06777

There is a positive bias
## Walking in Delaunay triangulations

<table>
<thead>
<tr>
<th>Path Type</th>
<th>Expected Length (Experiments)</th>
<th>Expected Length (Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclidean length</td>
<td>1</td>
<td>$\geq 1 + 10^{-11}$</td>
</tr>
<tr>
<td>Shortest path</td>
<td>1.04</td>
<td>1.16</td>
</tr>
<tr>
<td>Compass walk</td>
<td>1.07</td>
<td>$\frac{35}{3\pi^2} \approx 1.18$</td>
</tr>
<tr>
<td>Shortened V. path</td>
<td>1.16</td>
<td>$\frac{4}{\pi} \approx 1.27$</td>
</tr>
<tr>
<td>Upper path</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>Voronoi path</td>
<td>1.27</td>
<td></td>
</tr>
</tbody>
</table>

References:
- [Baccelli et al., 2000]
- [D. & Noizet, 2018]
- [Chenavier & D., 2018]
Walking in Delaunay triangulations

Expected length (experiments)       theory

- Euclidean length                  1
- Shortest path                     1.04
- Compass walk                      1.07
- Shortened V. path                 $\geq 1 + 10^{-11}$
- Upper path                        [Chenavier & D., 2018]
- Voronoi path                      &ACEMENT

Locally defined path

"easy" to analyze

(computation may be difficult)
Walking in Delaunay triangulations

Expected length (experiments)  

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Incrementally defined path dependency issues
the tree can be analyzed

[D. & Noizet, 2018]

[Chenavier & D., 2018]
Walking in Delaunay triangulations

Expected length (experiments)

Euclidean length
Shortest path
Compass walk
Shortened V. path
Upper path
Voronoi path

Theory

Expected length (theory)

Euclidean length

1

Incrementally defined path
also dependency issues
no idea about tight bounds

[Baccelli et al., 2000]

[Chenavier & D., 2018]

[D. & Noizet, 2018]
Thank you