Probability and Delaunay triangulations

Randomized algorithms for Delaunay triangulations

Poisson Delaunay triangulation

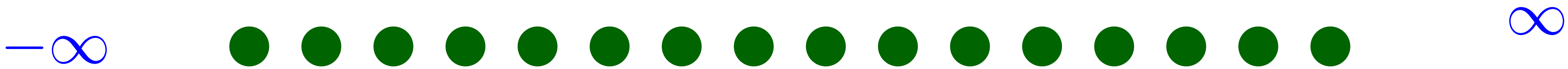
Randomized algorithms for Delaunay triangulations

- Randomized backward analysis of binary trees
- Randomized incremental construction of Delaunay
- Jump and walk
- The Delaunay hierarchy
- Biased randomized incremental order
- Chew algorithm for convex polygon

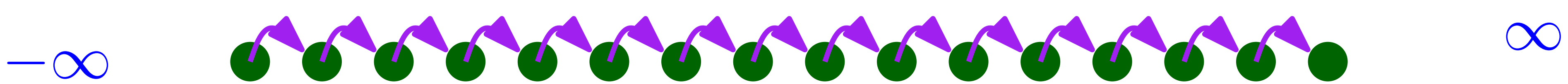
Poisson Delaunay triangulation

- Poisson distribution
- Slivnyak-Mecke formula
- Blaschke-Petkanschin variables substitution
- Stupid analysis of the expected degree
- Straight walk expected analysis
- Catalog of properties

Sorting



Sorting

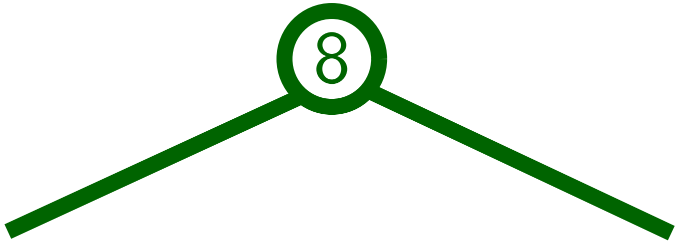


Sorting

Binary tree

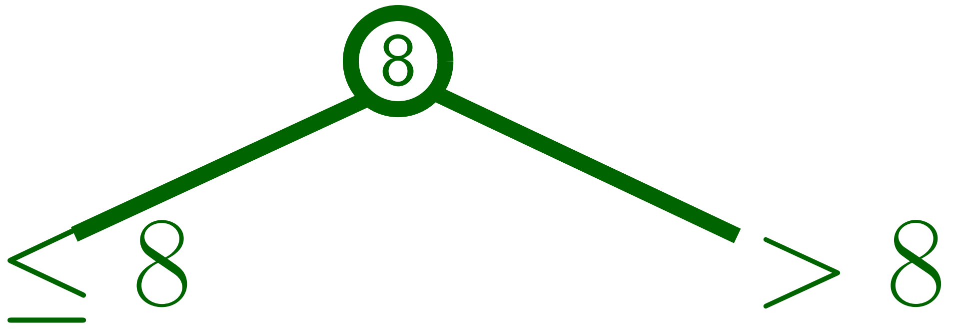
Sorting

Binary tree



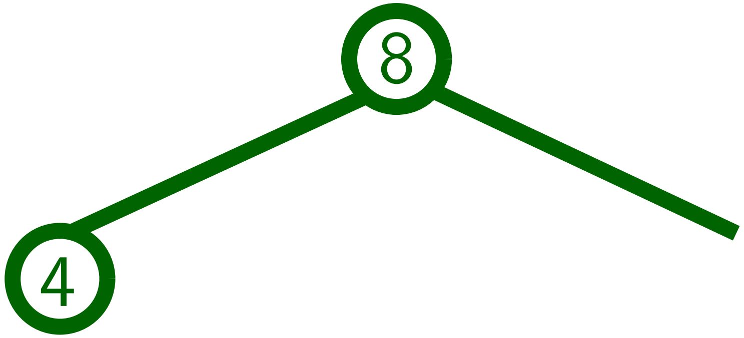
Sorting

Binary tree



Sorting

Binary tree



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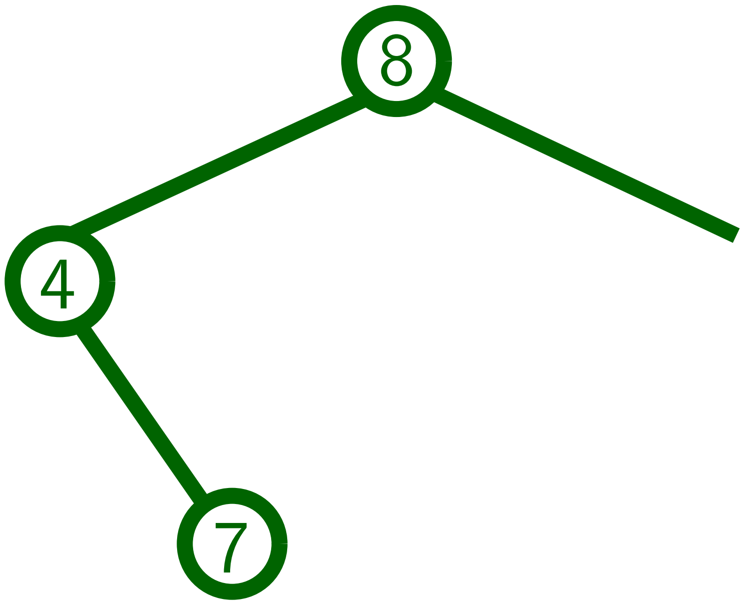
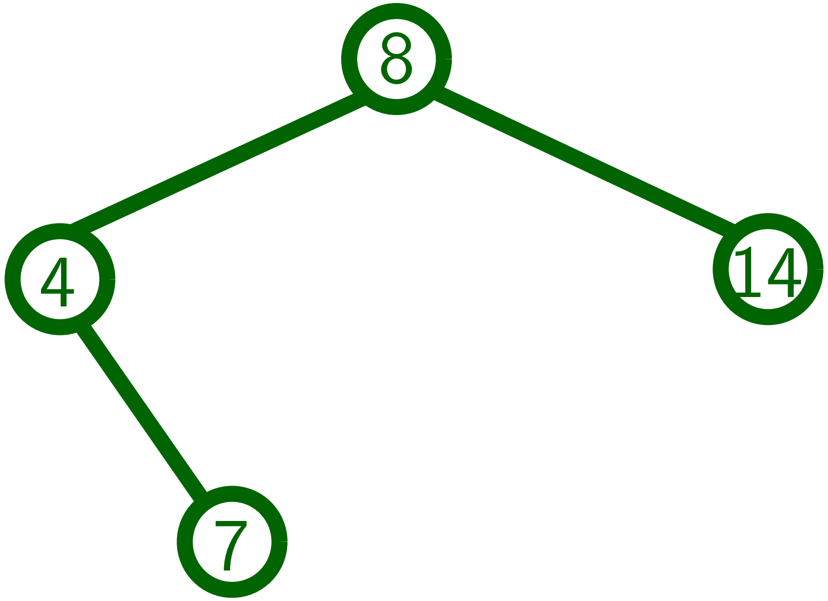
Binary tree



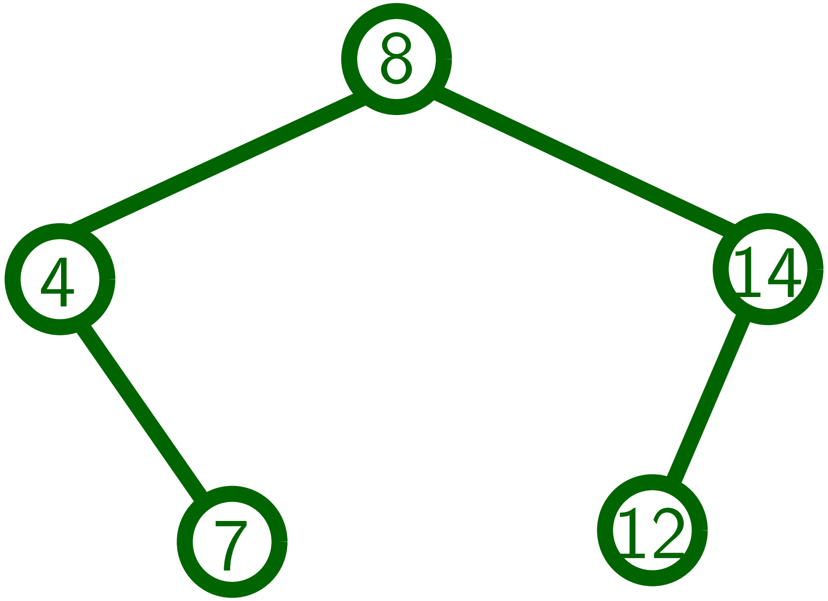
Sorting

Binary tree



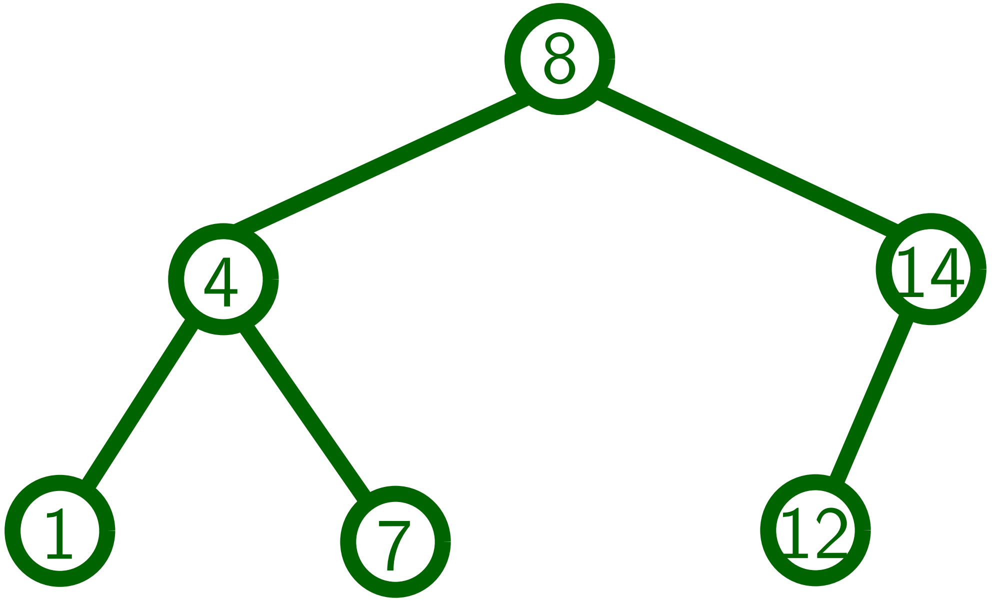
Sorting

Binary tree



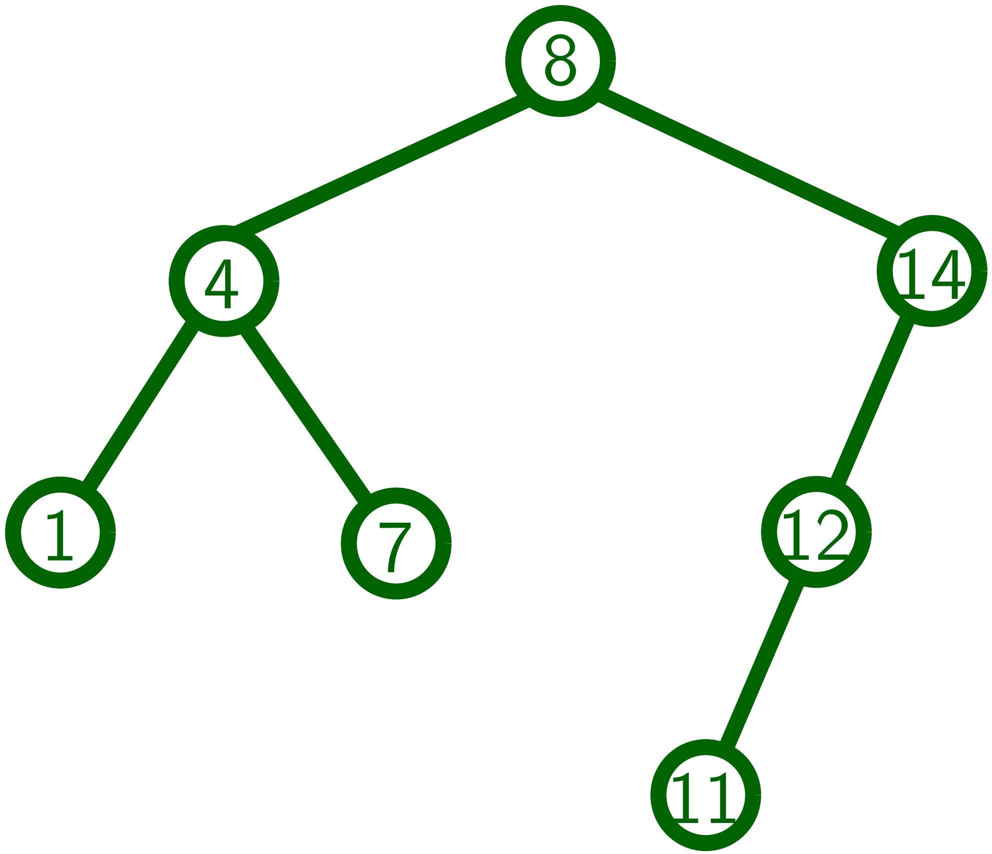
Sorting

Binary tree



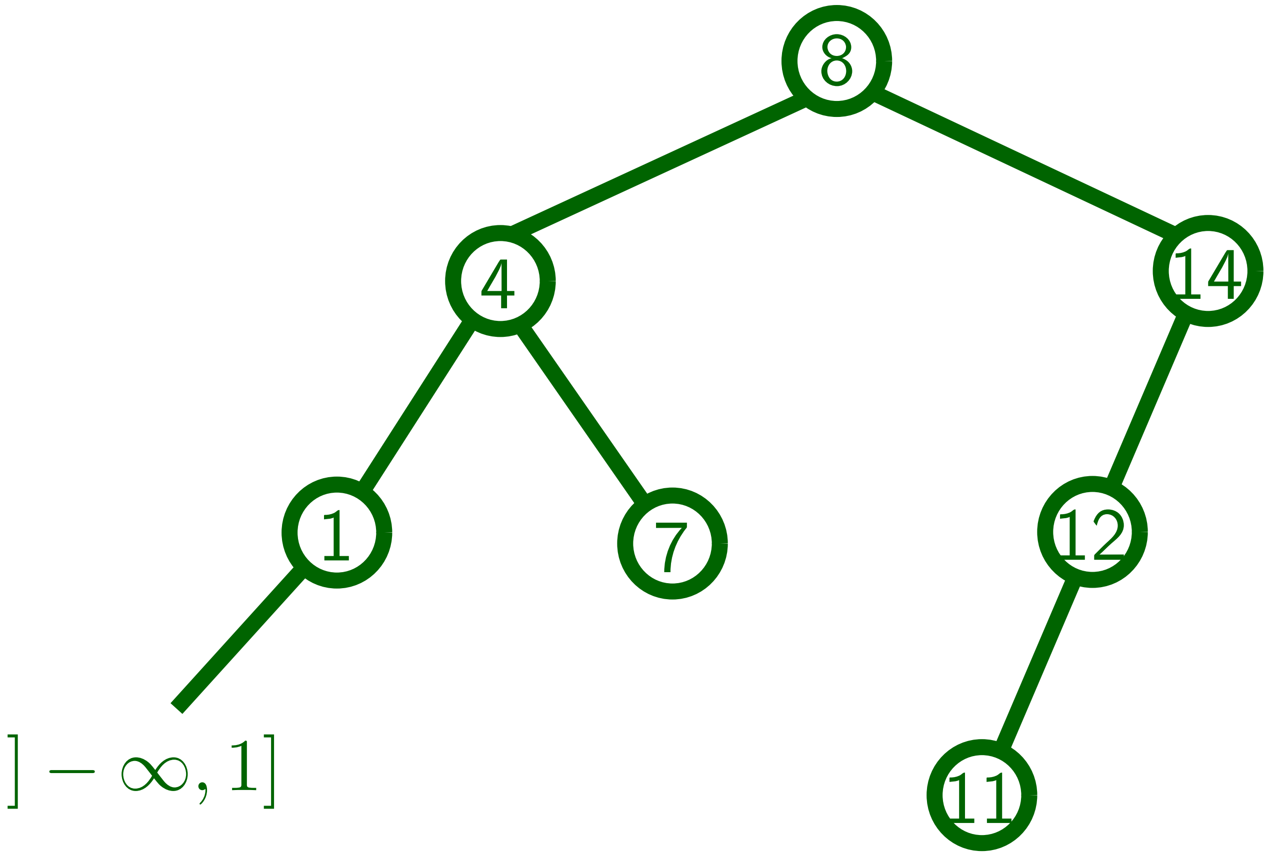
Sorting

Binary tree



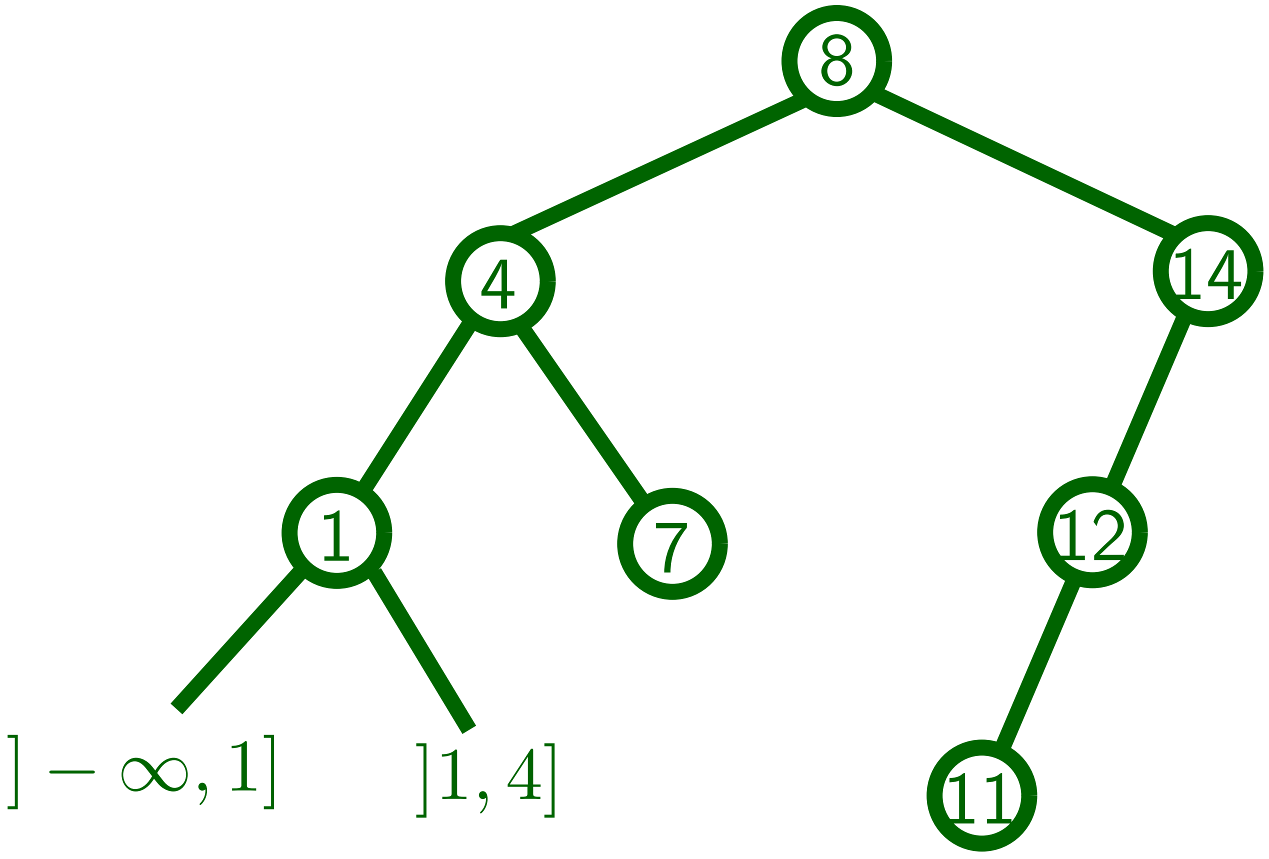
Sorting

Binary tree



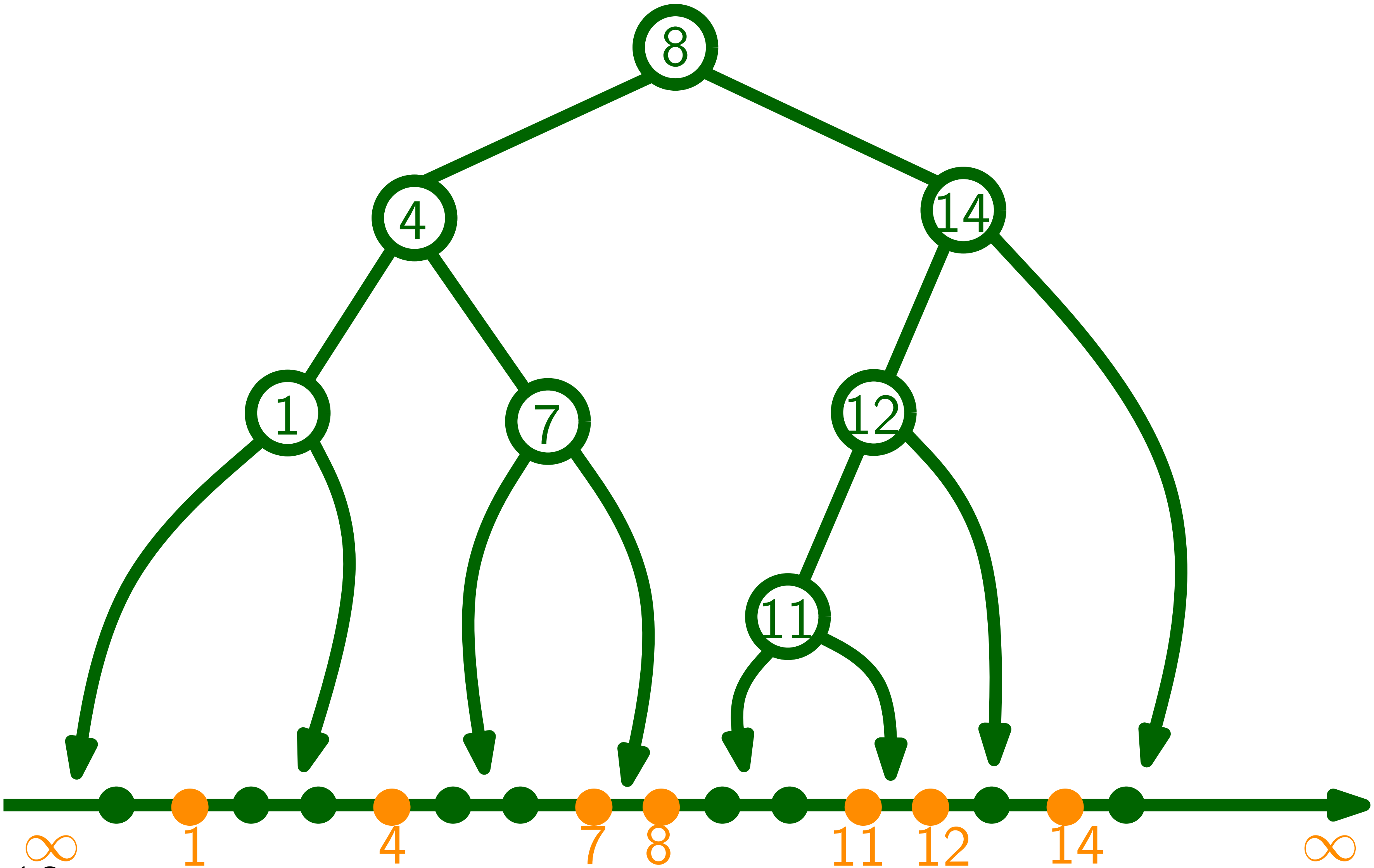
Sorting

Binary tree



Sorting

Binary tree



Sorting

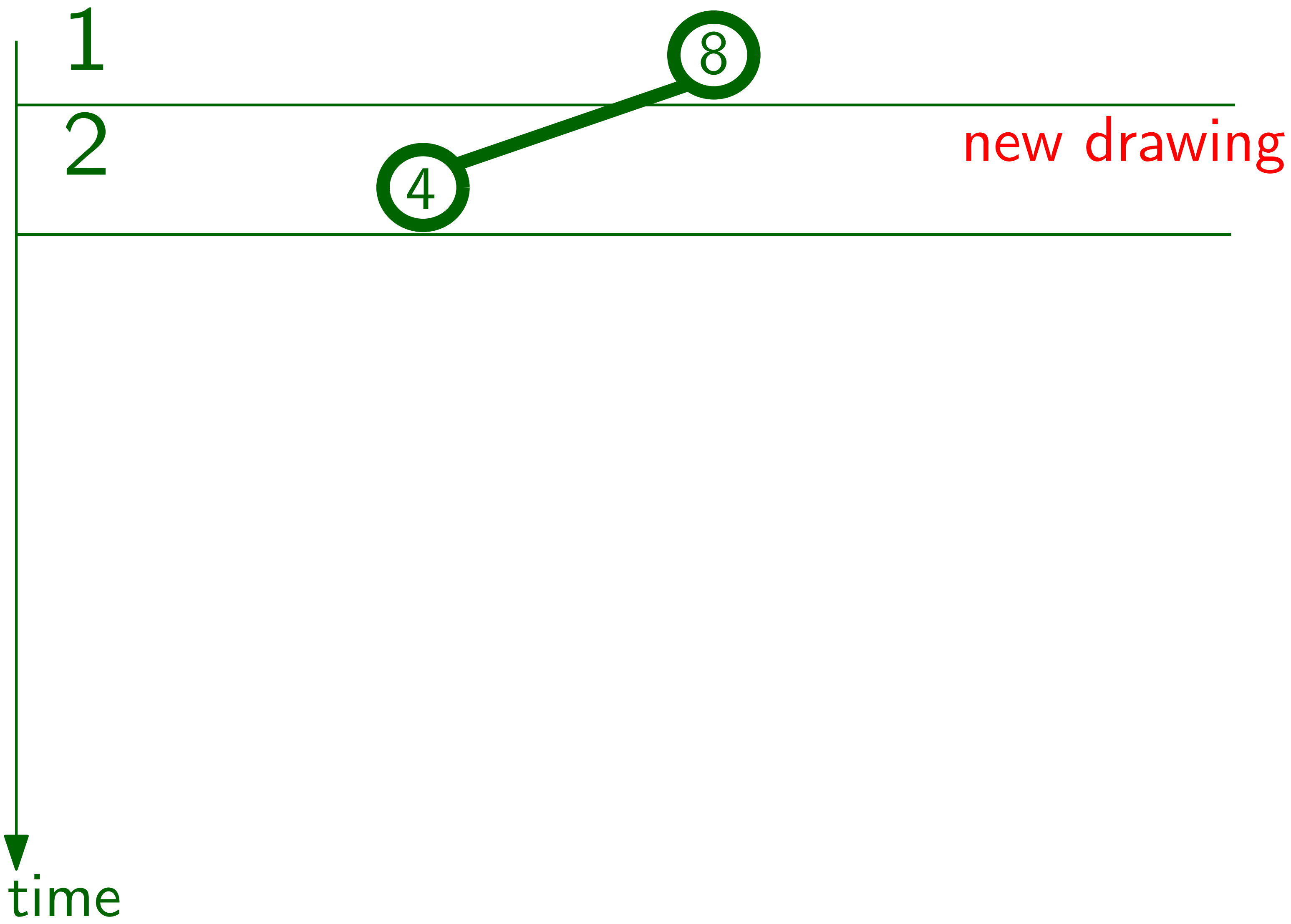
1

8

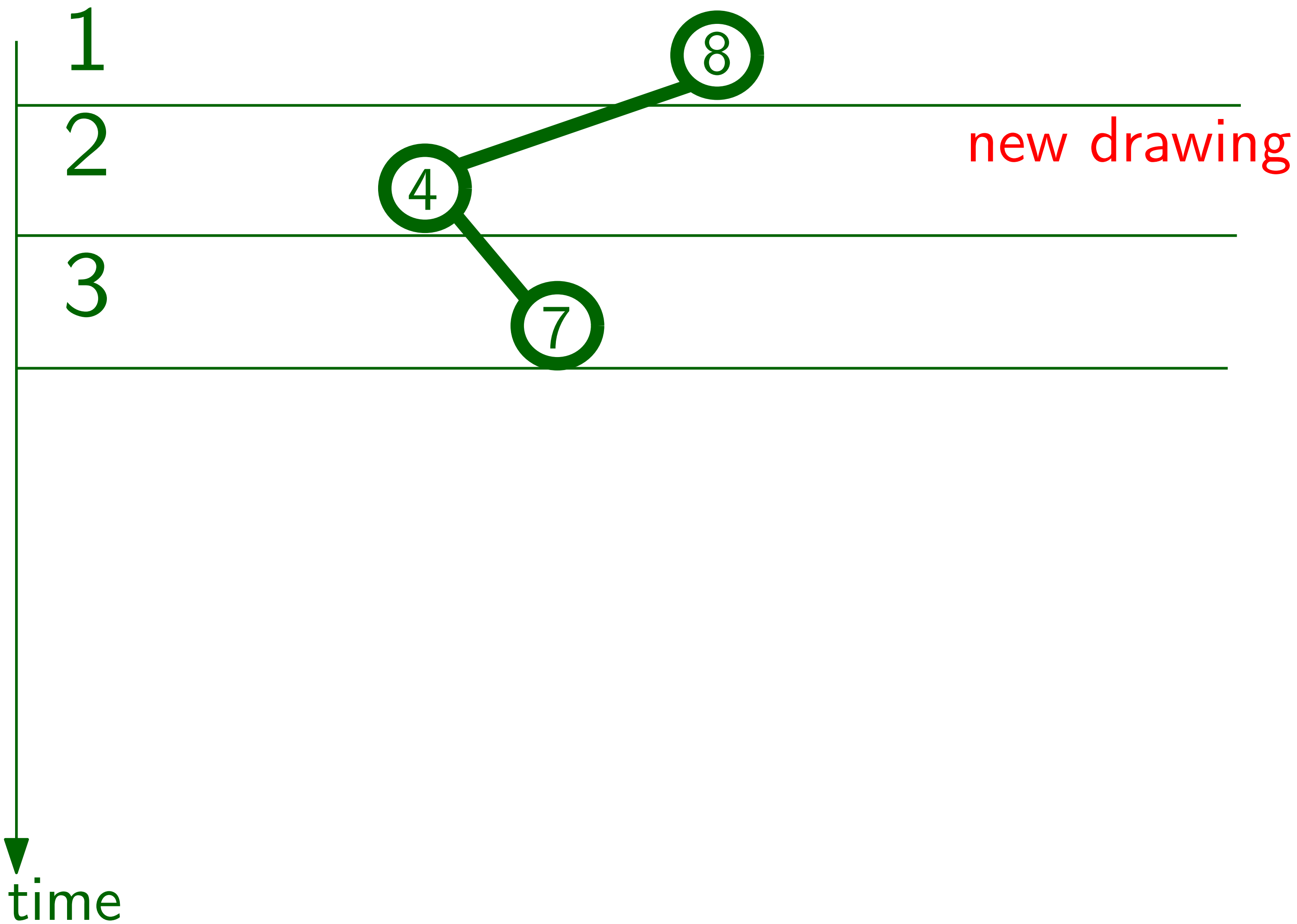
new drawing

time

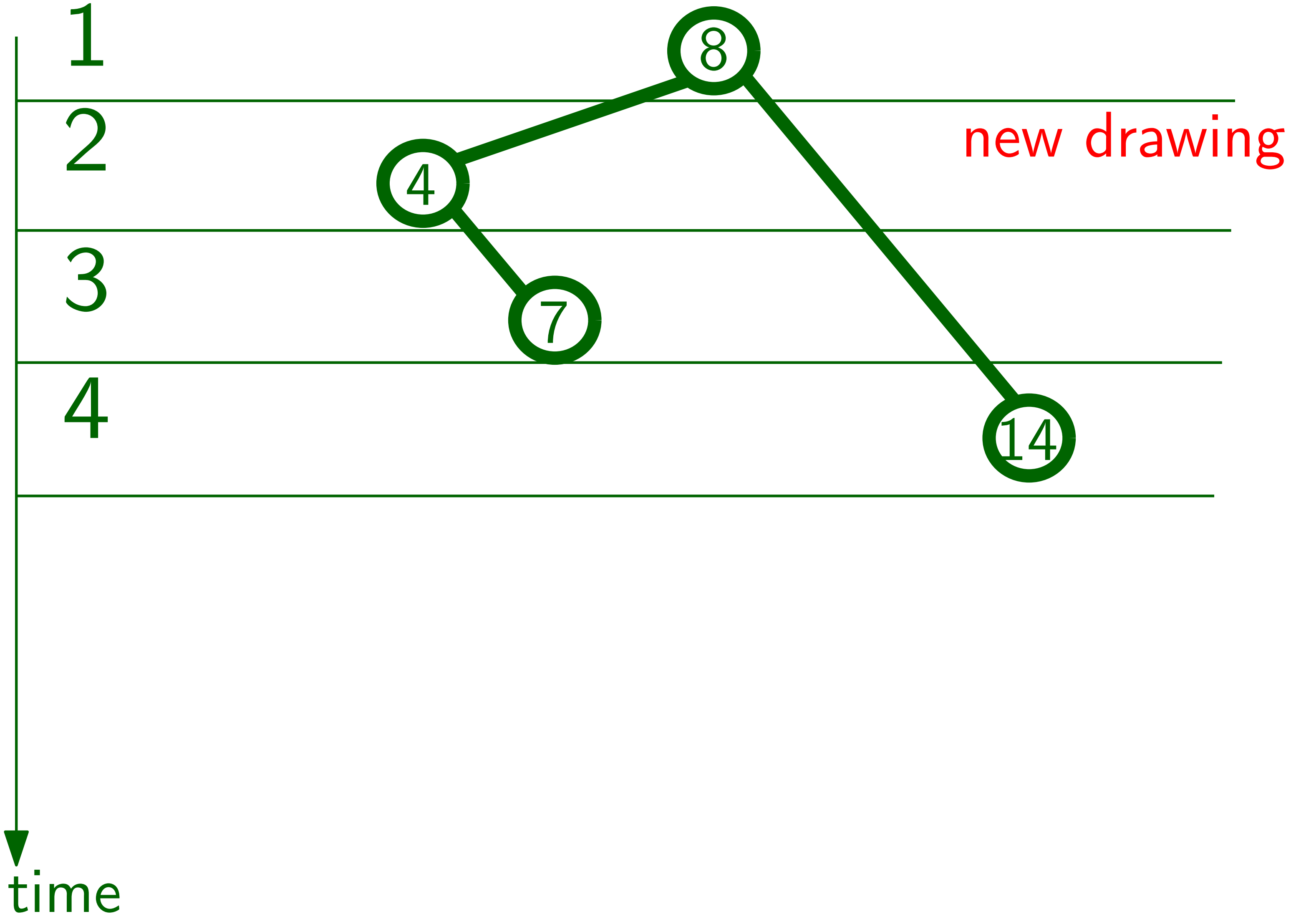
Sorting



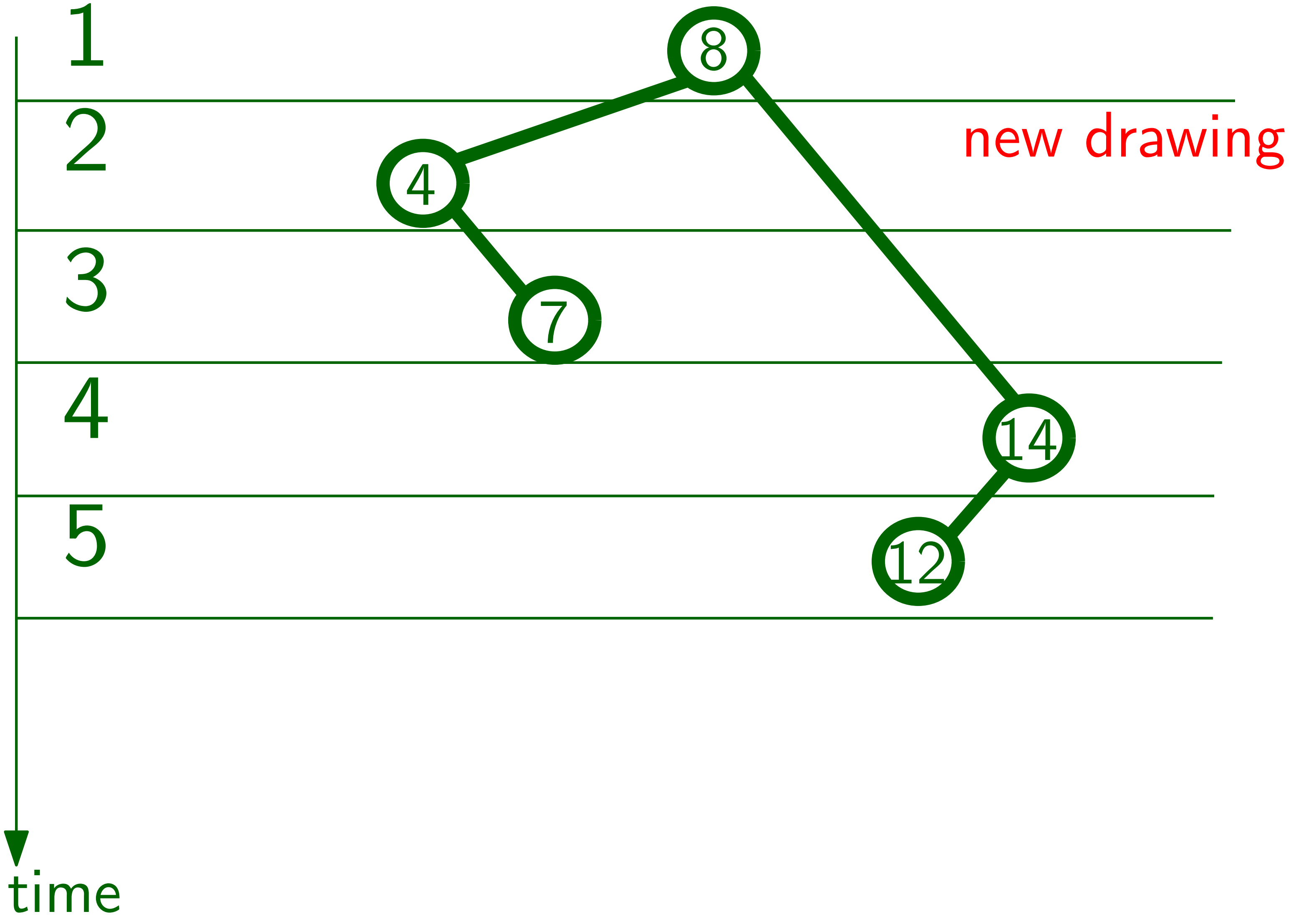
Sorting



Sorting



Sorting



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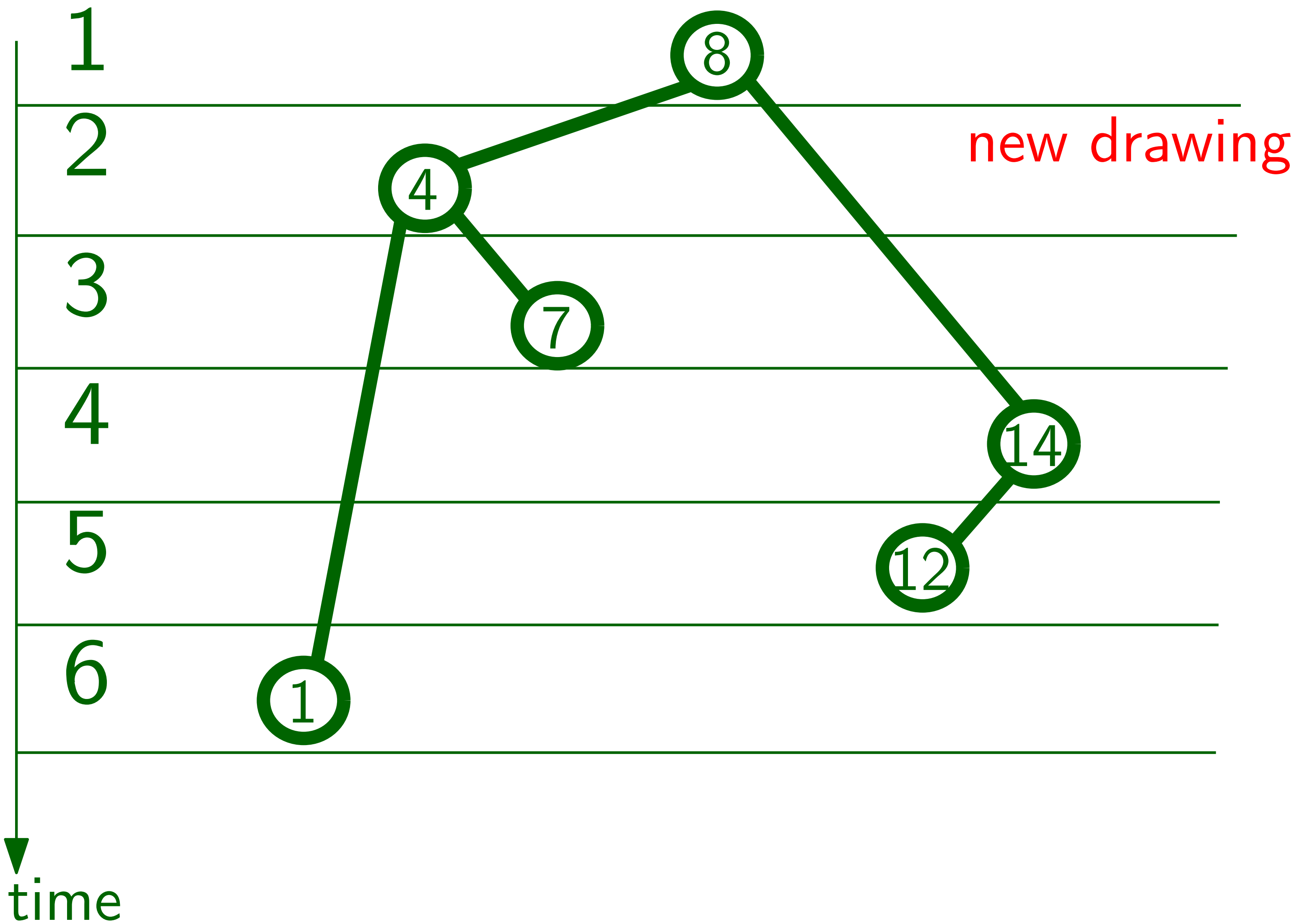
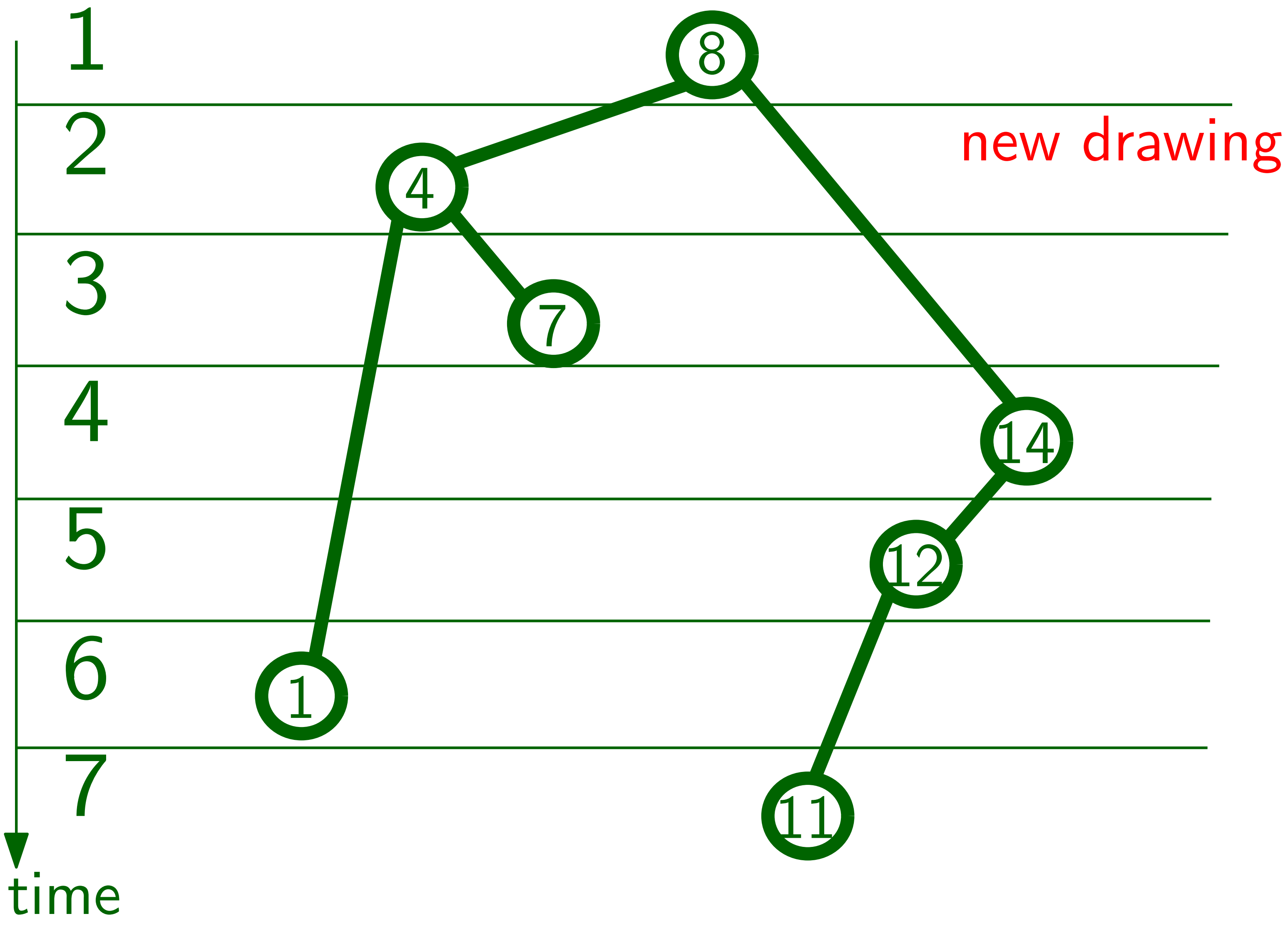
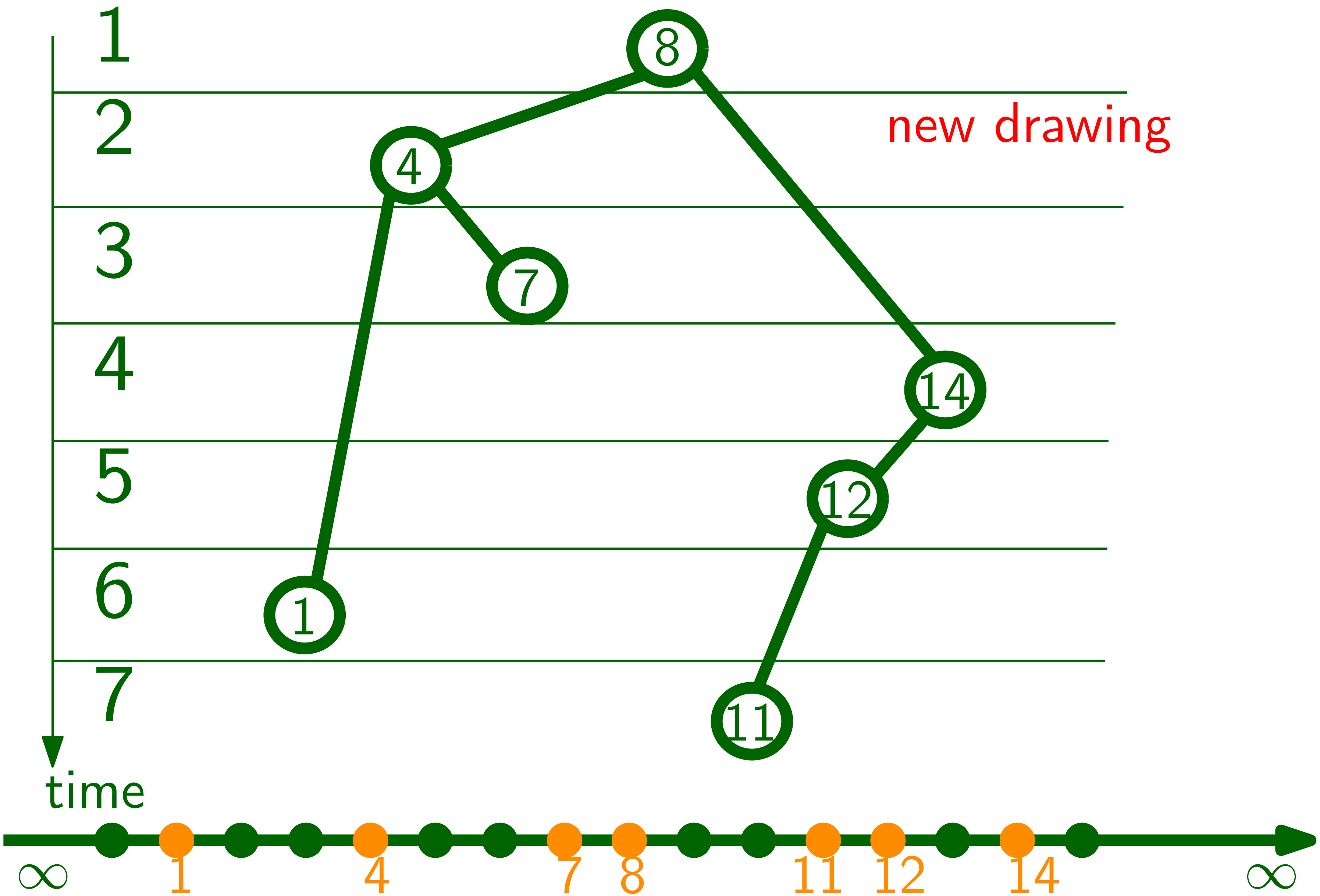


Sorting



Sorting



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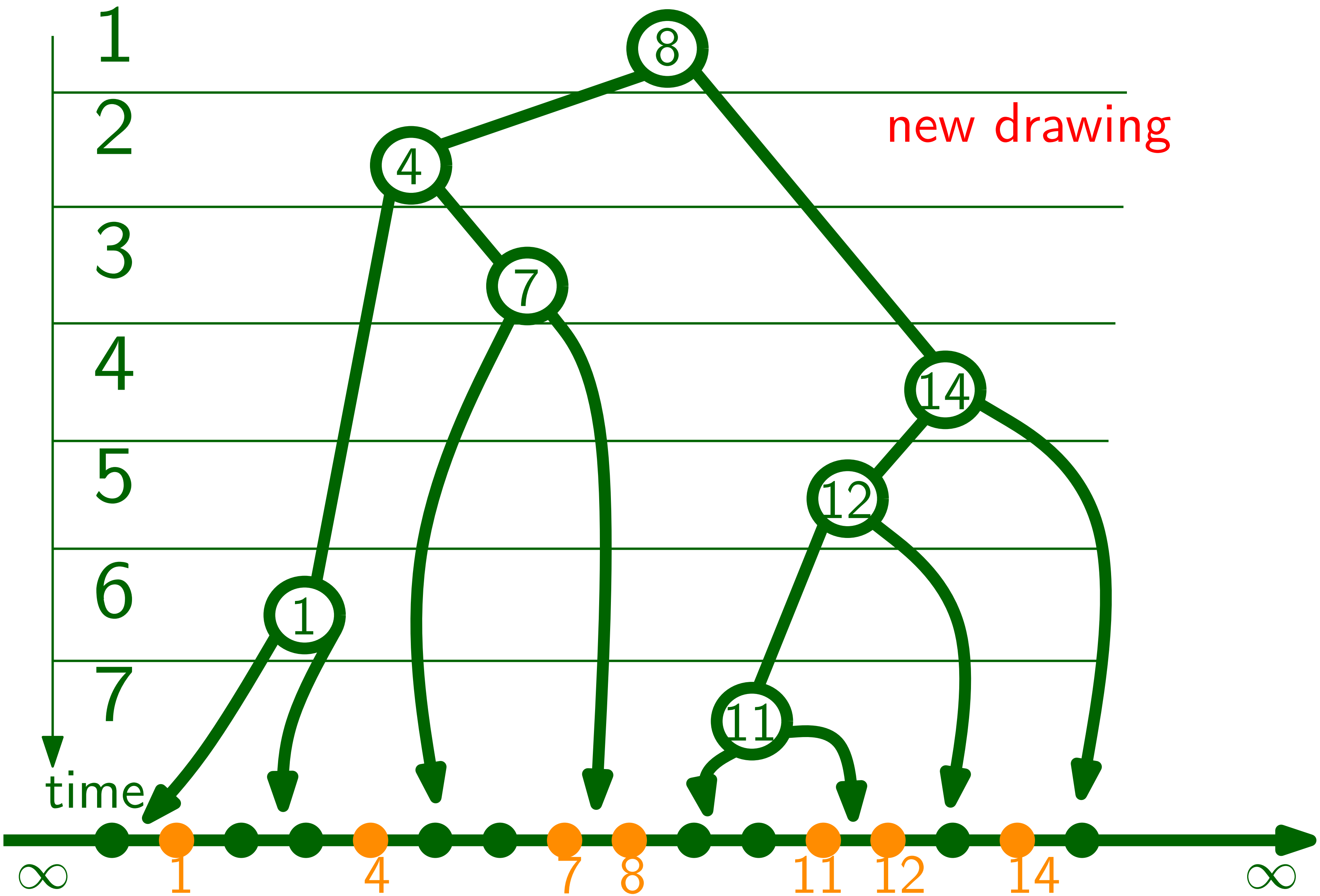

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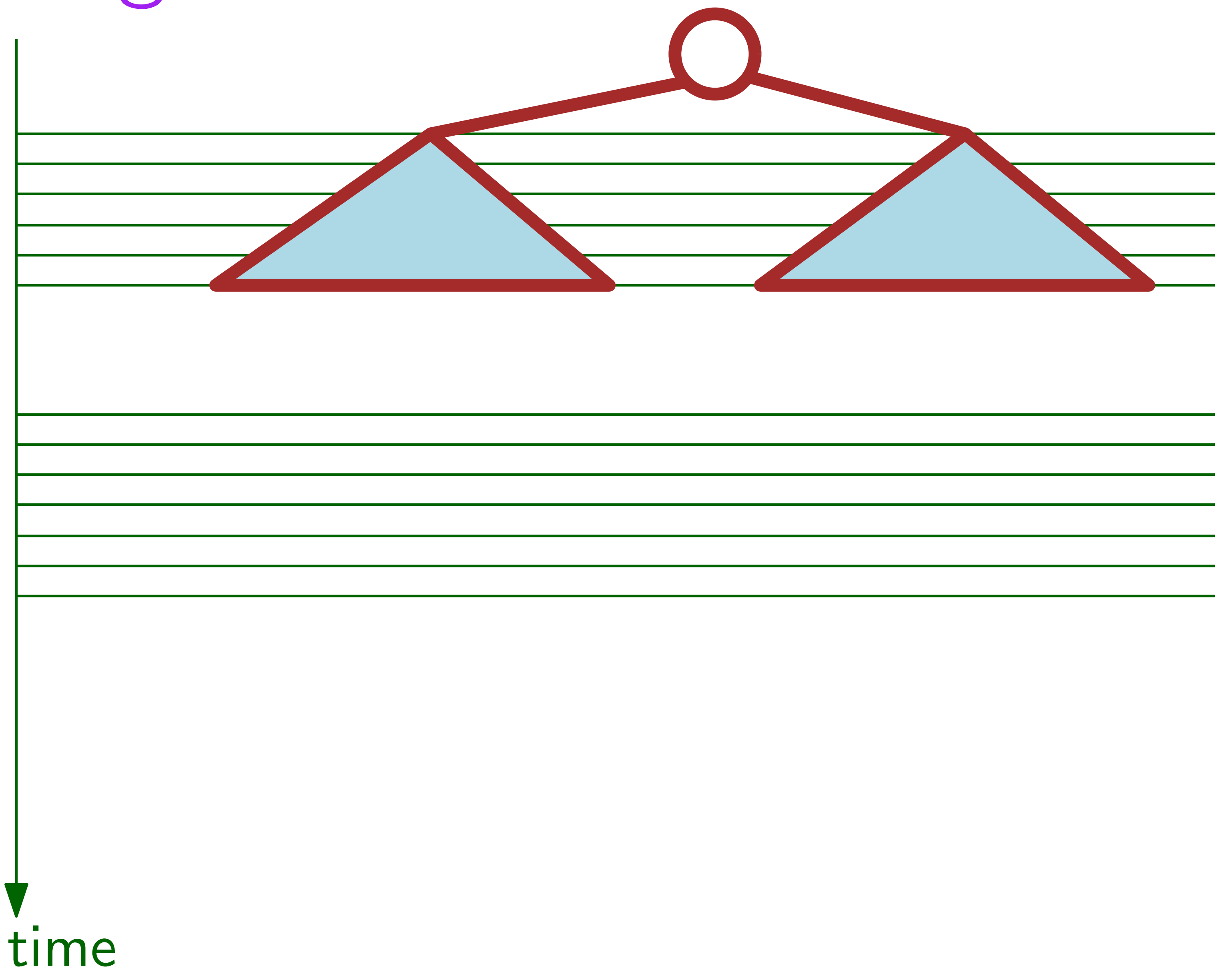

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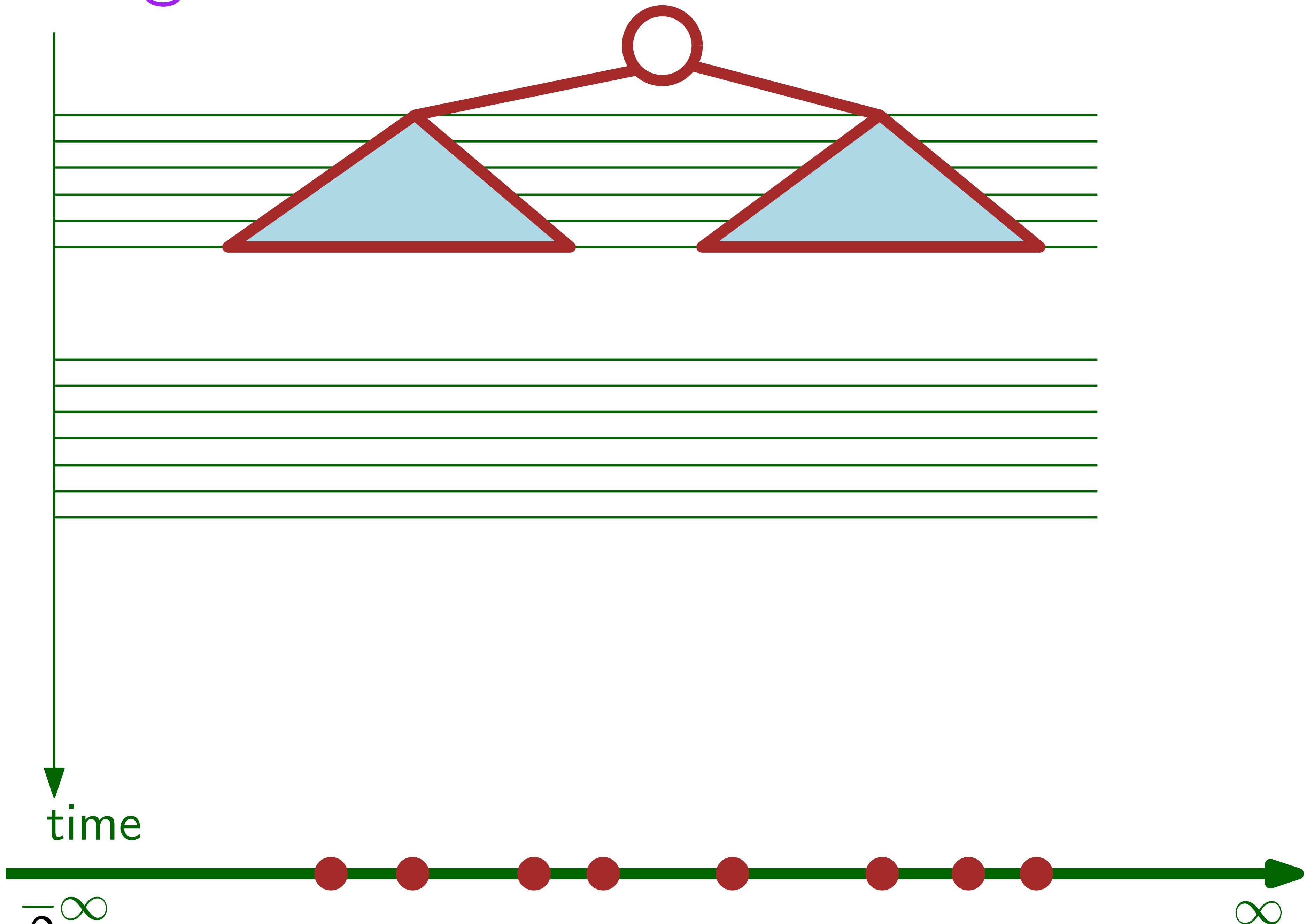

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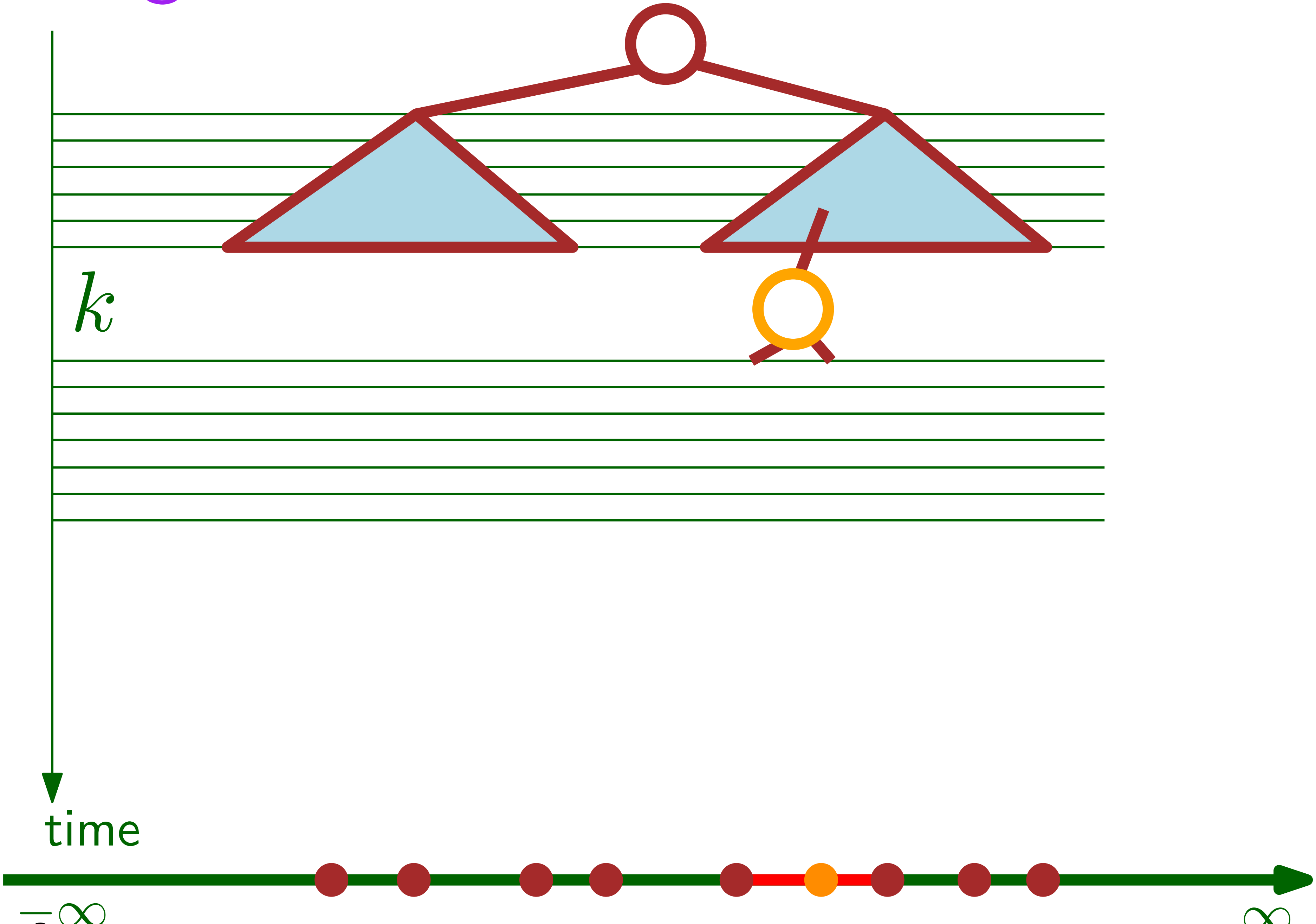

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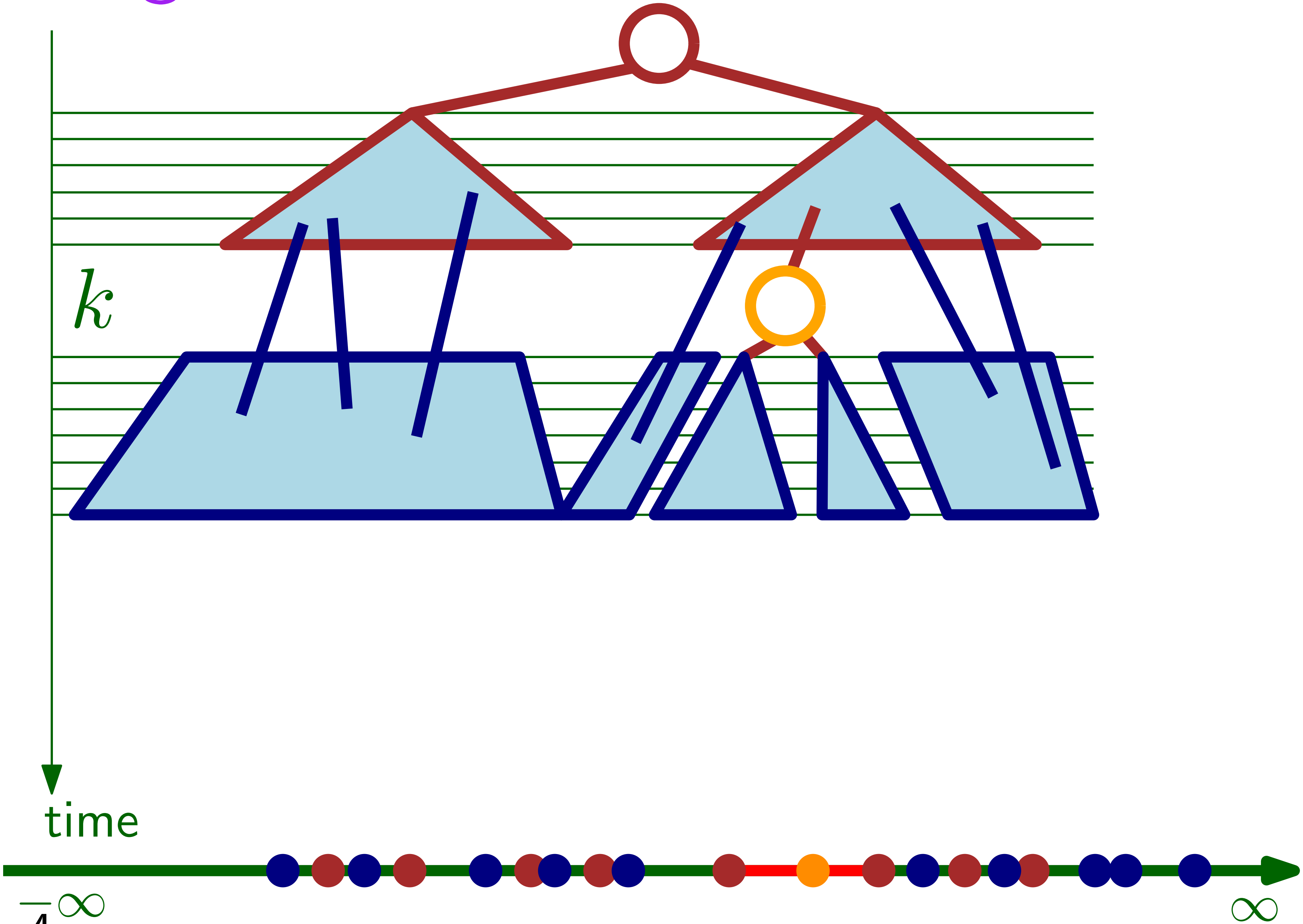
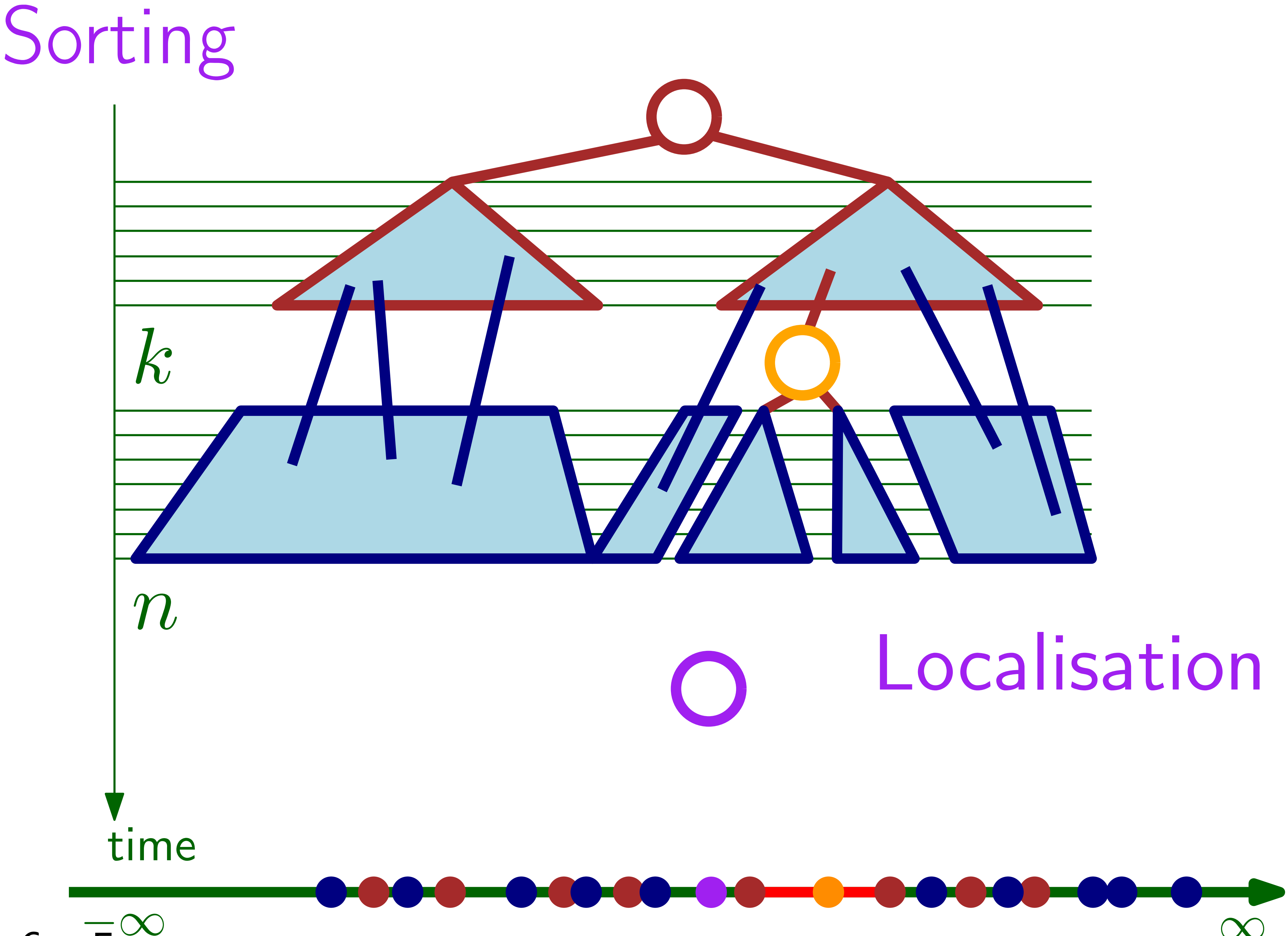
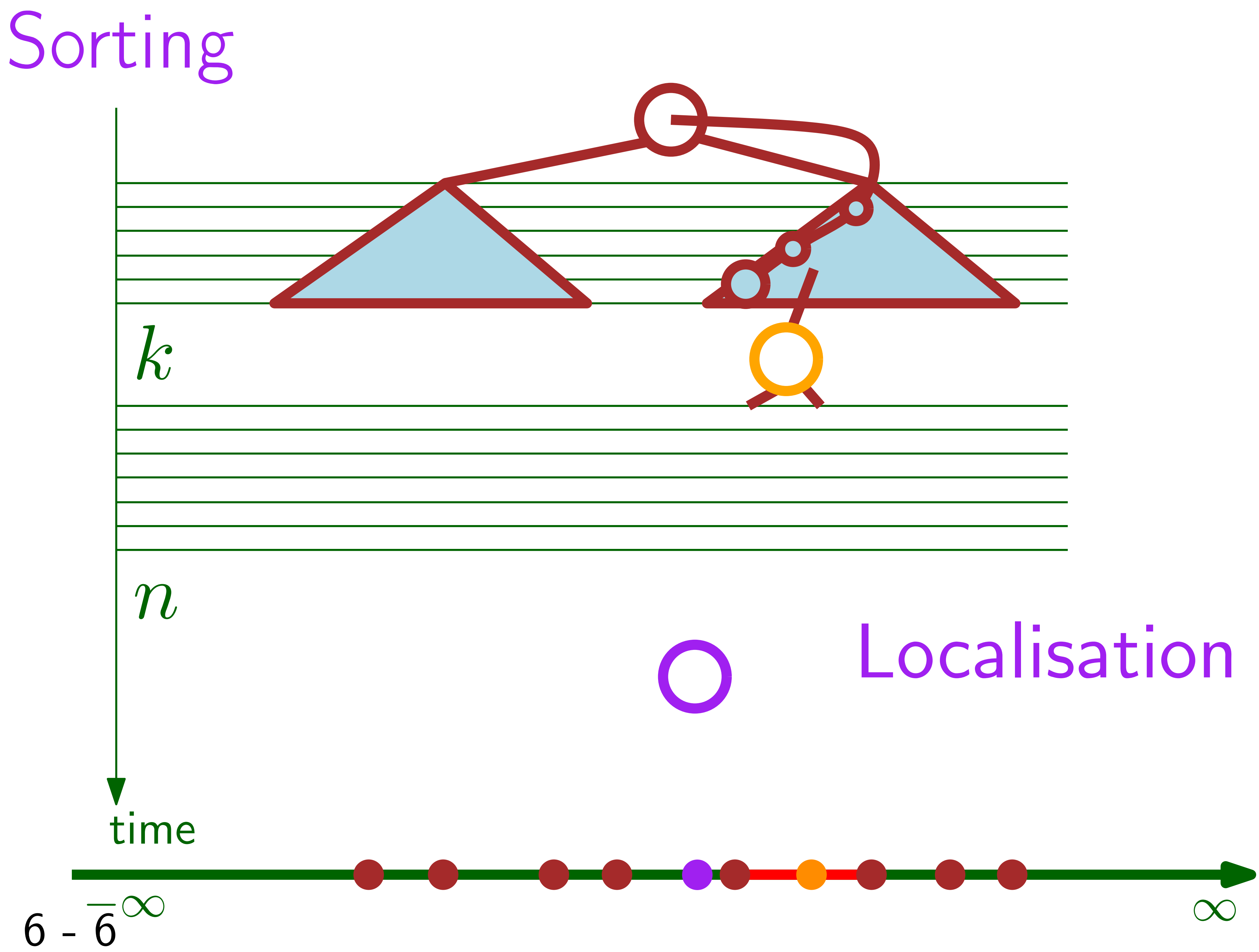
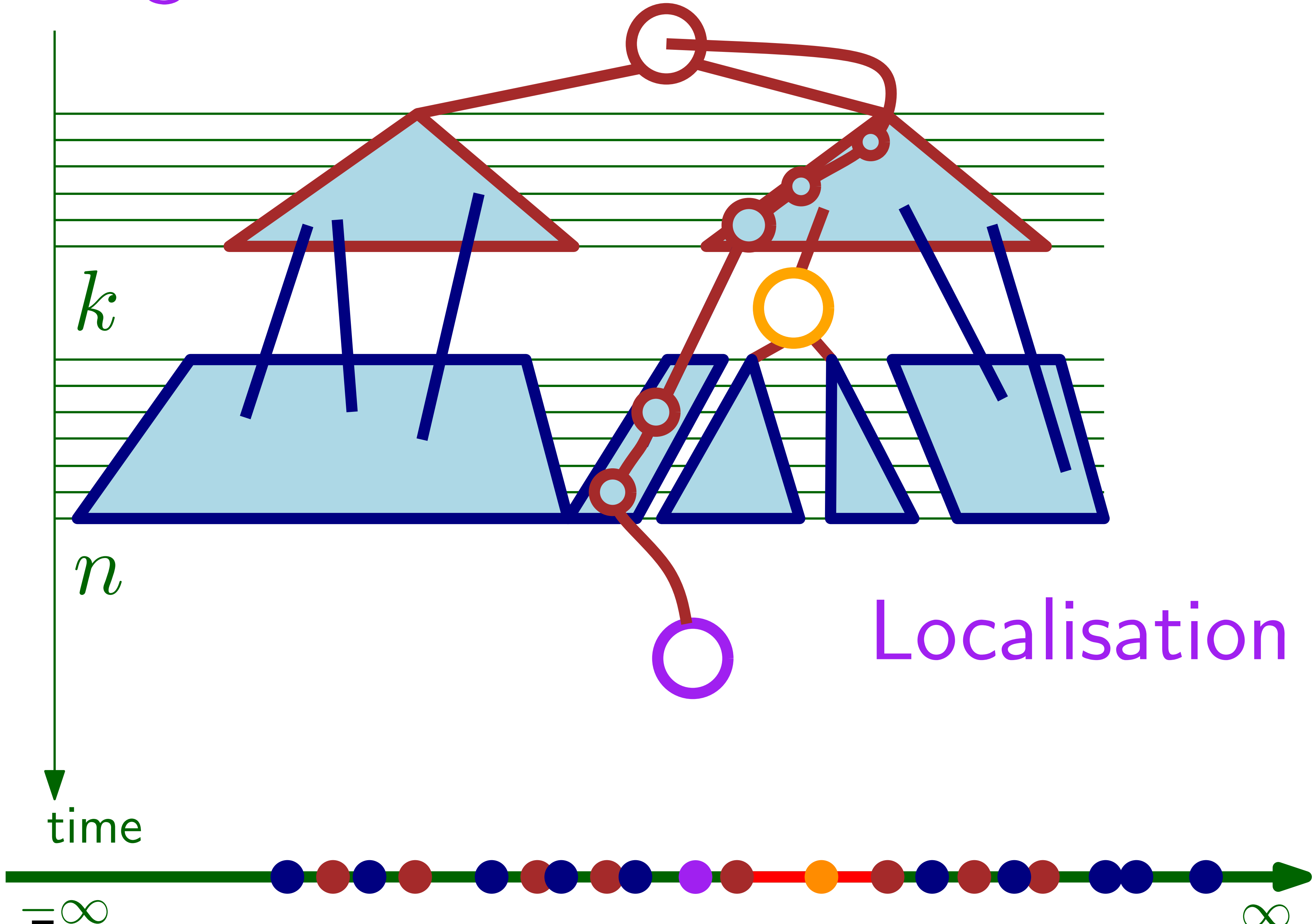




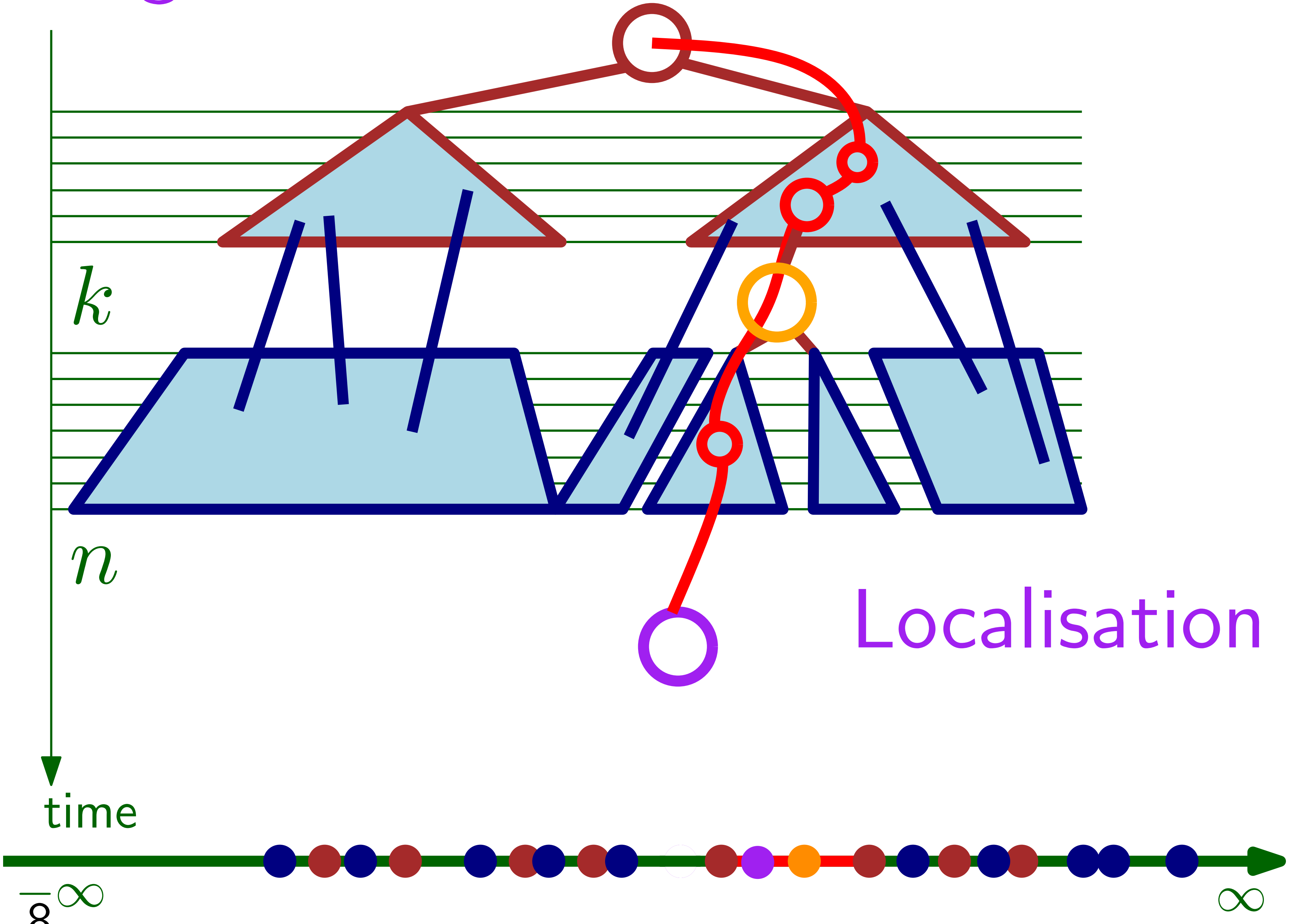
Sorting
k
n
Localisation
time
$-\infty$
∞


Sorting
k
n
Localisation
time
6 - 6
$-\infty$
∞

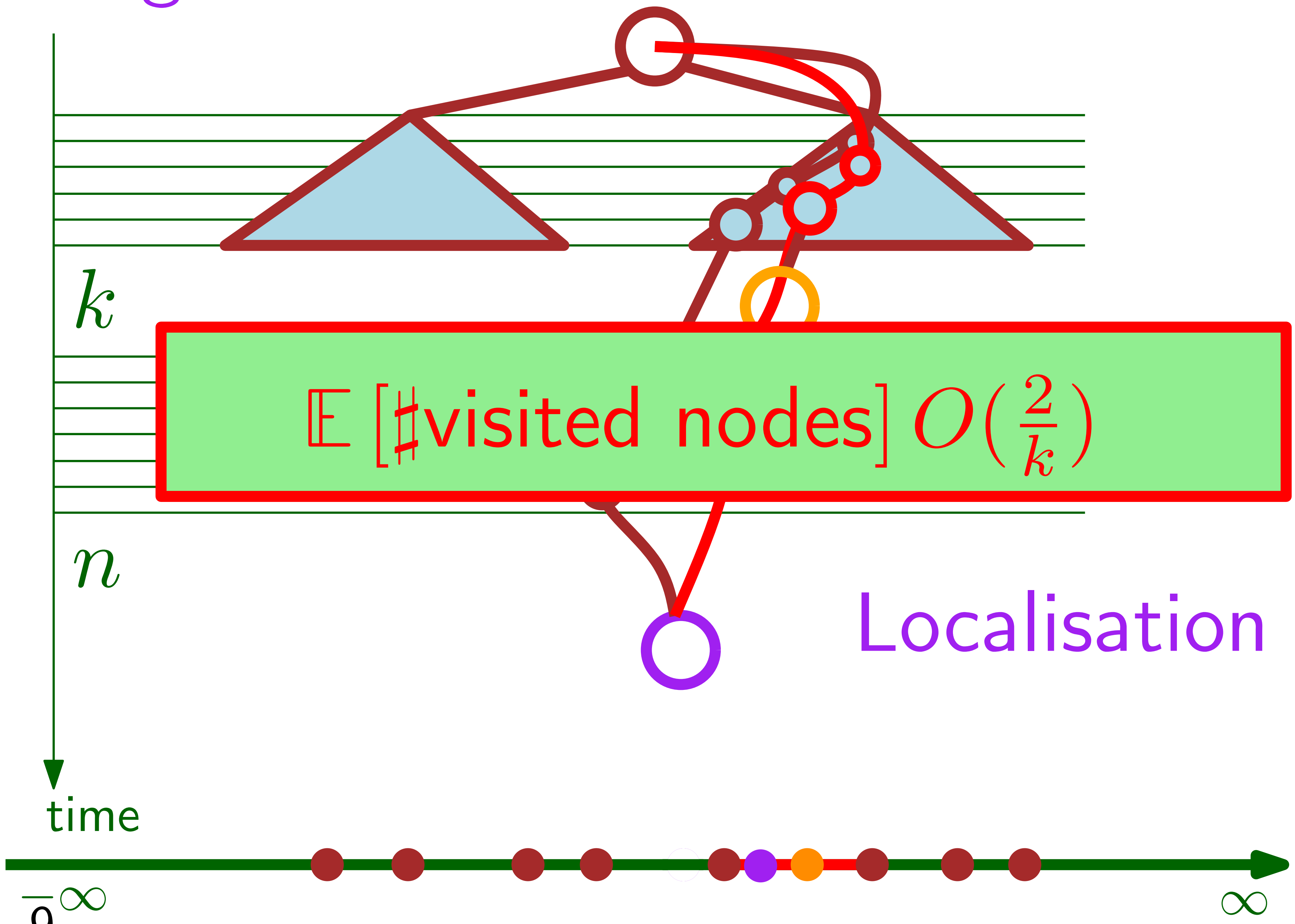
Sorting



Sorting

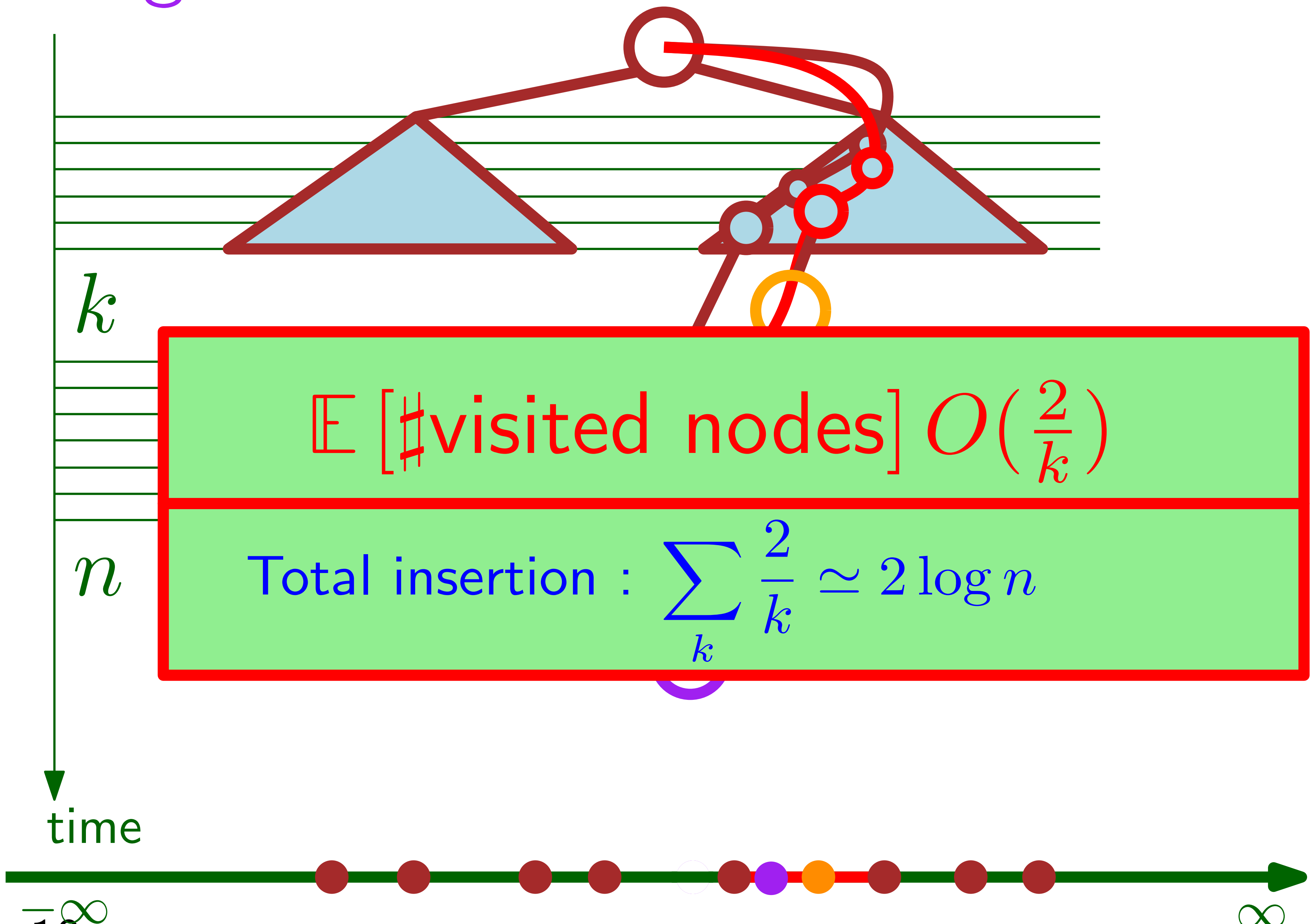


Sorting



k
n
$\mathbb{E}\left[\sharp\text{visited nodes}\right] O(\frac{2}{k})$
Localisation
time
$-\infty$
∞

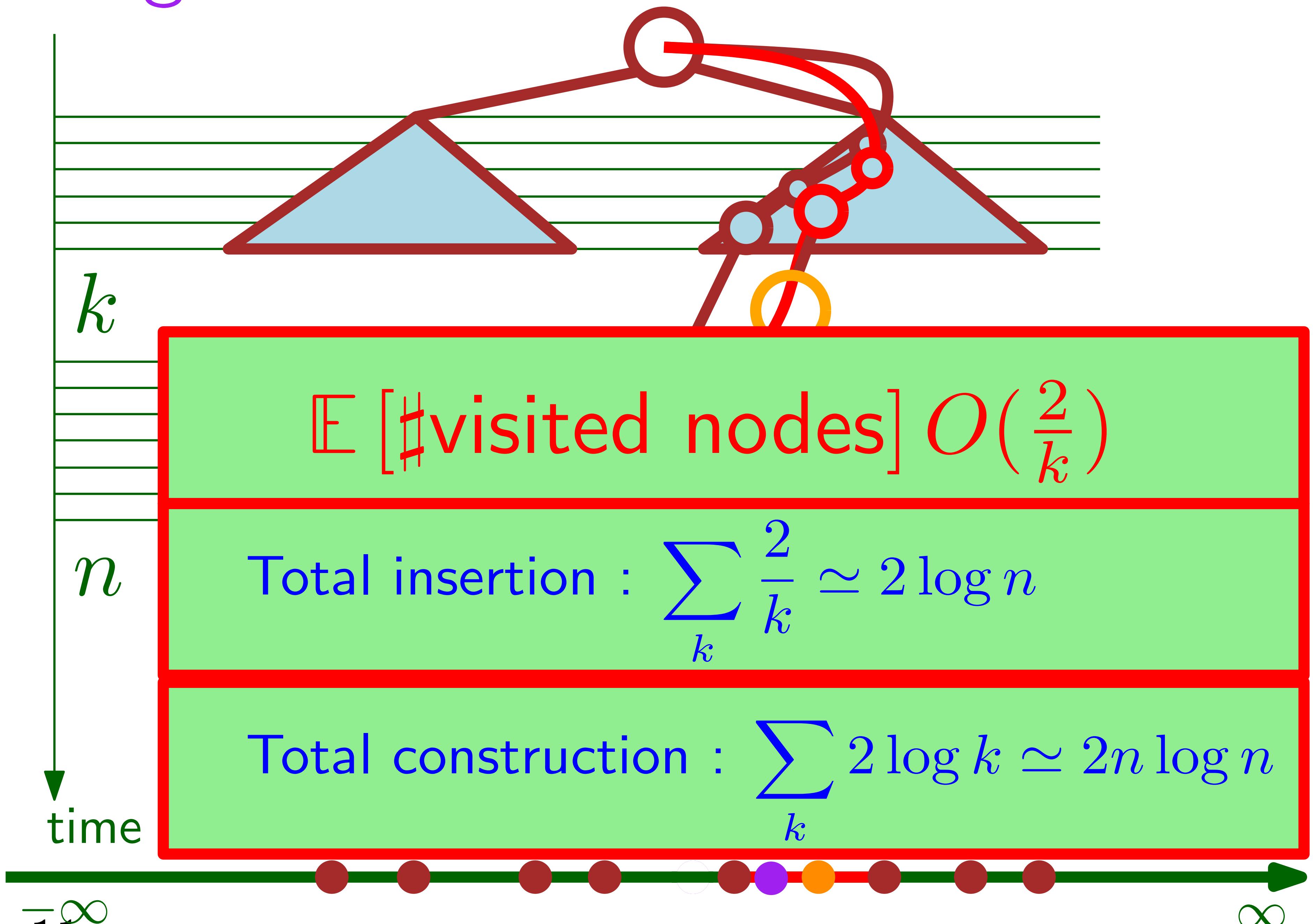
Sorting



$\mathbb{E}\left[\sharp\text{visited nodes}\right] O(\frac{2}{k})$

Total insertion : $\sum_k \frac{2}{k} \simeq 2\log n$

Sorting

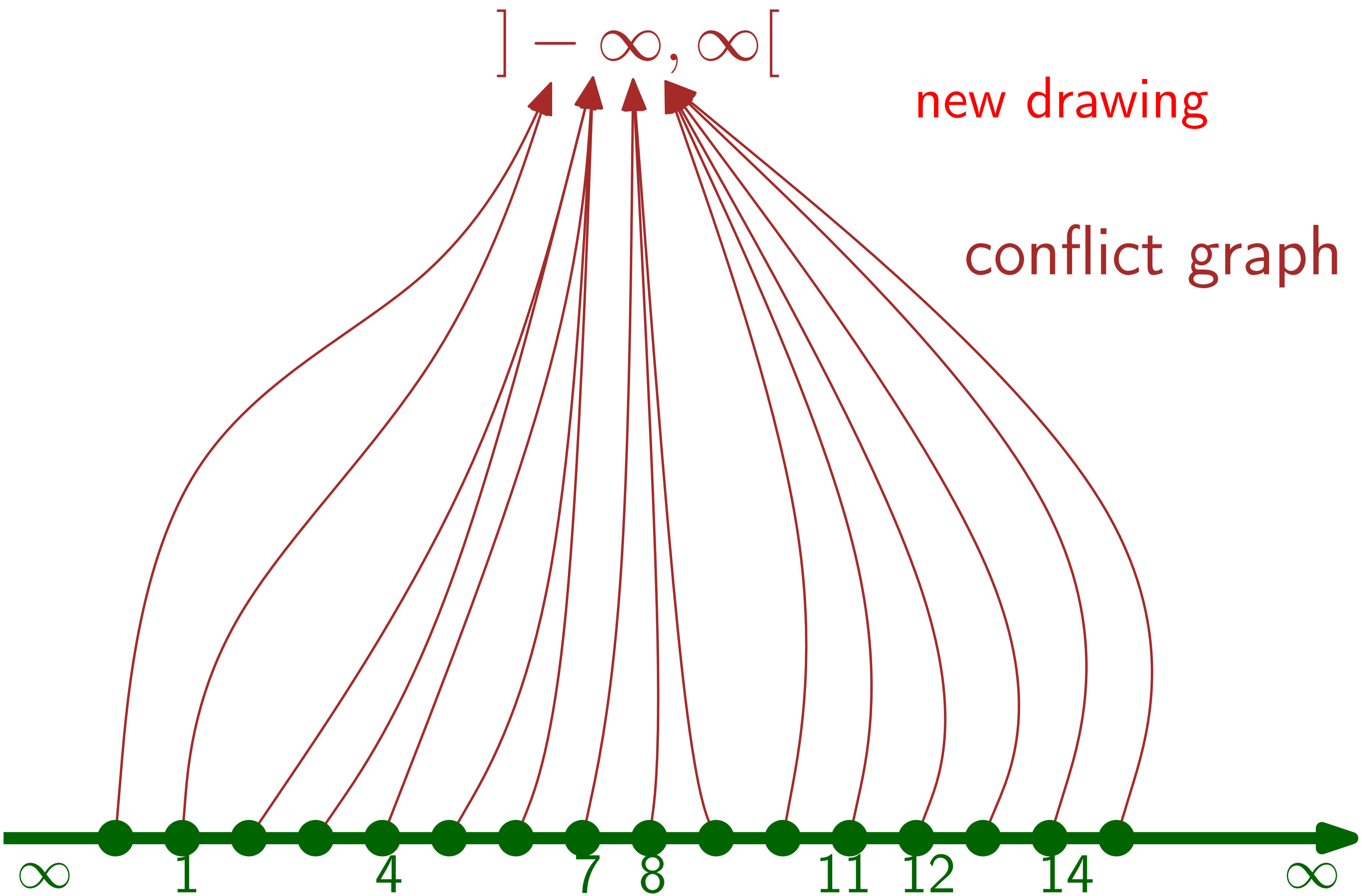


$\mathbb{E}\left[\sharp\text{visited nodes}\right] O(\frac{2}{k})$

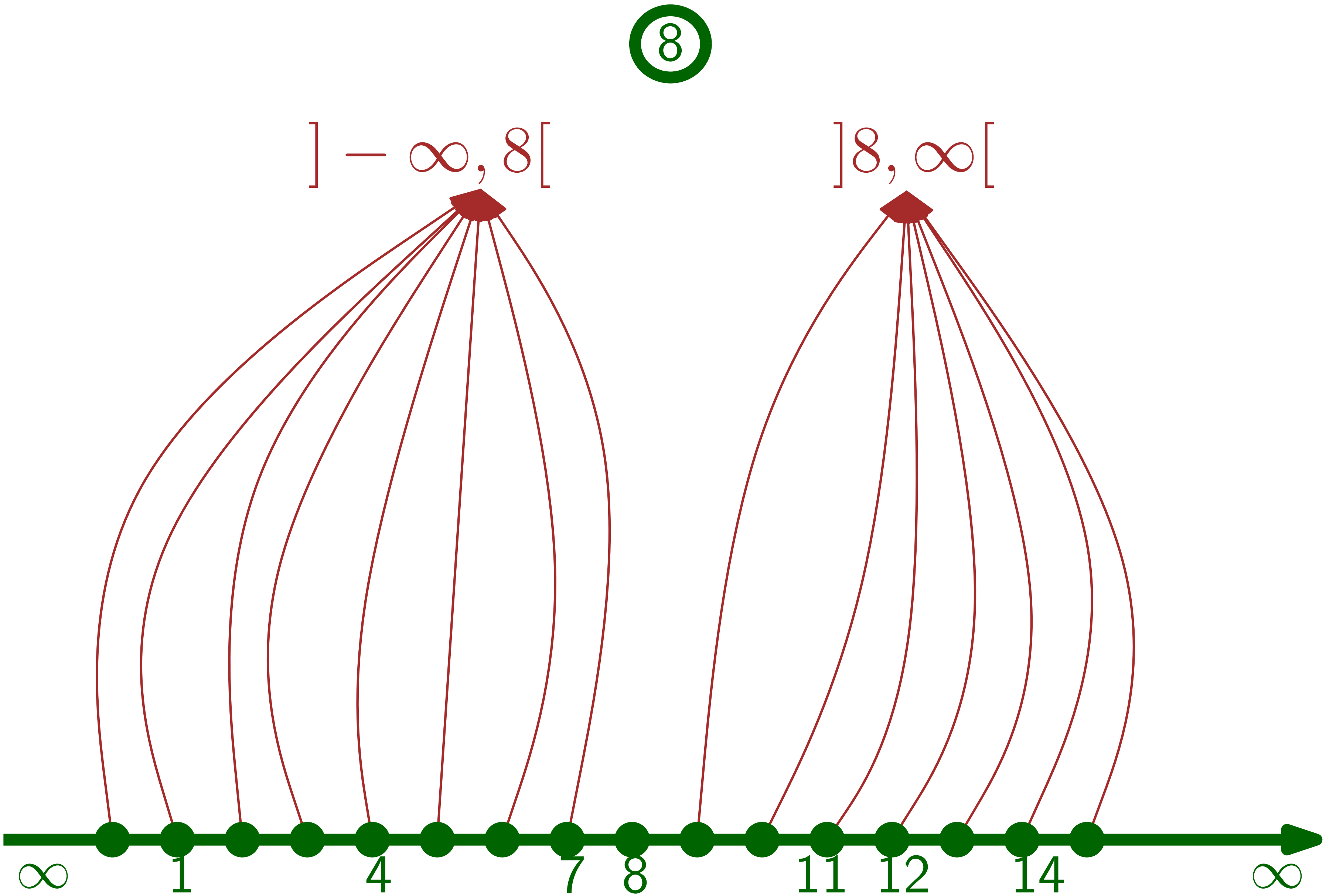
Total insertion : $\sum_k \frac{2}{k} \simeq 2 \log n$

Total construction : $\sum_k 2 \log k \simeq 2n \log n$

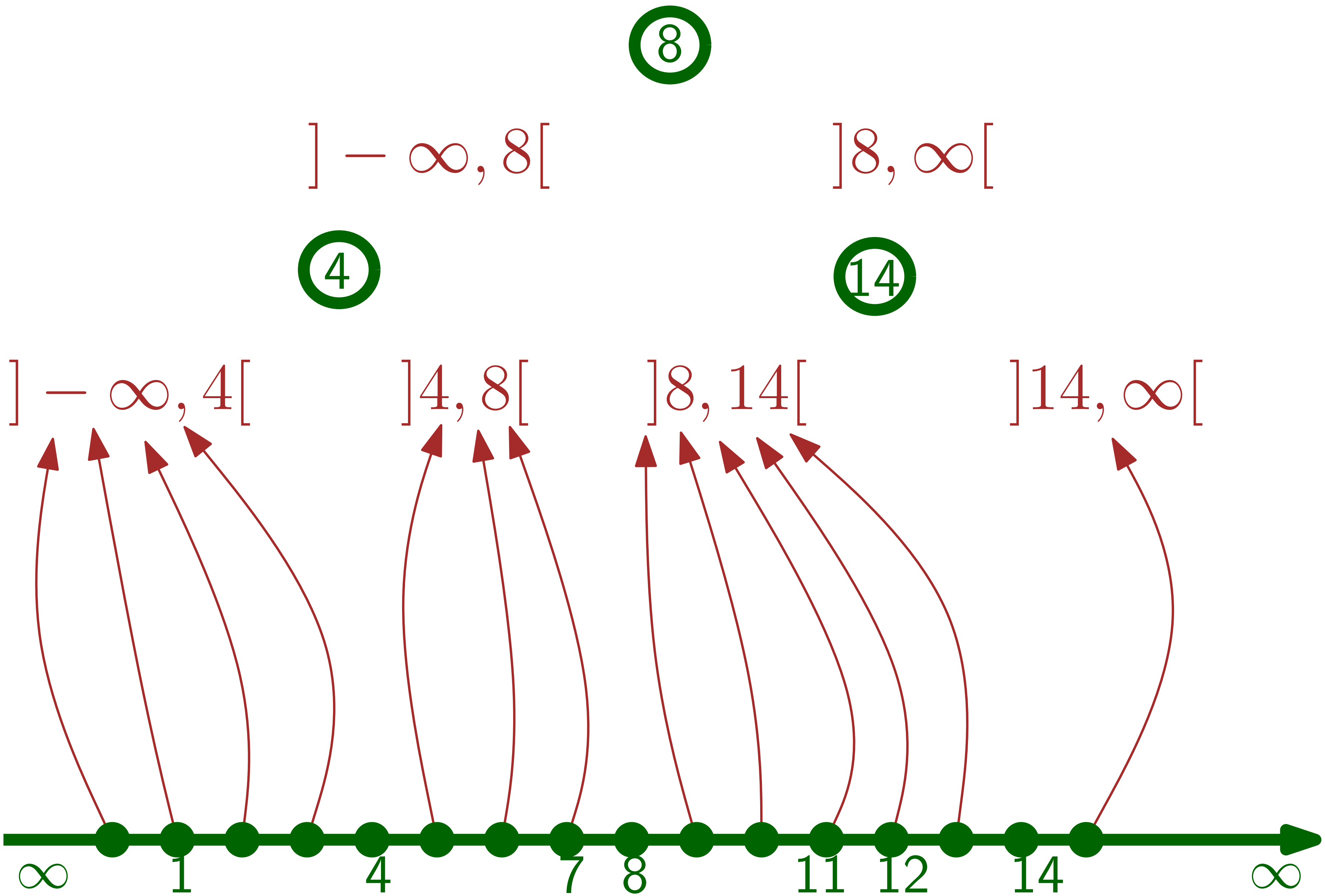
Sorting


$]-\infty,\infty[$
new drawing
conflict graph
∞
1
4
7
8
11
12
14
∞

Sorting



Sorting


8
$]-\infty, 8[$
$]8, \infty[$
4
14
$]-\infty, 4[$
$]4, 8[$
$]8, 14[$
$]14, \infty[$
∞
1
4
7
8
11
12
14
∞

Sorting

Unbalanced binary tree

Quicksort

$O(n \log n)$

History graph

Conflict graph

Same analysis

Backwards analysis

Analyse last insertion and sum

Last object is a random object

Randomization

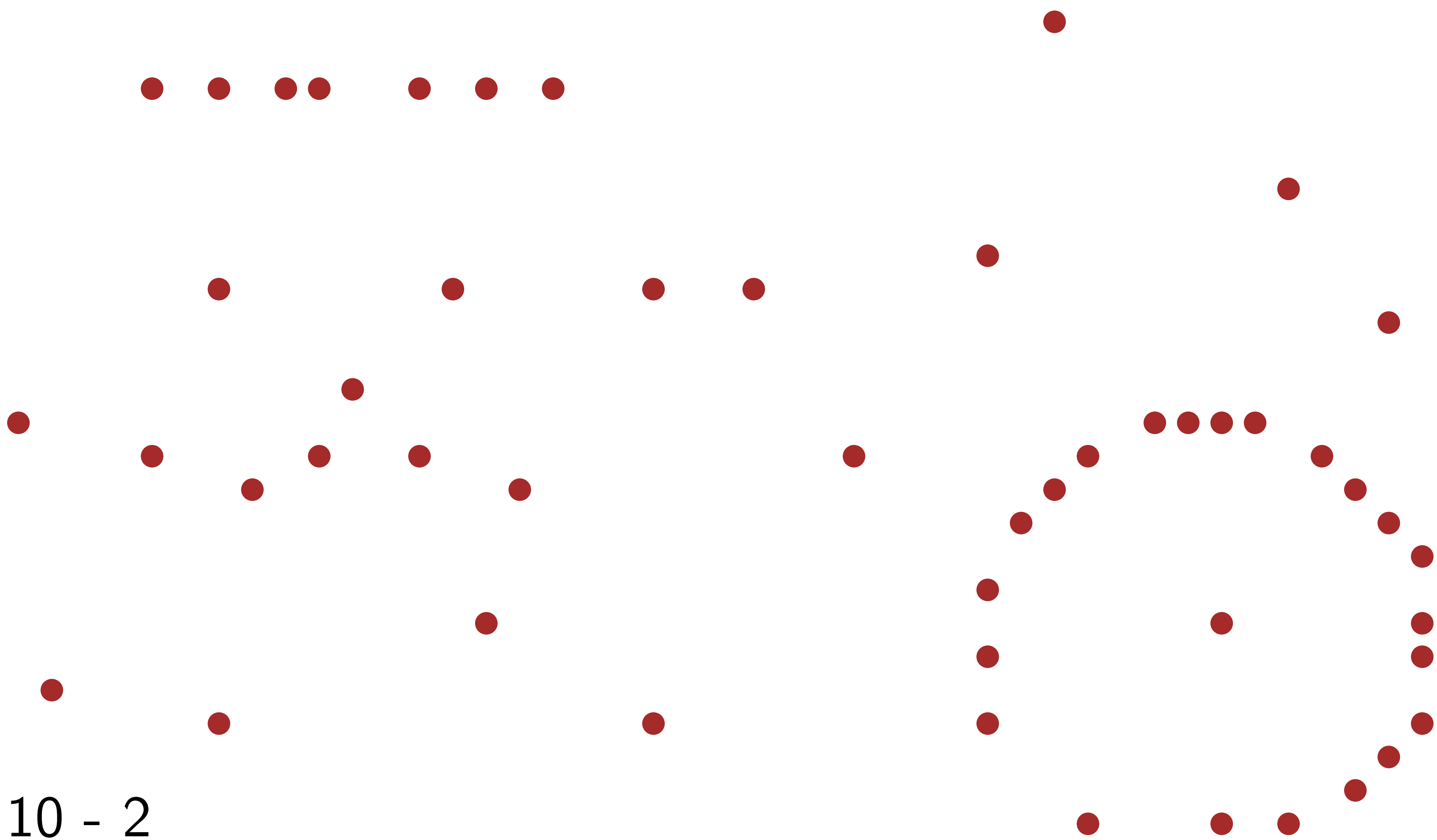
Backwards analysis for Delaunay triangulation

Delaunay triangulation

$\sharp$ of triangles during incremental construction?

Delaunay triangulation

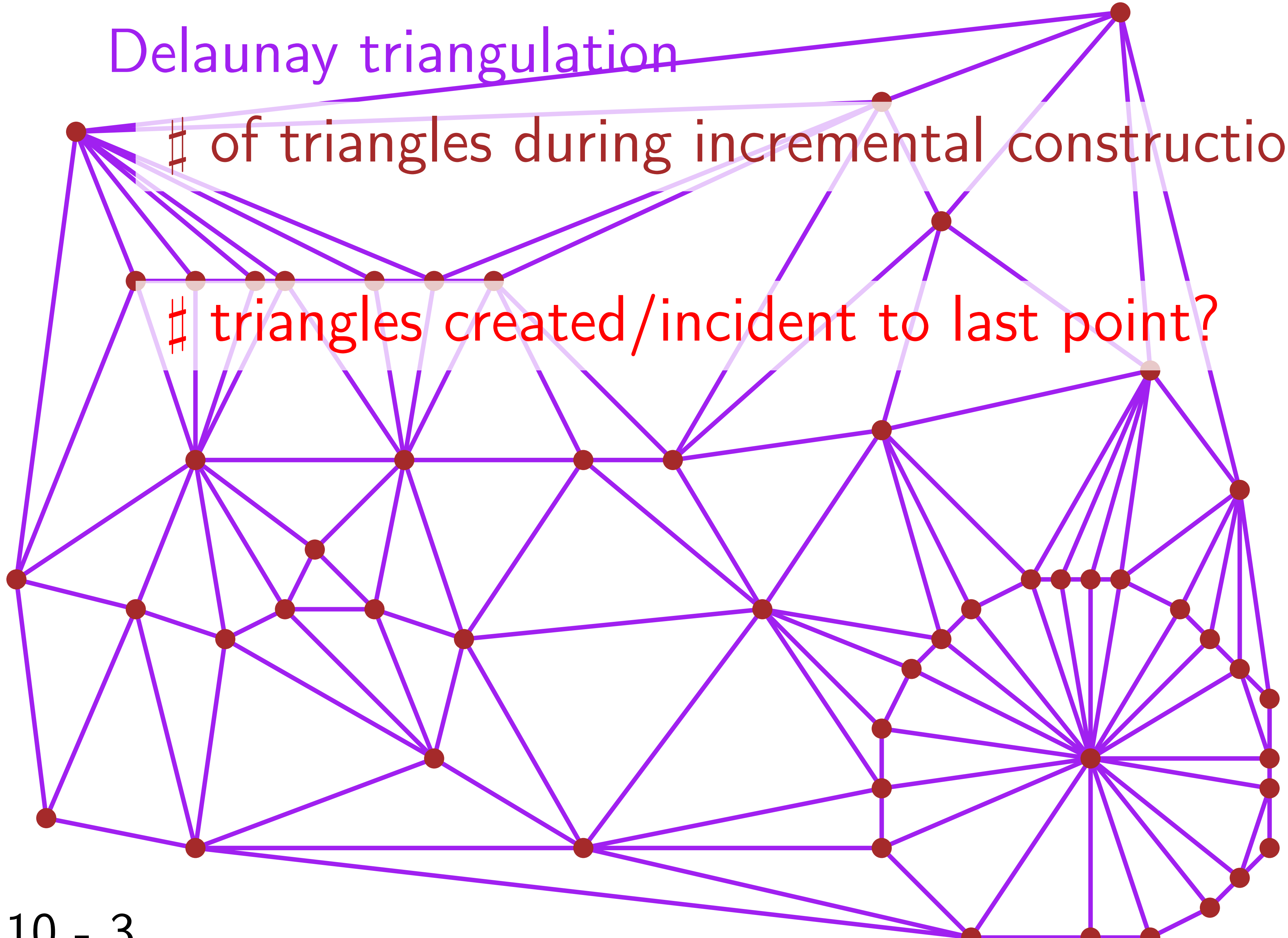
- $\sharp$ of triangles during incremental construction?

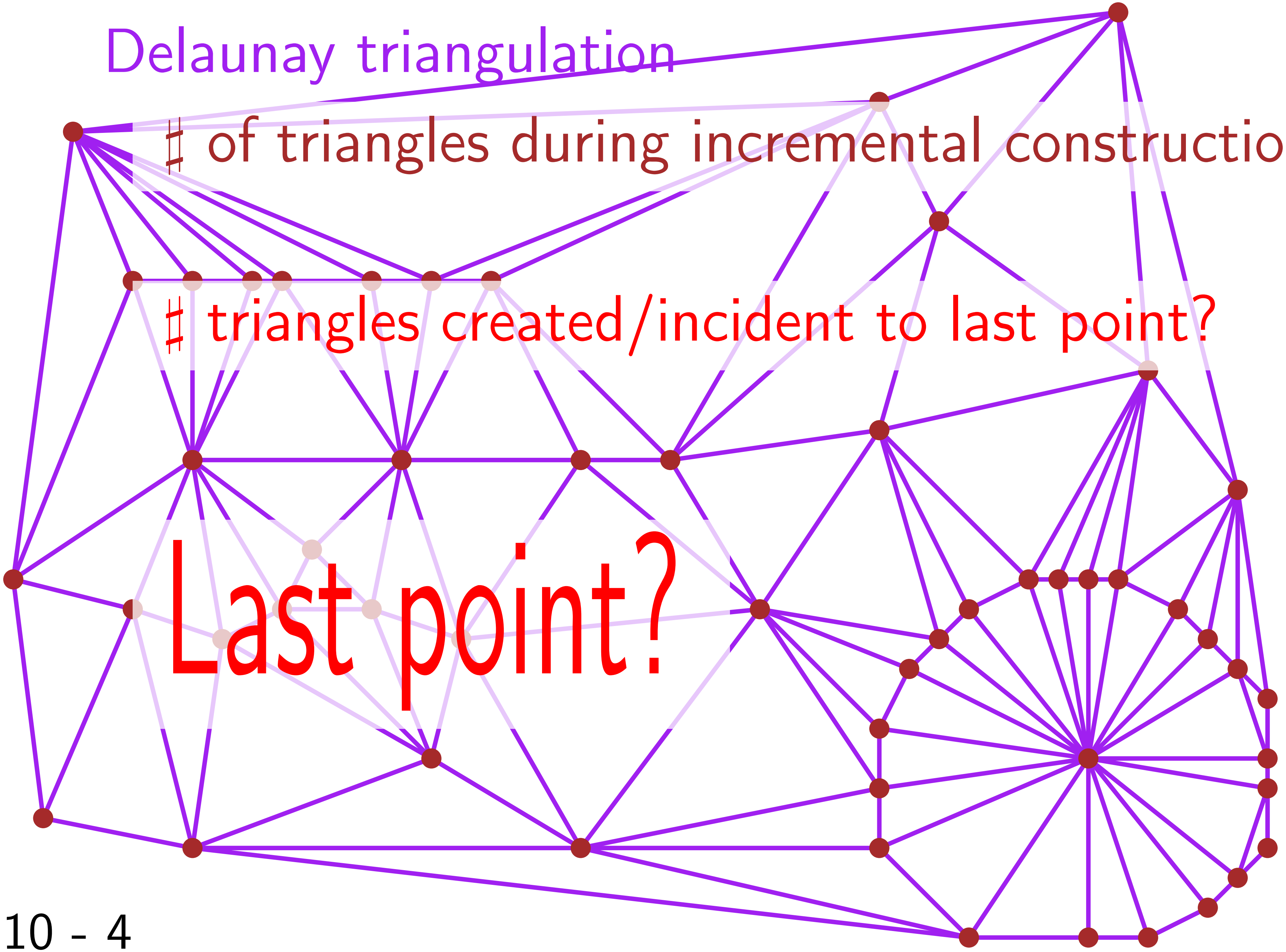


Delaunay triangulation

$\sharp$ of triangles during incremental construction?

$\sharp$ triangles created/incident to last point?



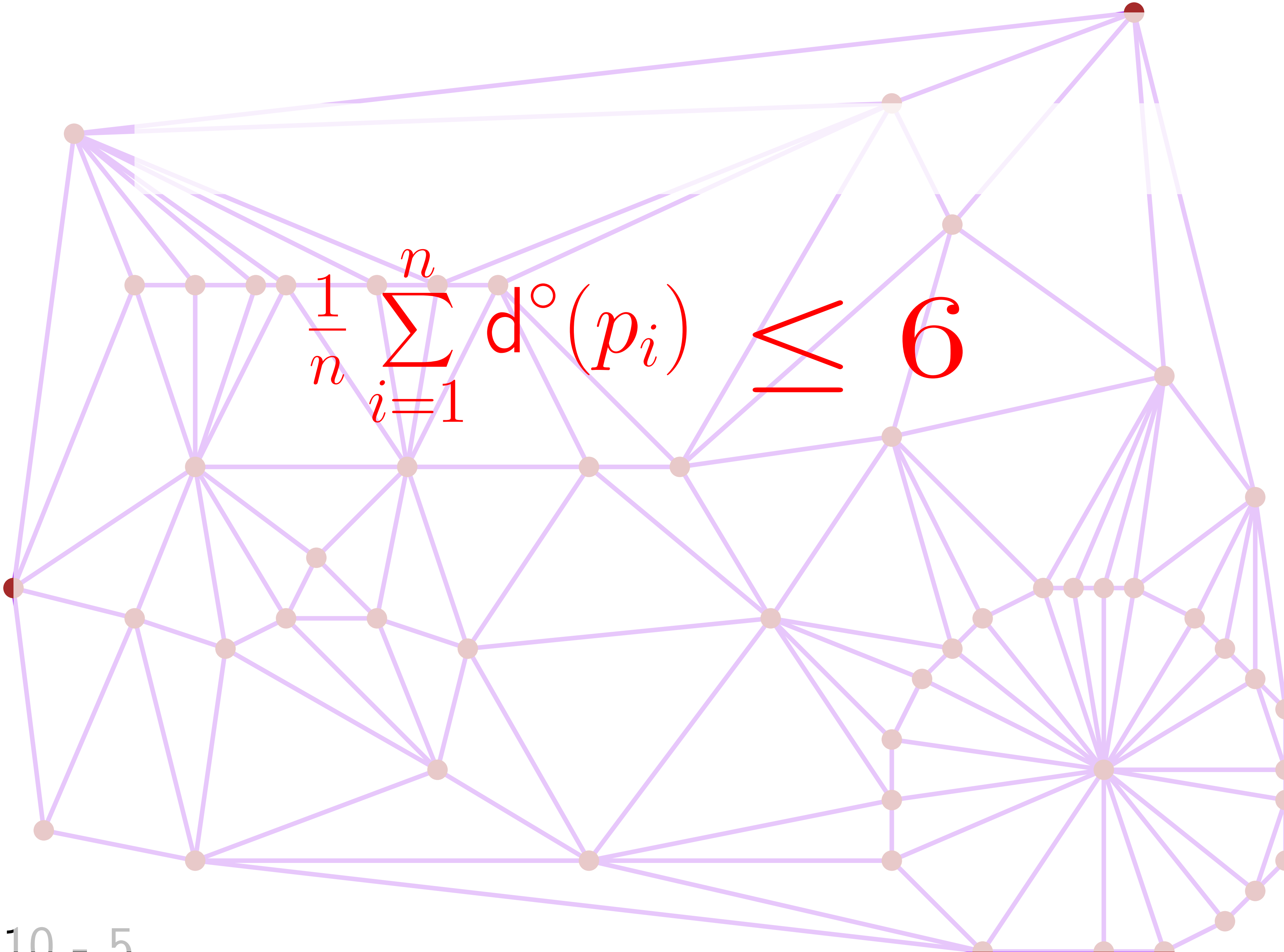


Delaunay triangulation

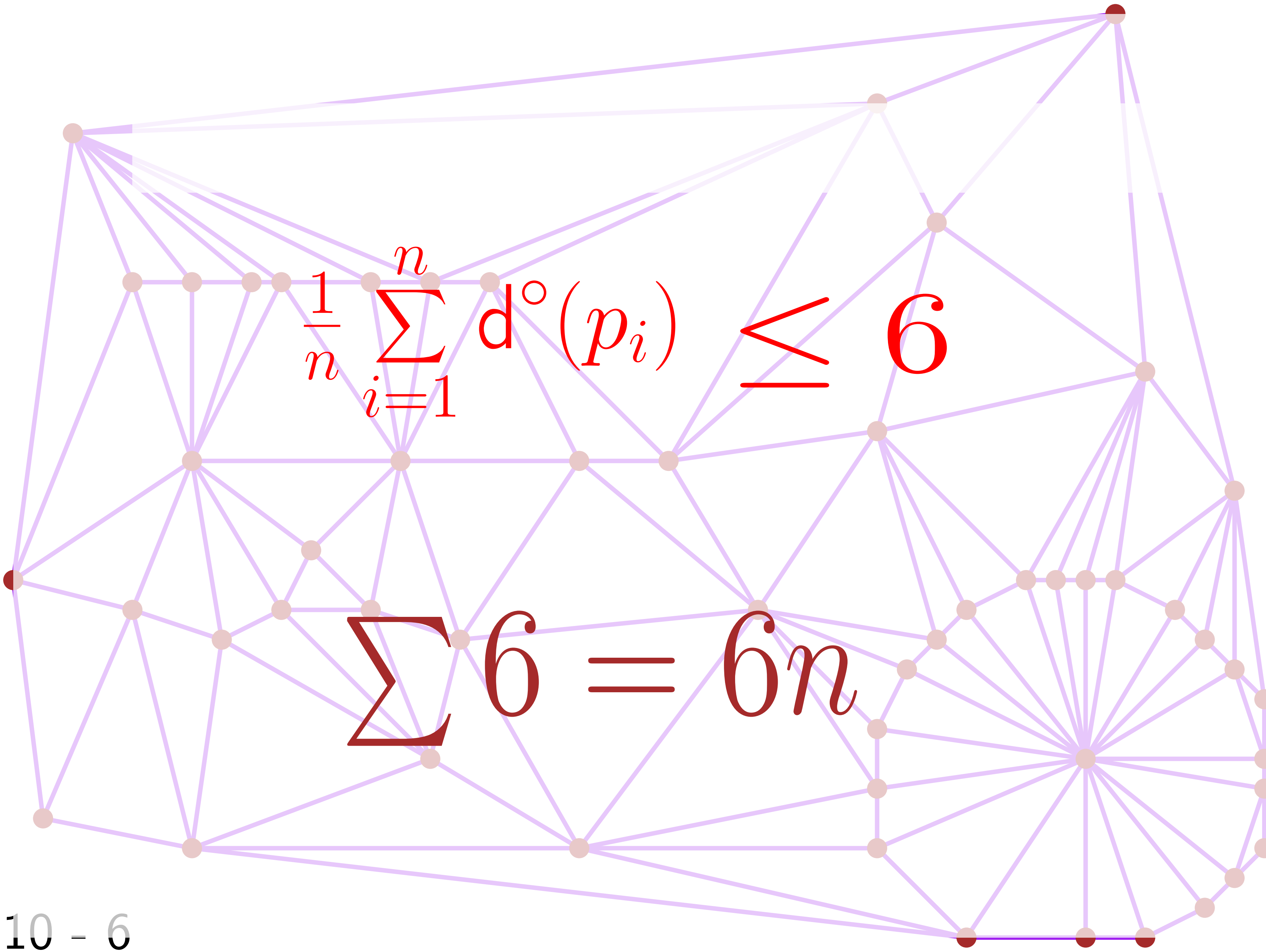
$\sharp$ of triangles during incremental construction?

$\sharp$ triangles created/incident to last point?

Last point?



$$\frac{1}{n}\sum_{i=1}^{n} \mathsf{d}^{\circ}(p_i) \leq 6$$

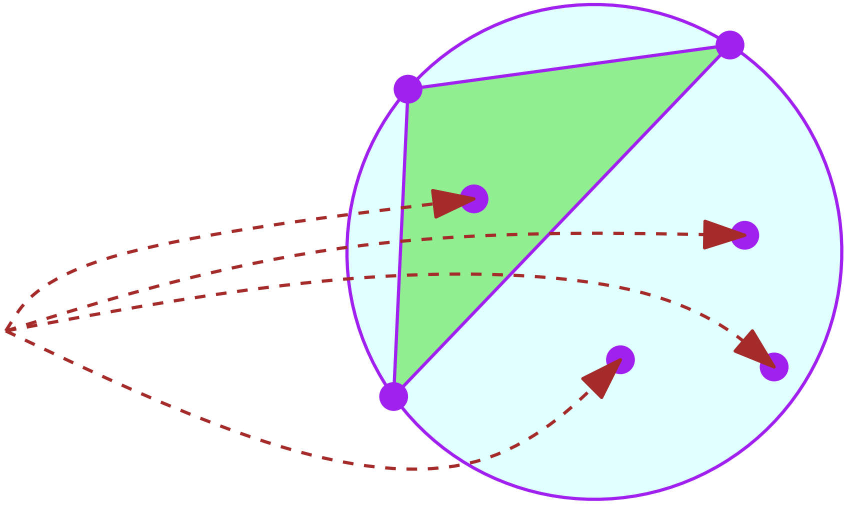


$$\frac{1}{n}\sum_{i=1}^{n} \mathsf{d}^\circ(p_i) \leq 6$$

$$\sum 6 = 6n$$

Alternative analysis

Triangle Δ with j stoppers



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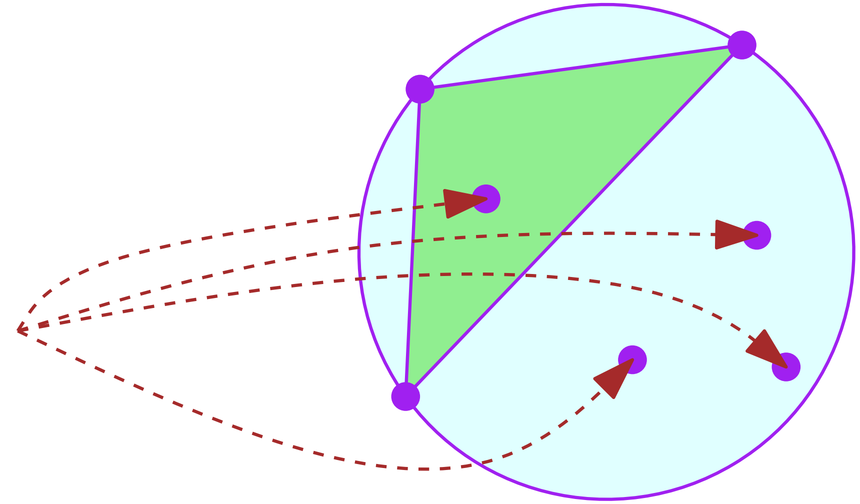
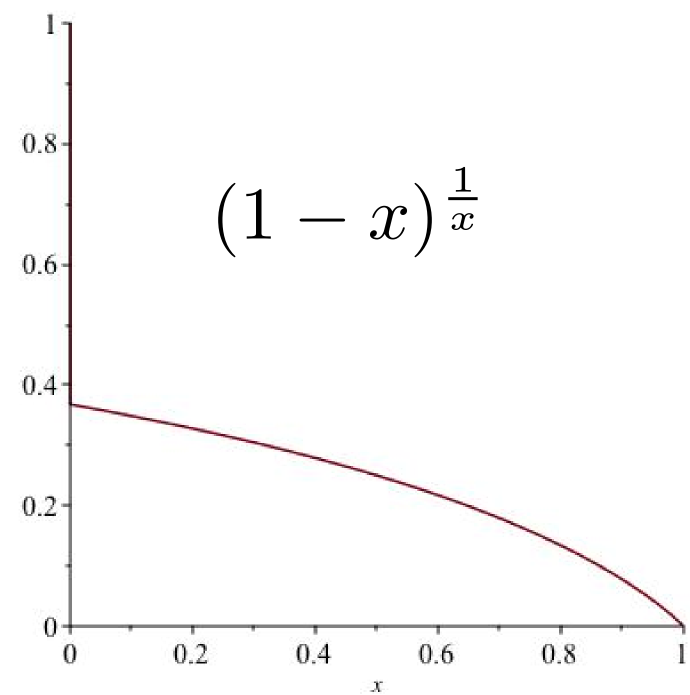
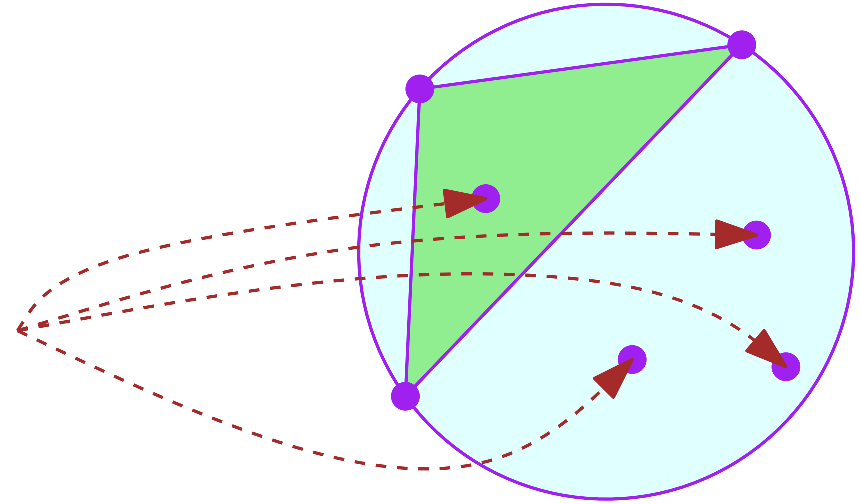


Probability that it exists in the triangulation of a sample of size αn

$$\simeq \alpha^3(1-\alpha)^j \geq \alpha^3(1-\alpha)^{\frac{1}{\alpha}} \geq e^{-1}\alpha^3 \quad \text{if } 2 \leq j \leq \frac{1}{\alpha}$$



Alternative analysis



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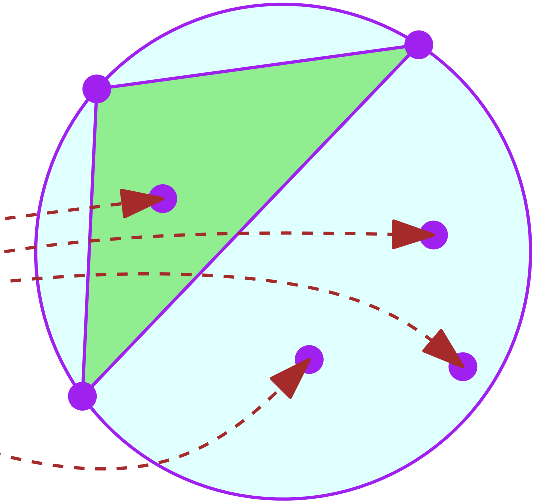
Size of the triangulation of the sample

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers is there}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



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Alternative analysis

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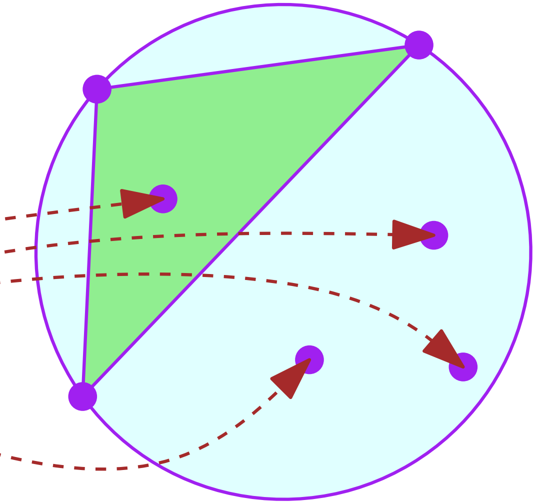

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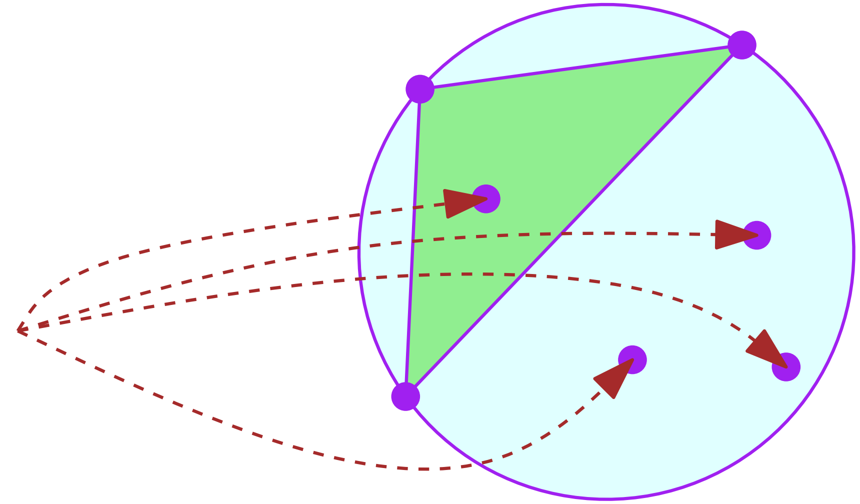
$$\geq \sum_{j=0}^{1/\alpha} e^{-1}\alpha^3 \times \sharp\Delta \text{ with } j \text{ stoppers} = \alpha^3 \sharp\Delta \text{with} \leq \frac{1}{\alpha} \text{stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



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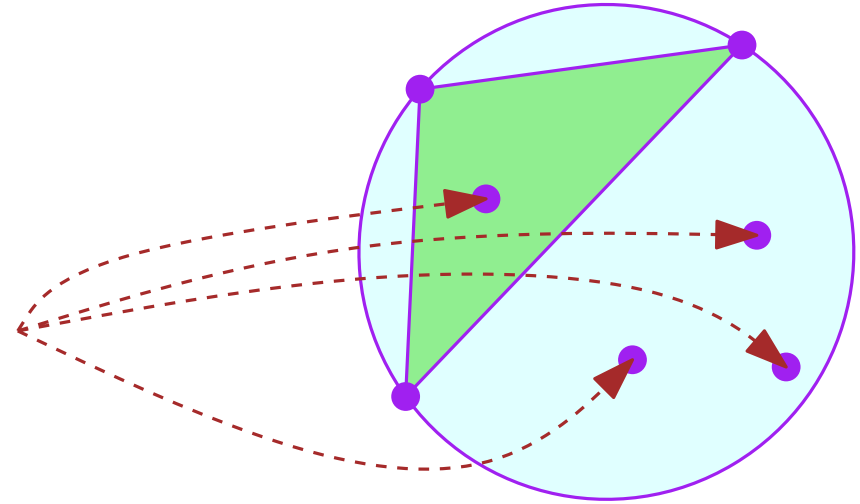
Alternative analysis



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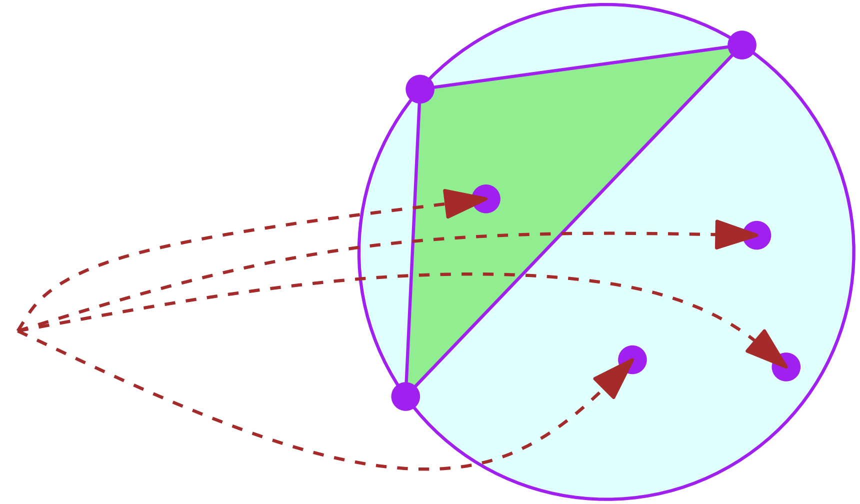
Size (order $\leq k$ Voronoi) $\leq \frac{\alpha n}{\alpha^3} = nk^2$

Alternative analysis

Triangle Δ with j stoppers

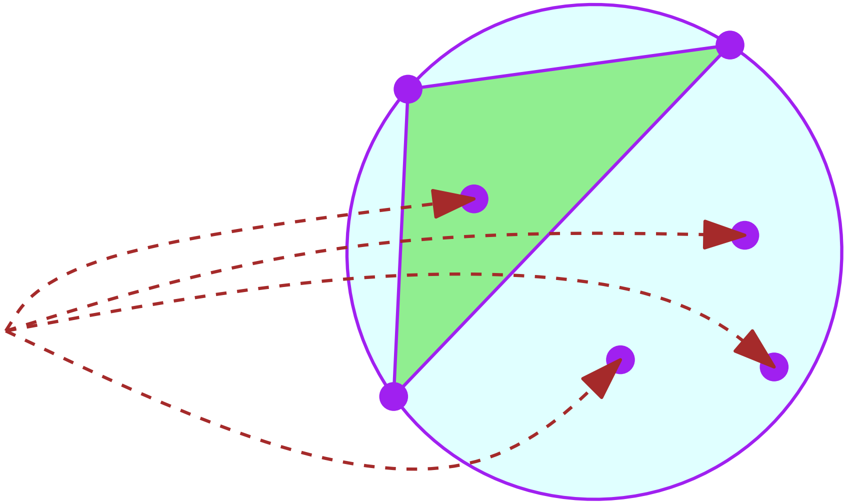


Probability that it exists during the construction

$$= \frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}$$

Alternative analysis

Triangle Δ with j stoppers



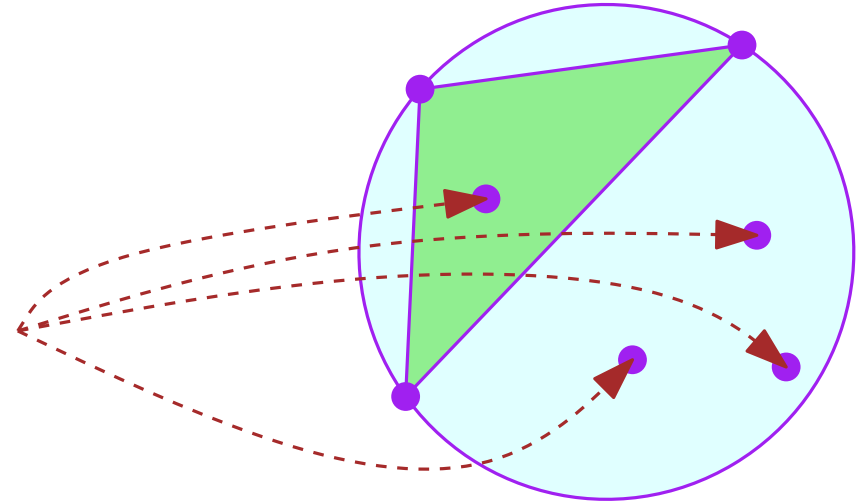
Probability that it exists during the construction

$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

$\sharp$ of created triangles

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

Alternative analysis



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$\sharp$ of created triangles

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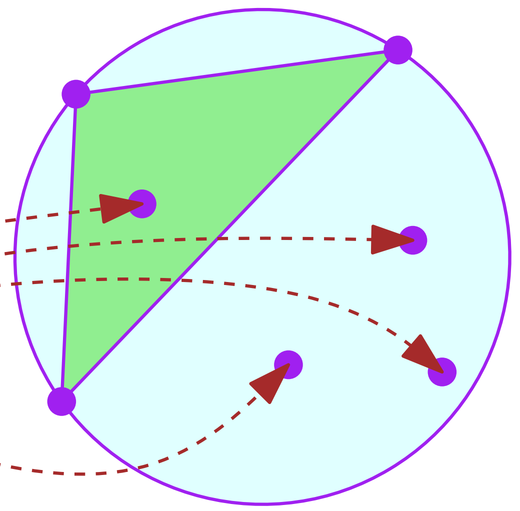
$$= \sum_{j=0}^{n} \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with} \leq j \text{ stoppers}$$

Alternative analysis



Triangle Δ with j stoppers

Probability that it exists during the construction

$$= \frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1} \qquad \frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1} - \frac{3}{j+4}\frac{2}{j+3}\frac{1}{j+2} = \frac{6(j+4)-6(j+1)}{(j+4)(j+3)(j+2)(j+1)}$$

$\sharp$ of created triangles

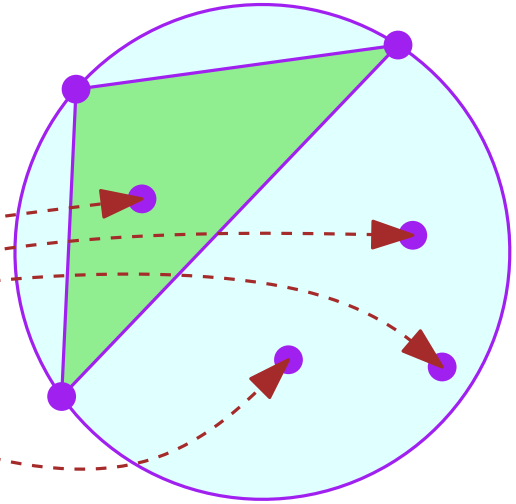
$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with} \leq j \text{ stoppers}$$

$$\simeq \sum_{j=0}^{n} \frac{18}{j^4} \times nj^2 = O\left(n \sum \frac{1}{j^2}\right) = O(n)$$

Alternative analysis

Triangle Δ with j stoppers

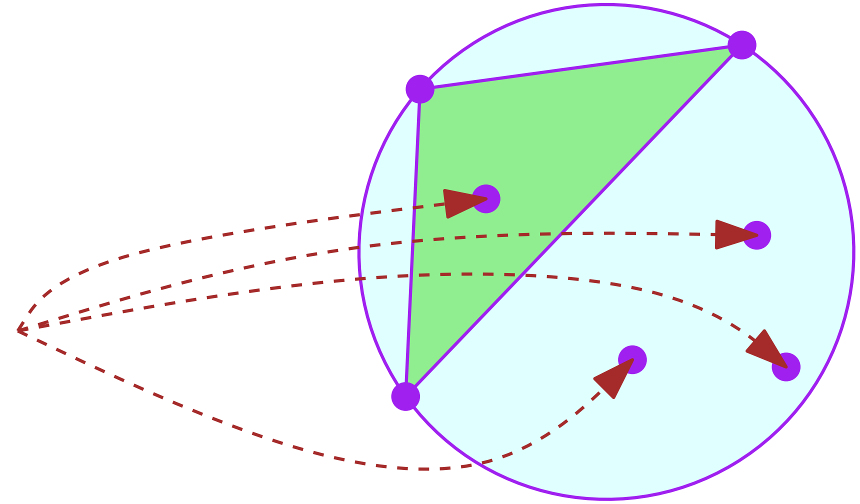


Conflict graph / History graph

It remains to analyze conflict location

Alternative analysis

Triangle Δ with j stoppers



Probability that it exists during the construction

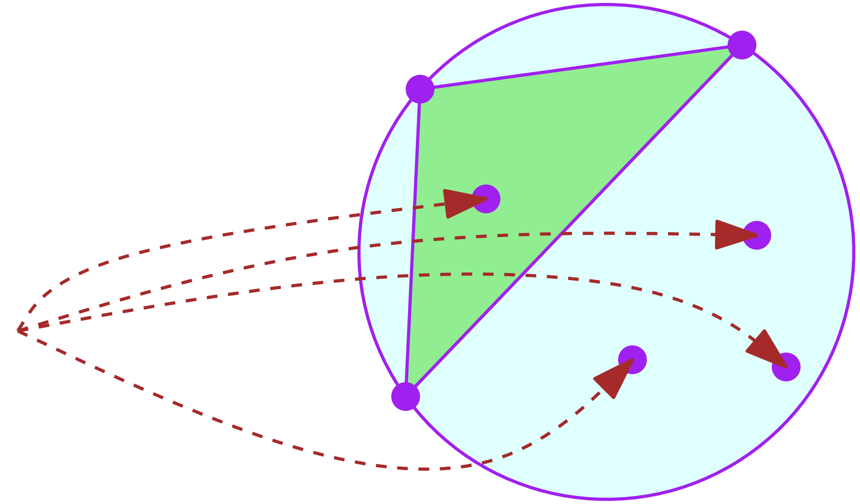
$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

$\sharp$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



Probability that it exists during the construction

$$= \frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}$$

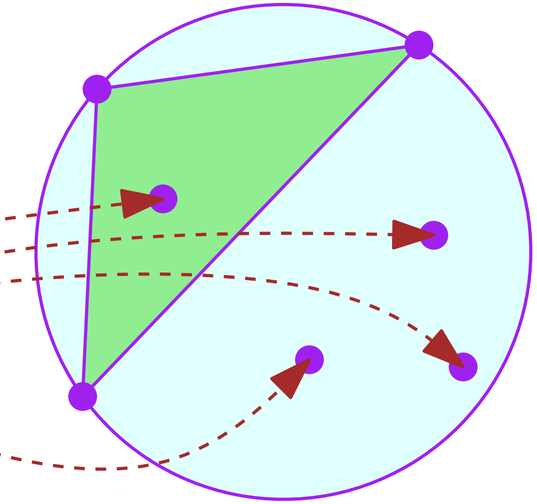
$\sharp$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$= \sum_{j=0}^{n} j \times \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with } \leq j \text{ stoppers}$$

Alternative analysis

Triangle Δ with j stoppers



Probability that it exists during the construction

$$= \frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}$$

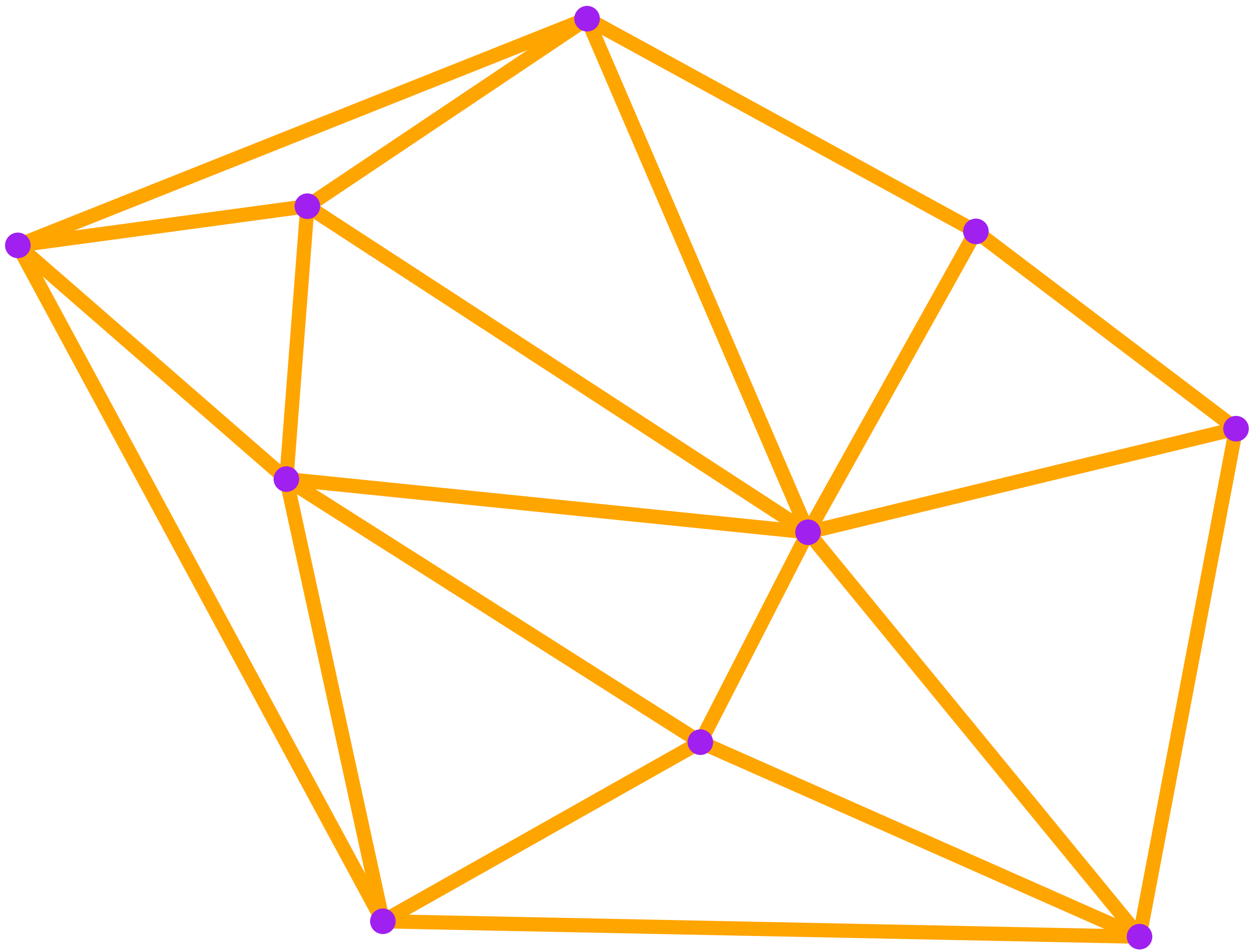
$\sharp$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

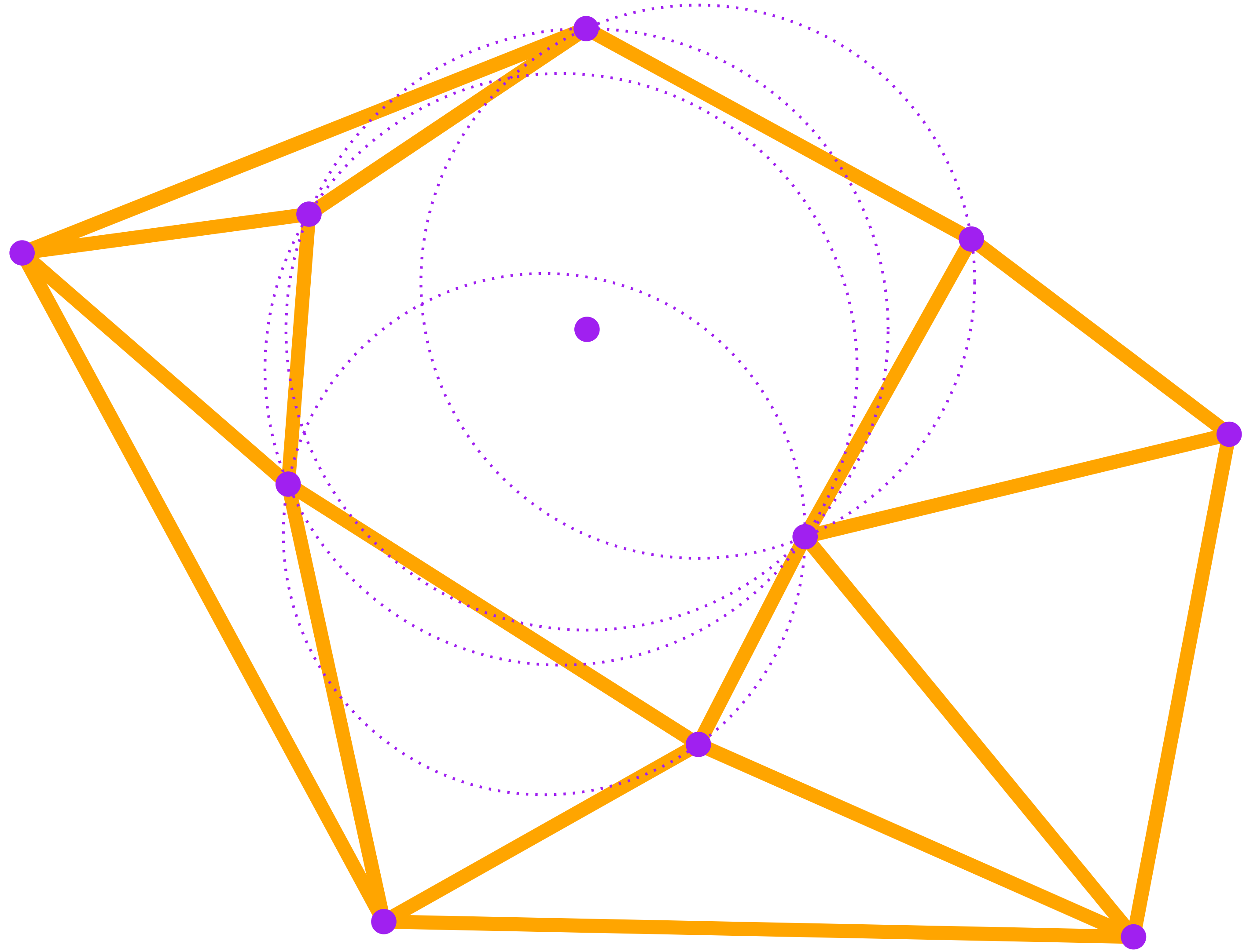
$$= \sum_{j=0}^{n} j \times \left(\mathbb{P}\left[\Delta \text{ with } j\right] - \mathbb{P}\left[\Delta \text{ with } j+1\right]\right) \times \sharp\Delta \text{ with } \leq j \text{ stoppers}$$

$$\simeq \sum_{j=0}^{n} j \times \frac{18}{j^4} \times nj^2 = O(n\sum\frac{1}{j}) = O(n\log n)$$

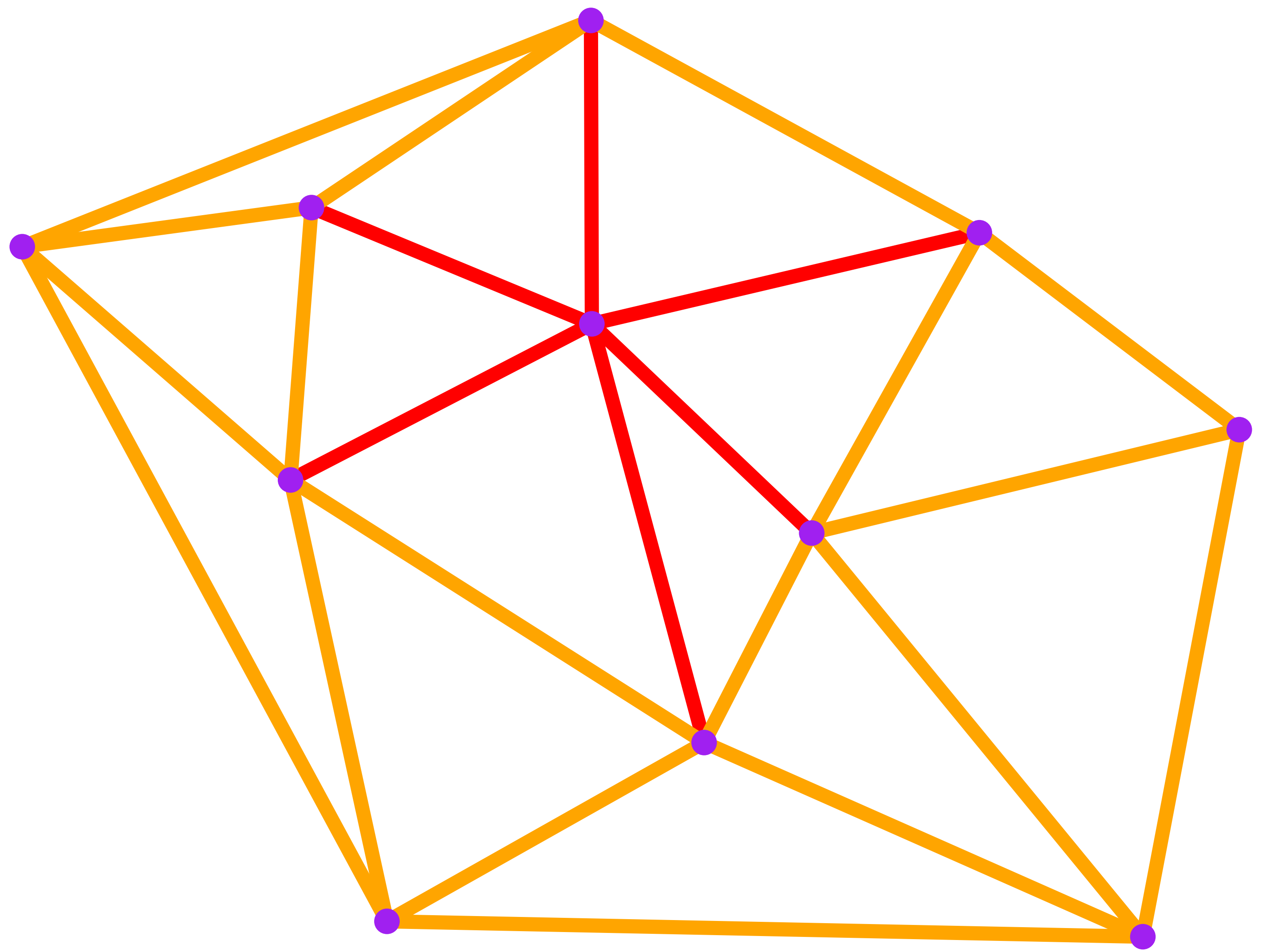
History graph



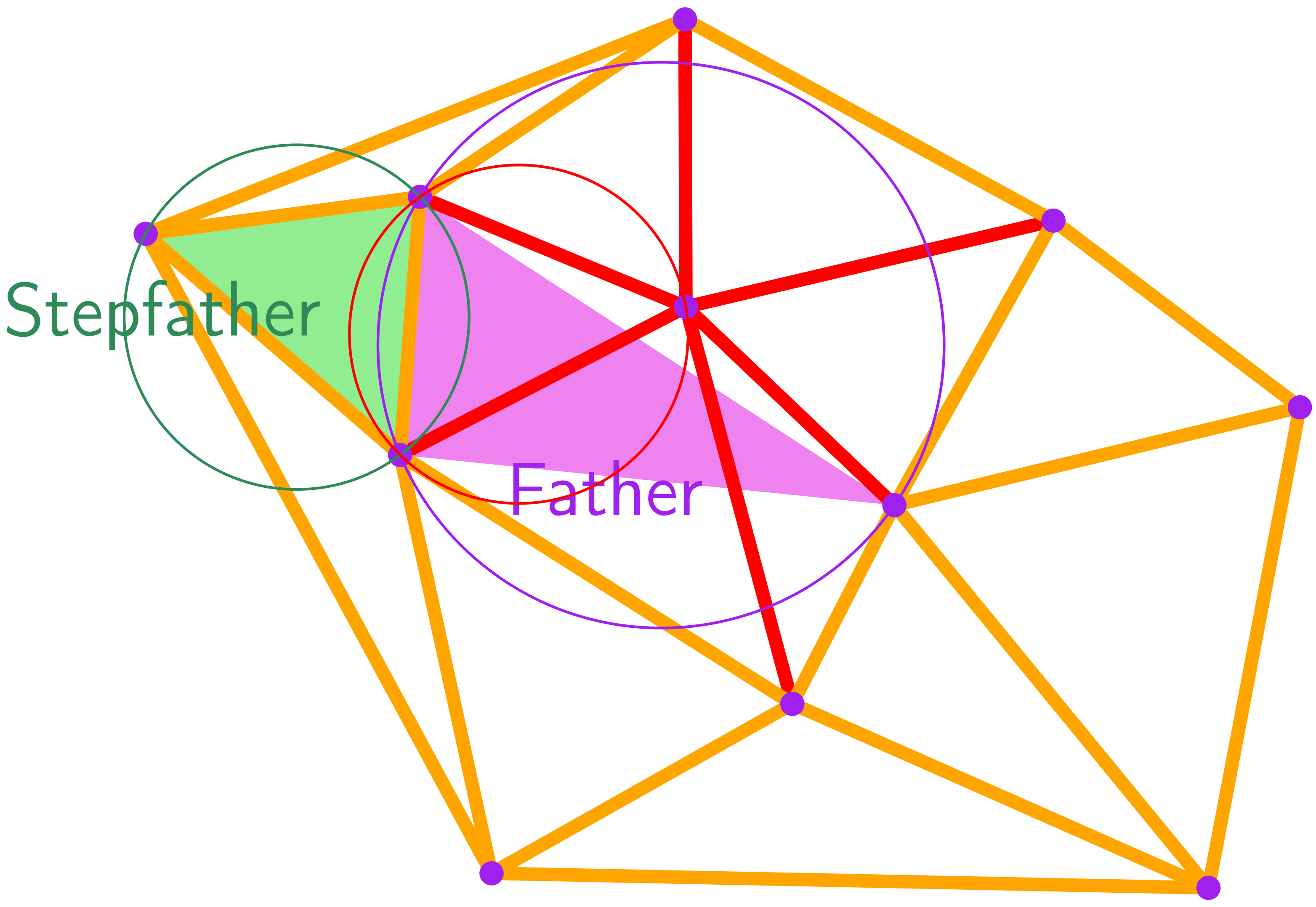
History graph



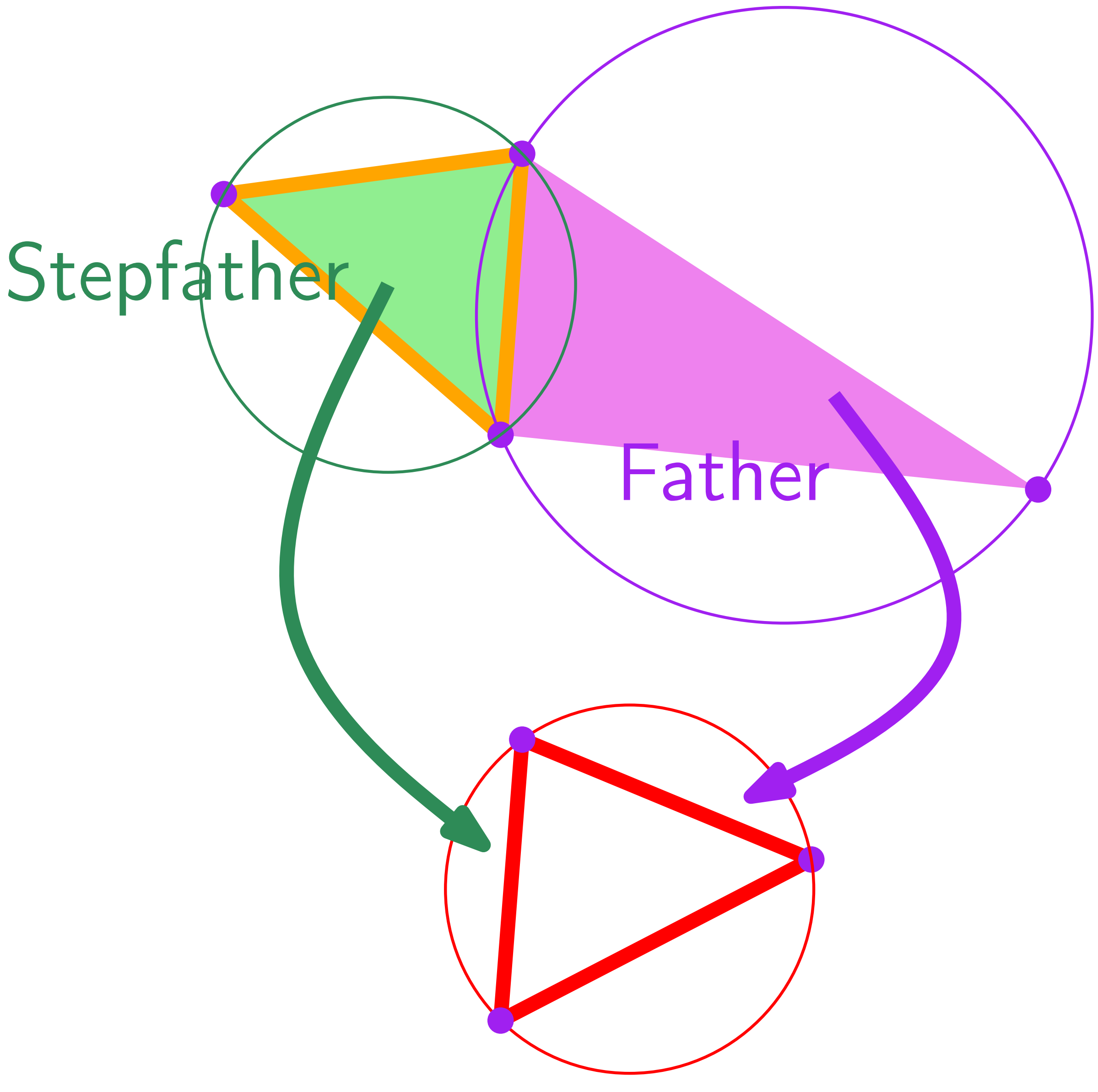
History graph

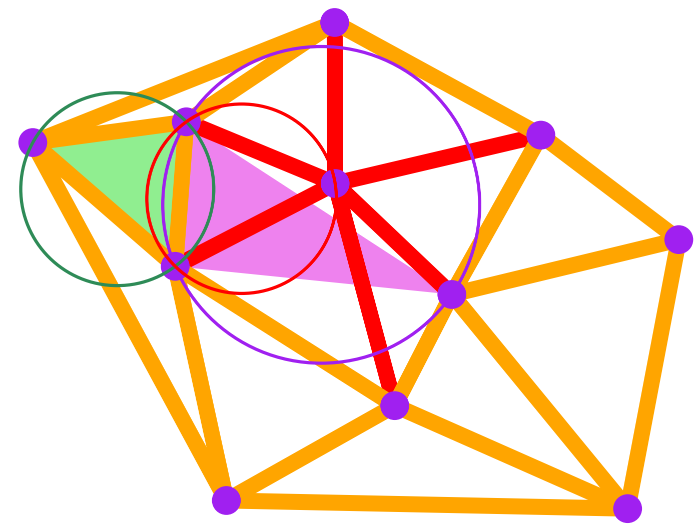


History graph

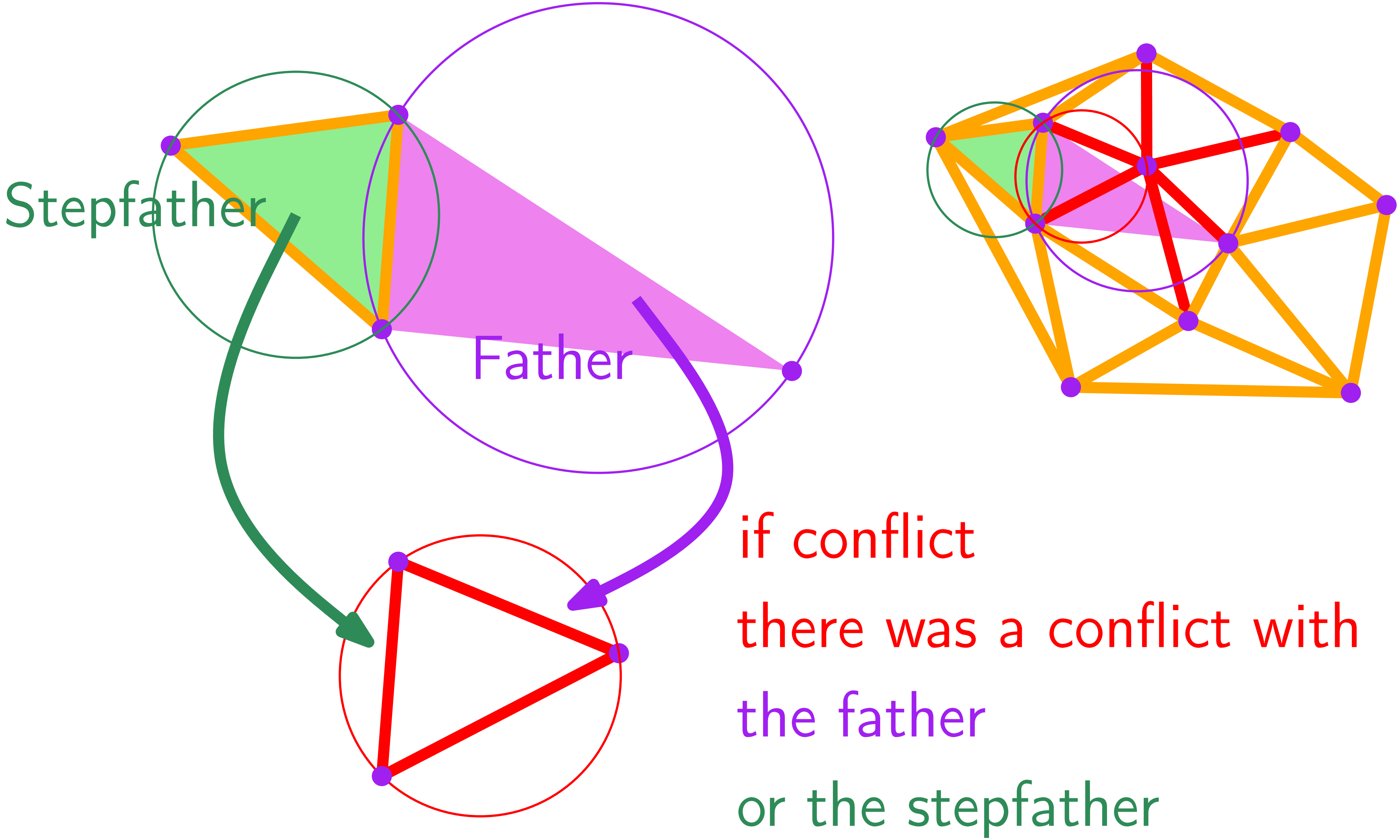


History graph (Delaunay tree)





History graph (Delaunay tree)



if conflict

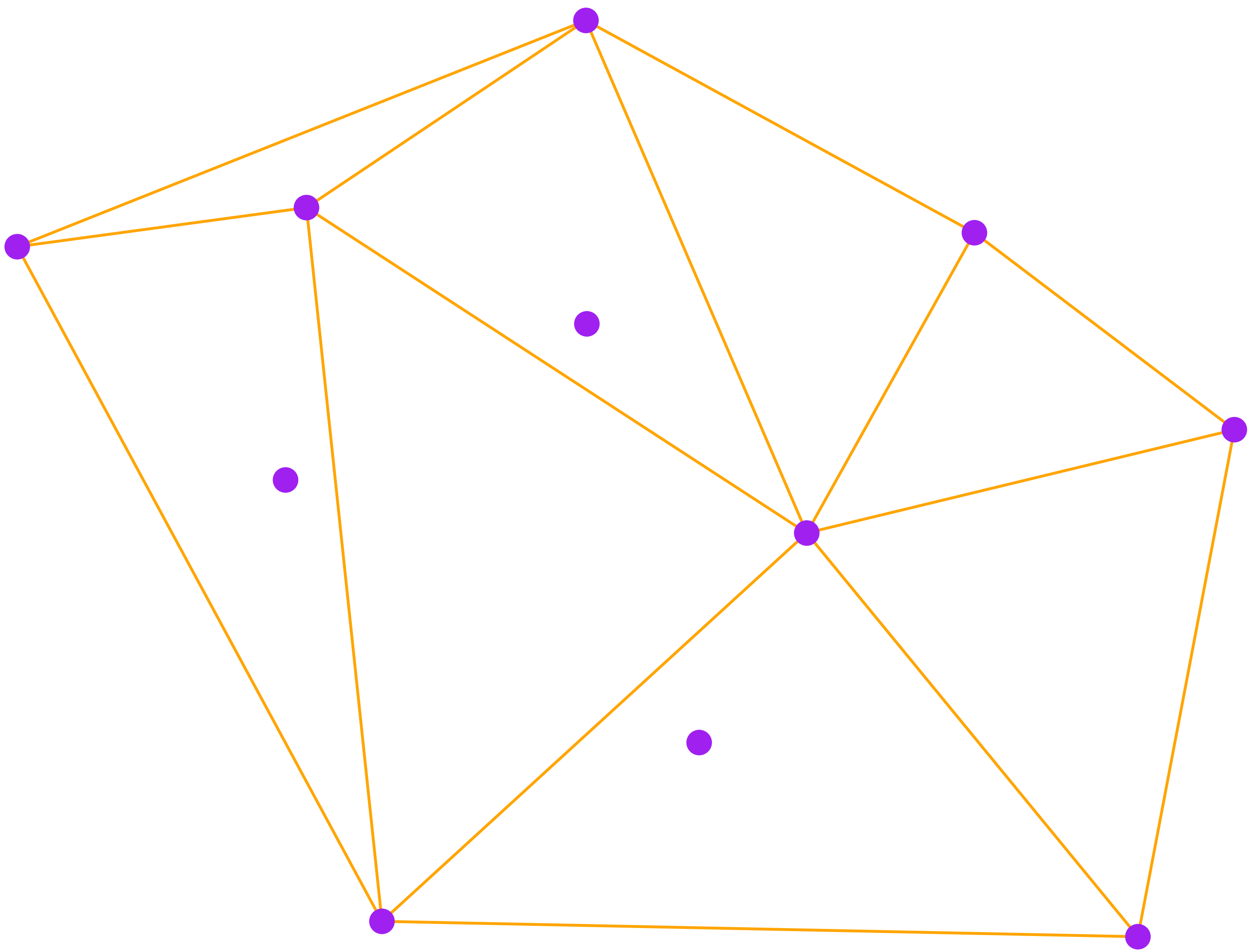
there was a conflict with

the father

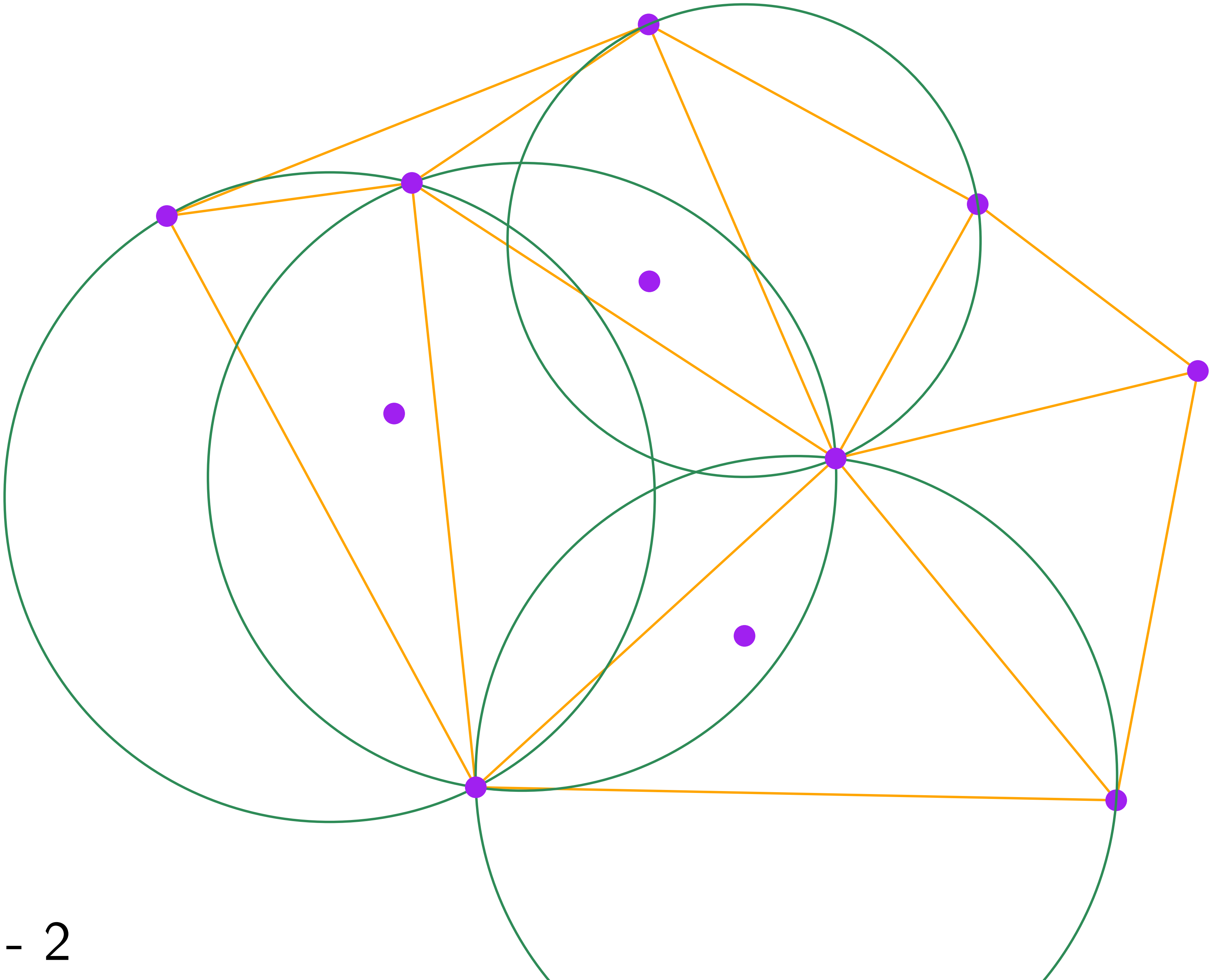
or the stepfather

or both

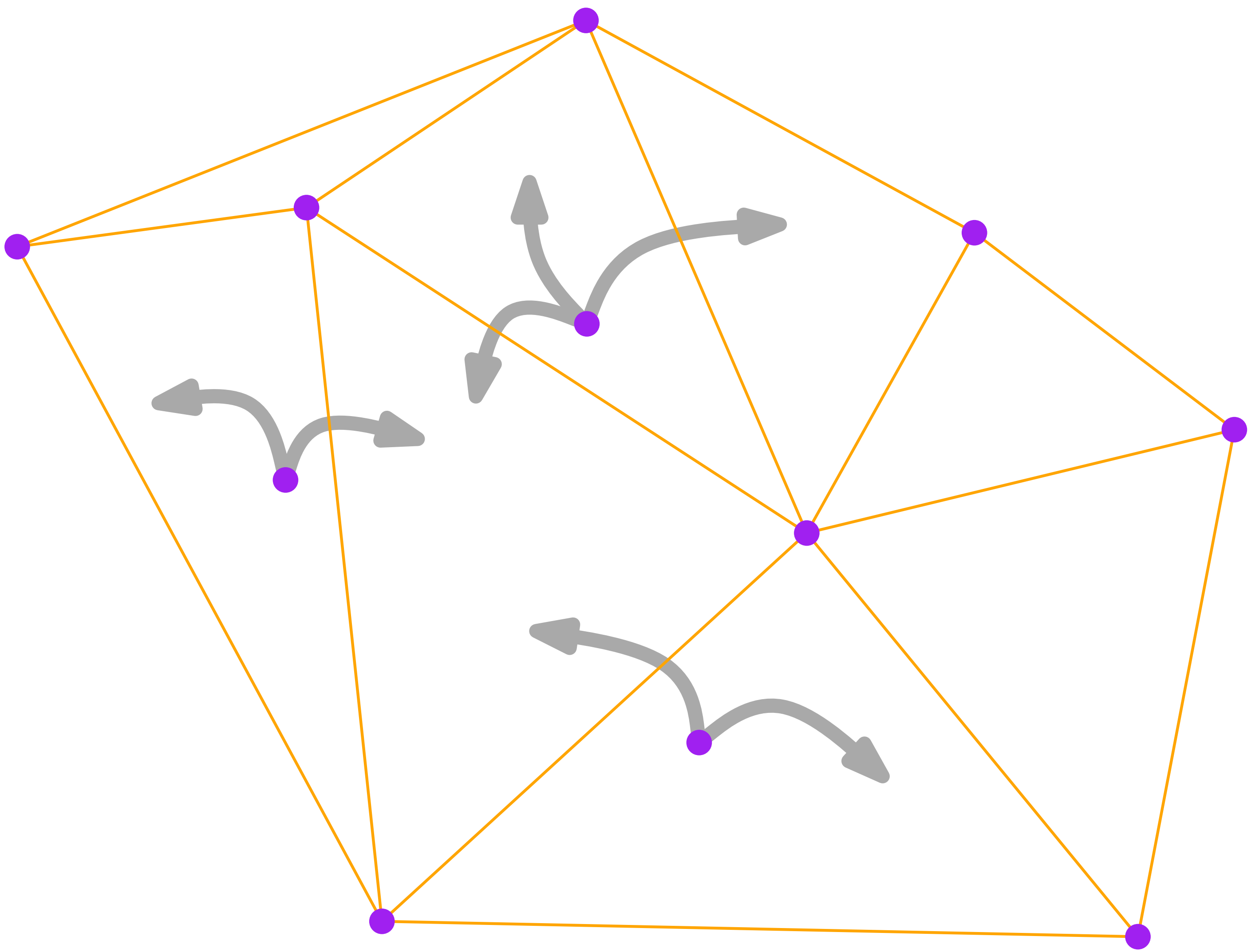
Conflict graph



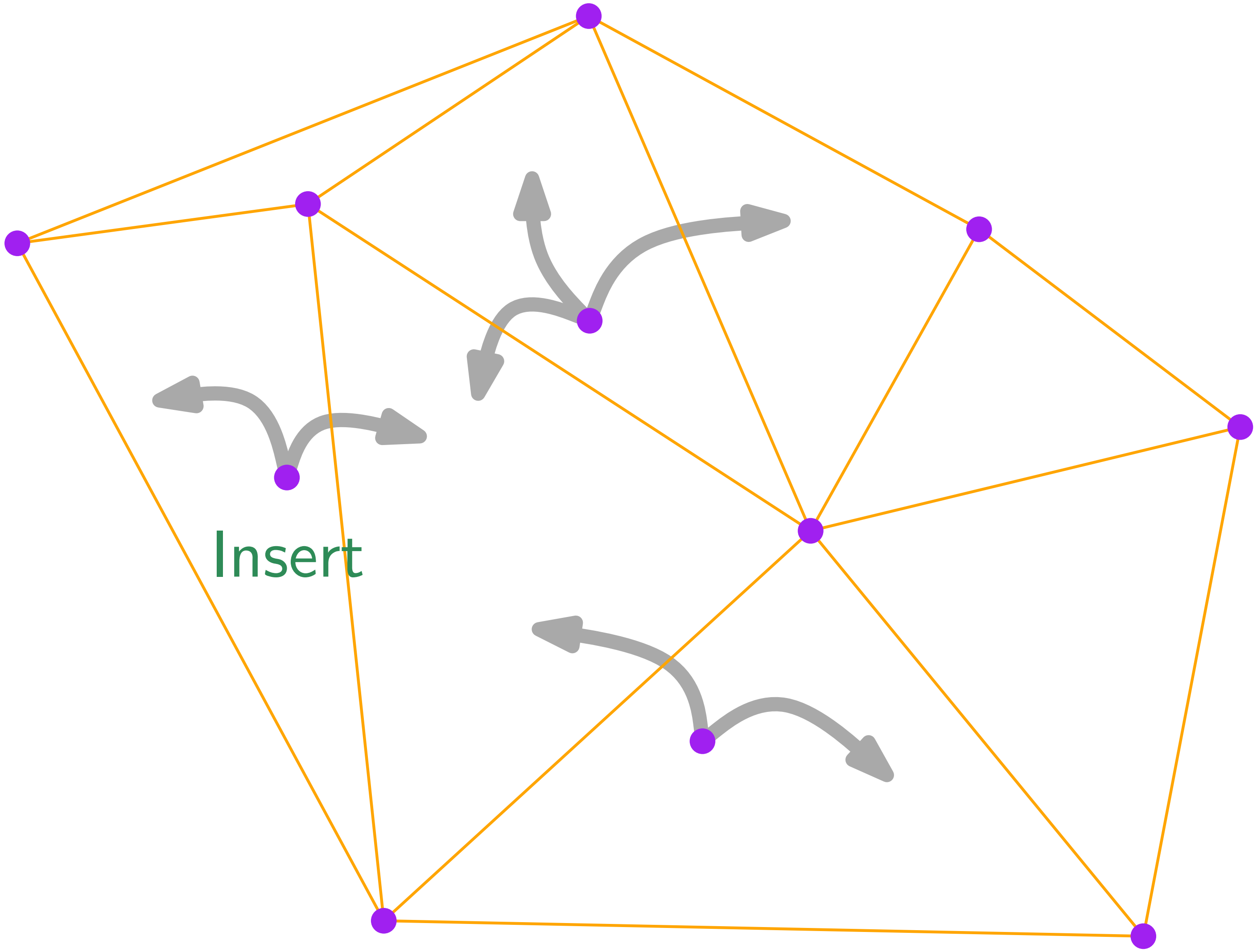
Conflict graph



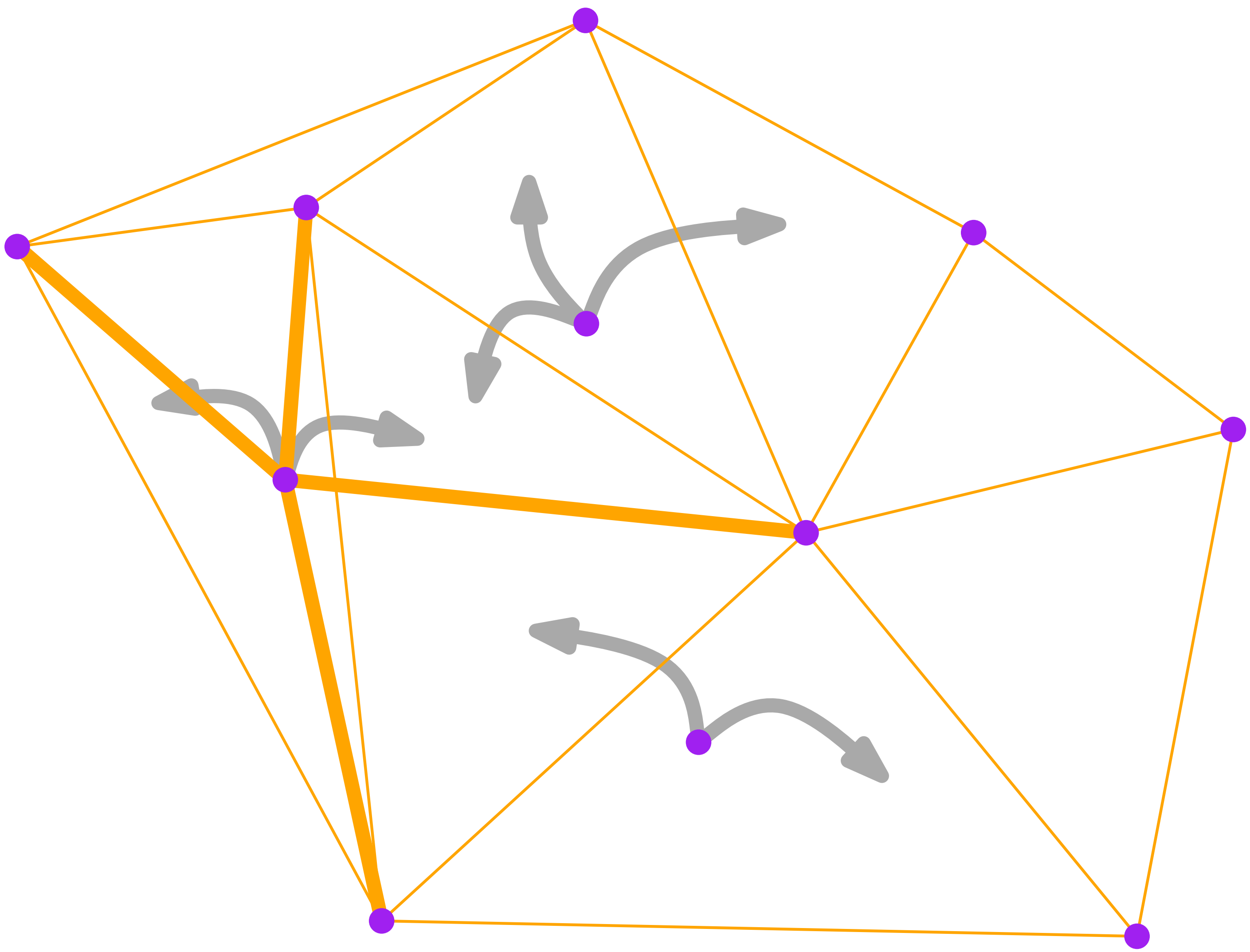
Conflict graph



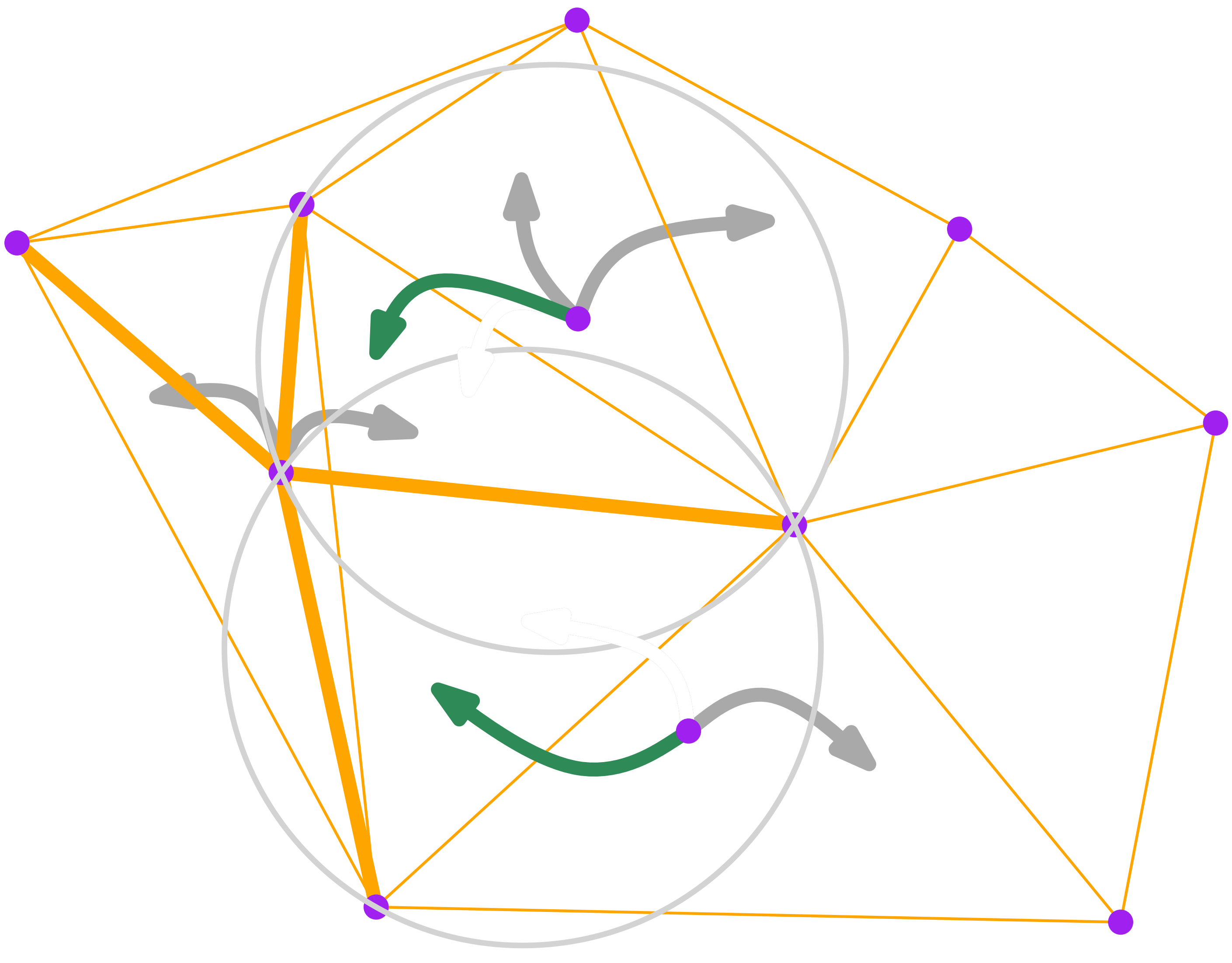
Conflict graph



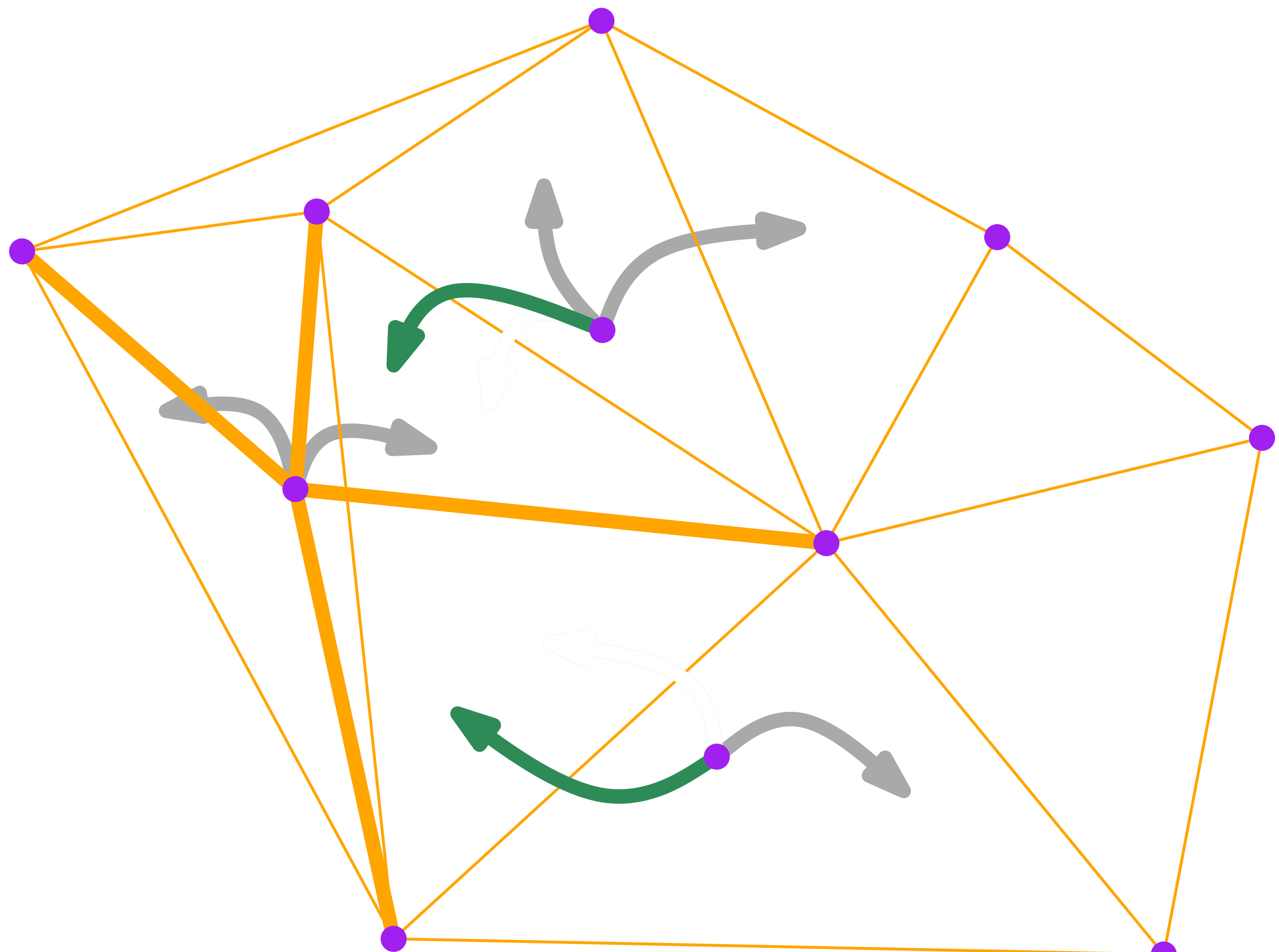
Conflict graph



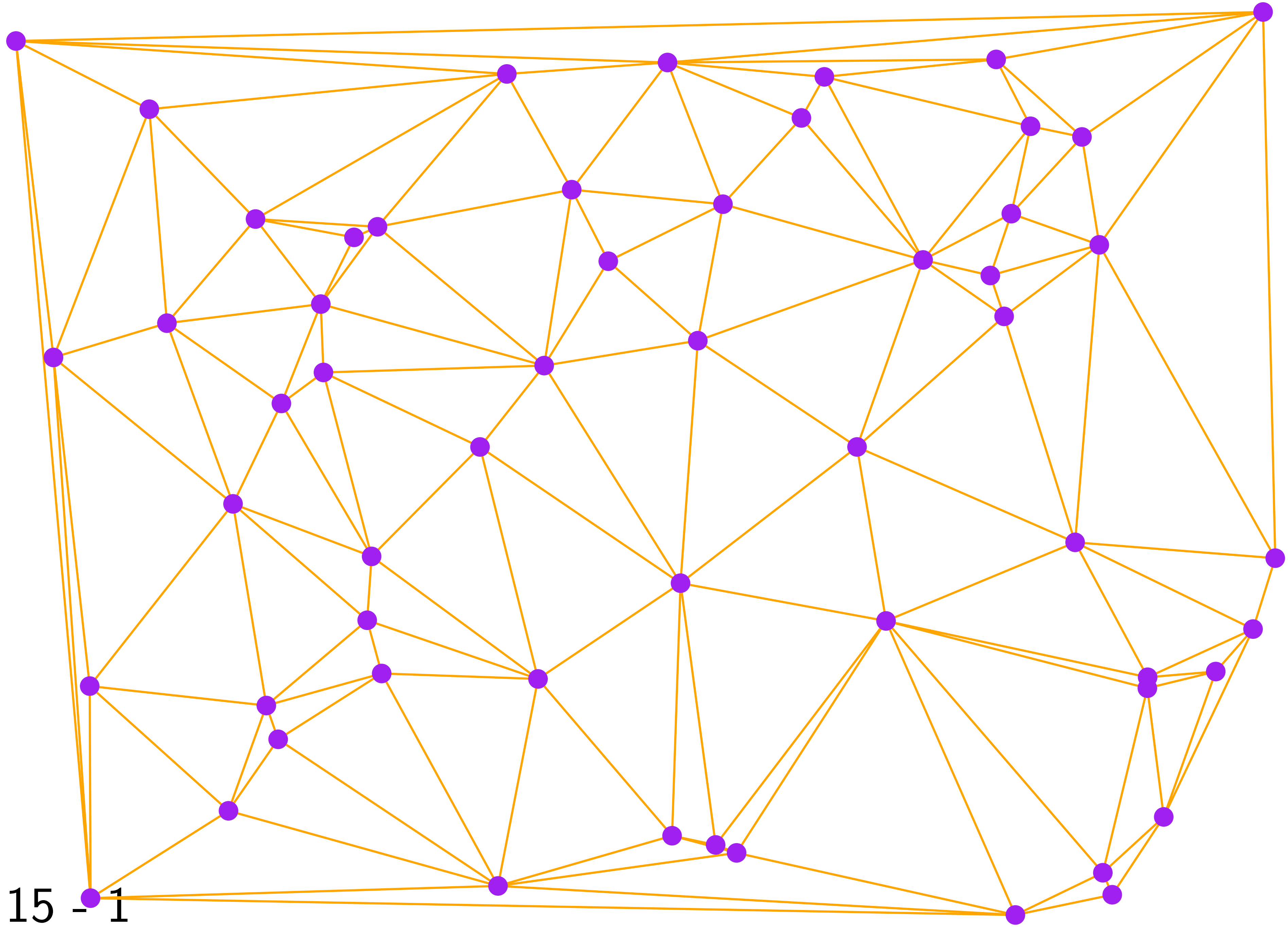
Conflict graph



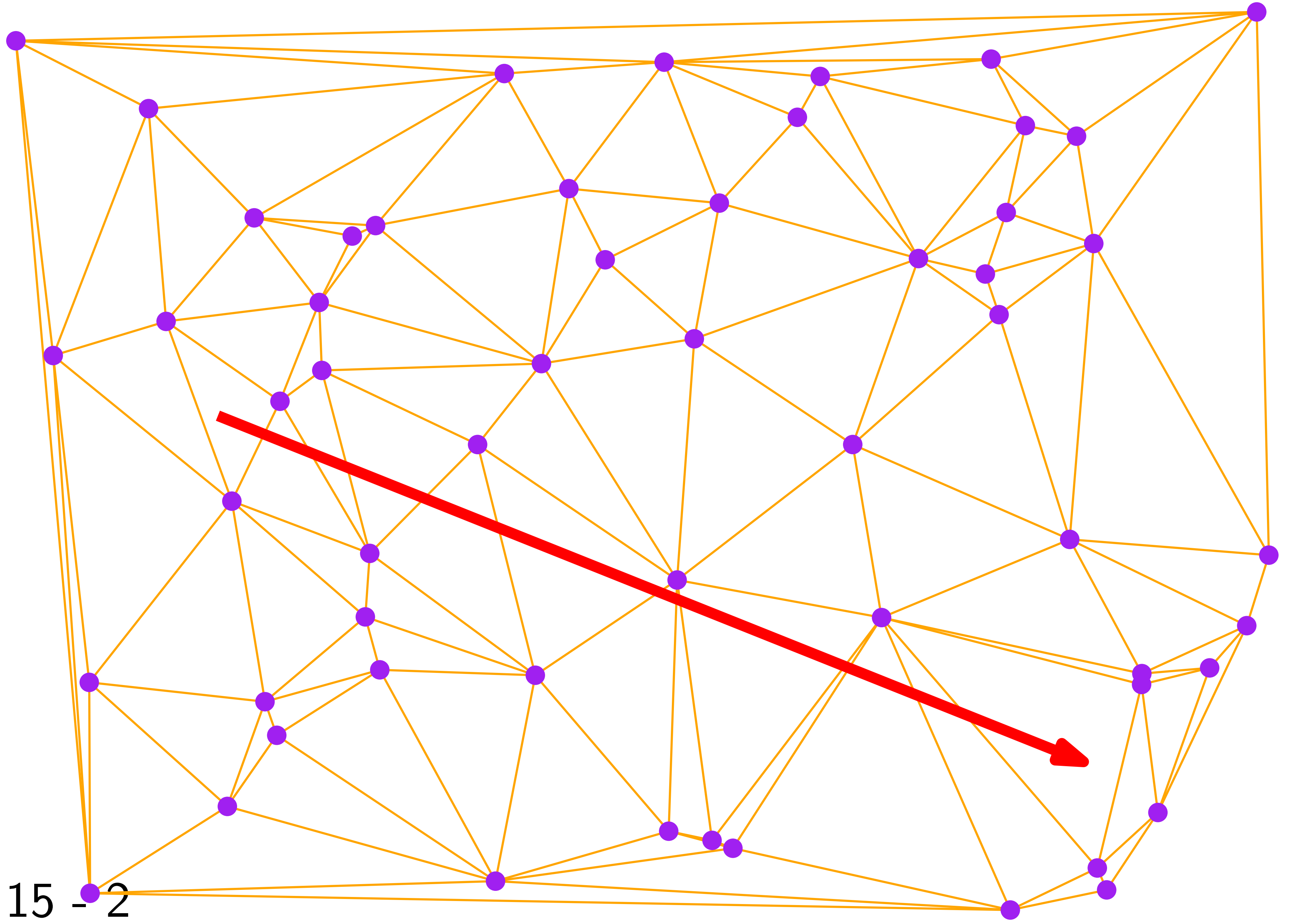
Conflict graph



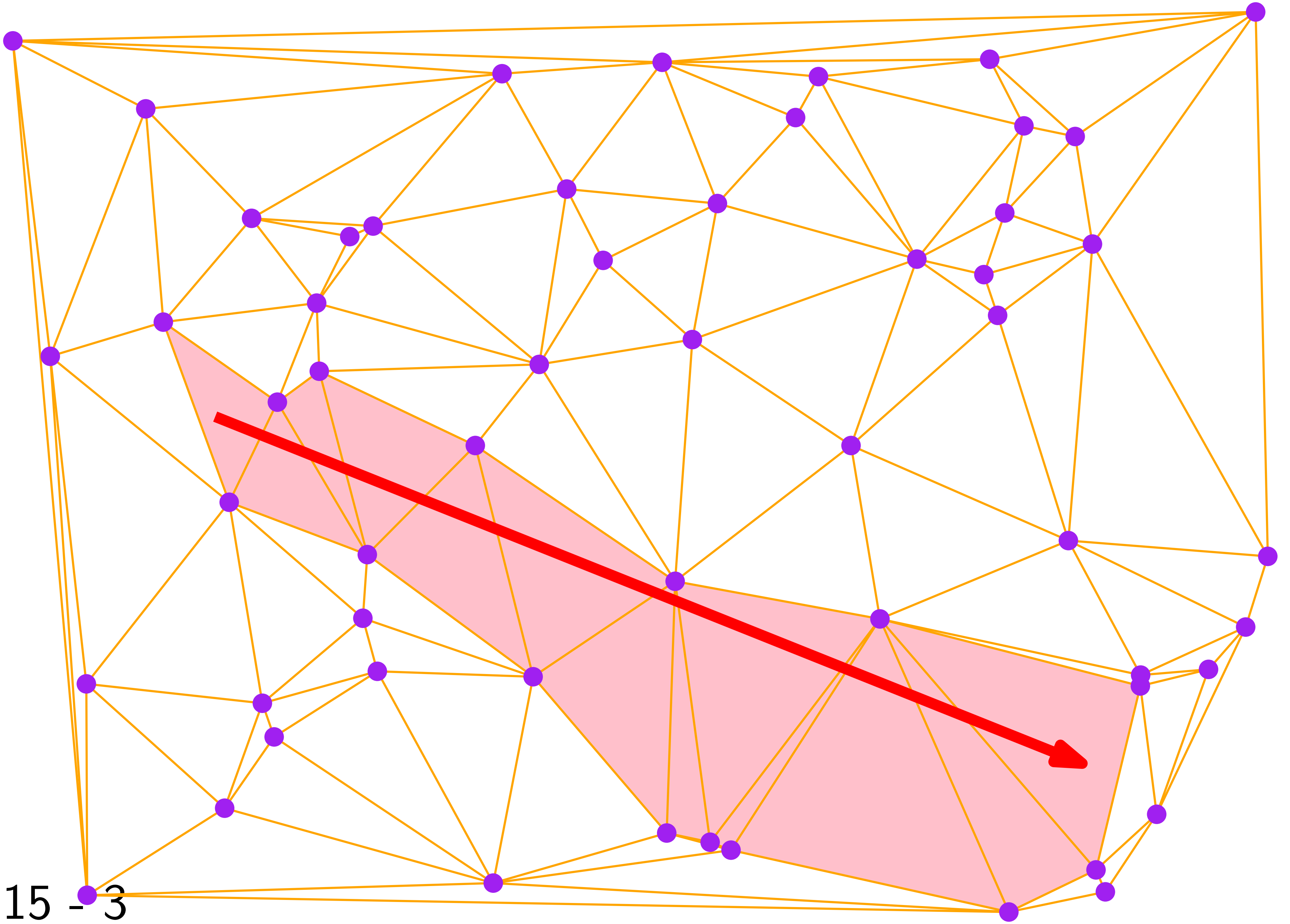
Walk



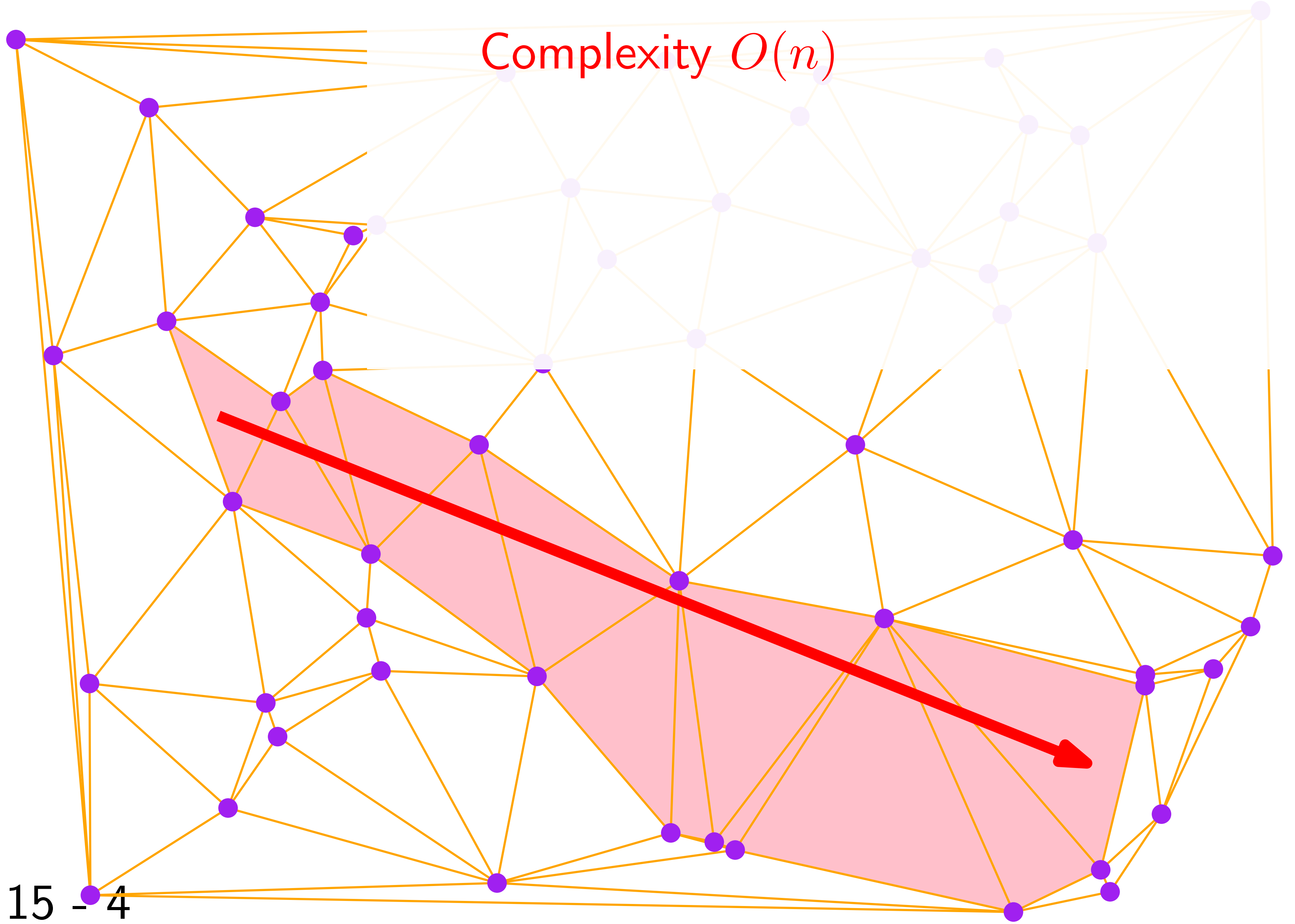
Walk



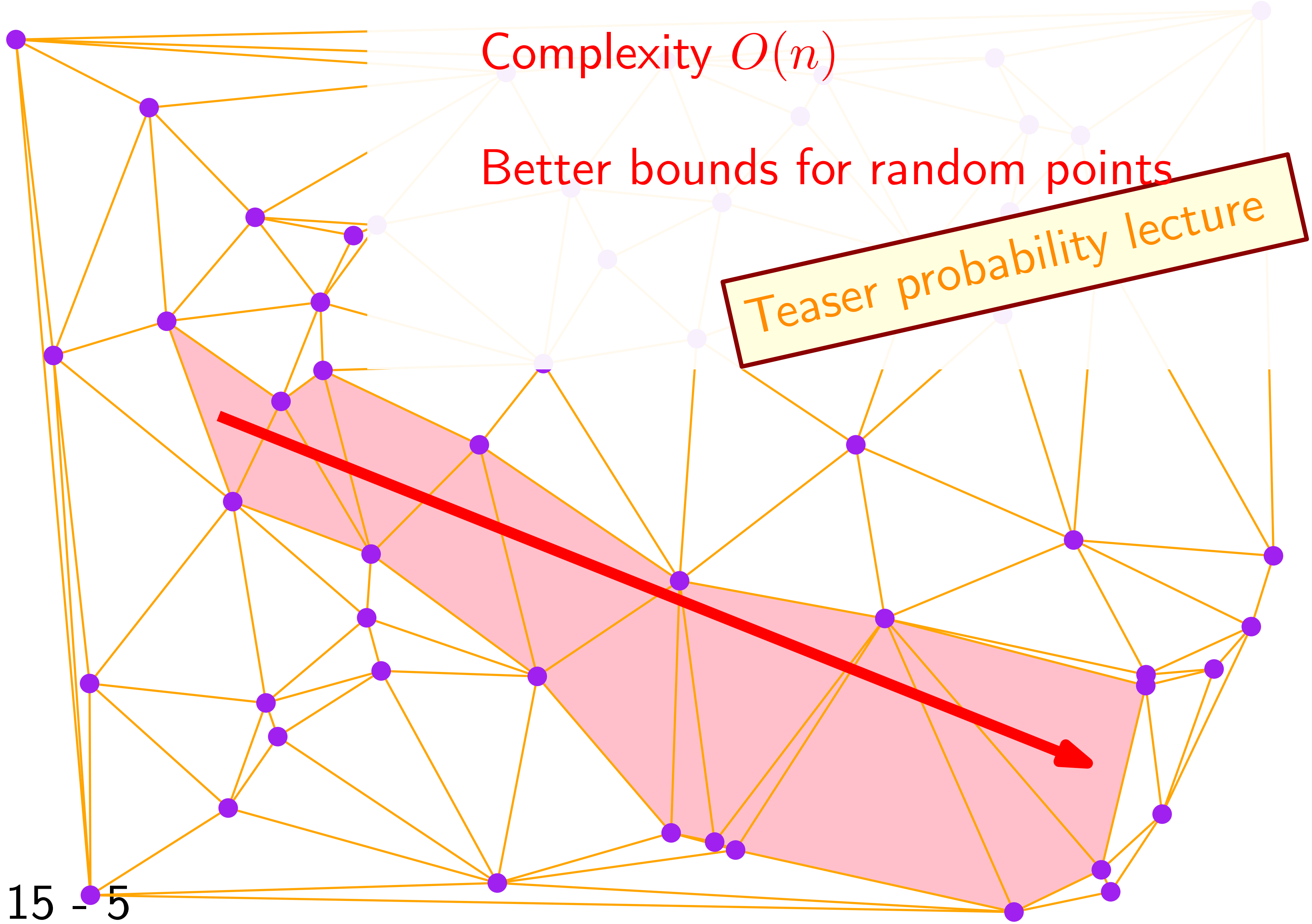
Walk



Walk



Walk

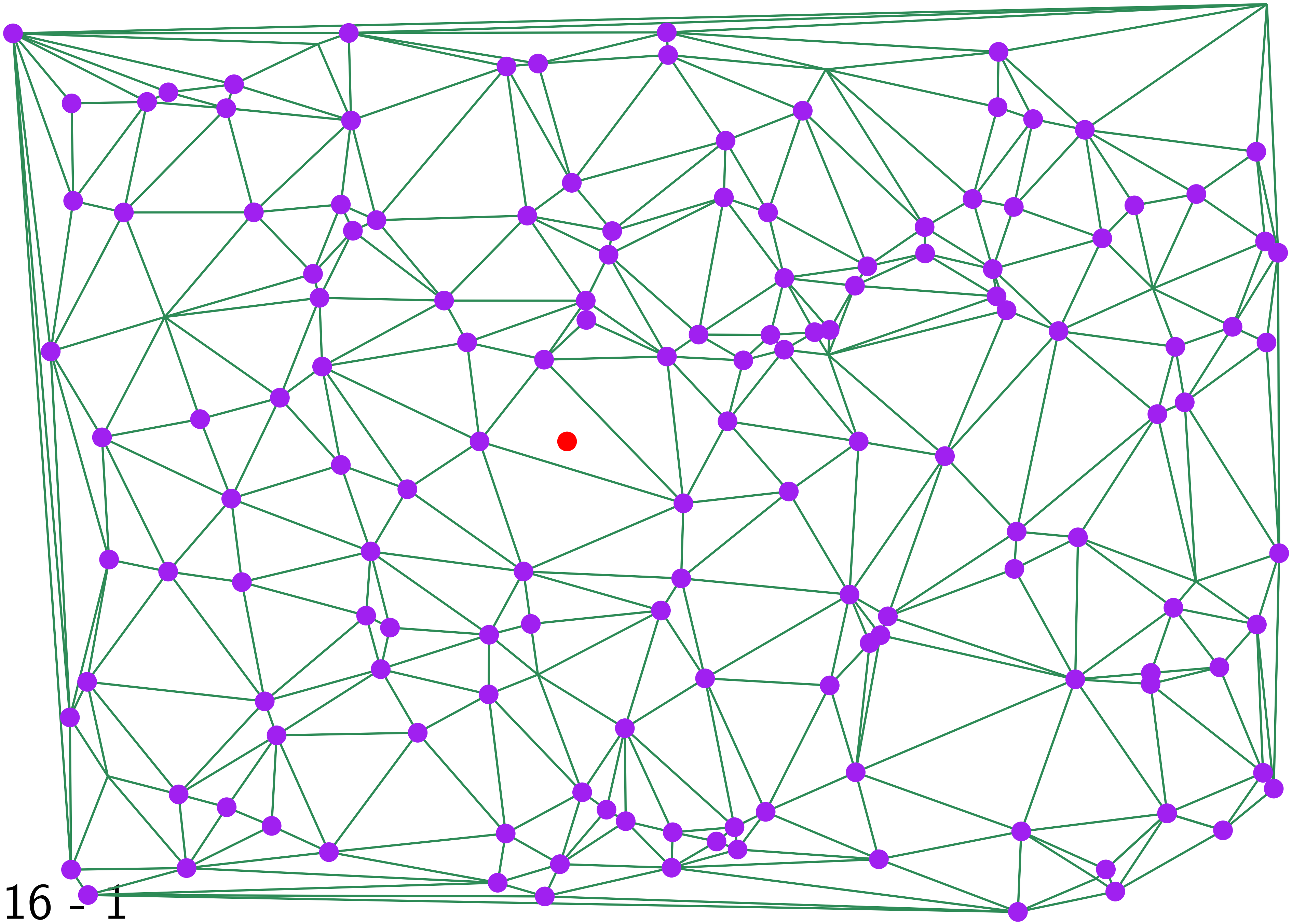


Complexity $O(n)$

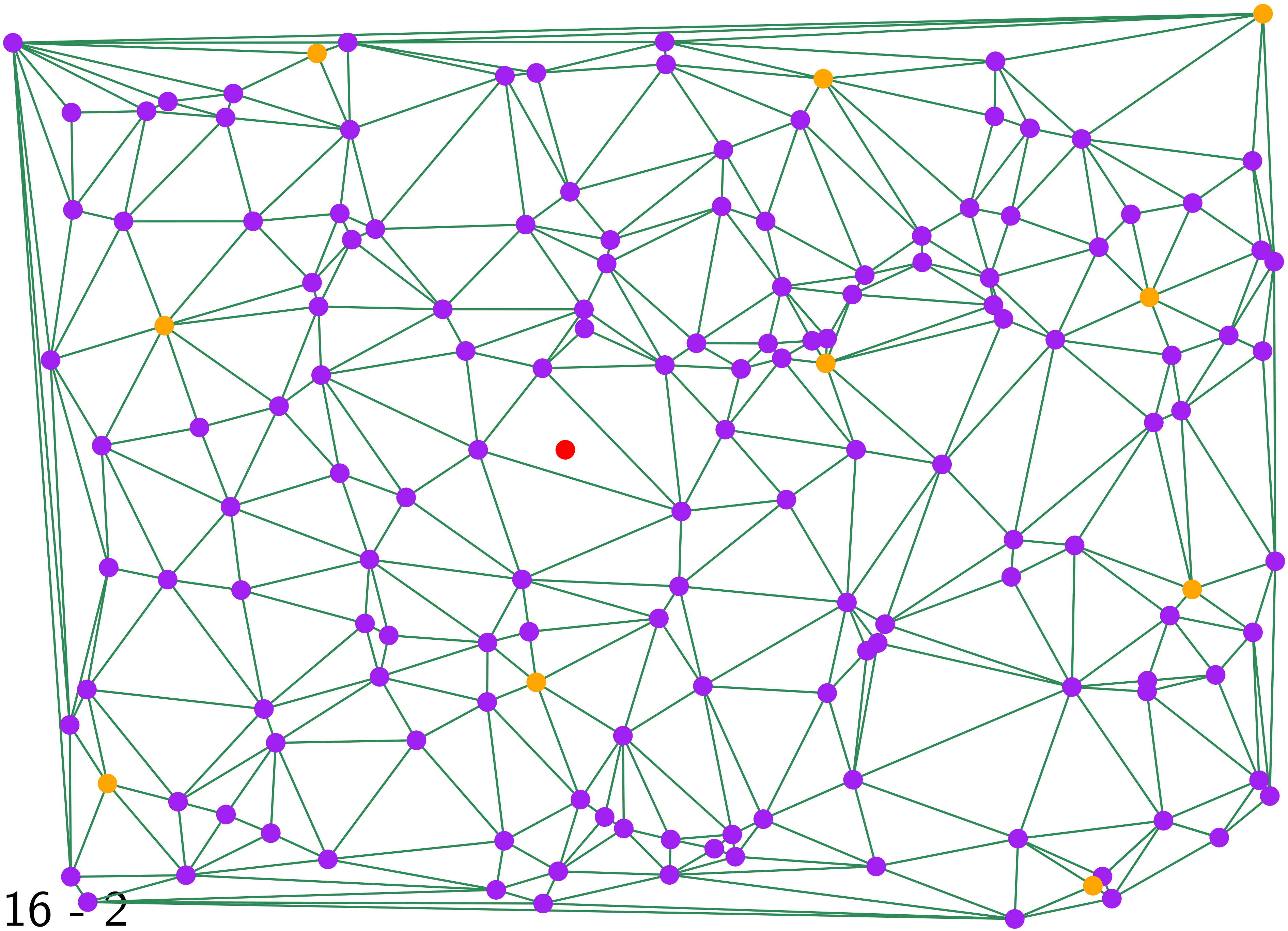
Better bounds for random points

Teaser probability lecture

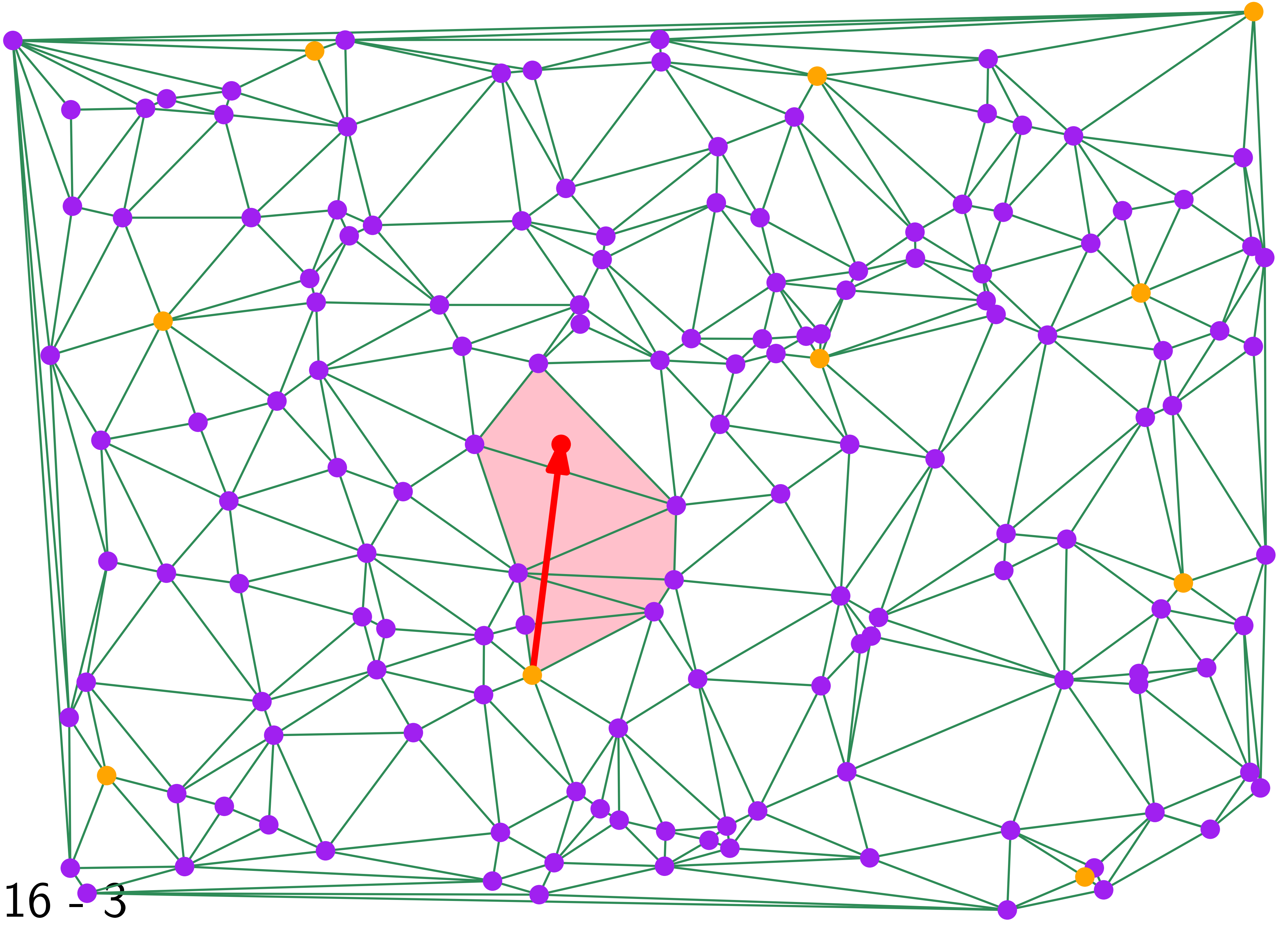
Jump and walk



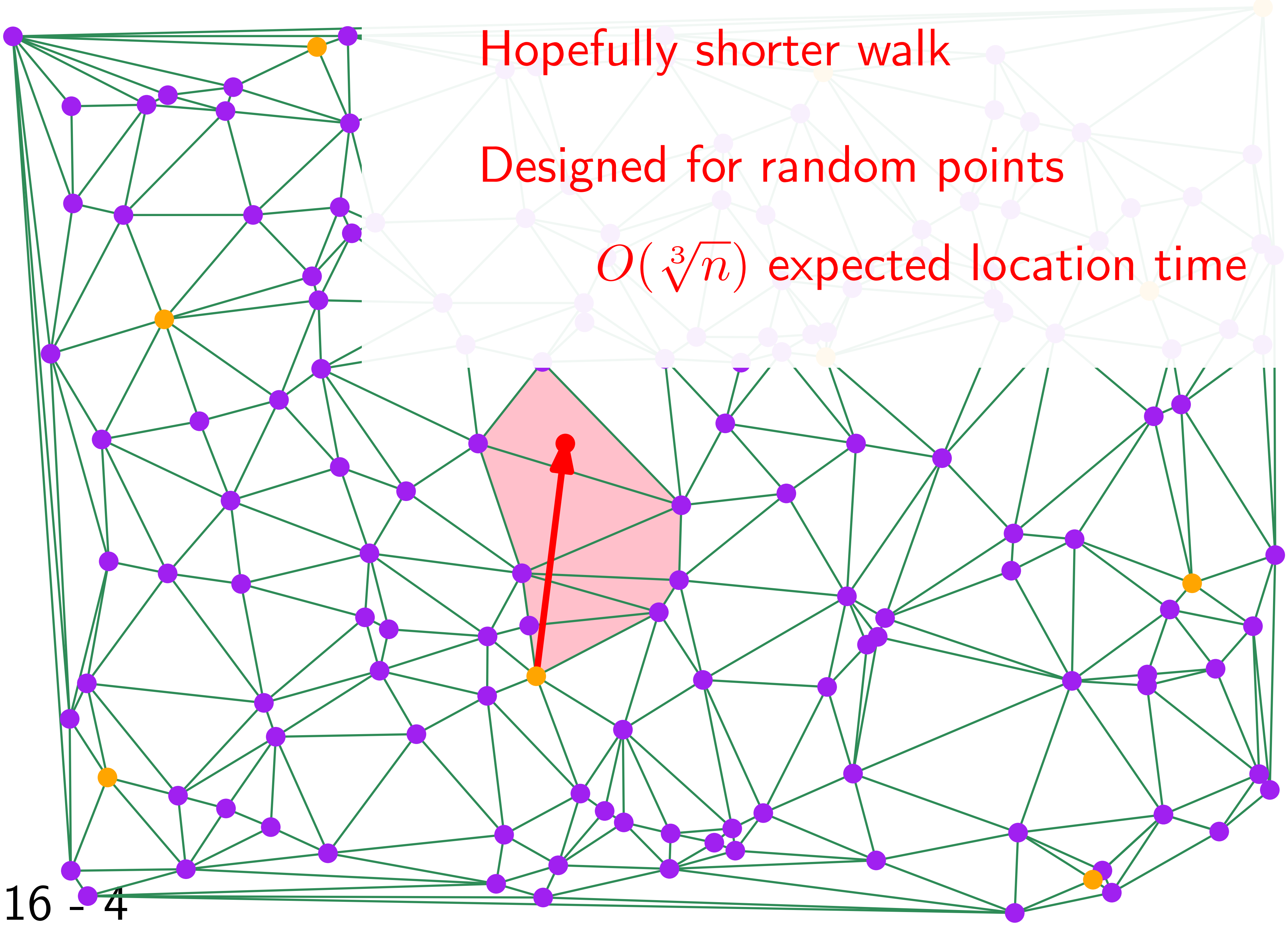
Jump and walk



Jump and walk



Jump and walk

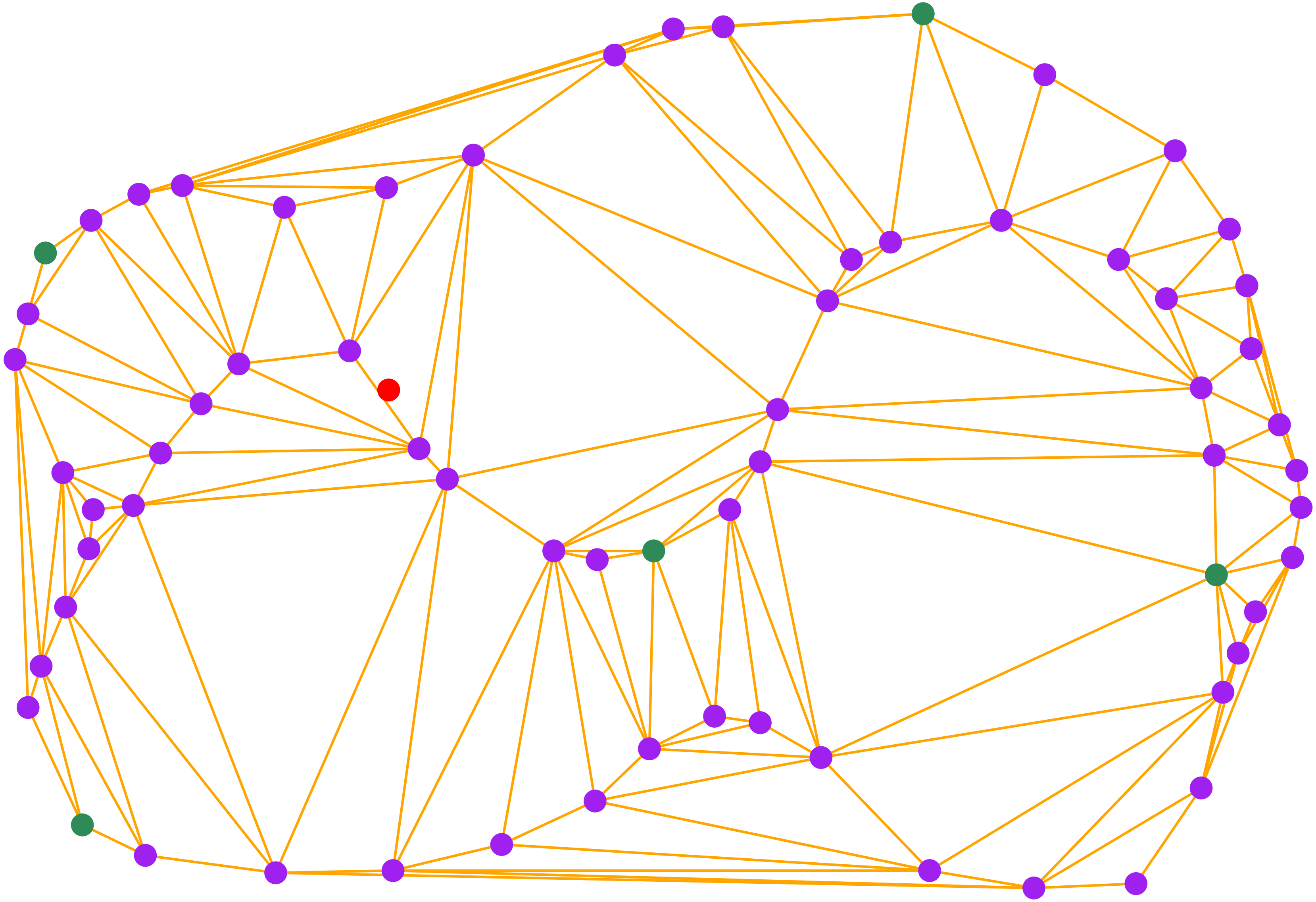


Hopefully shorter walk

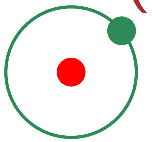
Designed for random points

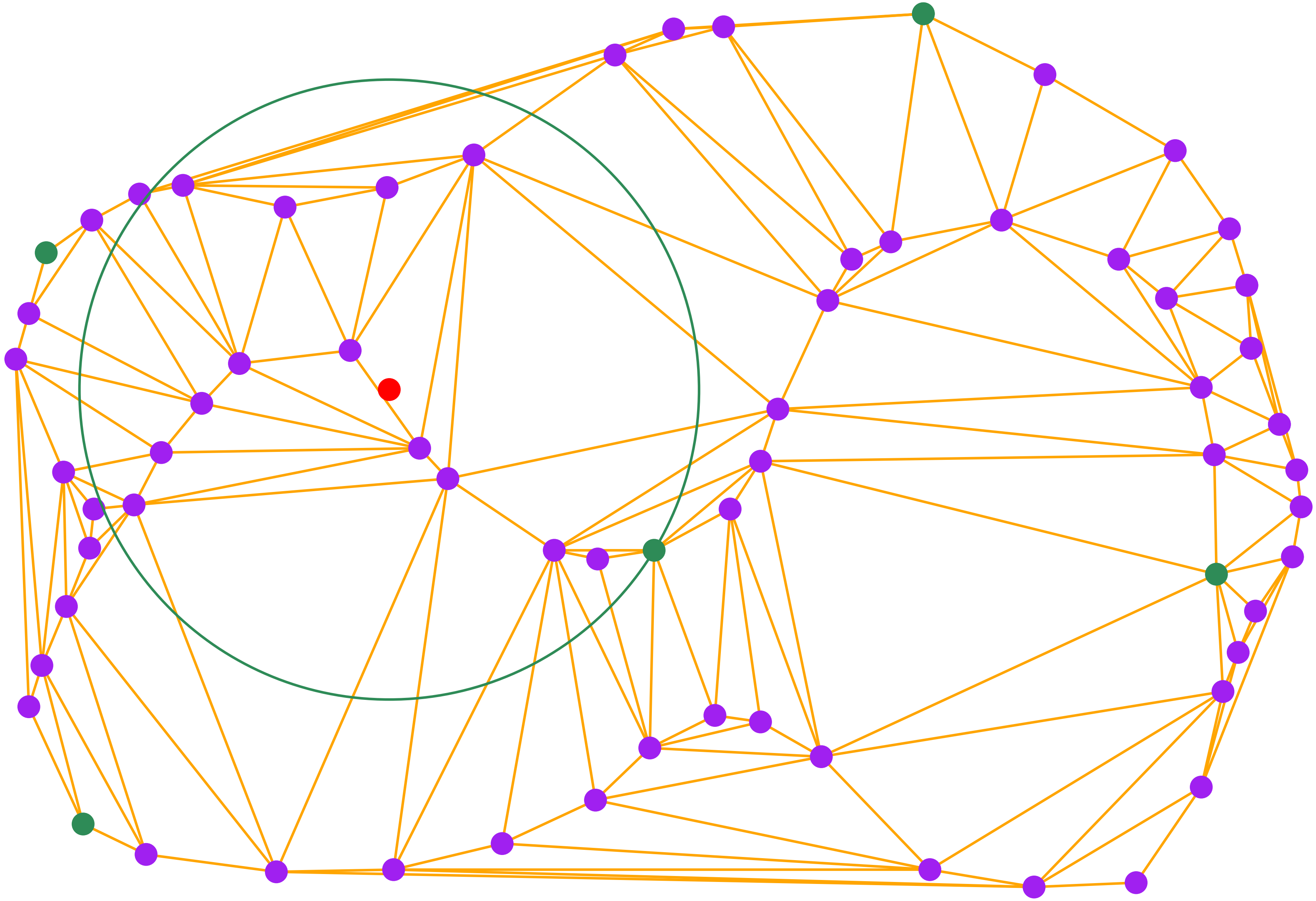
$O(\sqrt[3]{n})$ expected location time

Jump and walk (no distribution hypothesis)



Jump and walk (no distribution hypothesis)

$\mathbb{E}[\sharp$ of  in  $] = \frac{n}{k}$

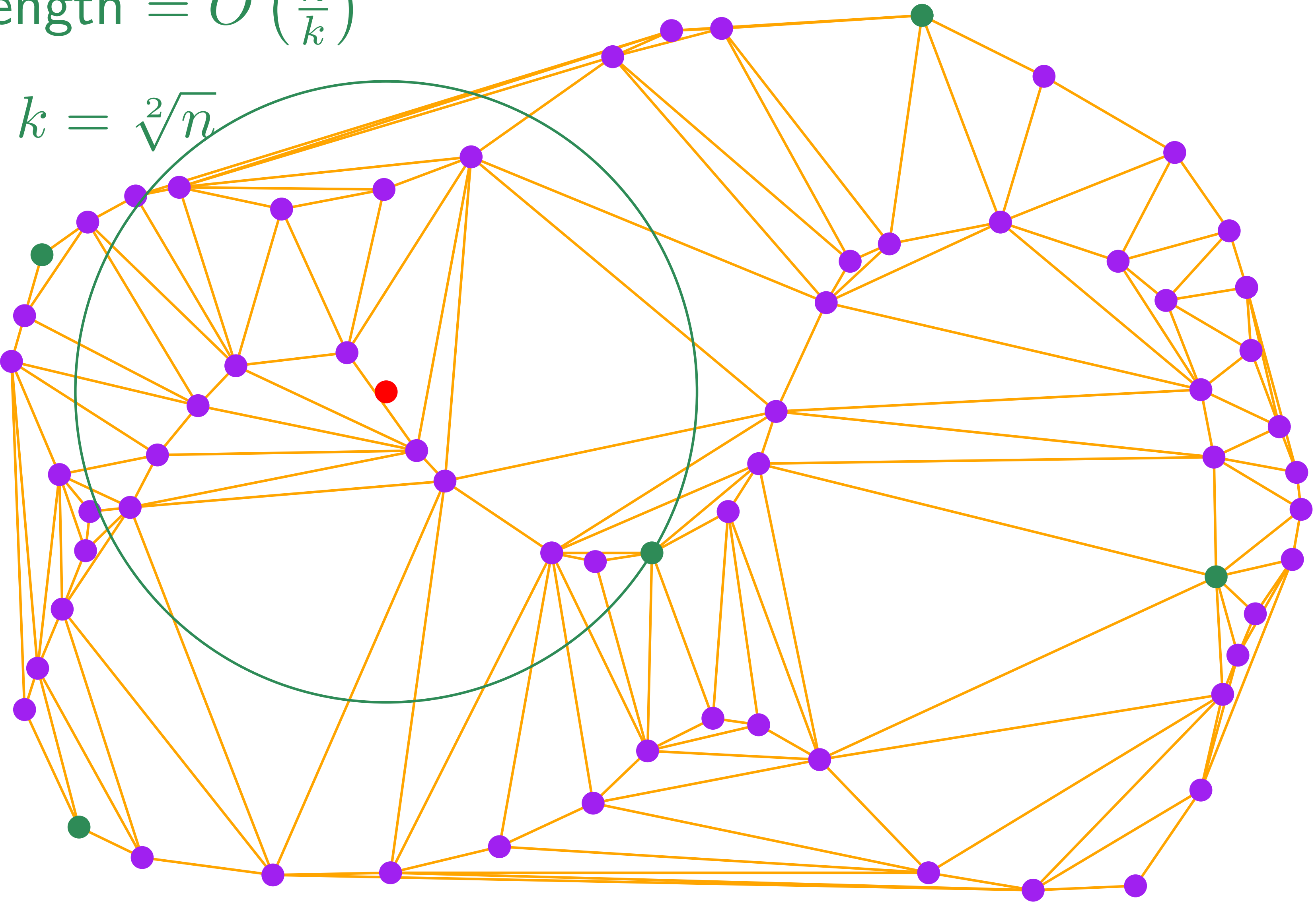


Jump and walk (no distribution hypothesis)

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \bigcirc\right] = \frac{n}{k}$

Walk length $= O\left(\frac{n}{k}\right)$

choose $k = \sqrt[2]{n}$

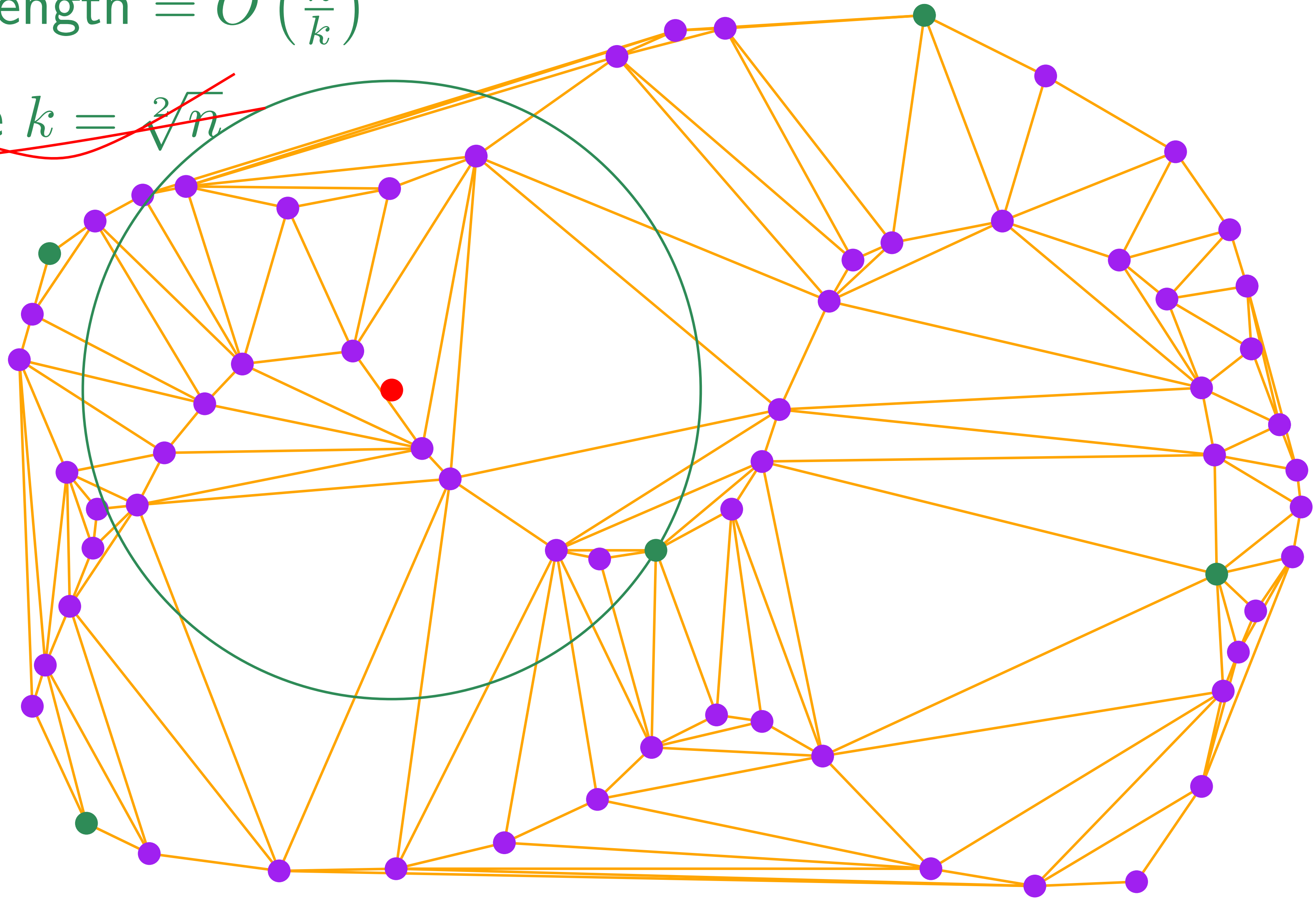


Jump and walk (no distribution hypothesis)Delaunay hierarchy

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \bigodot \right] = \frac{n}{k}$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~



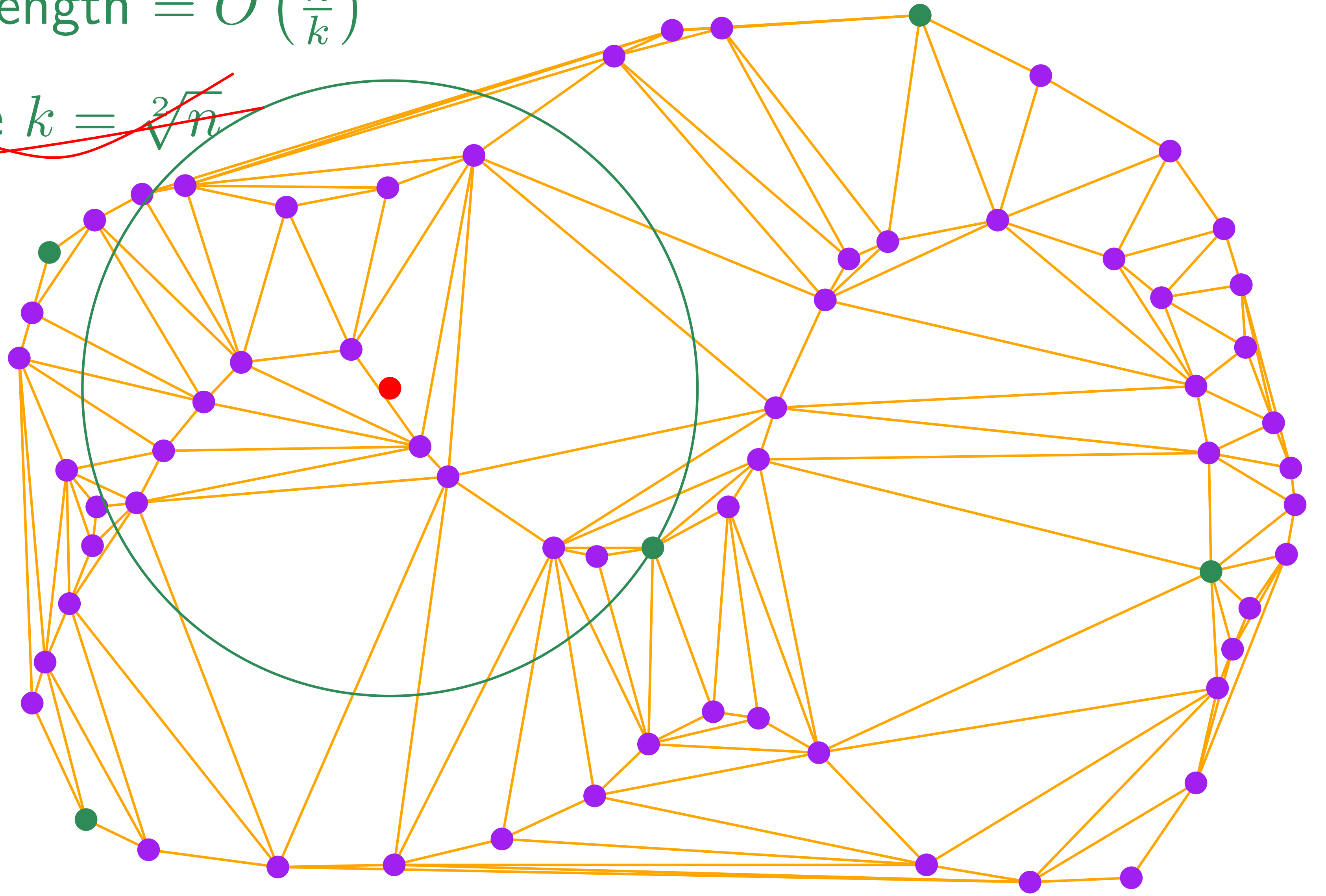
Jump and walk (no distribution hypothesis)Delaunay hierarchy

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \bigcirc\right] = \frac{n}{k}$

$\frac{n}{k_1}$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~

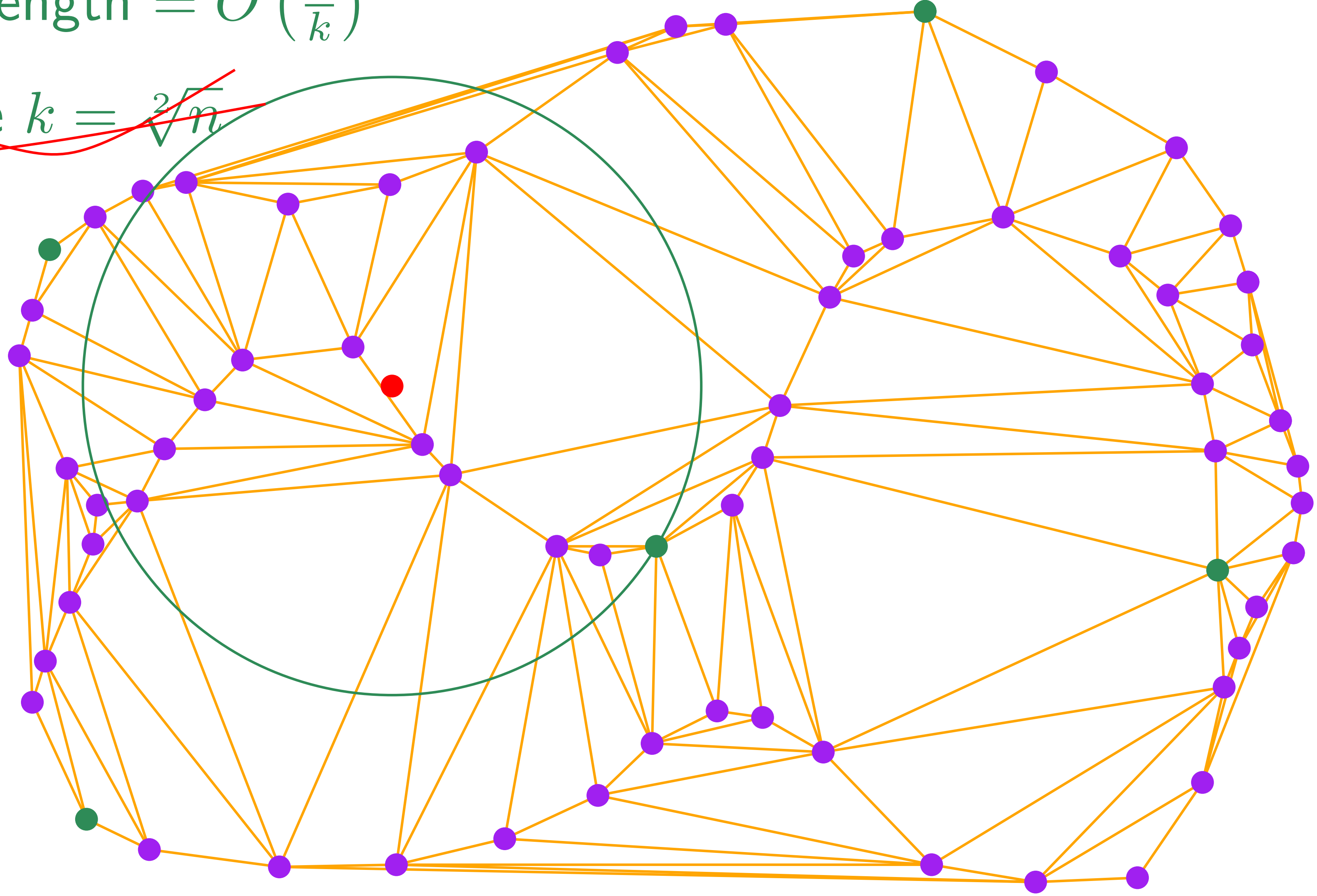


Jump and walk (no distribution hypothesis)Delaunay hierarchy

$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \bigcirc\right] = \frac{n}{k}$ $\qquad \frac{n}{k_1} + \frac{k_1}{k_2}$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~



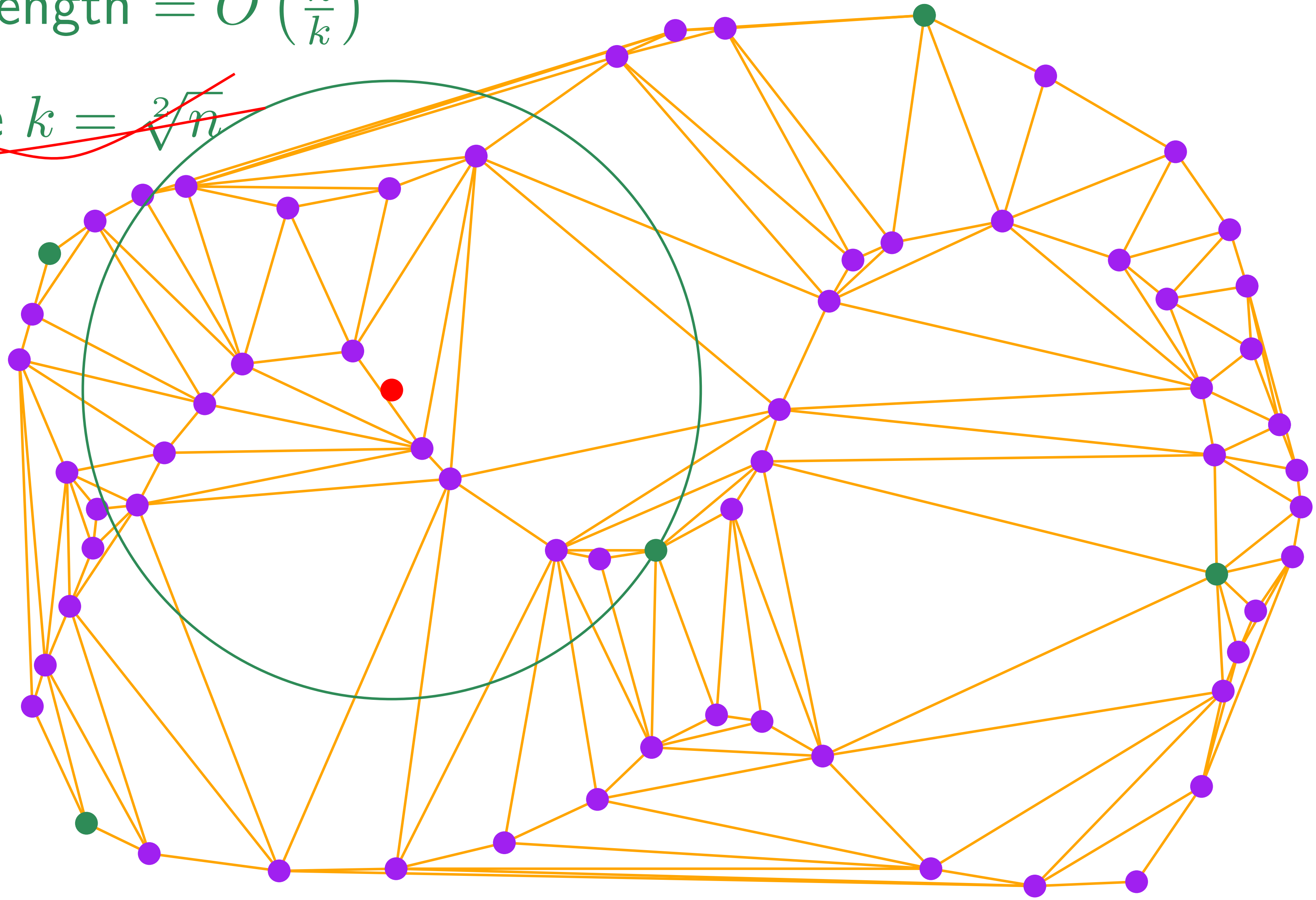
Jump and walk (no distribution hypothesis) Delaunay hierarchy

$\mathbb{E}\,[\sharp$ of $\bullet$ in $\bullet\,] = \frac{n}{k}$

$$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots$$

Walk length $= O\left(\frac{n}{k}\right)$

~~choose $k = \sqrt[2]{n}$~~



Jump and walk (no distribution hypothesis)Delaunay hierarchy

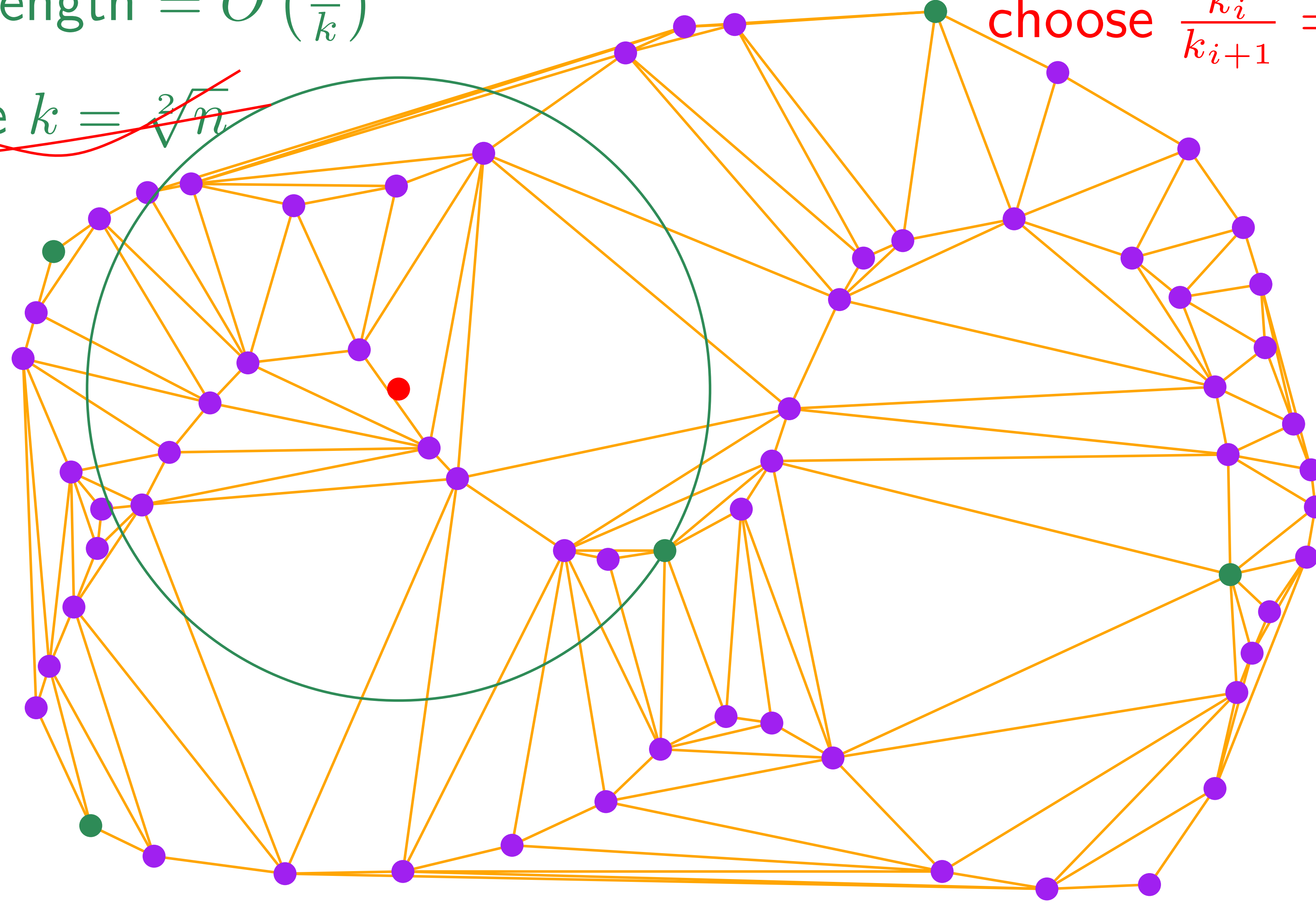
$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \odot\right] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots$

Walk length $= O\left(\frac{n}{k}\right)$

choose $\frac{k_i}{k_{i+1}} = \alpha$

~~choose $k = \sqrt[2]{n}$~~



Jump and walk (no distribution hypothesis)Delaunay hierarchy

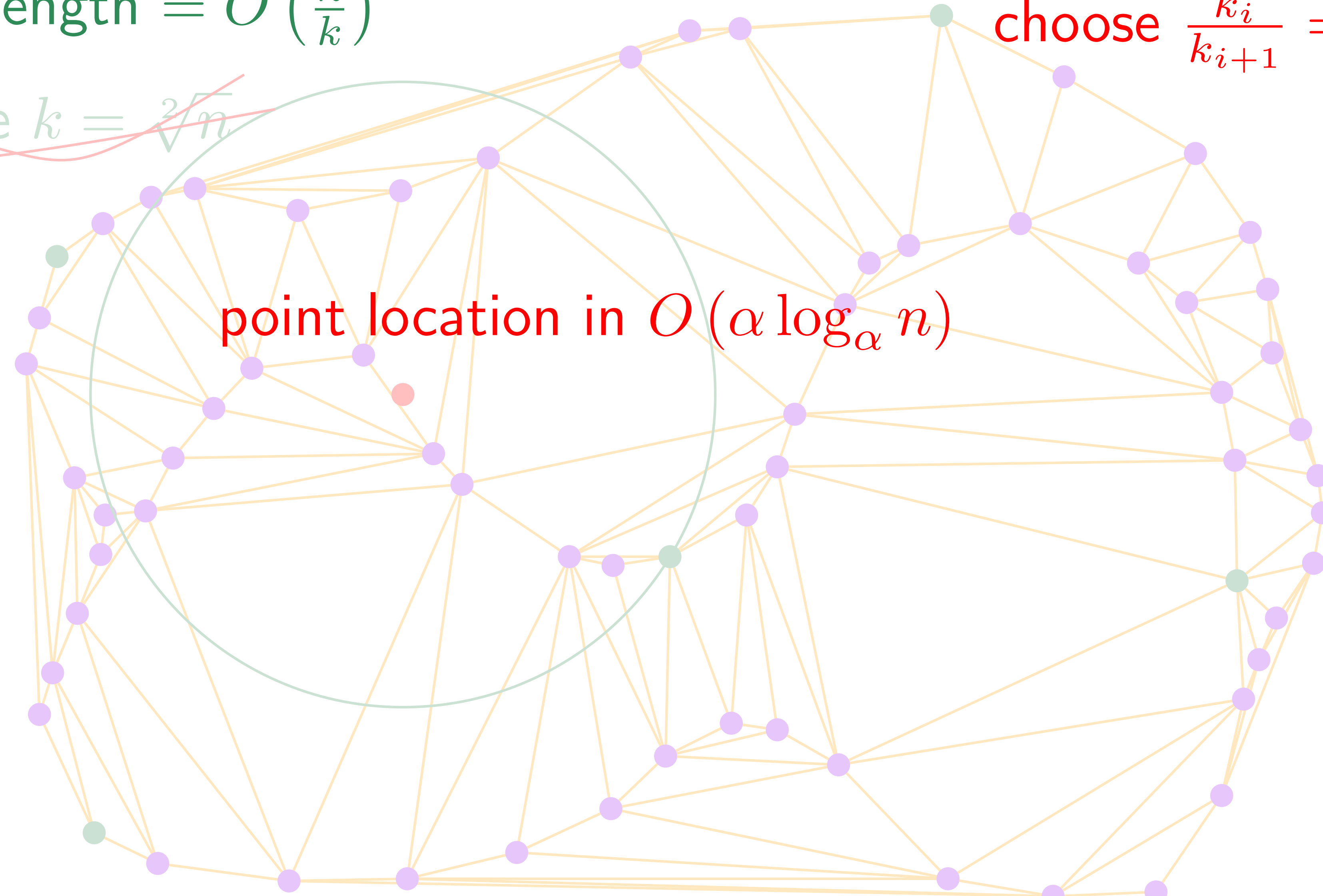
$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \bigcirc\right] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots$

Walk length $= O\left(\frac{n}{k}\right)$

choose $\frac{k_i}{k_{i+1}} = \alpha$

~~choose $k = \sqrt[2]{n}$~~



point location in $O\left(\alpha \log_\alpha n\right)$

Jump and walk (no distribution hypothesis)Delaunay hierarchy

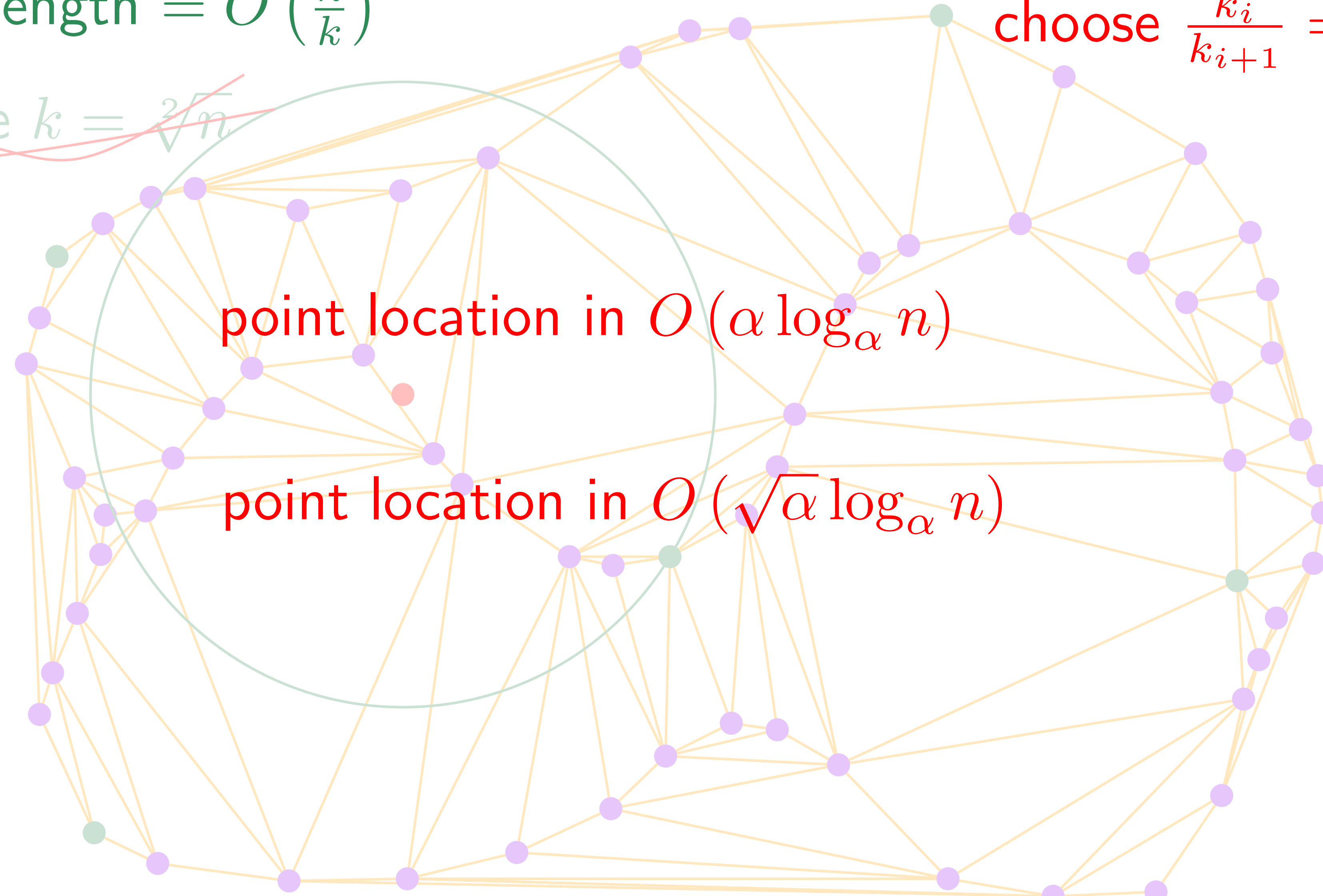
$\mathbb{E}\left[\sharp \text{ of } \bullet \text{ in } \odot\right] = \frac{n}{k}$

$\frac{n}{k_1} + \frac{k_1}{k_2} + \frac{k_2}{k_3} + \ldots$

Walk length $= O\left(\frac{n}{k}\right)$

choose $\frac{k_i}{k_{i+1}} = \alpha$

choose $k = \sqrt[2]{n}$

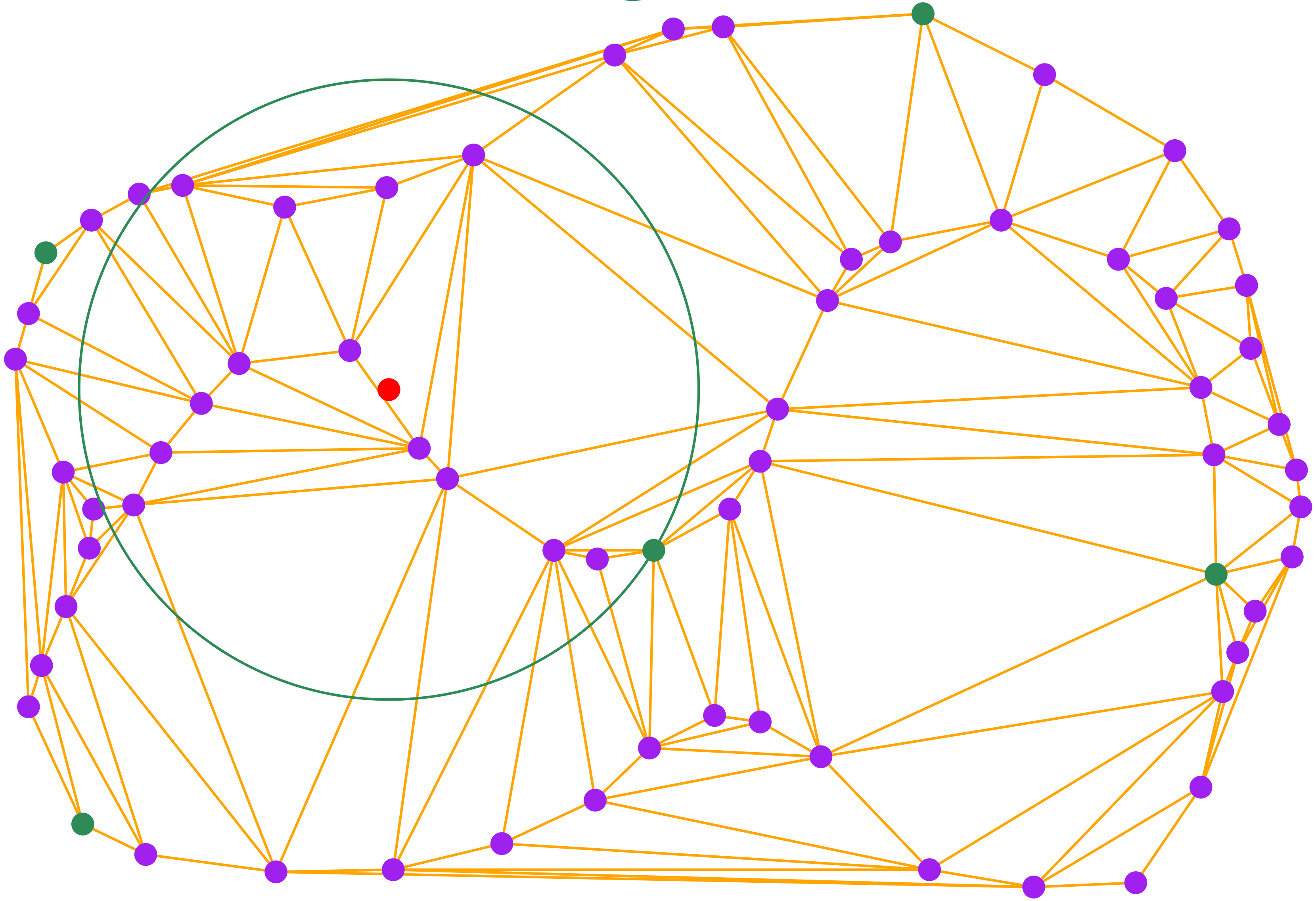


point location in $O\left(\alpha \log_\alpha n\right)$

point location in $O\left(\sqrt{\alpha} \log_\alpha n\right)$

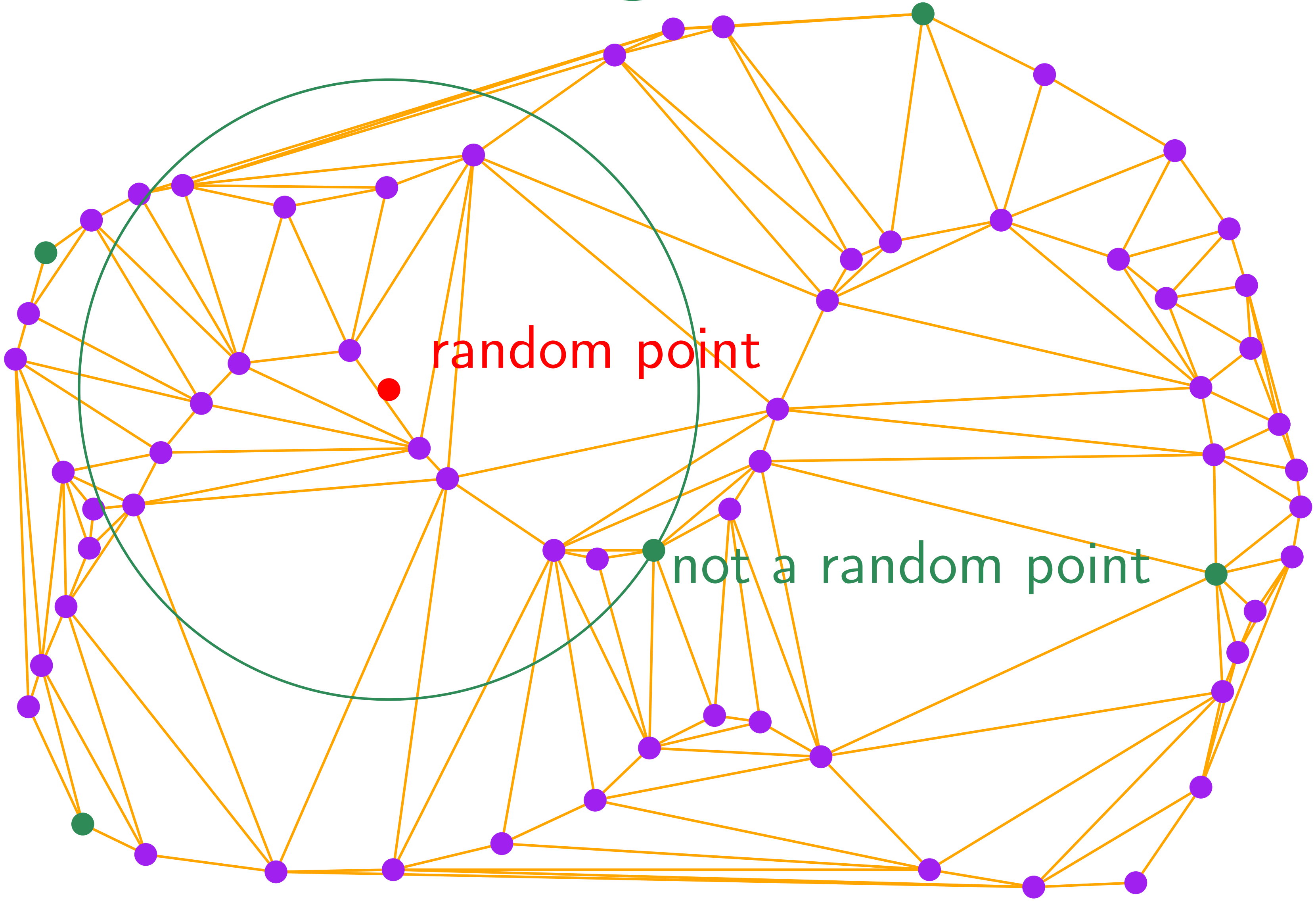
Technical detail

Walk length $= O\left(\sharp \text{ of } \bullet \text{ in } \right) = O\left(\frac{n}{k}\right)$



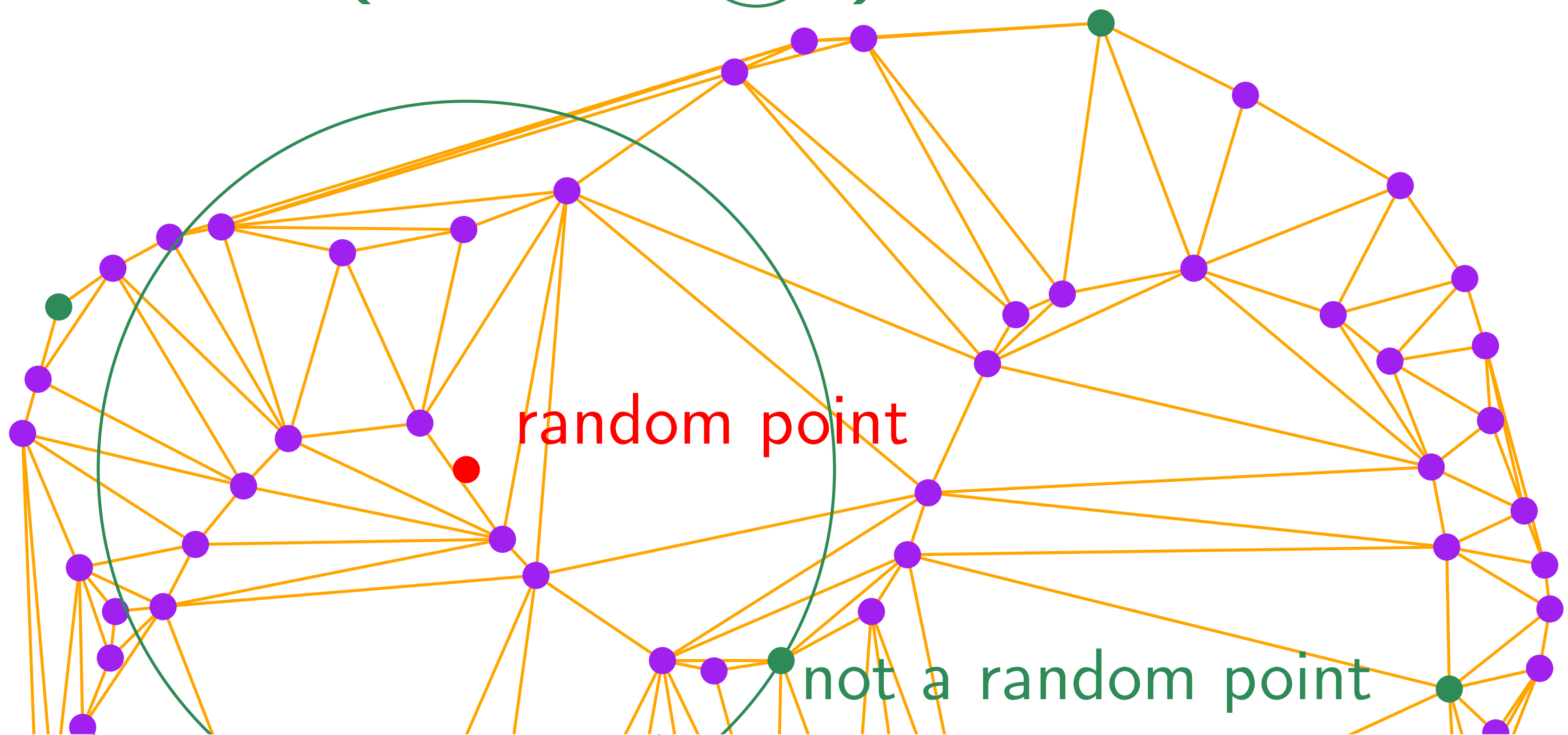
Technical detail

Walk length $= O\left(\sharp \text{ of } \bullet \text{ in } \bigcirc\right) = O\left(\frac{n}{k}\right)$



Technical detail

$$\text{Walk length} = O\left(\sharp \text{ of } \bullet \text{ in } \bigcirc \right) = O\left(\frac{n}{k}\right)$$

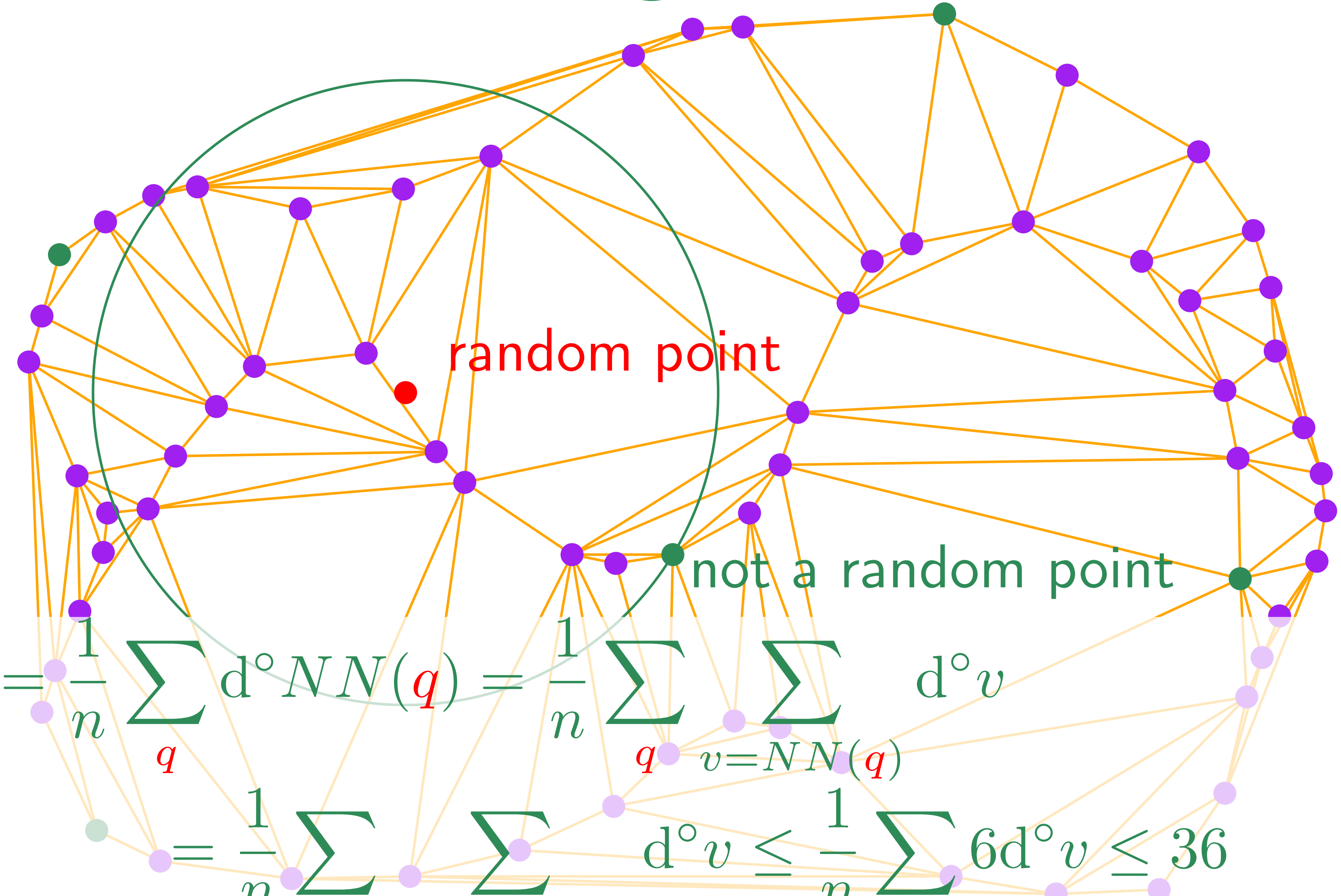


$$\mathbb{E}\left[\mathrm{d}^\circ \bullet\right] = \frac{1}{n}\sum_{q} \mathrm{d}^\circ NN(q) = \frac{1}{n}\sum_{q} \sum_{v=NN(q)} \mathrm{d}^\circ v$$

$$= \frac{1}{n}\sum_{v} \sum_{q;v=NN(q)} \mathrm{d}^\circ v \leq \frac{1}{n}\sum_{v} 6\mathrm{d}^\circ v \leq 36$$

Technical detail

$$\text{Walk length} = O\Big(\overset{\sum d^\circ}{\sharp} \text{ of } \bullet \text{ in } \bigcirc\Big) = O\left(\frac{n}{k}\right)$$



$$\mathbb{E}\left[\mathrm{d}^\circ \bullet\right] = \frac{1}{n}\sum_{q} \mathrm{d}^\circ NN(q) = \frac{1}{n}\sum_{q}\sum_{v=NN(q)} \mathrm{d}^\circ v$$

$$= \frac{1}{n}\sum_{v}\sum_{q;v=NN(q)} \mathrm{d}^\circ v \le \frac{1}{n}\sum_{v} 6\mathrm{d}^\circ v \le 36$$

Randomization

How many randomness is necessary?

If the data are not known in advance

shuffle locally

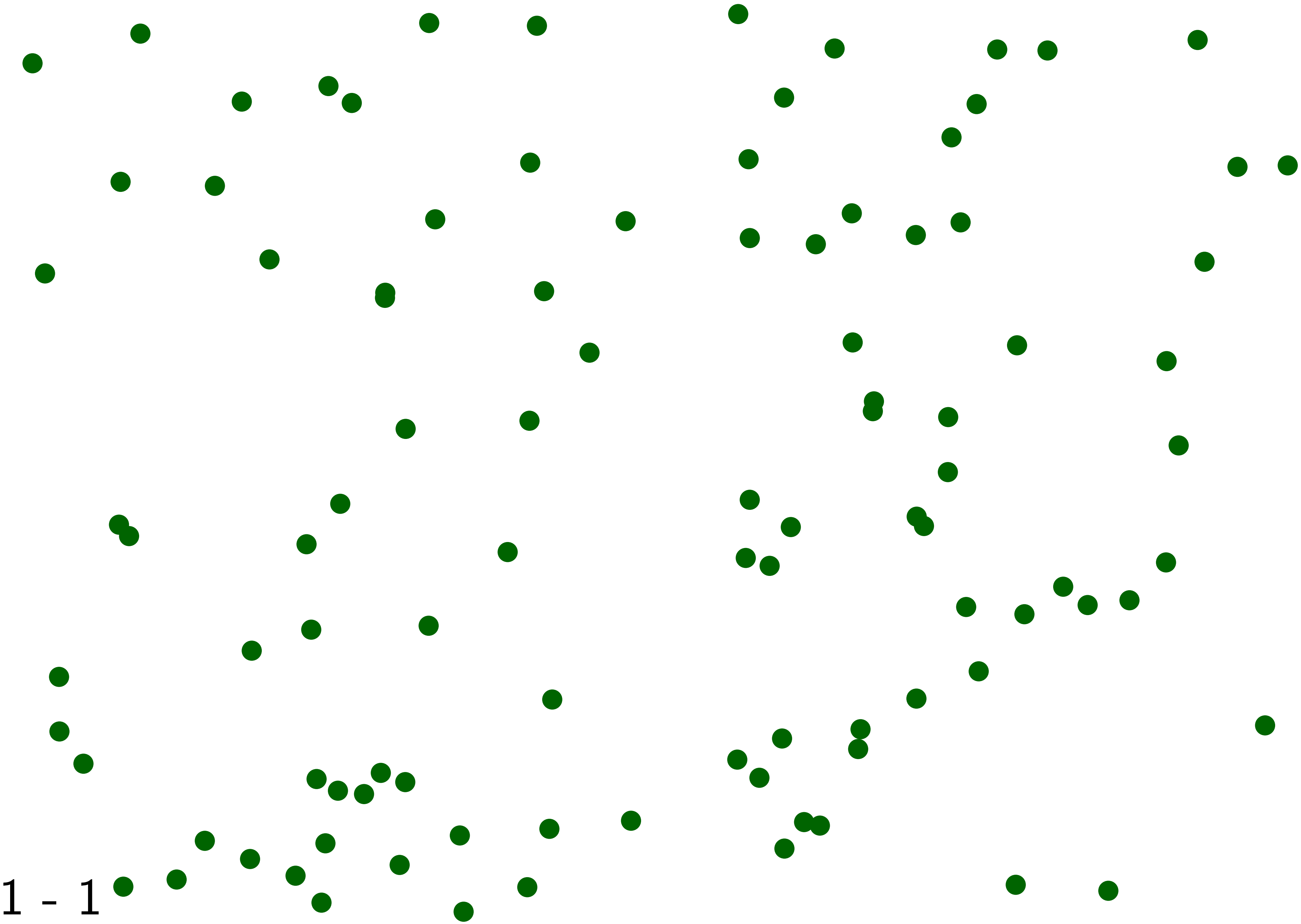
Randomization

Drawbacks of random order

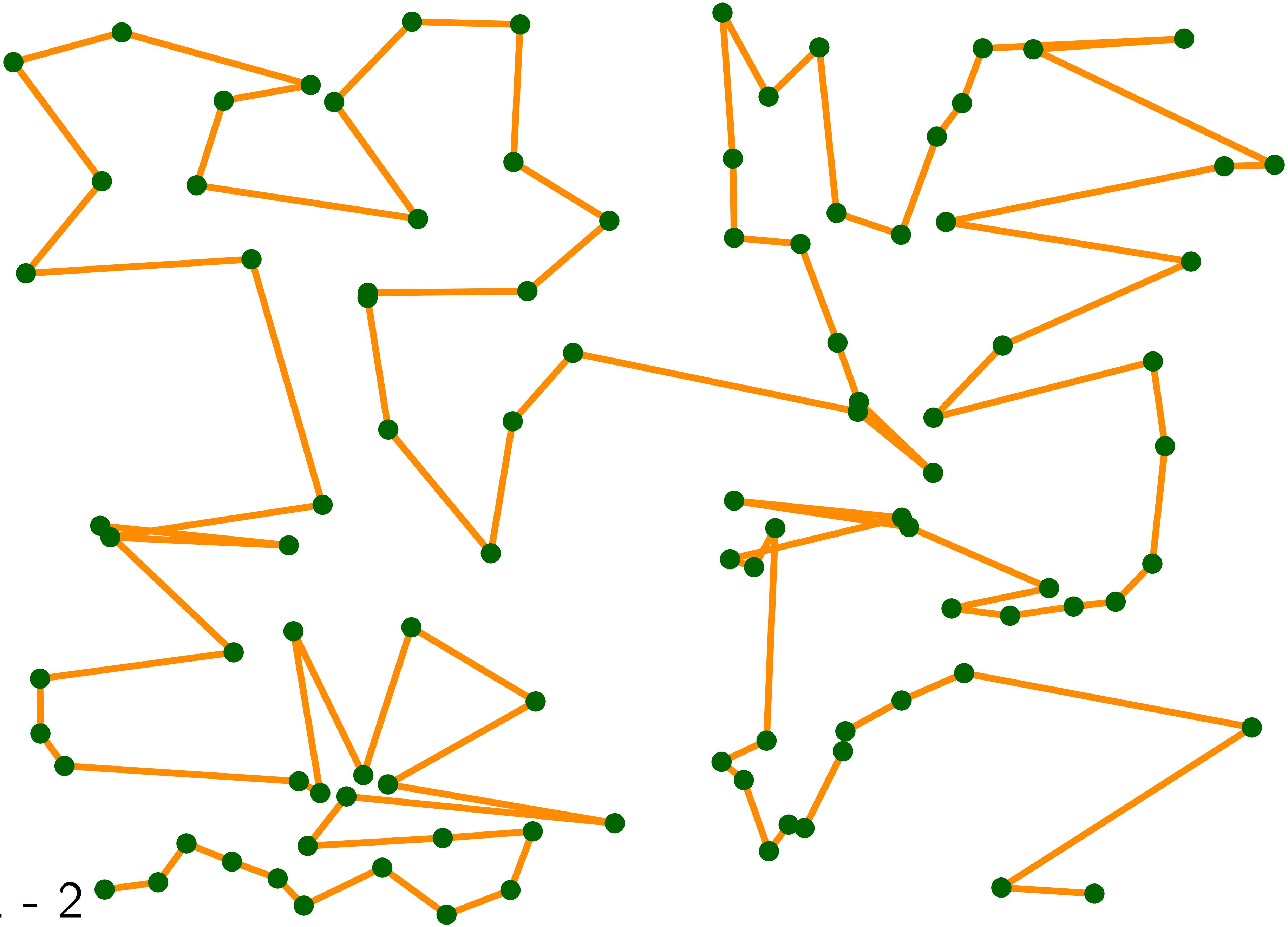
non locality of memory access

data structure for point location

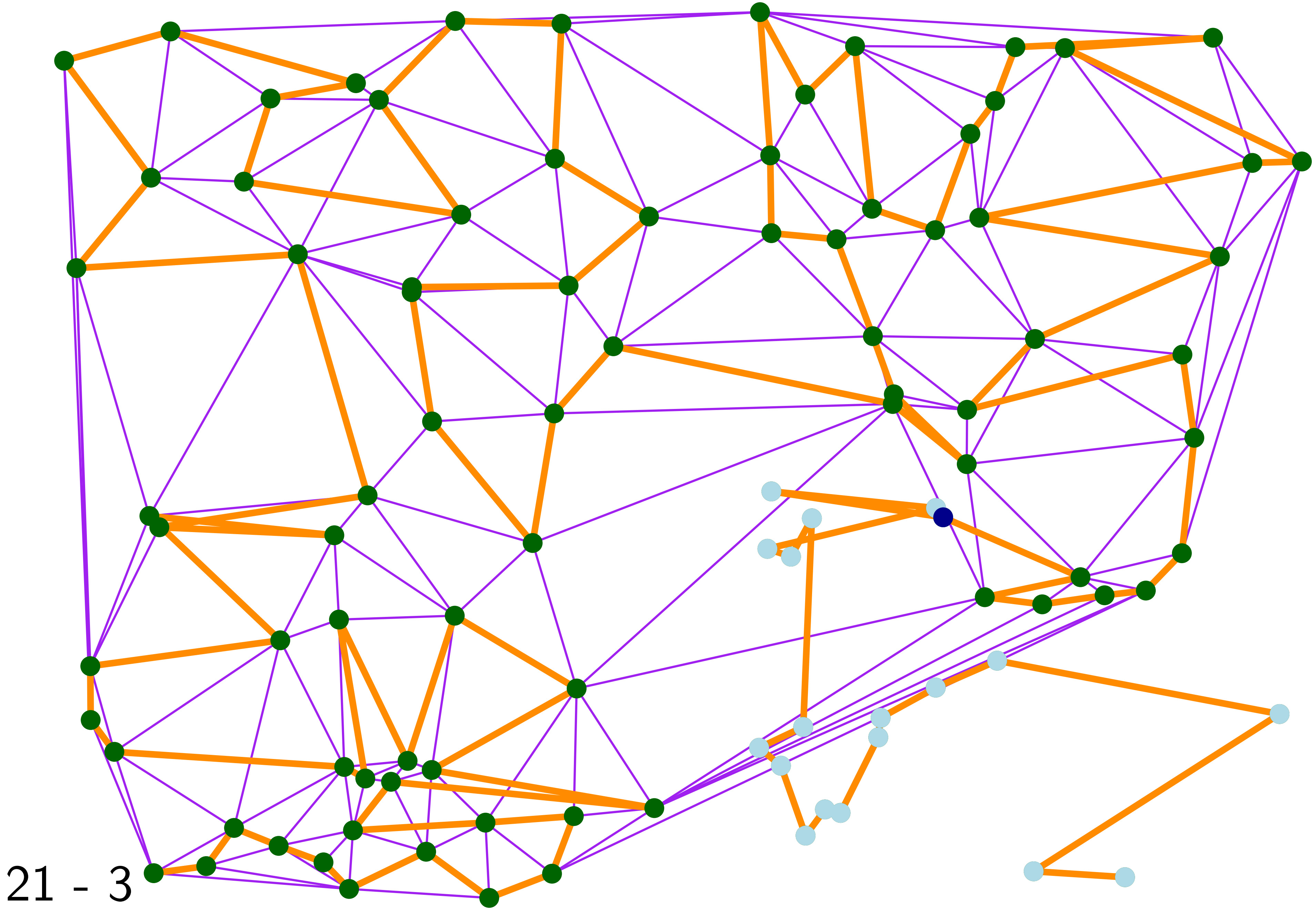
$\longrightarrow$ Hilbert sort

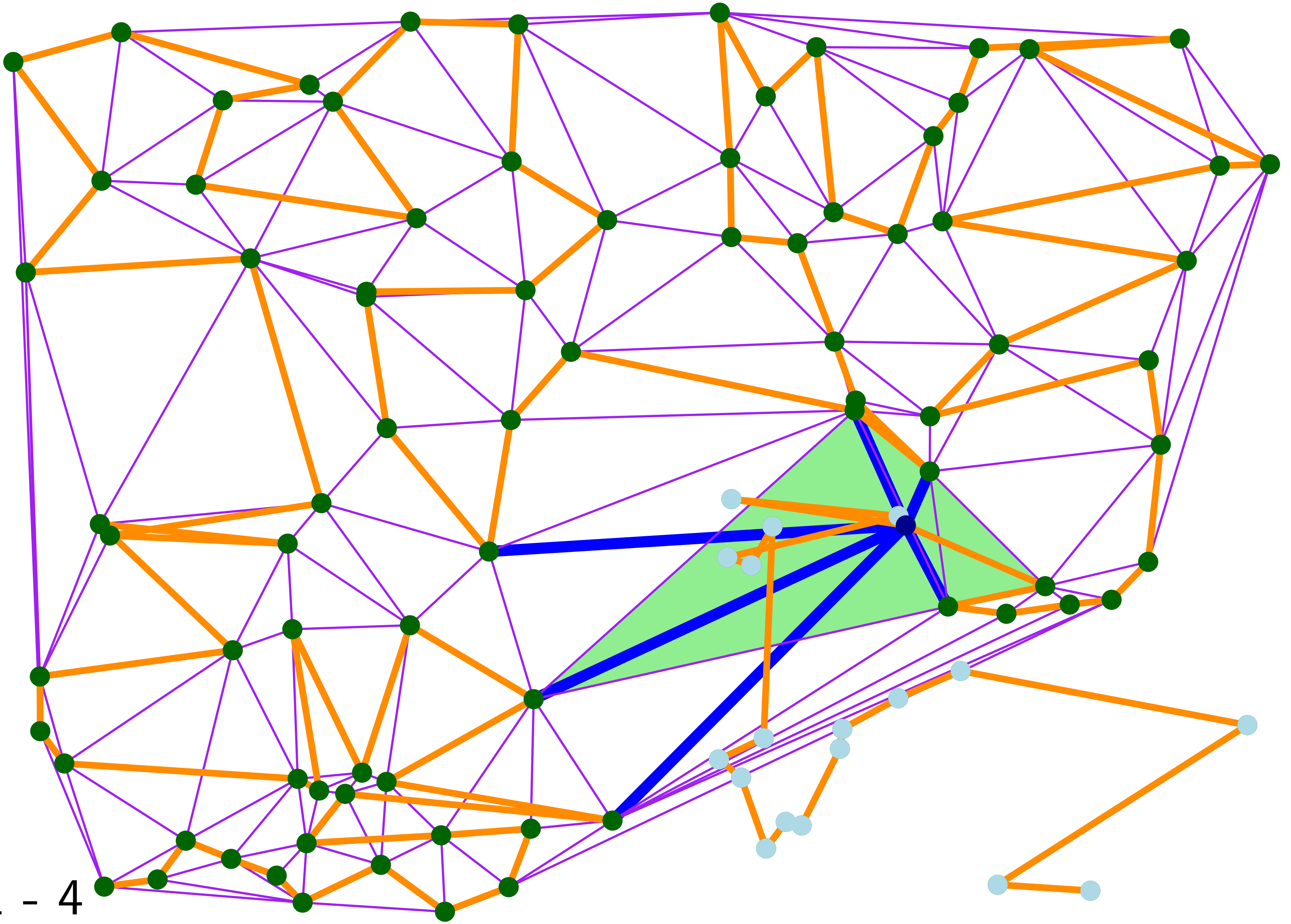


21 - 1



21 - 2


21 - 3



21 - 4

Drawbacks of random order

non locality of memory access

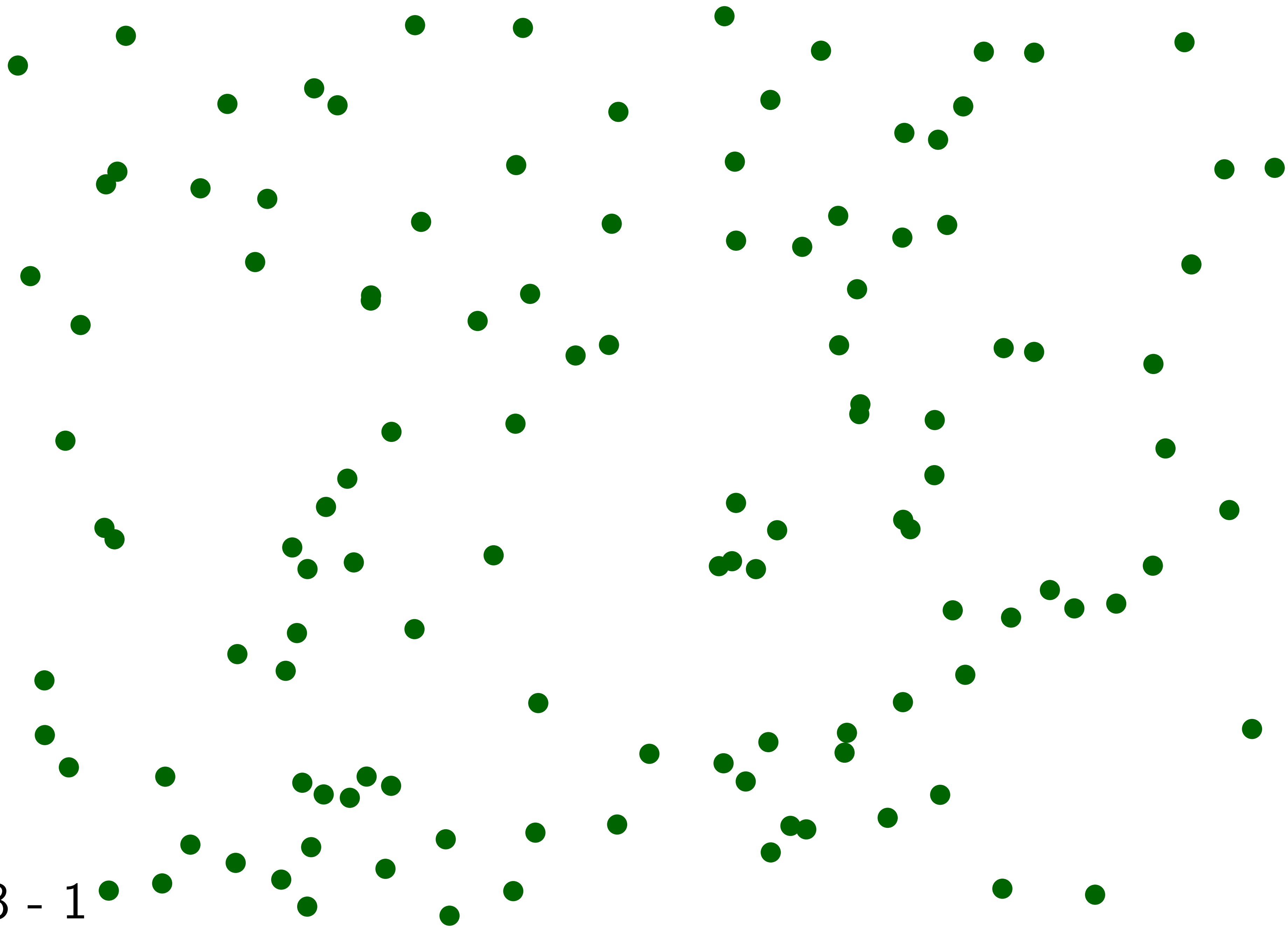
data structure for point location

$\longrightarrow$ Hilbert sort

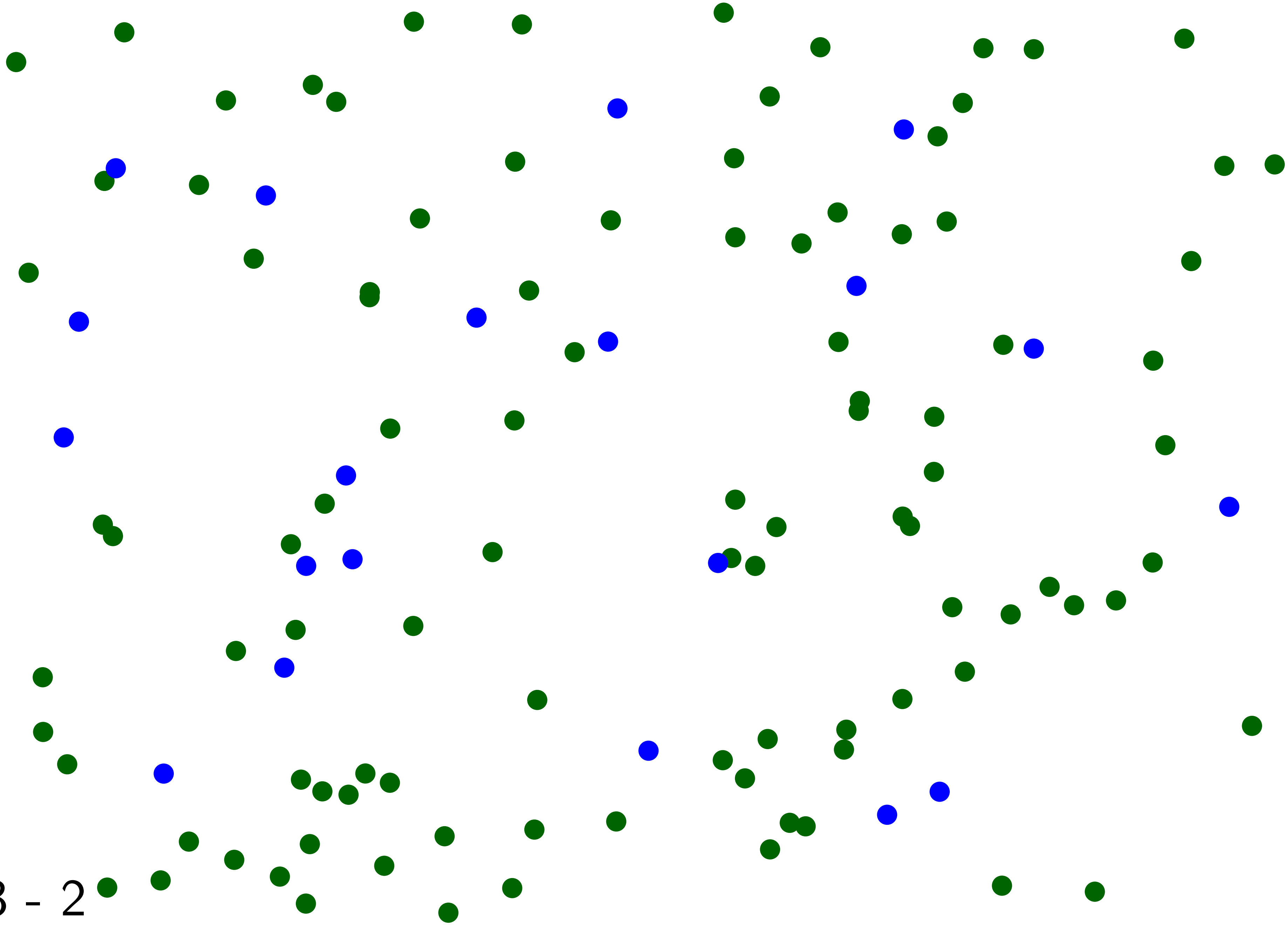
Walk should be fast

Last point is not at all a random point

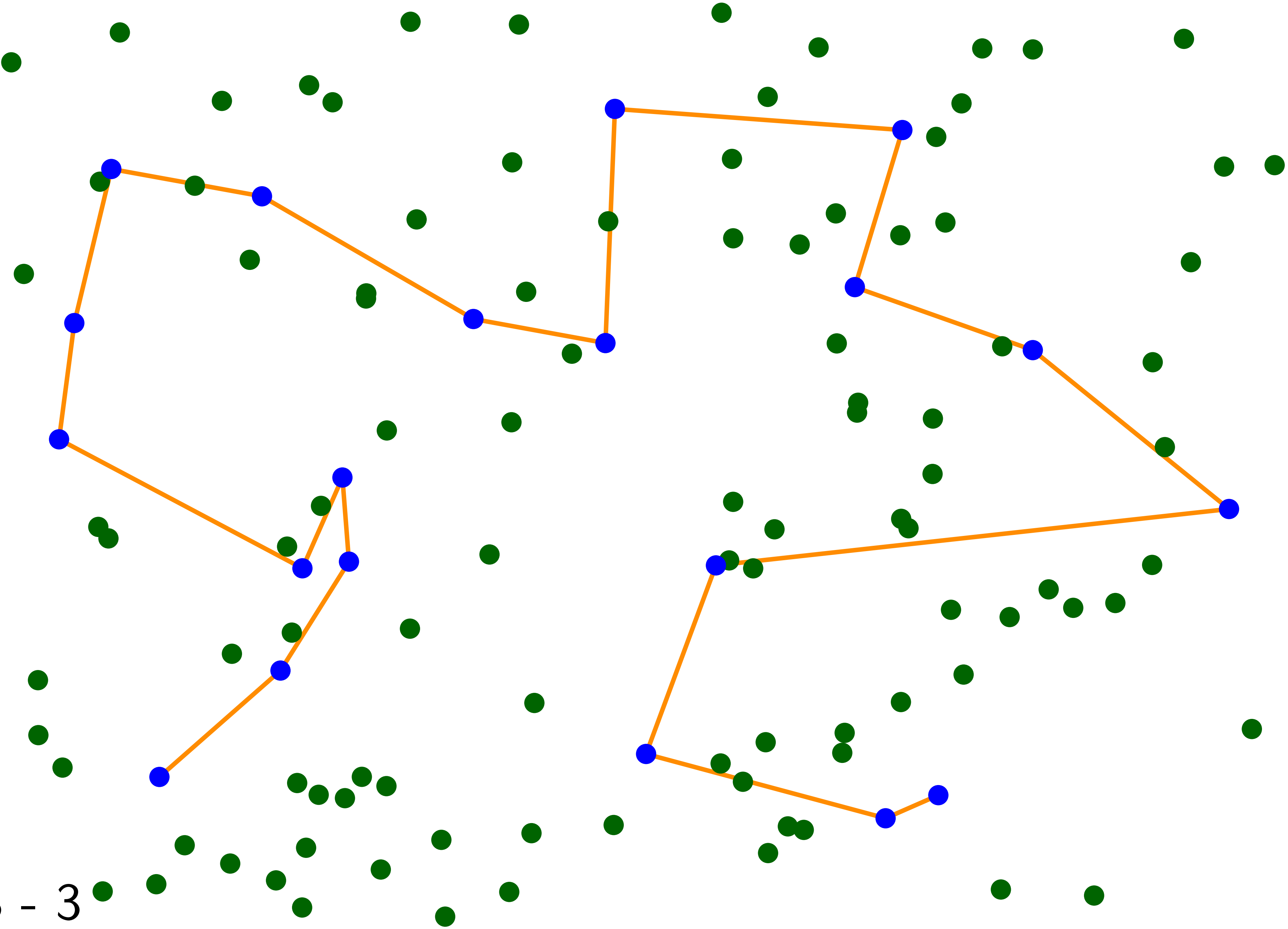
$\longrightarrow$ no control of degree of last point



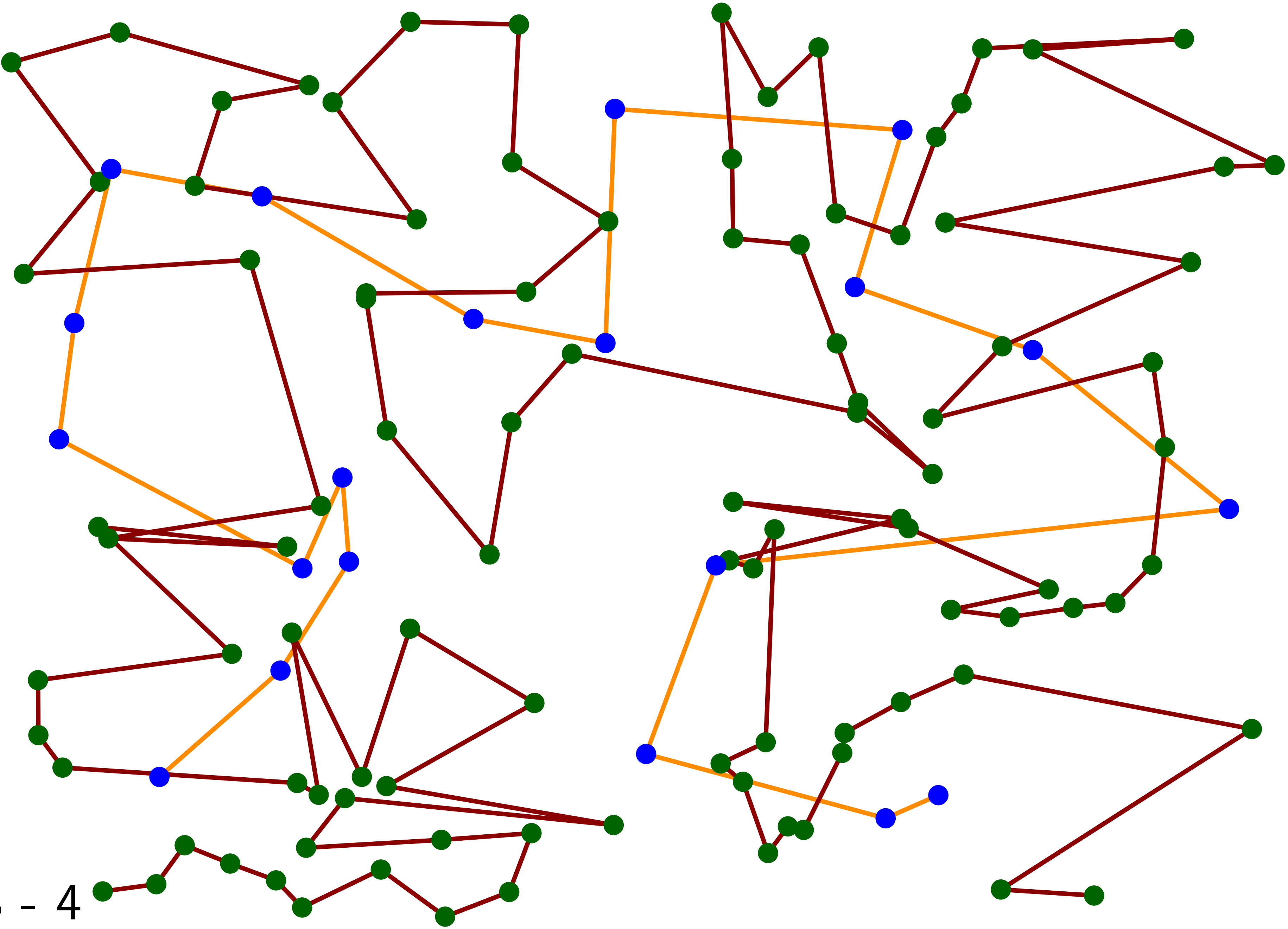
23 - 1



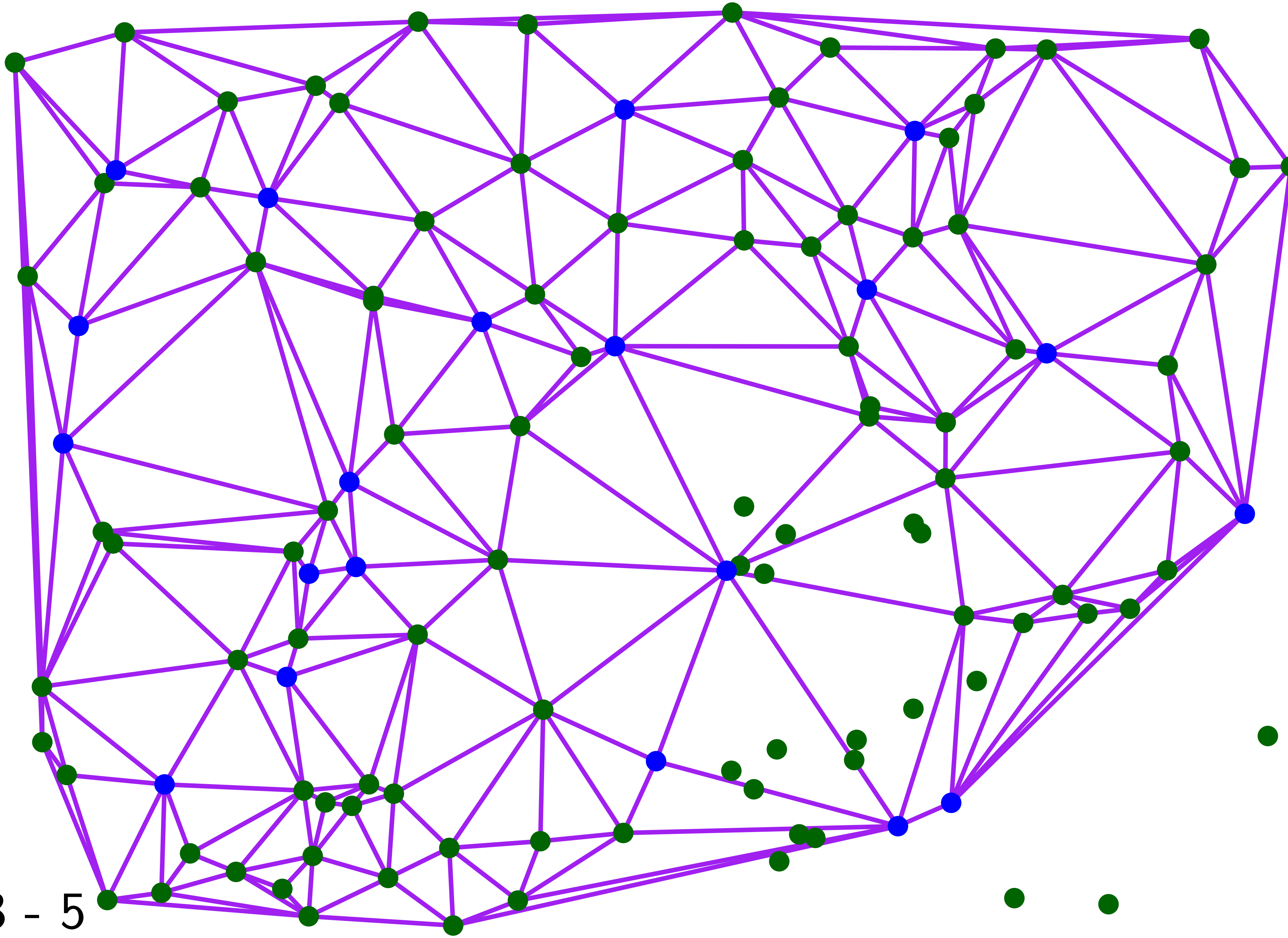
23 - 2



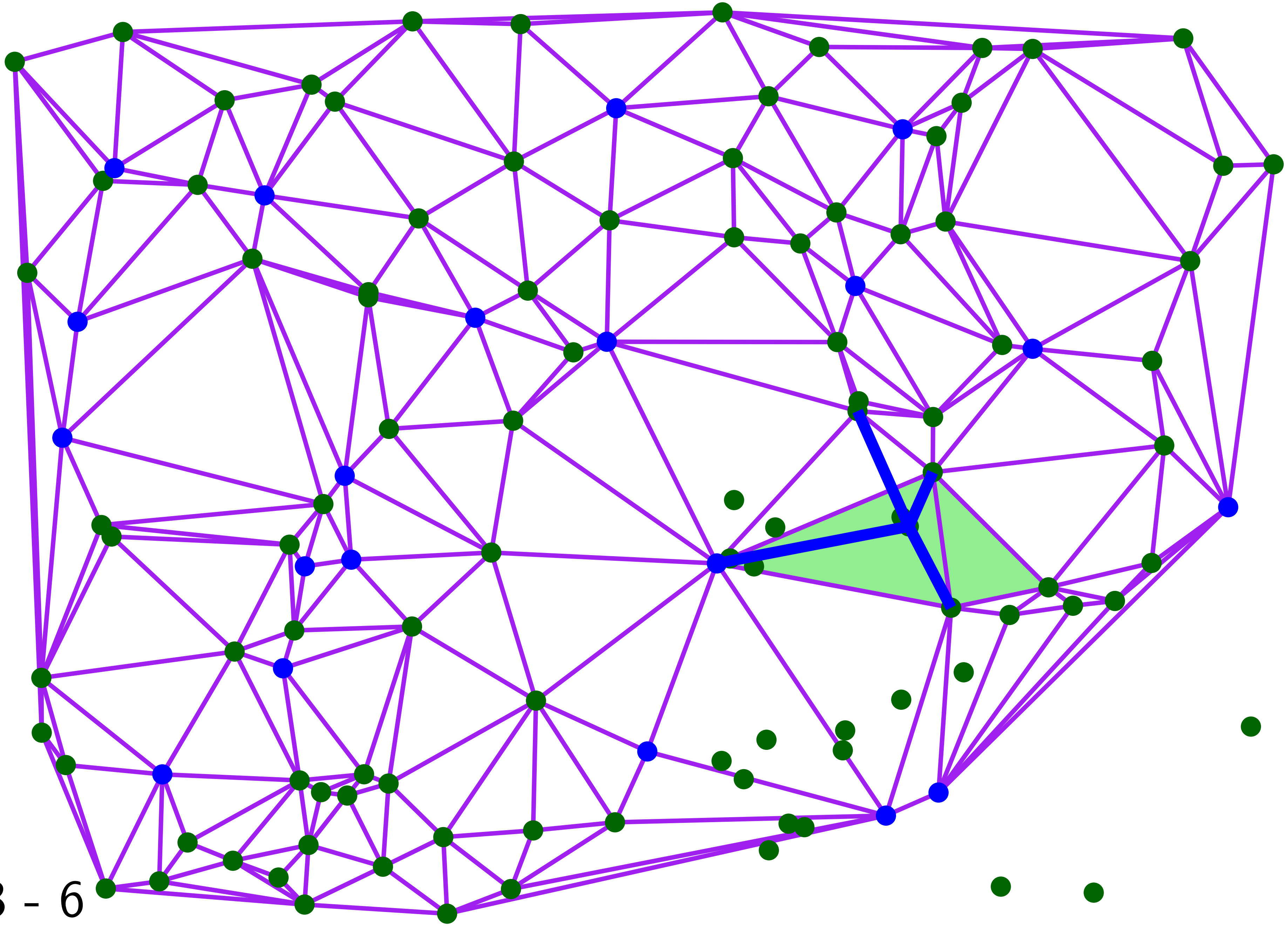
23 - 3



23 - 4

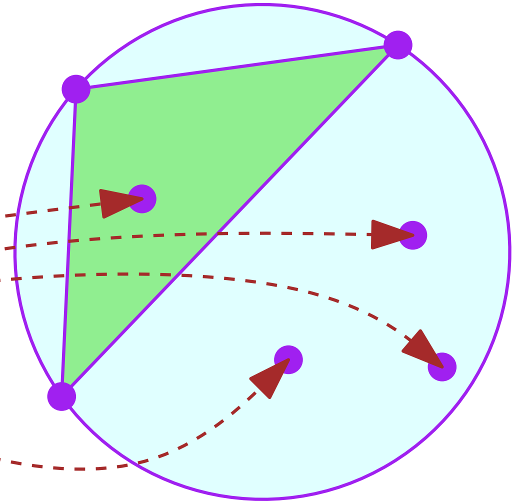


23 - 5

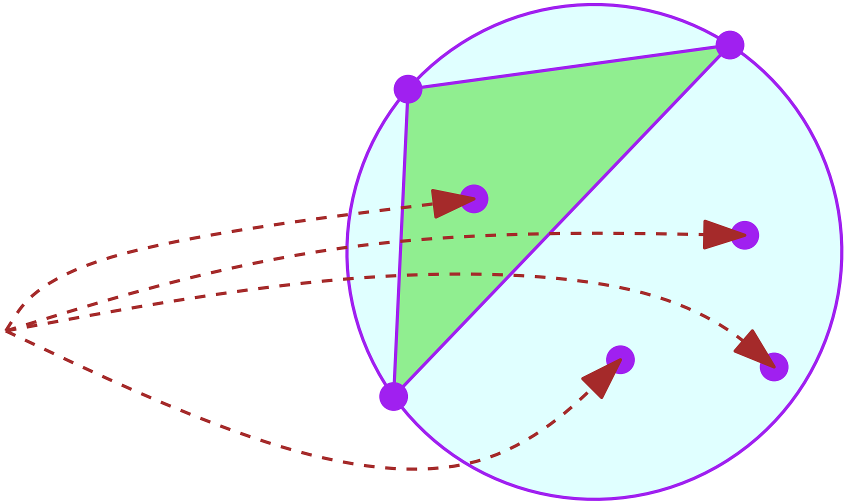


23 - 6

Triangle Δ with j stoppers

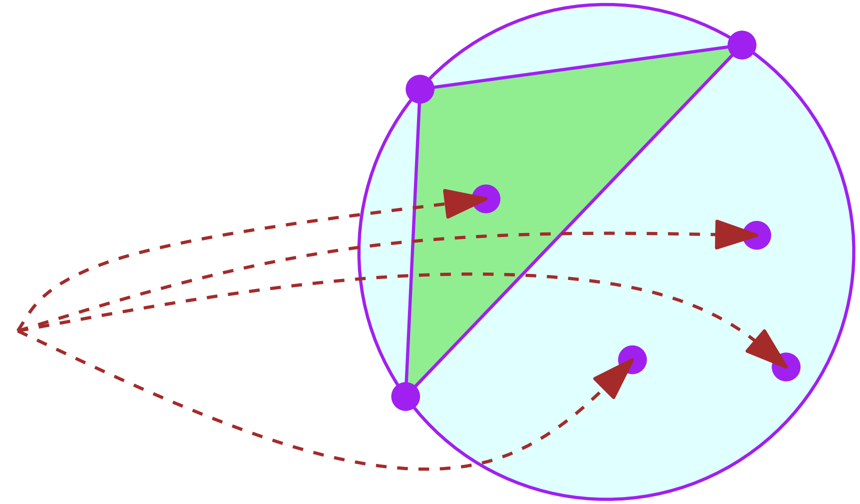


Triangle Δ with j stoppers



Size (order $\leq k$ Voronoi) $\leq \frac{\alpha n}{\alpha^3} = nk^2$

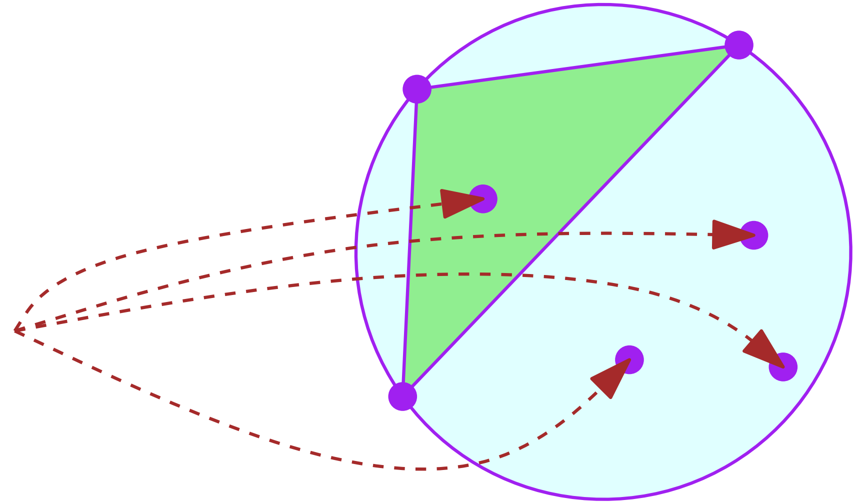
Triangle Δ with j stoppers



Probability that it exists during the construction

$$= \frac{3}{j+3}\,\frac{2}{j+2}\,\frac{1}{j+1}$$

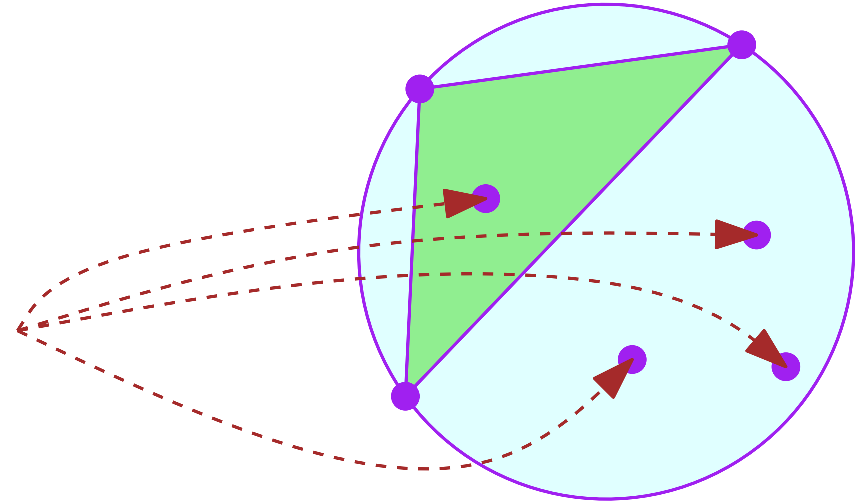
Triangle Δ with j stoppers



Probability that it exists during the construction

$$= \cancel{\frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}} \quad \text{remains } \Theta(j^{-3})$$

Triangle Δ with j stoppers



Probability that it exists during the construction

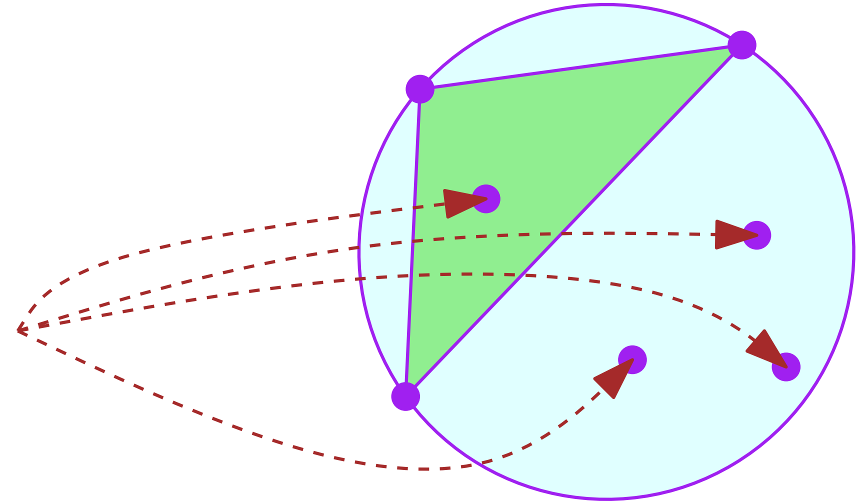
$$= \cancel{\frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}} \quad \text{remains } \Theta(j^{-3})$$

$\sharp$ of created triangles

$$= \sum_{j=0}^{n} \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$\simeq O\left(\sum \frac{nj^2}{j^4}\right) = O(n)$$

Triangle Δ with j stoppers



Probability that it exists during the construction

$$= \cancel{\frac{3}{j+3}\frac{2}{j+2}\frac{1}{j+1}} \quad \text{remains } \Theta(j^{-3})$$

$\sharp$ of conflicts occuring

$$= \sum_{j=0}^{n} j \times \mathbb{P}\left[\Delta \text{ with } j \text{ stoppers appears}\right] \times \sharp\Delta \text{ with } j \text{ stoppers}$$

$$\simeq O\left(\sum j \frac{nj^2}{j^4}\right) = O(n \log n)$$

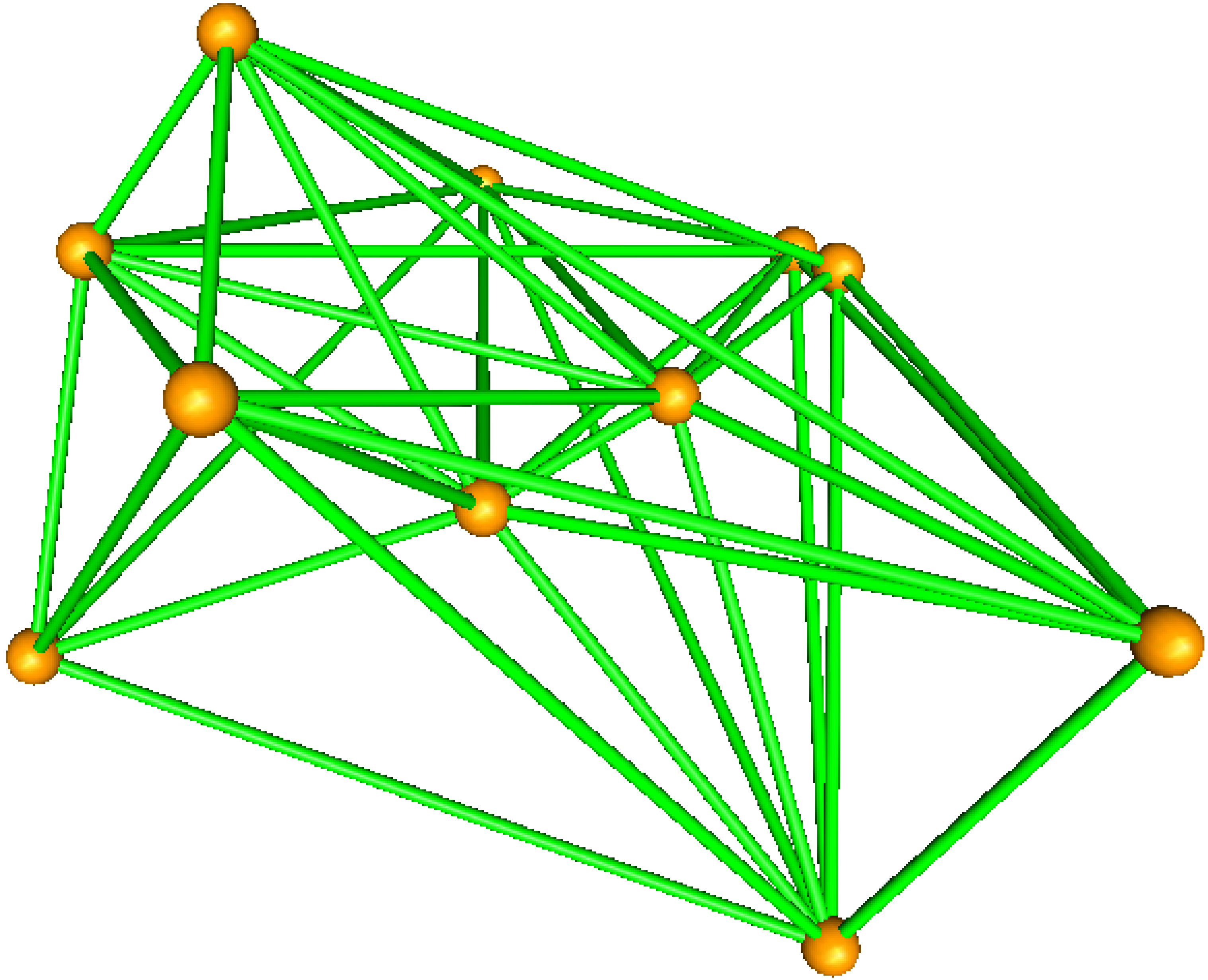
Delaunay 2D 1M random points

locate using Delaunay hierarchy	6 seconds
random order (visibility walk)	157 seconds
x-order	3 seconds
Hilbert order	0.8 seconds
Biased order (Spatial sorting)	0.7 seconds

Delaunay 2D 100K parabola points

locate using Delaunay hierarchy	0.3 seconds
random order (visibility walk)	128 seconds
x-order	632 seconds
Hilbert order	46 seconds
Biased order (Spatial sorting)	0.3 seconds

3D



3D

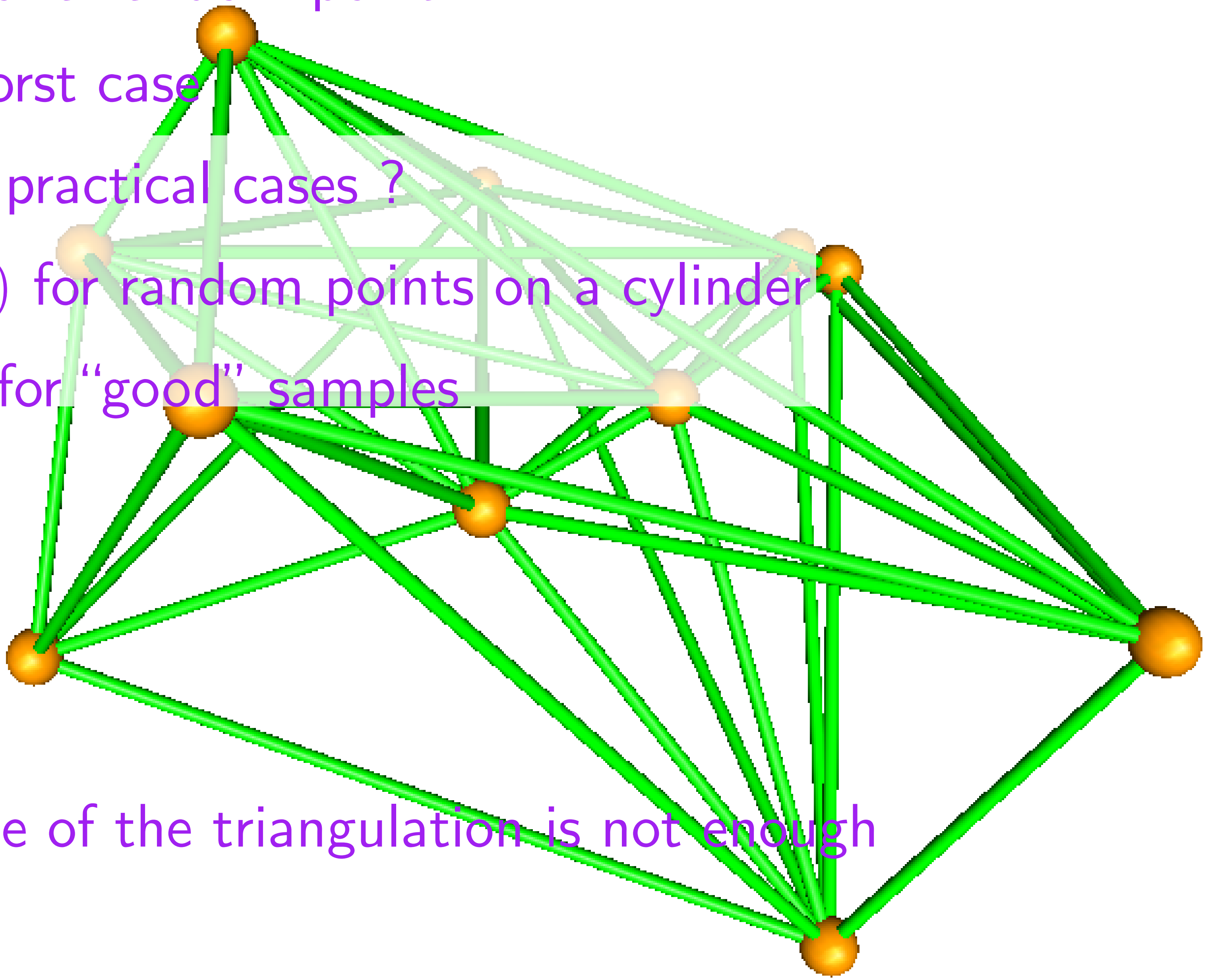
Degree of a random point?

$O(n)$ worst case

$O(1)$ in practical cases ?

$O(\log n)$ for random points on a cylinder

$O(\sqrt{n})$ for “good” samples



Final size of the triangulation is not enough

Randomization

Avoiding point location

Delaunay randomized construction

$O(n)$

Delaunay randomized construction

$O(n)$ + point location

Delaunay randomized construction

$O(n)$ + point location

Use additional information to save on point location

Delaunay randomized construction

$O(n)$ + point location

Use additional information to save on point location

e.g. points are sorted by spatial sort

Delaunay randomized construction

$O(n)$ + point location

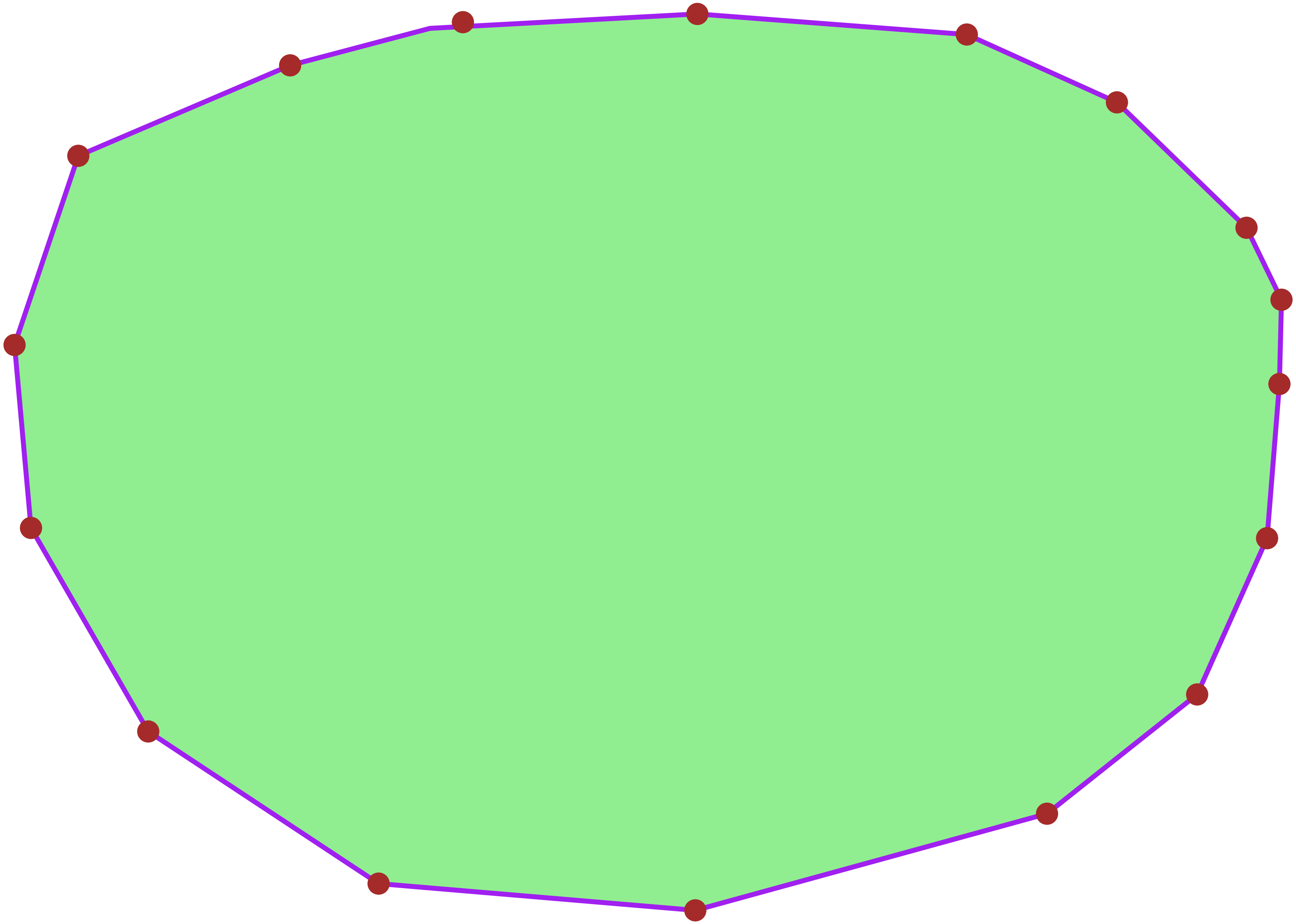
Use additional information to save on point location

e.g. points are sorted by spatial sort

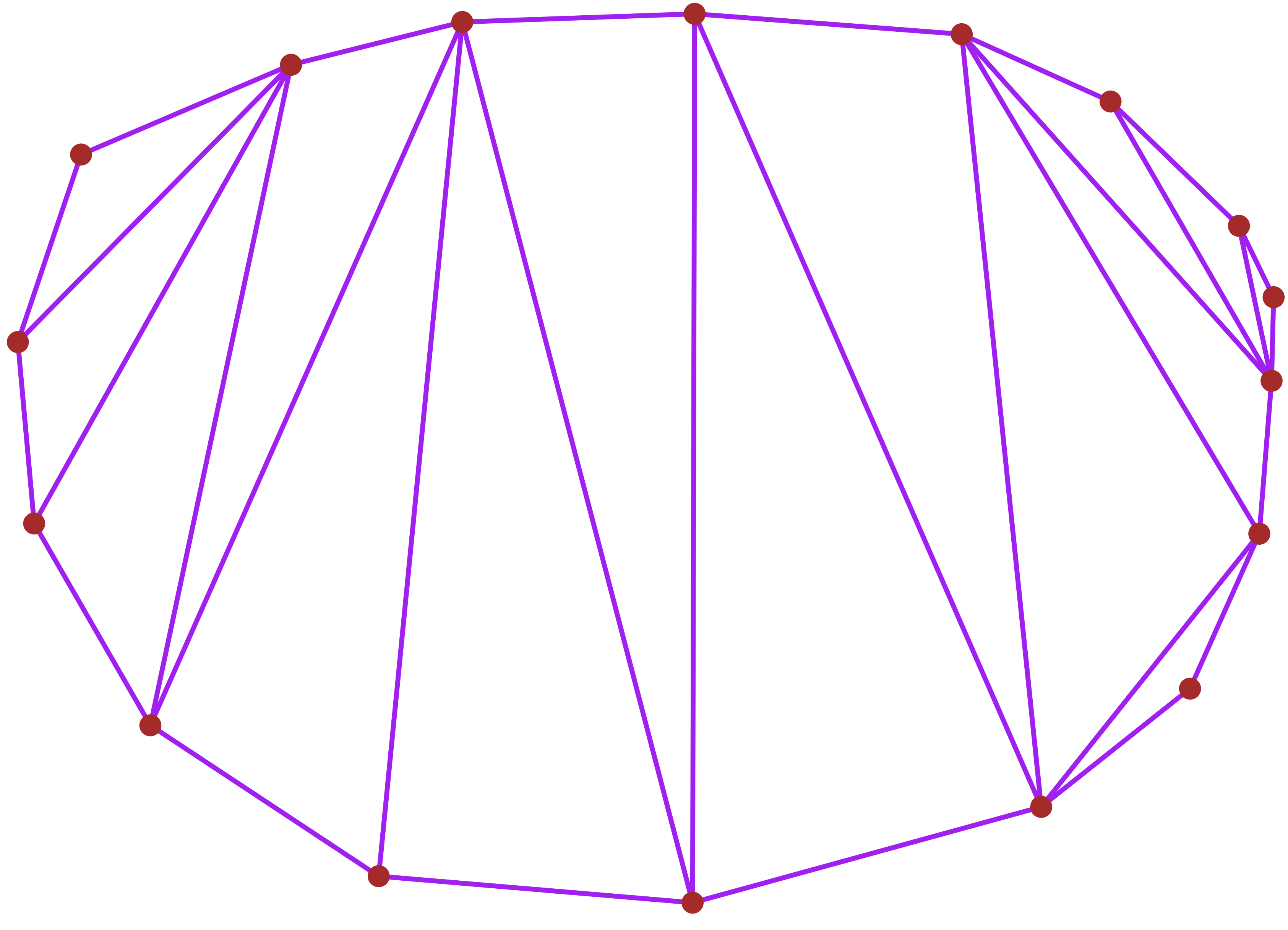
Delaunay of points in convex position

Splitting Delaunay

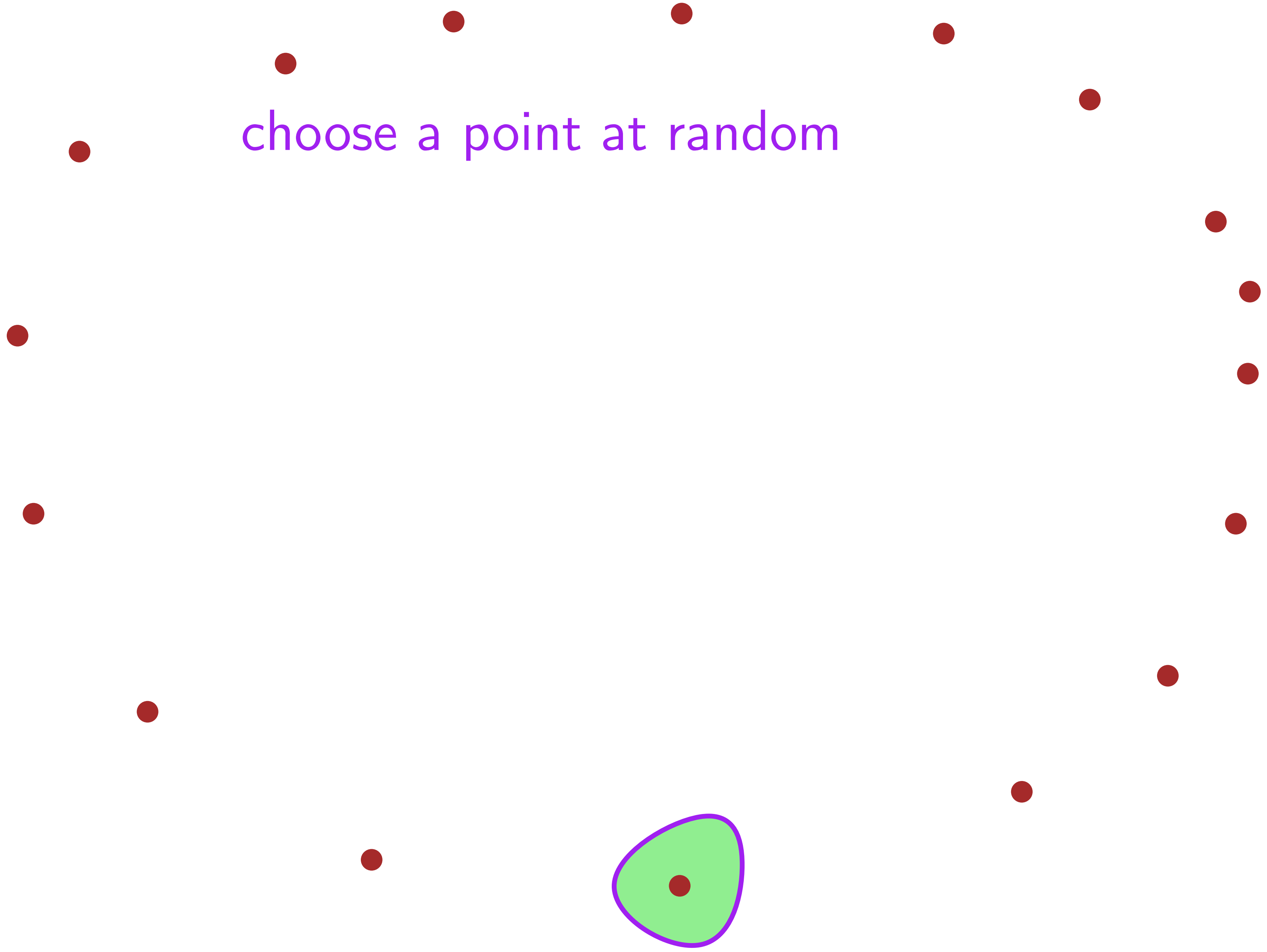
Delaunay of points in convex position



Delaunay of points in convex position



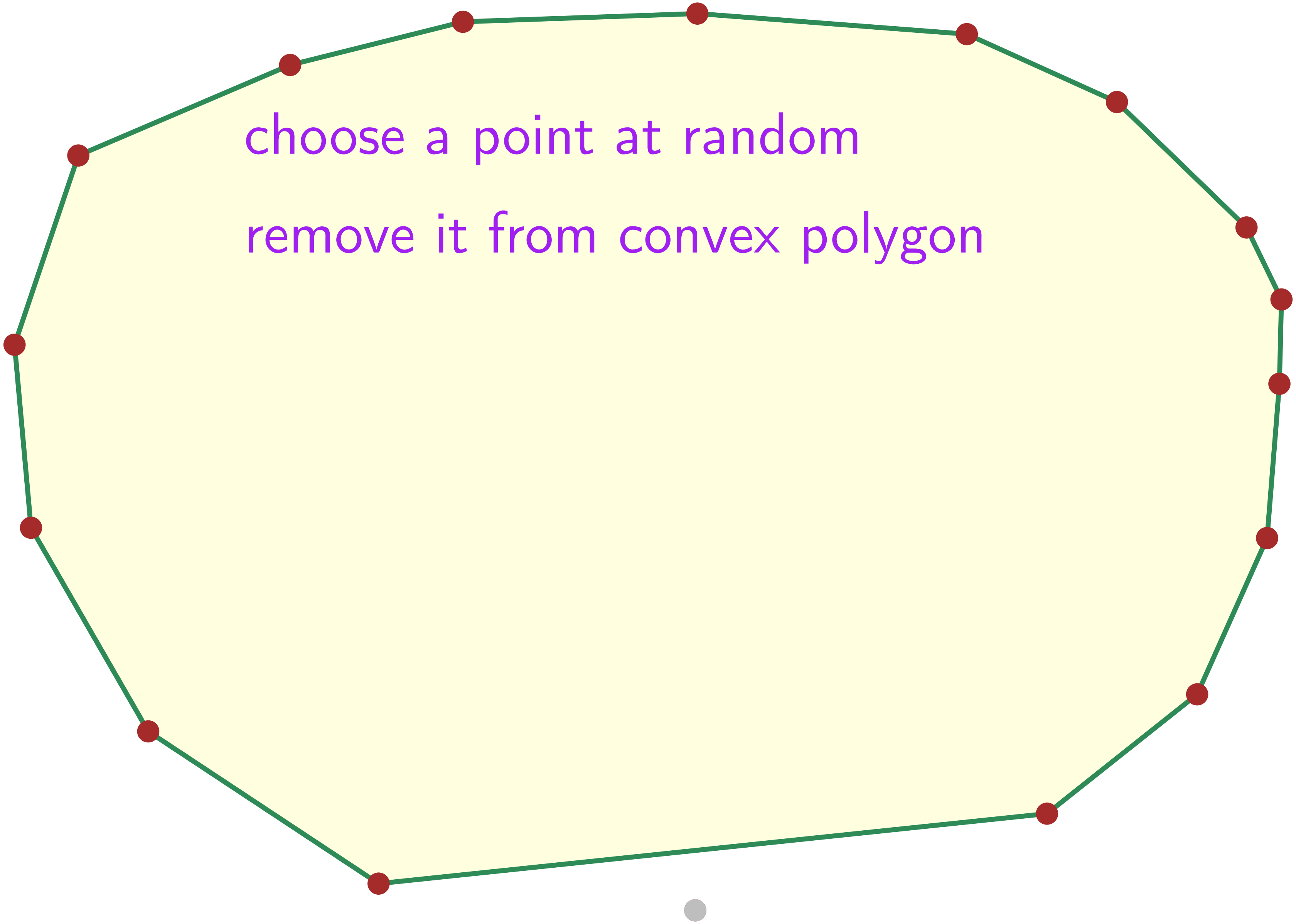
Delaunay of points in convex position



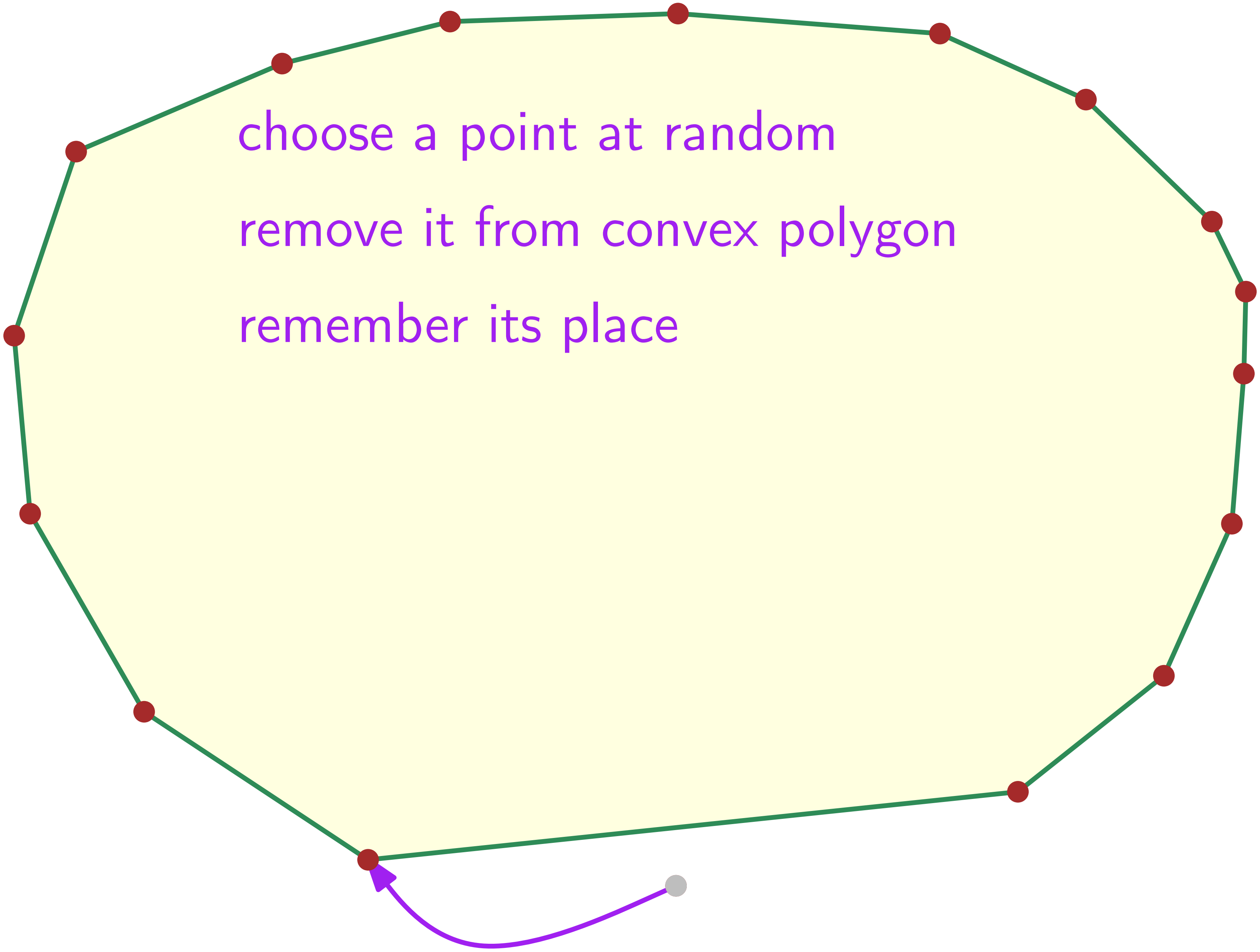
Delaunay of points in convex position

choose a point at random

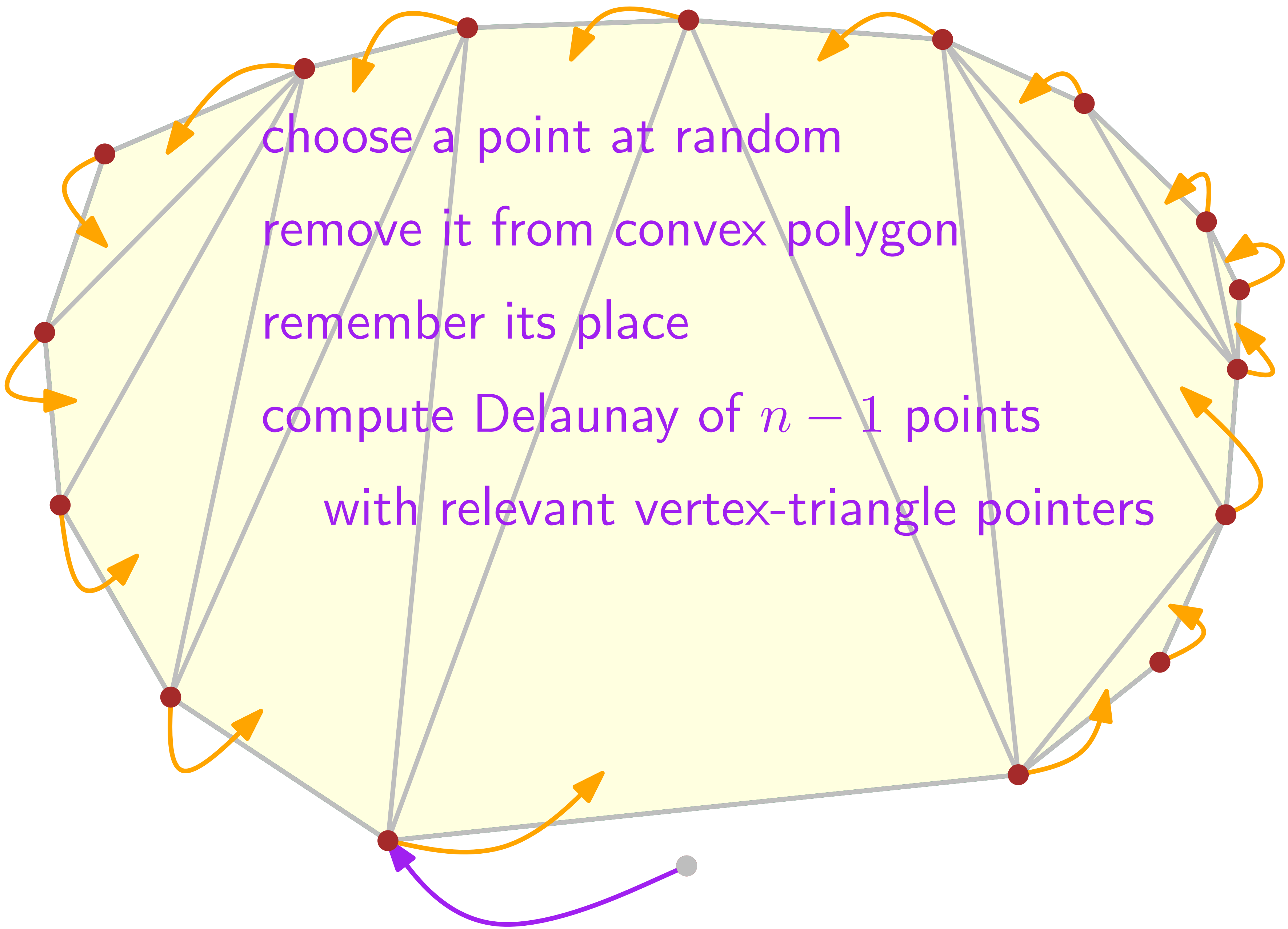
remove it from convex polygon



Delaunay of points in convex position



Delaunay of points in convex position



choose a point at random

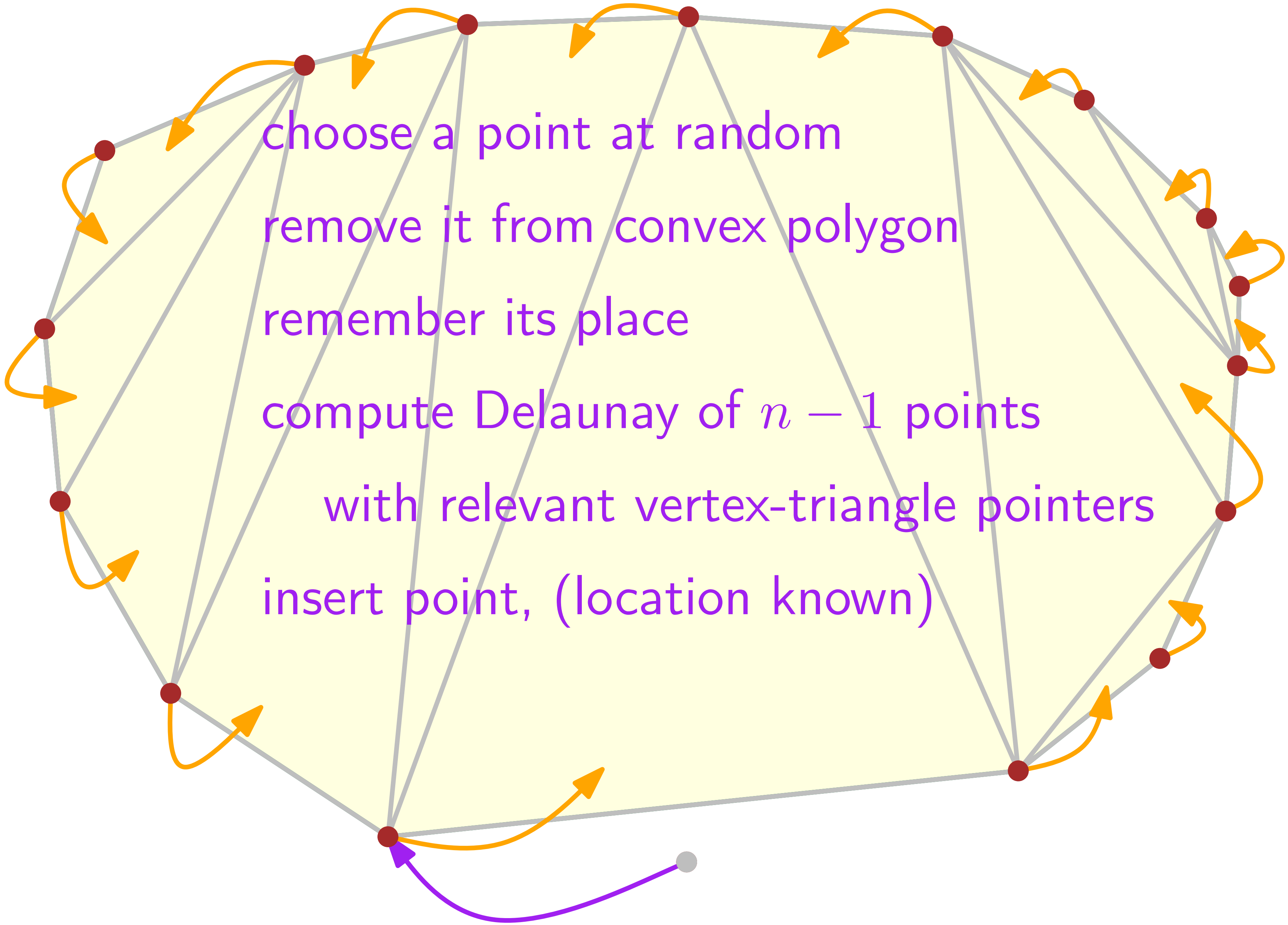
remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

Delaunay of points in convex position



choose a point at random

remove it from convex polygon

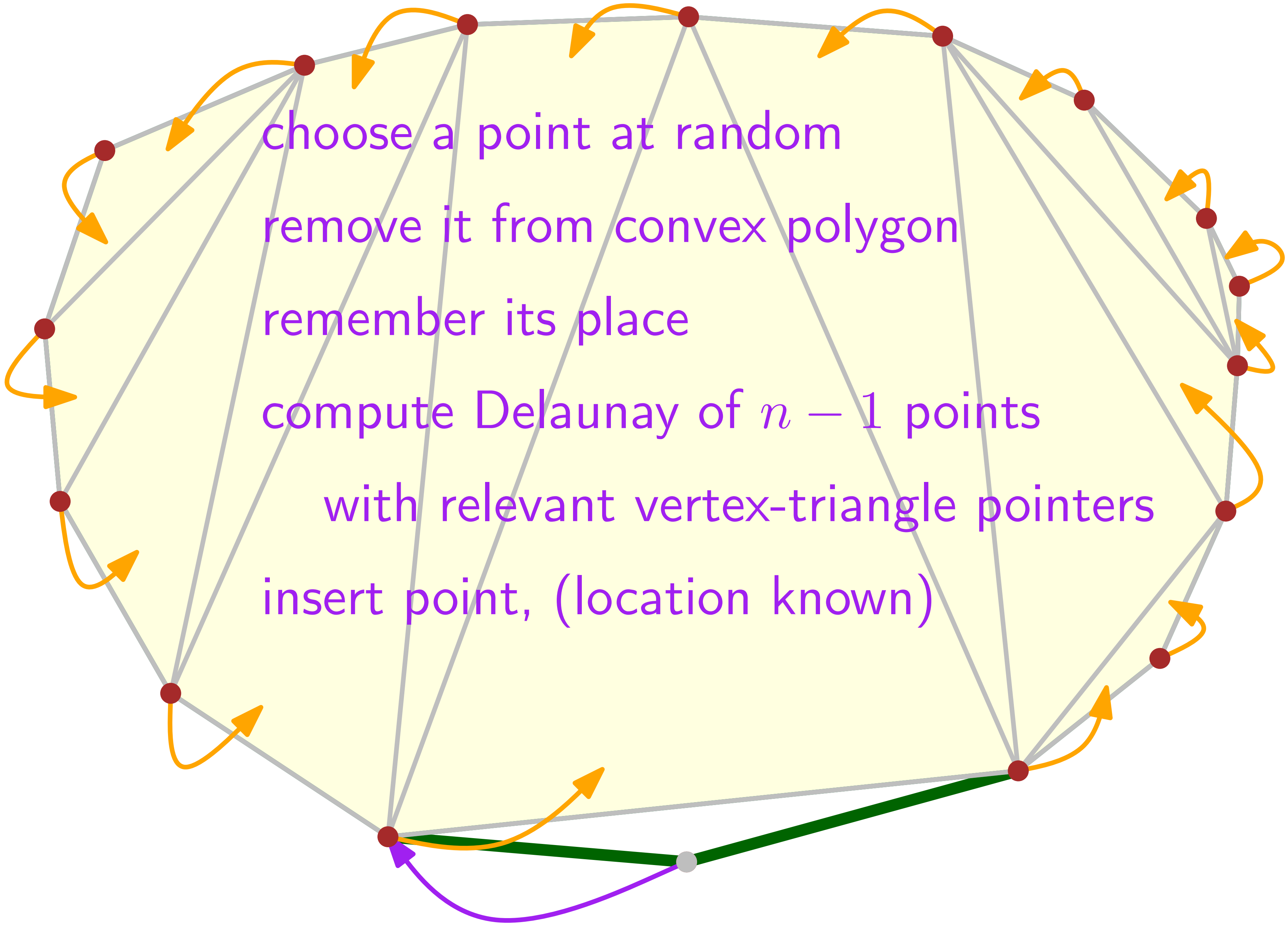
remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position



choose a point at random

remove it from convex polygon

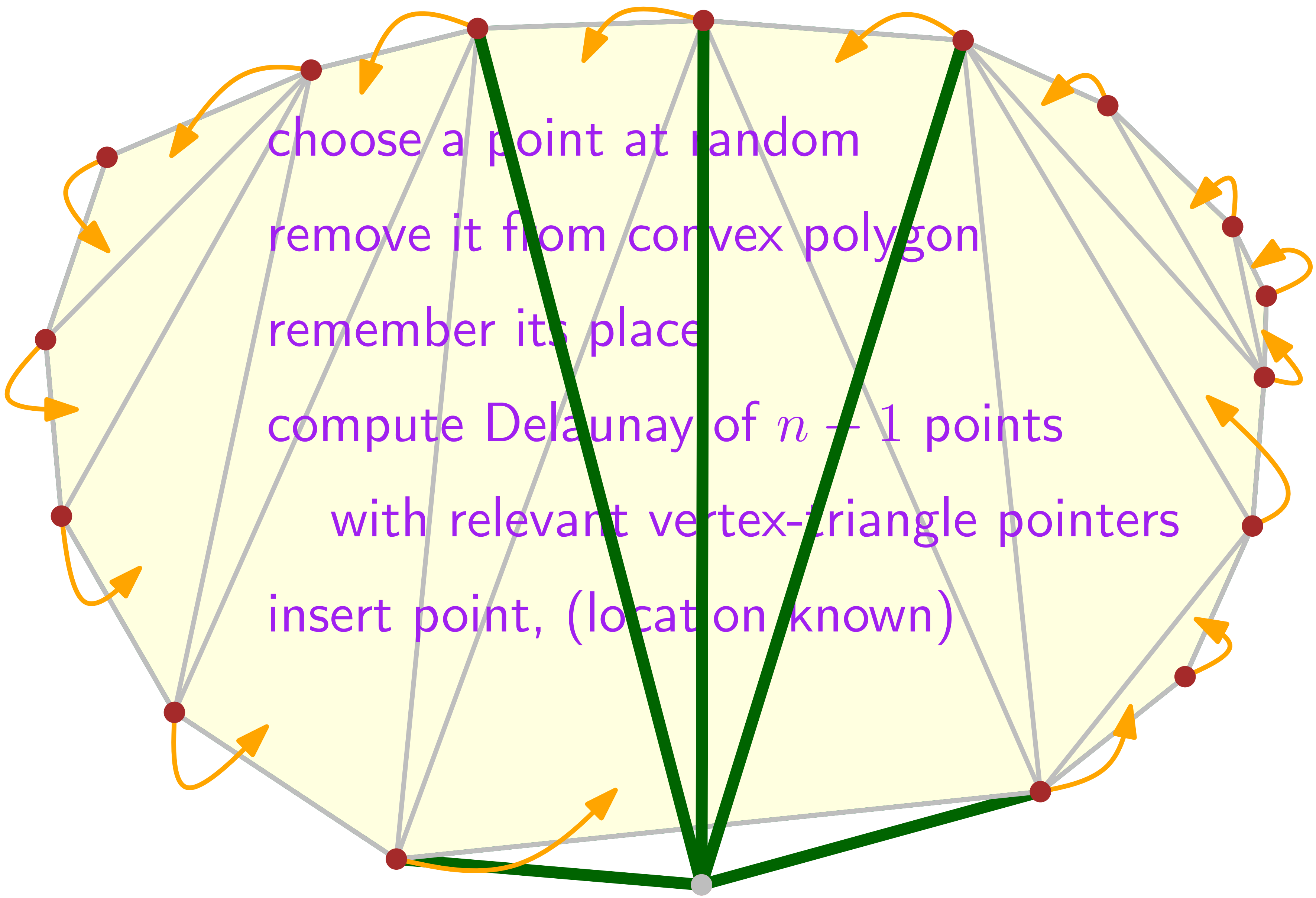
remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position



choose a point at random

remove it from convex polygon

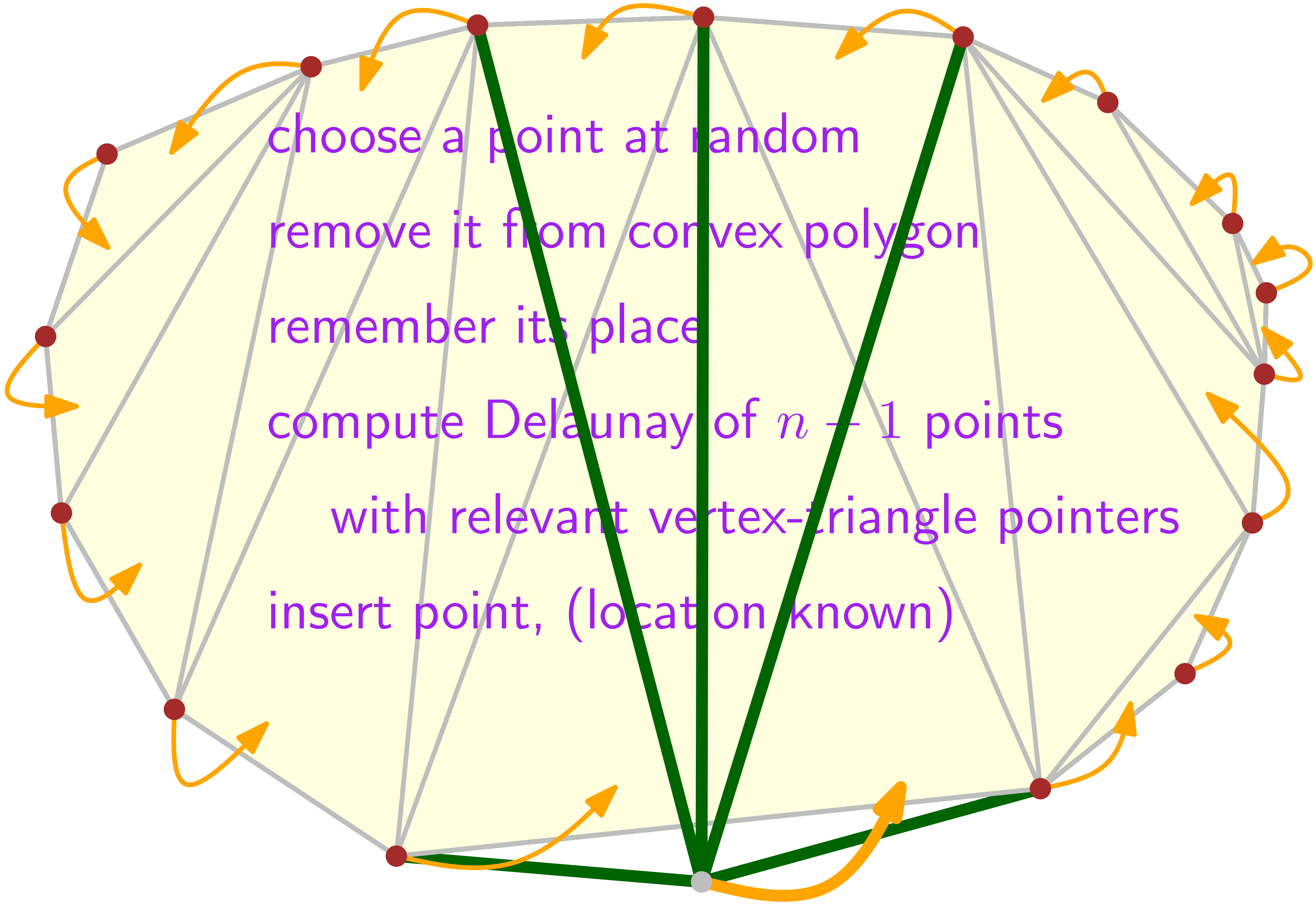
remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position



choose a point at random

remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

choose a point at random

remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

choose a point at random $O(1)$ [model]

remove it from convex polygon

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known)

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known) $O(d^\circ p)$

Delaunay of points in convex position

Analysis

choose a point at random

remove it from convex polygon $\qquad O(1)$

remember its place

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers

insert point, (location known) $\qquad O(d^\circ p) = O(1)$

Delaunay of points in convex position

Analysis

choose a point at random

remove it from convex polygon

remember its place

$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers $\quad f(n-1)$

insert point, (location known) $\quad O(d^\circ p) = O(1)$

Delaunay of points in convex position

Analysis

choose a point at random

remove it from convex polygon

remember its place

$\left.\right\} O(1)$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers $\quad f(n-1)$

insert point, (location known) $\quad O(d^\circ p) = O(1)$

$$f(n) = f(n-1) + O(1)$$

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

compute Delaunay of $n-1$ points

with relevant vertex-triangle pointers $\quad f(n-1)$

insert point, (location known) $\quad O(d^\circ p) = O(1)$

$$f(n) = f(n-1) + O(1) \ = O(n)$$

Delaunay of points in convex position

Analysis

$$\left.\begin{array}{l}\text{choose a point at random}\\ \text{remove it from convex polygon}\\ \text{remember its place}\end{array}\right\} O(1)$$

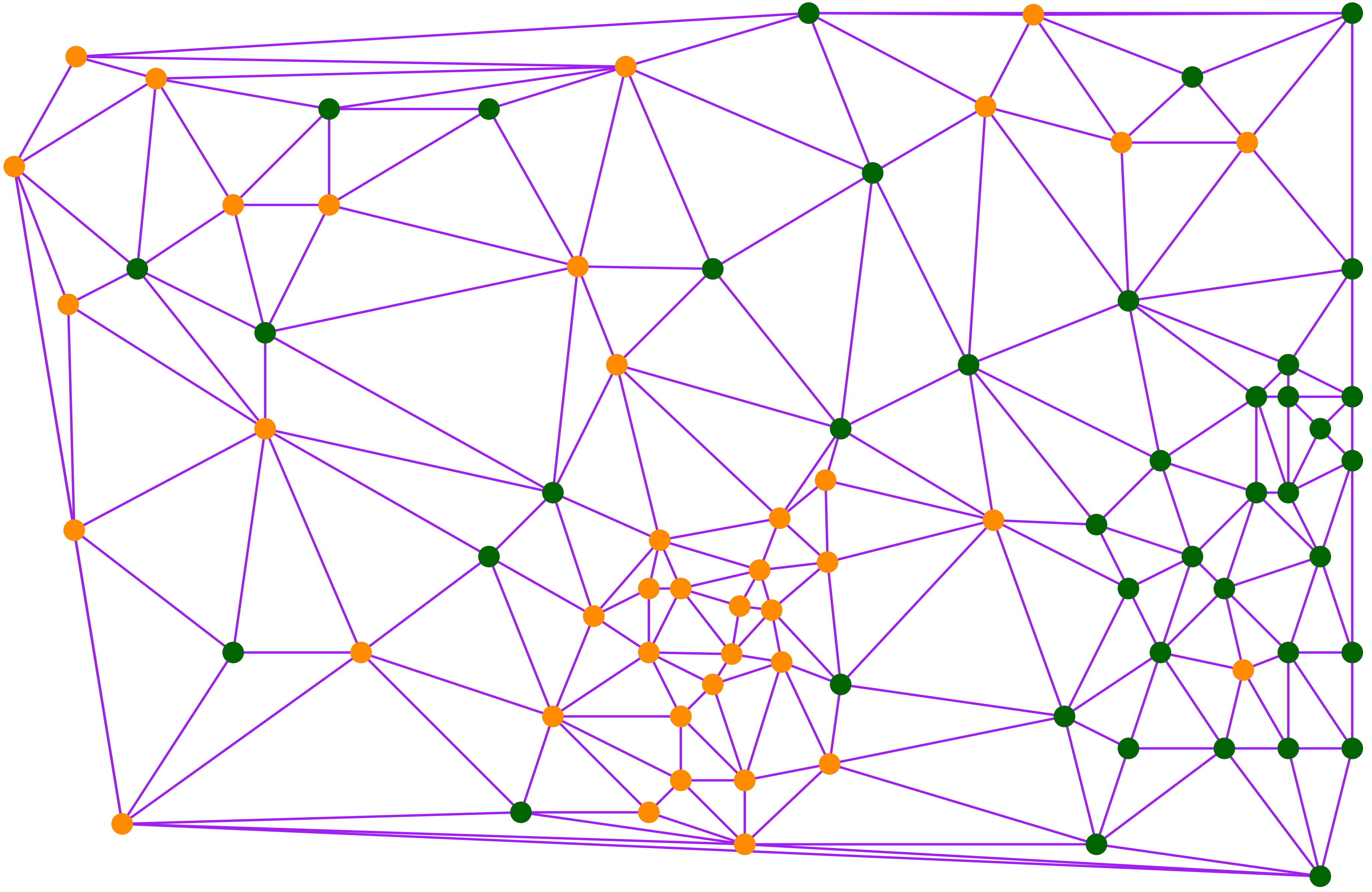
compute Delaunay of $n-1$ points
with relevant vertex-triangle pointers $\qquad f(n-1)$

insert point, (location known) $\qquad O(d^\circ p) = O(1)$

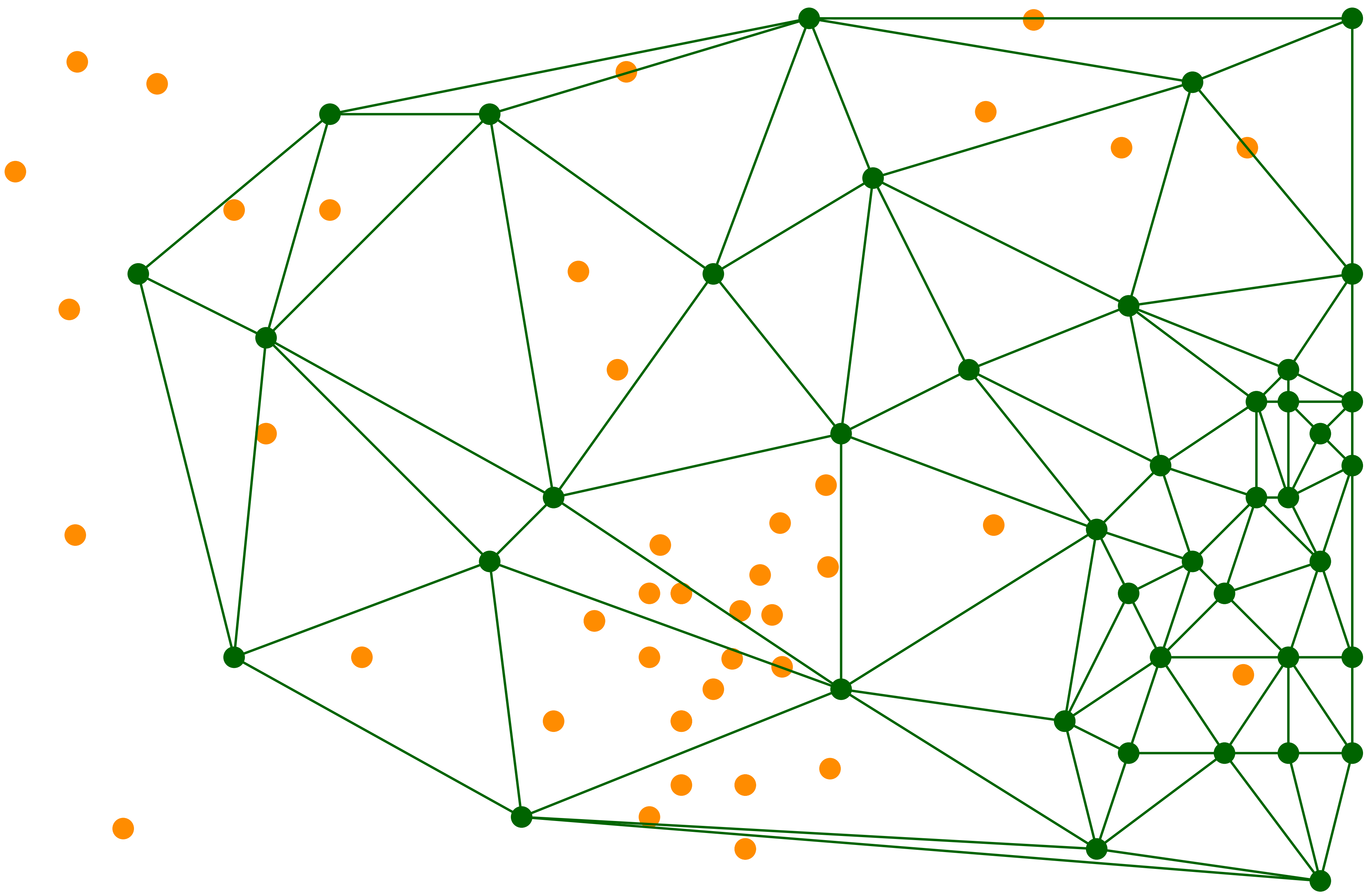
$$f(n) = f(n-1) + O(1) \; = O(n)$$

[Chew 86]

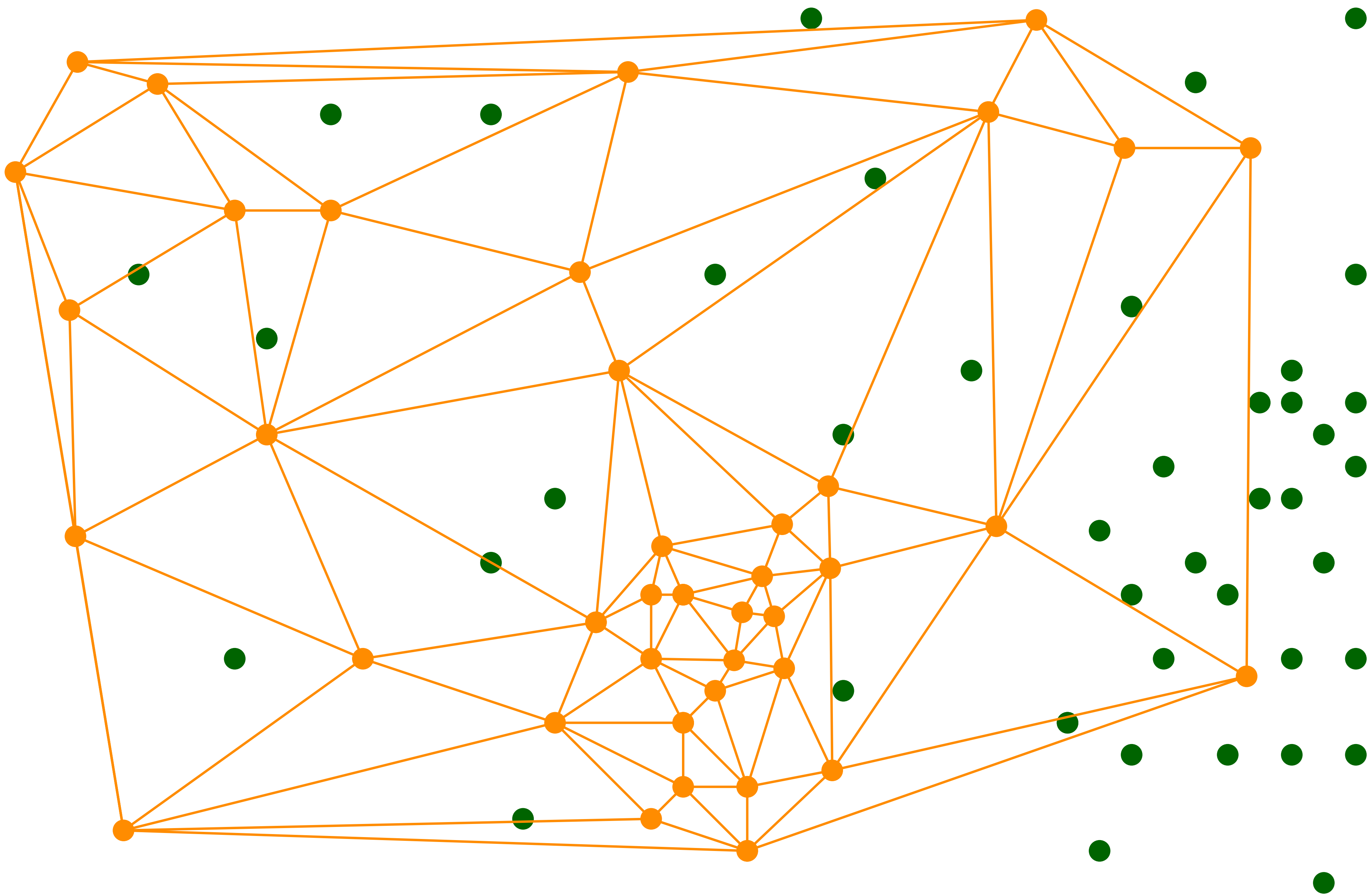
Splitting Delaunay



Splitting Delaunay

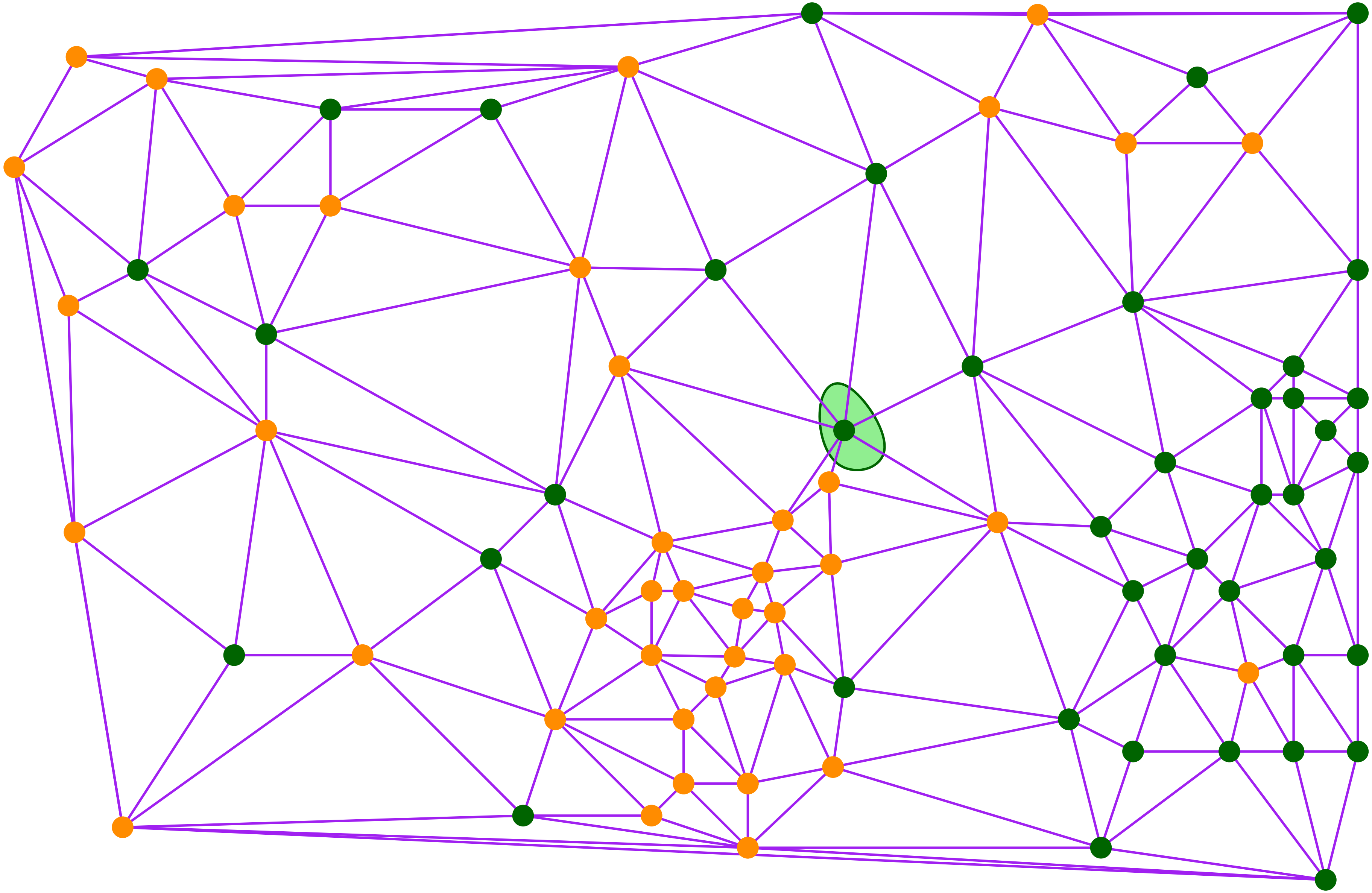


Splitting Delaunay



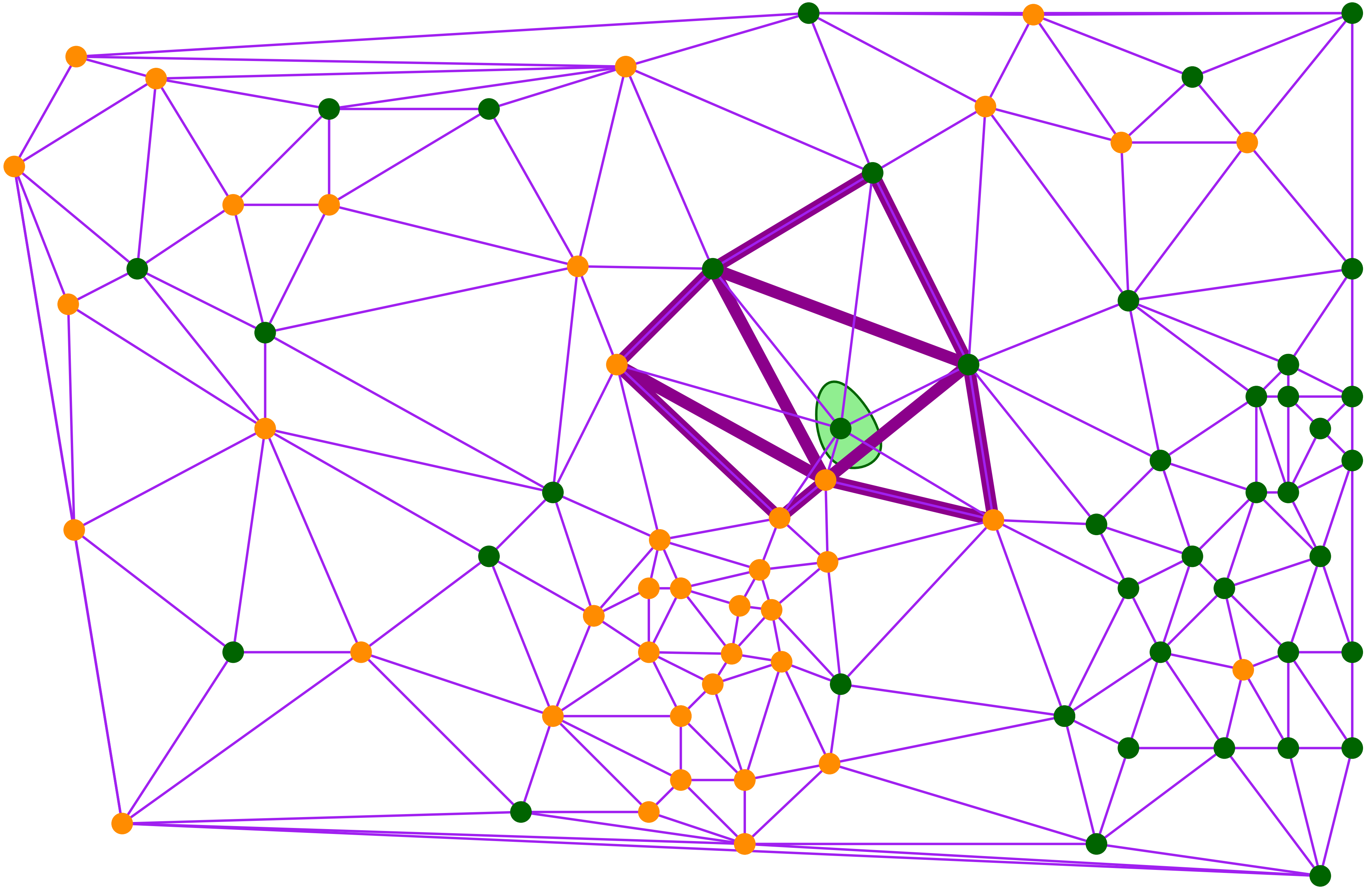
Splitting Delaunay

Remove random point



Splitting Delaunay

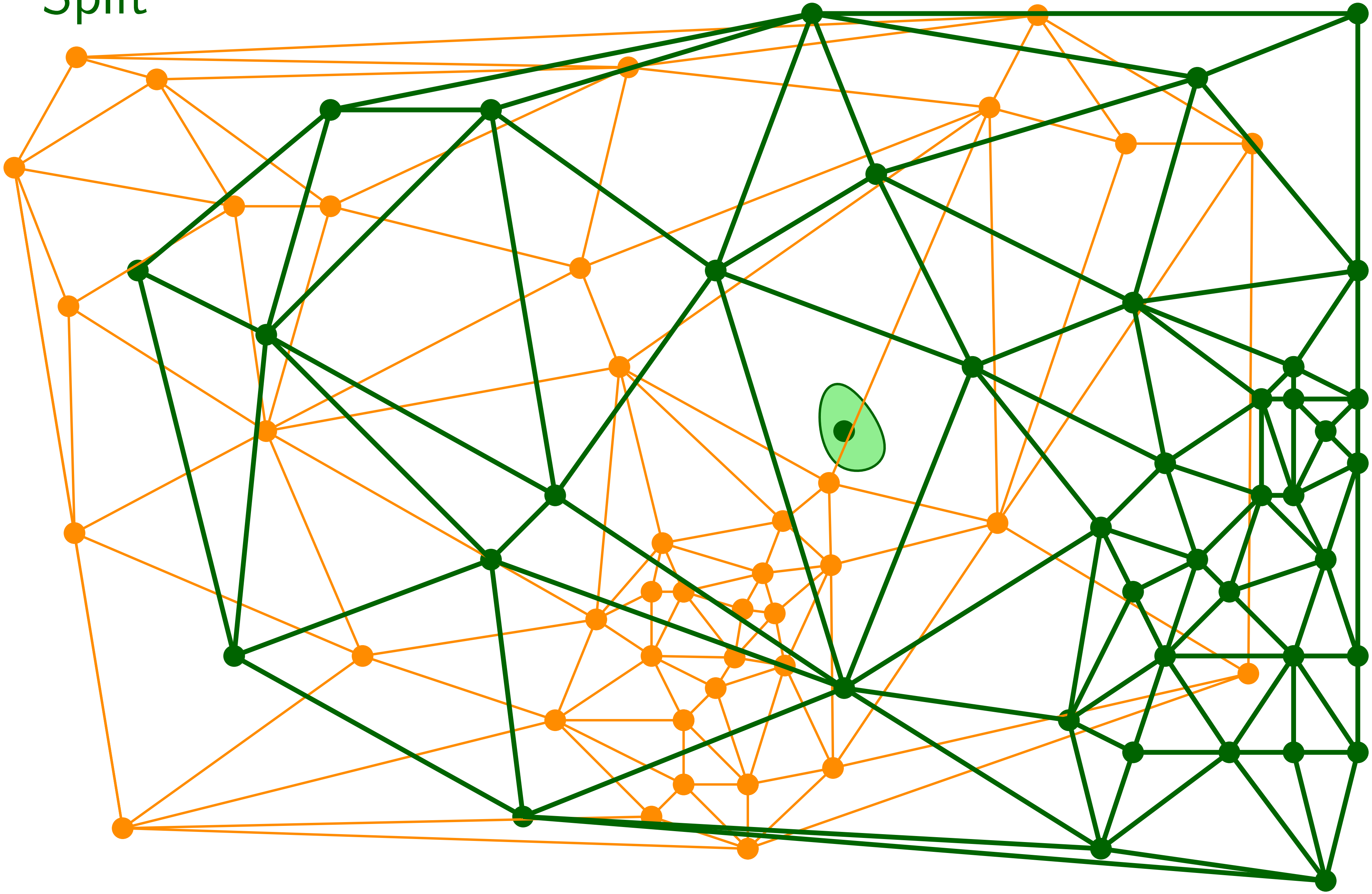
Remove random point



Splitting Delaunay

Remove random point

Split

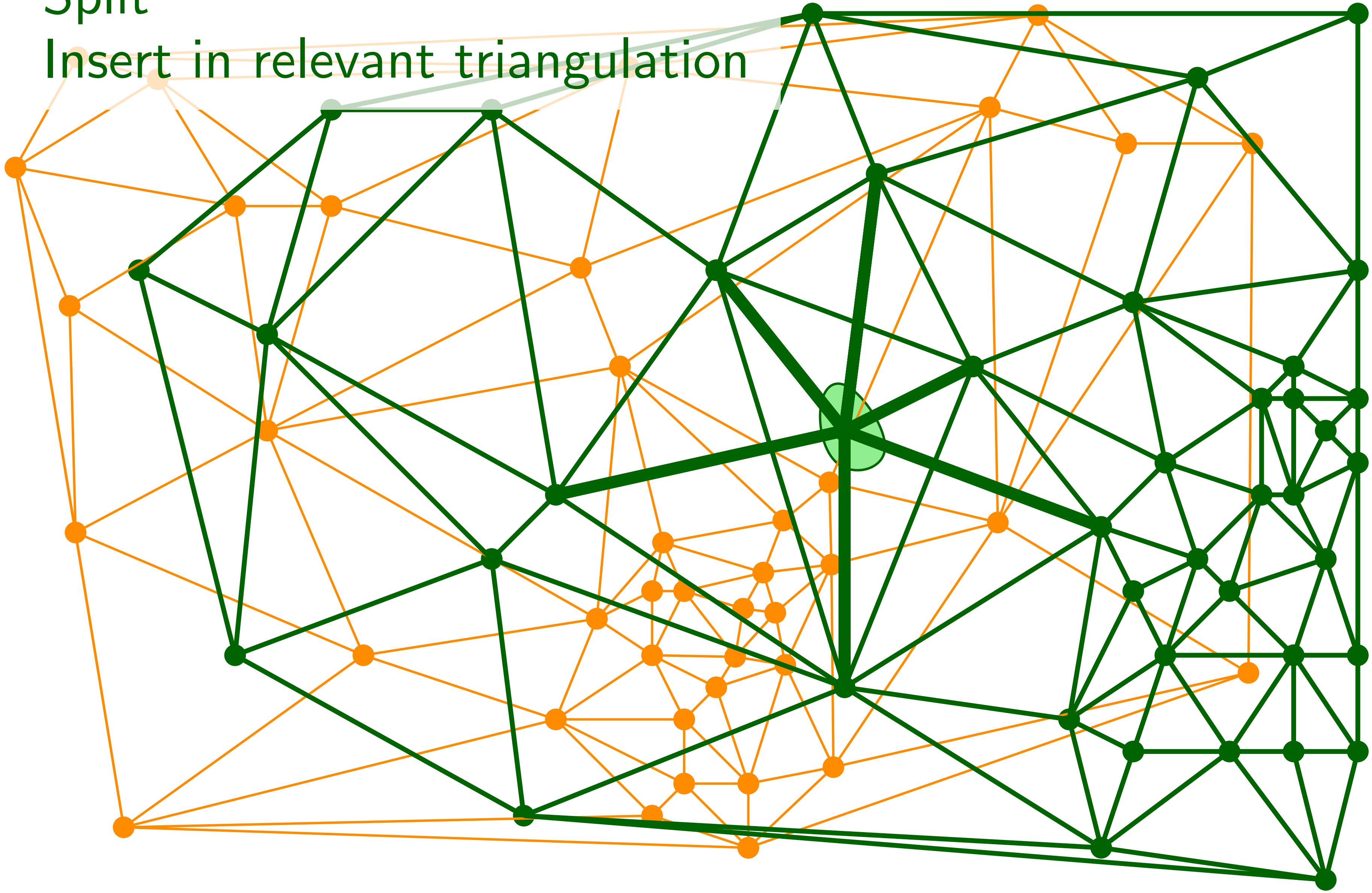


Splitting Delaunay

Remove random point

Split

Insert in relevant triangulation



Splitting Delaunay

Remove random point p $O(d^\circ p) = 6$

Split $f(n-1)$

Insert p in relevant triangulation

Splitting Delaunay

Remove random point p	$O(d^\circ p) = 6$
Split	$f(n-1)$
Insert p in relevant triangulation	still need to locate

Splitting Delaunay

Remove random point p $O(d^\circ p) = 6$

Compute and remember $NN(p)$ same color

Split $f(n-1)$

Insert p in relevant triangulation ~~still need to locate~~

locate $= O(1)$

Splitting Delaunay

Remove random point p $O(d^\circ p) = 6$

Compute and remember $NN(p)$ same color not so easy

Split $f(n-1)$

Insert p in relevant triangulation ~~still need to locate~~

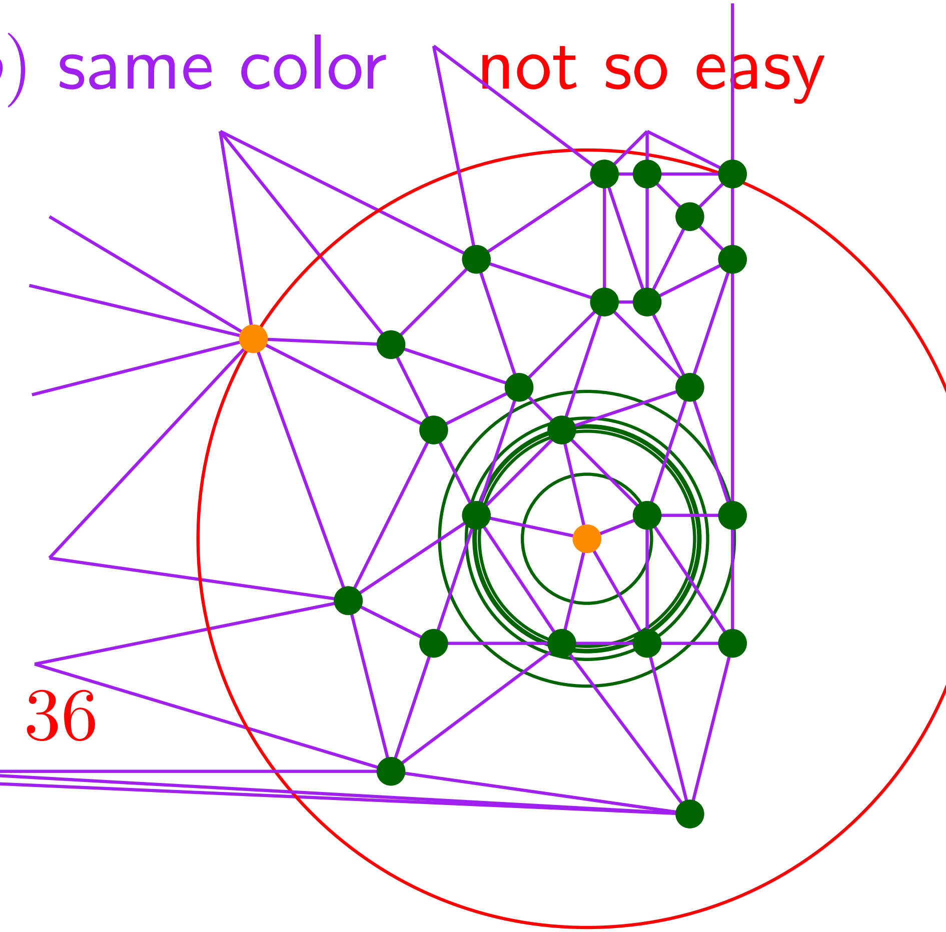
locate $= O(1)$

Compute and remember $NN(p)$ same color not so easy

$$E(T) = E\Big(\sum_{\bullet \in \bigcirc} d^\circ \bullet\Big)$$

$$= \frac{1}{n} \sum_{p} \sum_{q \in \bigcirc} d^\circ q$$

$$= \frac{1}{n} \sum_{q} \sum_{p} d^\circ q \leq \frac{1}{n} 6 \sum_{q} d^\circ q \leq 36$$



But finding this neighbor require at least $O(T \log T)$

and $E(T \log T)$ may be $> \Omega(1)$

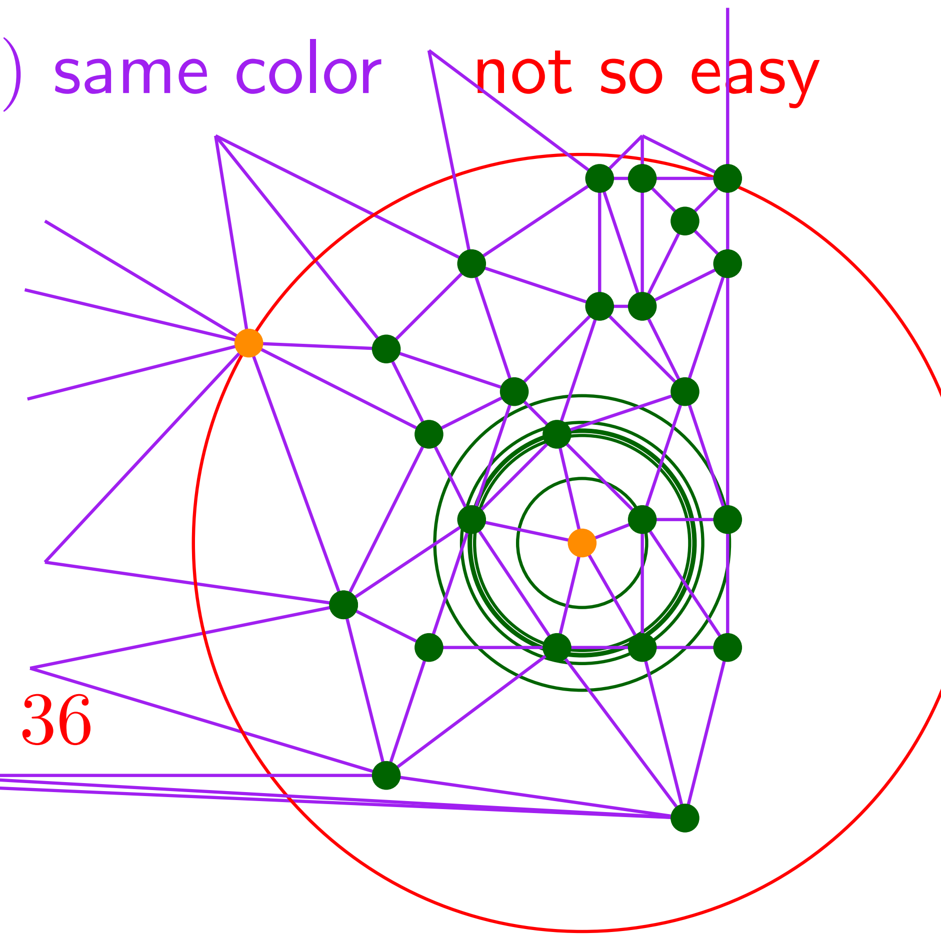
Compute and remember $NN(p)$ same color

not so easy

$$E(T) = E\left(\sum_{\bullet \in} d^\circ \bullet\right)$$

$$= \frac{1}{n} \sum_{p} \sum_{q \in} d^\circ q$$

$$= \frac{1}{n} \sum_{q} \sum_{p} d^\circ q \leq \frac{1}{n} 6 \sum_{q} d^\circ q \leq 36$$



Splitting Delaunay

find a trick

Remove random point p

Compute and remember $NN(p)$ same color

Split $f(n-1)$

Insert p in relevant triangulation locate $= O(1)$

Splitting Delaunay

find a trick

Take two random points q and q'

Remove ~~random~~ point $p \in \{q, q'\}$

Compute and remember $NN(p)$ same color

Split $f(n-1)$

Insert p in relevant triangulation locate $= O(1)$

Splitting Delaunay

find a trick

Take two random points q and q'

Remove ~~random~~ point $p \in \{q, q'\}$ $E(max(d^\circ q, d^\circ q'))$

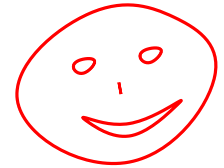
Compute and remember $NN(p)$ same color

Split $f(n-1)$

Insert p in relevant triangulation locate $= O(1)$

Splitting Delaunay

find a trick



Take two random points q and q'

Remove ~~random~~ point $p \in \{q, q'\}$ $\quad E(max(d^\circ q, d^\circ q'))$

Compute and remember $NN(p)$ same color

$E(min(T_q, T_{q'})^2)$

Split $\quad f(n-1)$

Insert p in relevant triangulation $\quad$ locate $= O(1)$

Splitting Delaunay

find a trick

Take two random points q and q'

Remove ~~random~~ point $p \in \{q, q'\}$ $E(max(d^\circ q, d^\circ q'))$

Compute and remember $NN(p)$ same color

$E(min(T_q, T_{q'})^2)$

Split $f(n-1)$

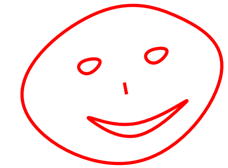
Insert p in relevant triangulation locate $= O(1)$

X random variable, Y independant copy of X

$$2E(X) = E(X+Y) = E(max(X,Y) + min(X,Y)) \geq E(max(X,Y))$$

Splitting Delaunay

find a trick



Take two random points q and q'

Remove ~~random~~ point $p \in \{q, q'\}$ $E(max(d^\circ q, d^\circ q')) \leq 12$

Compute and remember $NN(p)$ same color

$E(min(T_q, T_{q'})^2)$

Split $f(n-1)$

Insert p in relevant triangulation locate $= O(1)$

X random variable, Y independant copy of X

$$2E(X) = E(X+Y) = E(max(X,Y) + min(X,Y)) \geq E(max(X,Y))$$

Splitting Delaunay

find a trick

Take two random points q and q'

Remove ~~random~~ point $p \in \{q, q'\}$ $E(max(d^\circ q, d^\circ q')) \leq 12$

Compute and remember $NN(p)$ same color

$E(min(T_q, T_{q'})^2)$

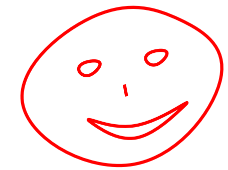
Split $f(n-1)$

Insert p in relevant triangulation locate $= O(1)$

$$E(min(X,Y)^2) \leq E(min(X,Y)max(X,Y)) = E(XY) = E(X)^2$$

Splitting Delaunay

find a trick



Take two random points q and q'

Remove ~~random~~ point $p \in \{q, q'\}$ $\qquad E(max(d^\circ q, d^\circ q')) \leq 12$

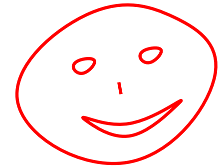
Compute and remember $NN(p)$ same color

$E(min(T_q, T_{q'})^2) \leq 36$

Split $\qquad f(n-1)$

Insert p in relevant triangulation $\qquad$ locate $= O(1)$

$$E(min(X,Y)^2) \leq E(min(X,Y)max(X,Y)) = E(XY) = E(X)^2$$

Splitting Delaunay find a trick 

Take two random points q and q'

Remove ~~random~~ point $p \in \{q, q'\}$ $E(max(d^\circ q, d^\circ q')) \leq 12$

Compute and remember $NN(p)$ same color

$E(min(T_q, T_{q'})^2) \leq 36$

Split $f(n-1)$

Insert p in relevant triangulation locate $= O(1)$

Thus overall $O(n)$ time

[Chazelle Devillers Hurtado Mora Sacristán Teillaud 2002]

Randomization

Randomization

Randomized incremental constructions

Simple algorithms

non trivial analysis

good complexities

efficient in practice

Randomization

Randomized incremental constructions

Simple algorithms

non trivial analysis

good complexities

efficient in practice

Delaunay hierarchy

Spatial sorting

Randomization

Randomized incremental constructions

Simple algorithms

non trivial analysis

good complexities

efficient in practice

Delaunay hierarchy

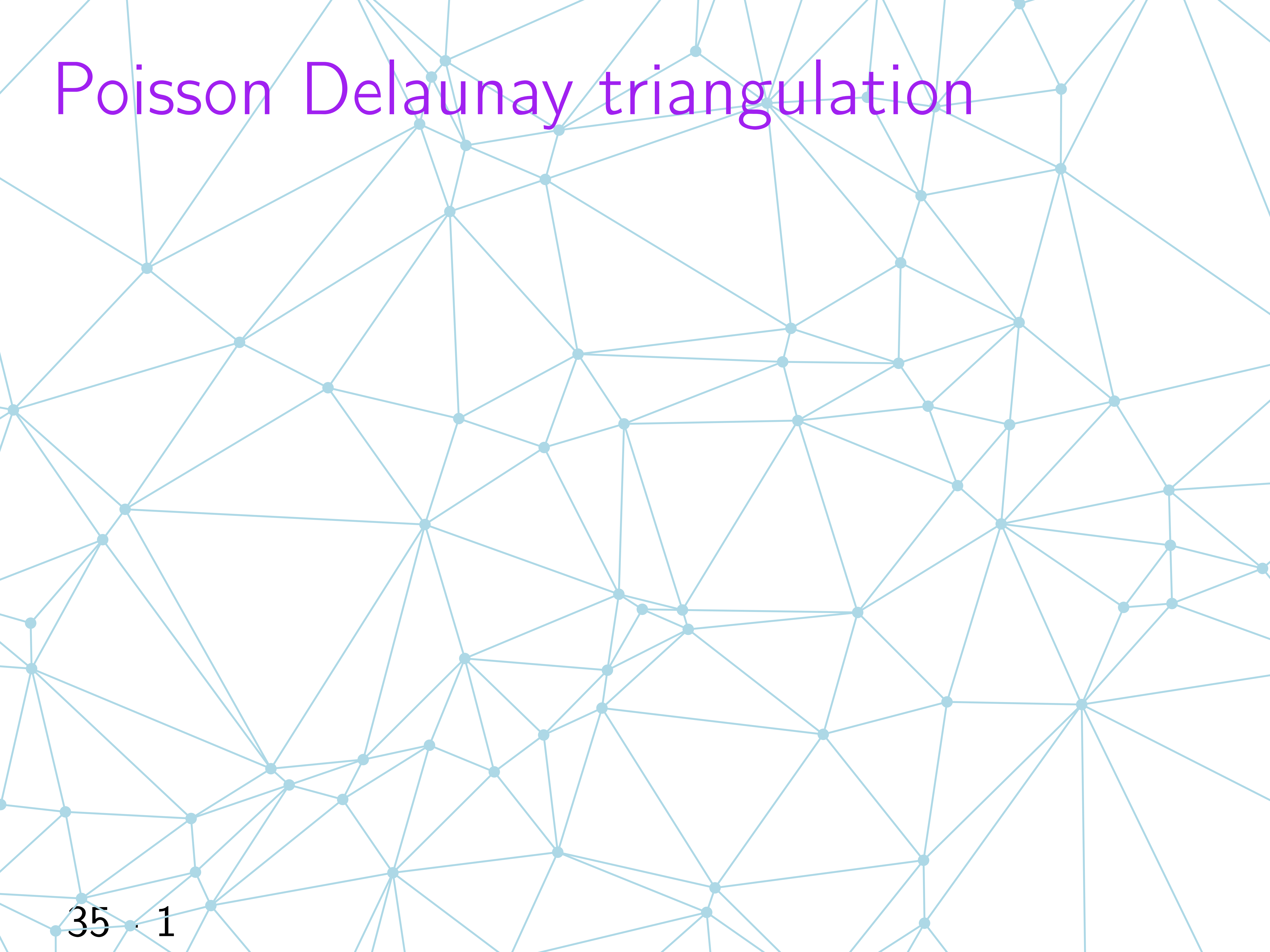
Spatial sorting

Other tools

divide and conquer

ϵ nets   Good sample with high probability

Poisson Delaunay triangulation



Poisson Delaunay triangulation

- Poisson distribution
- Slivnyak-Mecke formula
- Blaschke-Petkanschin variables substitution
- Stupid analysis of the expected degree
- Straight walk expected analysis
- Catalog of properties

Distribution in A independent from distribution in B.

when $A \cap B = \varnothing$

Unit uniform rate

$$\mathbb{P}\left[|X \cap A| = k\right] = \frac{\mathrm{vol}(A)^k}{k!} e^{-\mathrm{vol}(A)}$$

Distribution in A independent from distribution in B.

when $A \cap B = \varnothing$

Very convenient

Unit uniform rate

$$\mathbb{P}\left[|X \cap A| = k\right] = \frac{\mathrm{vol}(A)^k}{k!} e^{-\,\mathrm{vol}(A)}$$

X a Poisson point process

Distribution in A independent from distribution in B.

when $A \cap B = \varnothing$

Unit uniform rate

$$\mathbb{P}\left[|X \cap A| = k\right] = \frac{\mathrm{vol}(A)^k}{k!} e^{-\mathrm{vol}(A)}$$

$$\mathbb{P}\left[|X \cap A| = 0\right] = e^{-\mathrm{vol}(A)}$$

$$\mathbb{E}\left[|X \cap A|\right] = \sum_0^\infty k \frac{\mathrm{vol}(A)^k}{k!} e^{-\mathrm{vol}(A)} = \mathrm{vol}(A)$$

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum $\longrightarrow$ Integral

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum $\longrightarrow$ Integral

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[P(X,q)]}\right]$$

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum ⟶ Integral

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[P(X,q)]}\right] = n\int_{\mathbb{R}^2} \mathbb{P}\left[P(X\cup\{q\},q)\right]\,\mathrm{d}q$$

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum $\longrightarrow$ Integral

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[P(X,q)]}\right] = n \int_{\mathbb{R}^2} \mathbb{P}\left[P(X \cup \{q\}, q)\right] \mathrm{d}q$$

e.g.,

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[NN_X(0)=q]}\right]$$

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum ⟶ Integral

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[P(X,q)]}\right] = n\int_{\mathbb{R}^2} \mathbb{P}\left[P(X\cup\{q\},q)\right]\,\mathrm{d}q$$

e.g.,

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[NN_X(0)=q]}\right] = n\int_{\mathbb{R}^2} \mathbb{P}\left[D(0,\|q\|)\cap X=\emptyset\right]\,\mathrm{d}q$$

Slivnyak-Mecke formula

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$$= n\int_{\mathbb{R}^2} e^{-n\pi\|q\|^2}\,\mathrm{d}q$$

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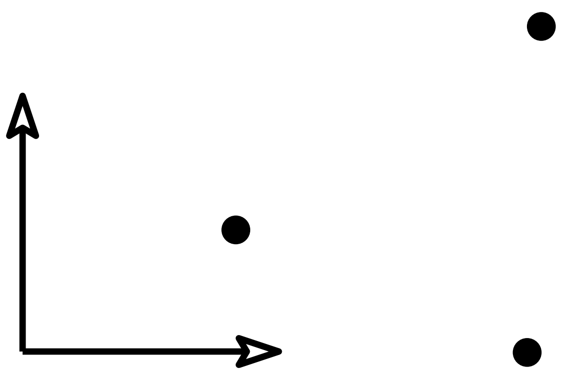
$$= n\int_{\mathbb{R}^2} e^{-n\pi\|q\|^2} \mathrm{d}q$$

$$= n\int_0^{2\pi}\int_0^{\infty} e^{-n\pi r^2} r \mathrm{d}\theta\, \mathrm{d}r = n \times 2\pi \times \frac{1}{2n\pi} = 1$$

Blaschke-Petkantschin variable substitution

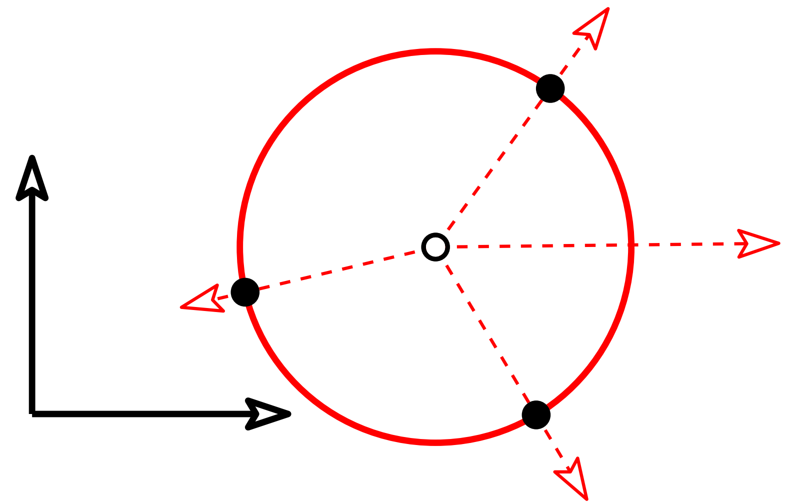
$$\int_{(\mathbb{R}^2)^3} f(p, q, t)\, dp\, dq\, dt$$



Blaschke-Petkantschin variable substitution

$$\int_{(\mathbb{R}^2)^3} f(p,q,t)\,dp\,dq\,dt$$

$$= \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(p,q,t)|det(J)|\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}x\mathrm{d}y\mathrm{d}r$$



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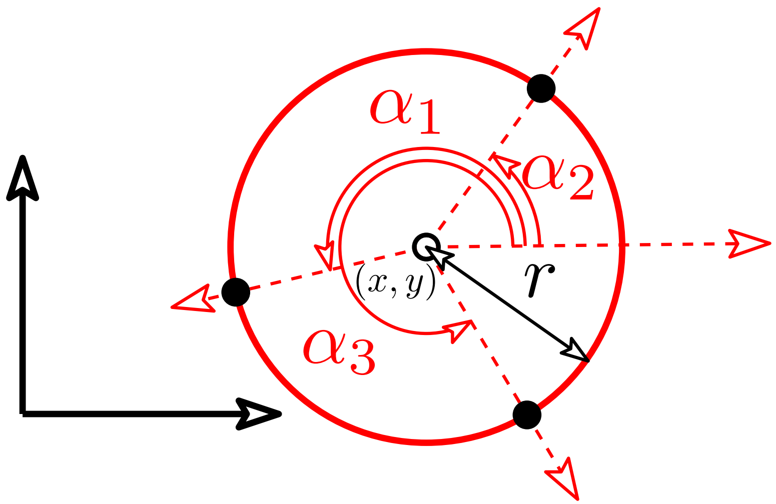
$$= \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f(p,q,t)2r^3 area(\alpha_1\alpha_2\alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}x\mathrm{d}y\mathrm{d}r$$



Expected number of triangles in conflict with origin

X a Poisson point process of density n

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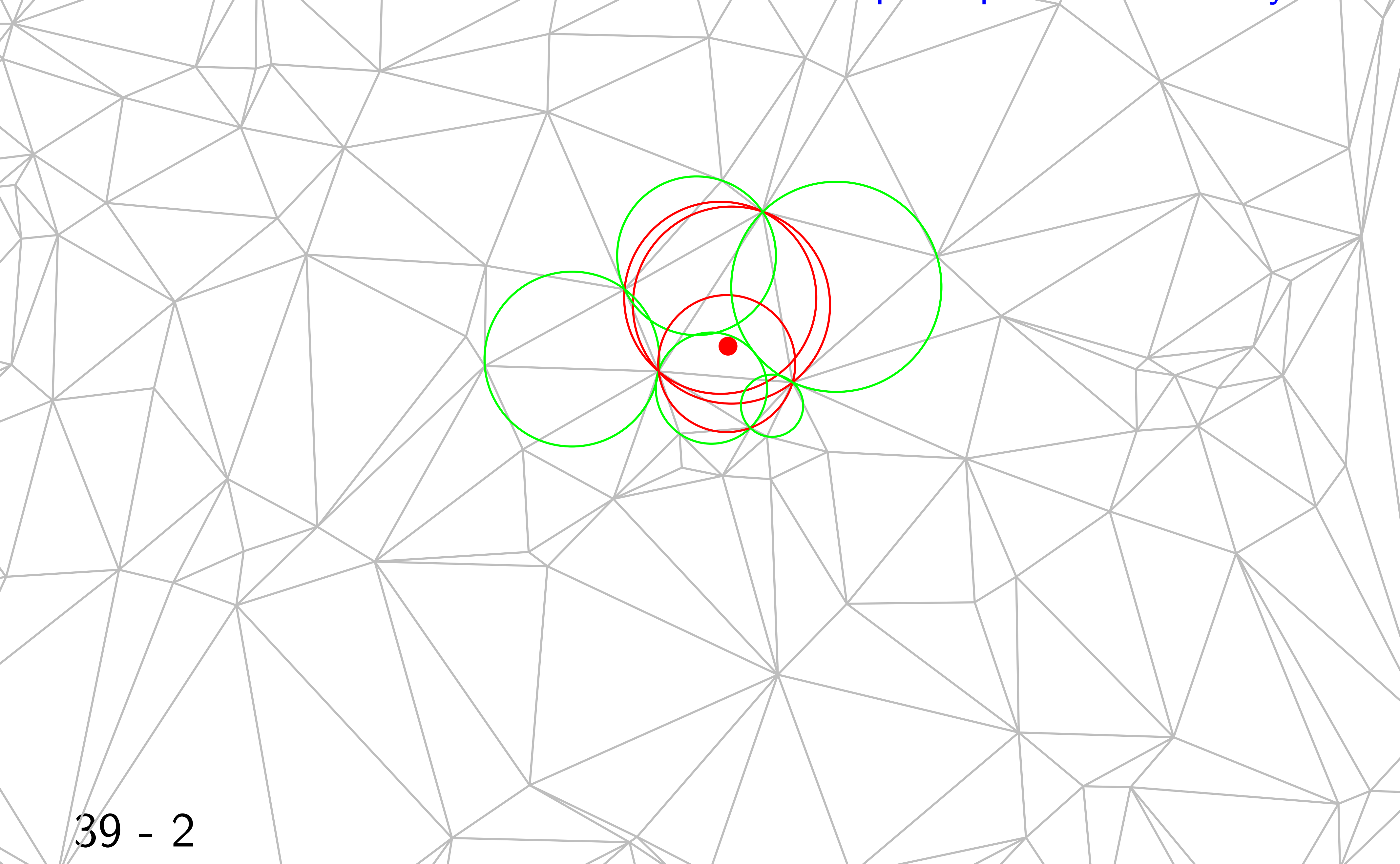

Expected number of triangles in conflict with origin

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$$\mathbb{E}\left[\frac{1}{3}\sum_{p,q,t\in X^3} \mathbb{1}_{[pqt\in DT(X)]}\mathbb{1}_{[pqt\ \textsf{CCW}]}\mathbb{1}_{[O\in Disk(pqt)]}\right]$$

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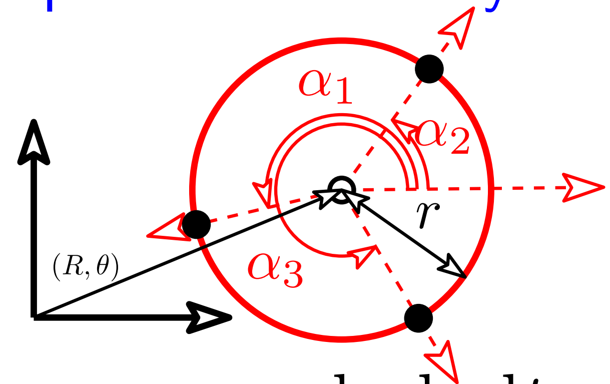

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```
Maple computation:
> assume(n>0):with(LinearAlgebra):
> int( exp(-n*Pi*r^ 2)*r^ 5,r=0..infinity);
```

$$\frac{1}{n^3\pi^3}$$

```
> 6*int(int(int(Determinant([[          1,            1,
                             [cos(alpha1),cos(alpha2),cos
                             [sin(alpha1),sin(alpha2),sin
     alpha3=alpha2..alpha1+2*Pi),alpha2=alpha1..alpha1+2
```

$$12\pi^2$$

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$$= \tfrac{n^3}{3}\int_0^\infty e^{-n\pi r^2}r^3 2\pi\frac{r^2}{2}\mathrm{d}r\ \ 12\pi^2 = \tfrac{n^3}{3}\pi\frac{1}{n^3\pi^3}12\pi^2 = 4$$

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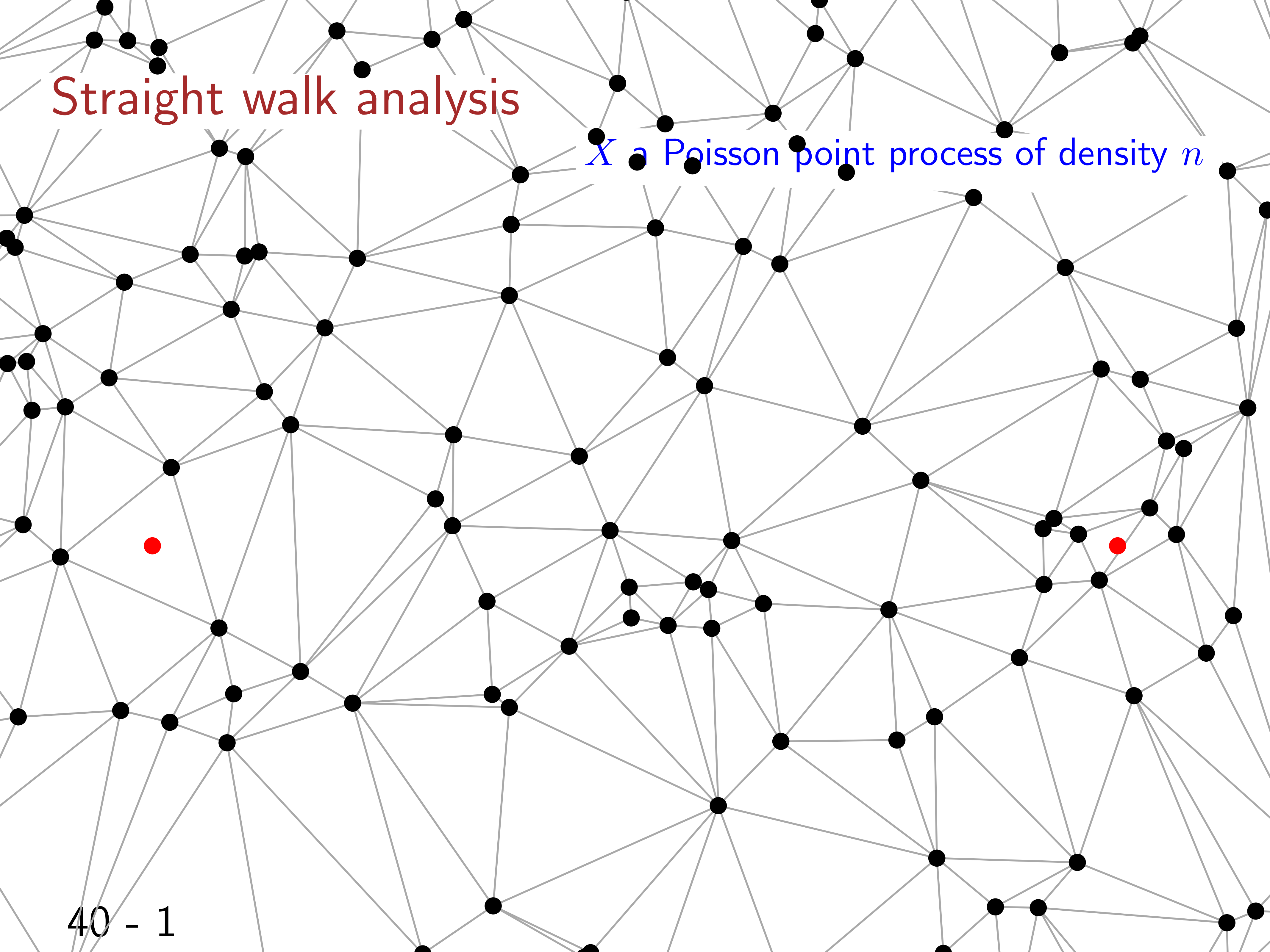
$$\Rightarrow \mathbb{E}\left[d^\circ_{DT(X\cup\{0\})}(0)\right] = 6$$

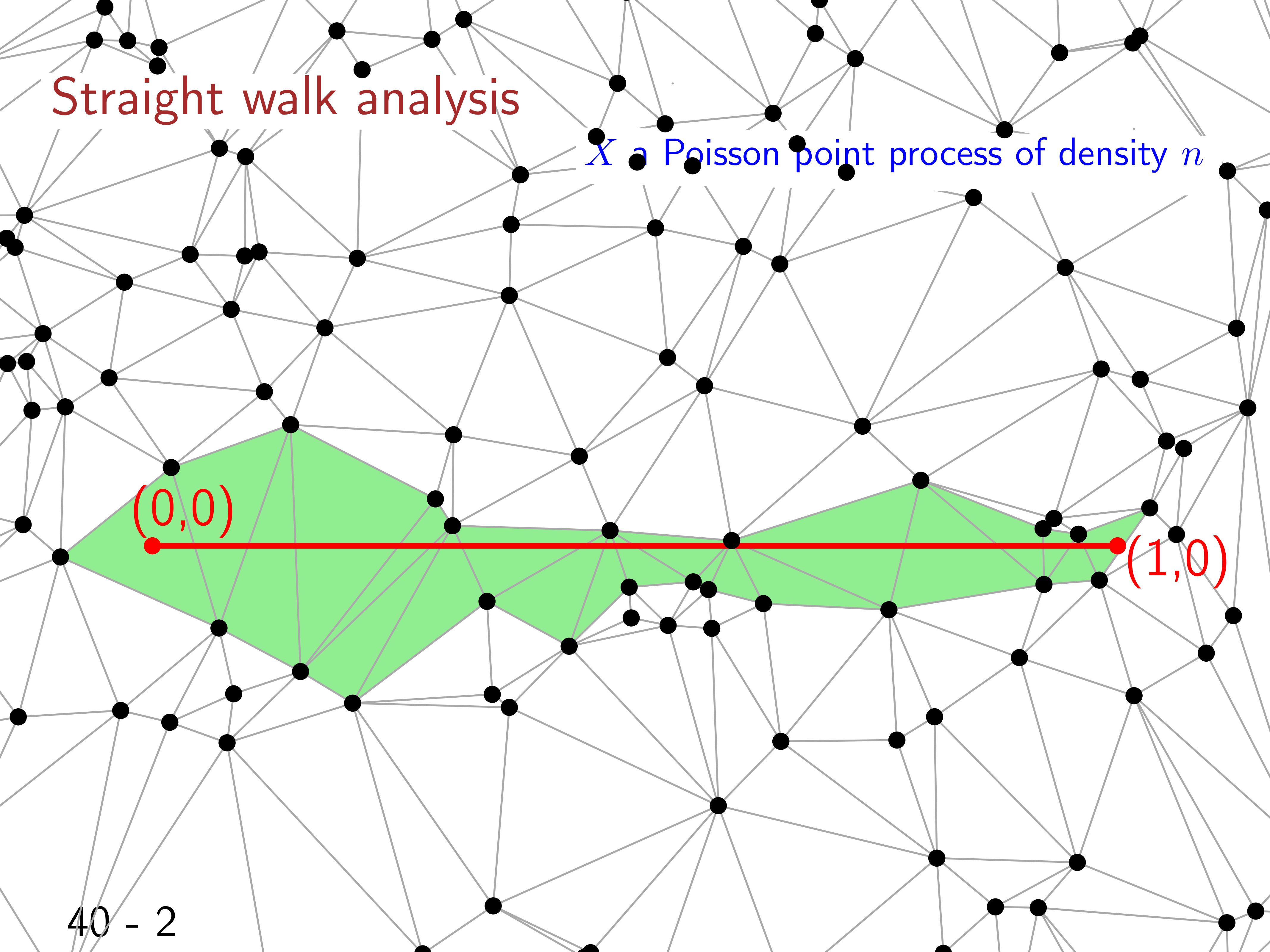
(you already know ☺)

Straight walk analysis

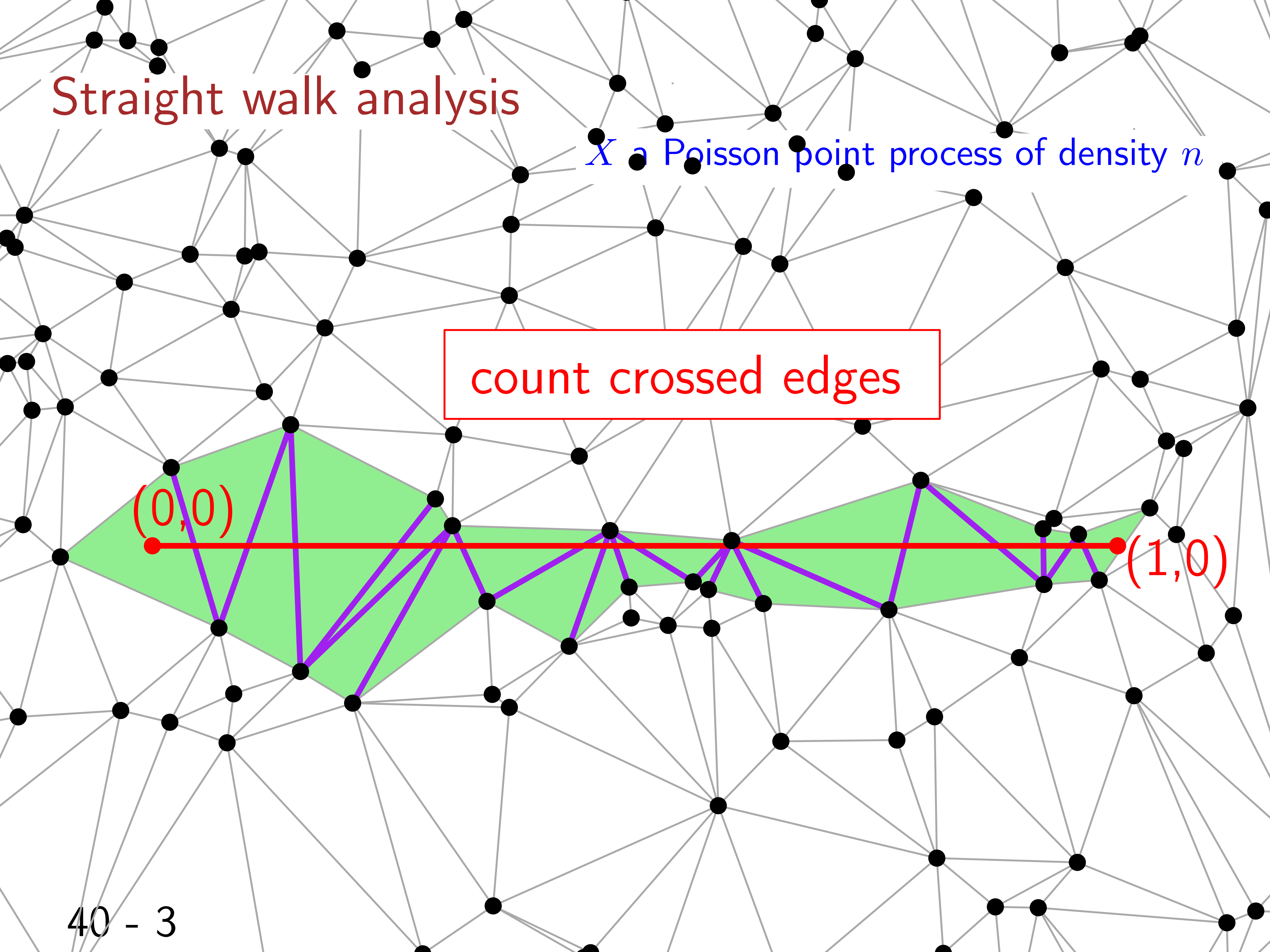


X a Poisson point process of density n

Straight walk analysis



Straight walk analysis



Straight walk analysis

$$\mathbb{E}\left[\overset{\color{blue}{X \text{ a Poisson point process of density } n}}{\frac{1}{2}\sum_{p,q,t\in X^3} \mathbb{1}_{[pqt\in DT(X)]}\mathbb{1}_{[p \text{ below}, q \text{ above}]}\mathbb{1}_{[pq \text{ intersects segment}]}}\right]$$

Straight walk analysis

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$$+\mathbb{E}\left[\frac{1}{2}\sum_{p,q,t\in X^3}\mathbb{1}_{[pqt\in DT(X)]}\mathbb{1}_{[p,t\text{ below},q\text{ above}]}\mathbb{1}_{[pq\text{ intersects segment}]}\right]$$

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$$= n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P}\left[X \cap B(pqt) = \emptyset\right] \mathbb{1}_{[\text{"position"}]} \, dp \, dq \, dt$$

Slivnyak-Mecke formula

Straight walk analysis

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$$\simeq n^3 \int_0^\infty \int_0^1 \int_{-r}^{r} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} e^{-n\pi r^2} \mathbb{1}_{[\text{"position"}]}$$

$$\cdot\, r^3 2area(\alpha_1\alpha_2\alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}x\mathrm{d}y\mathrm{d}r$$

Blaschke-Petkantschin formula

Straight walk analysis

X a Poisson point process of density n

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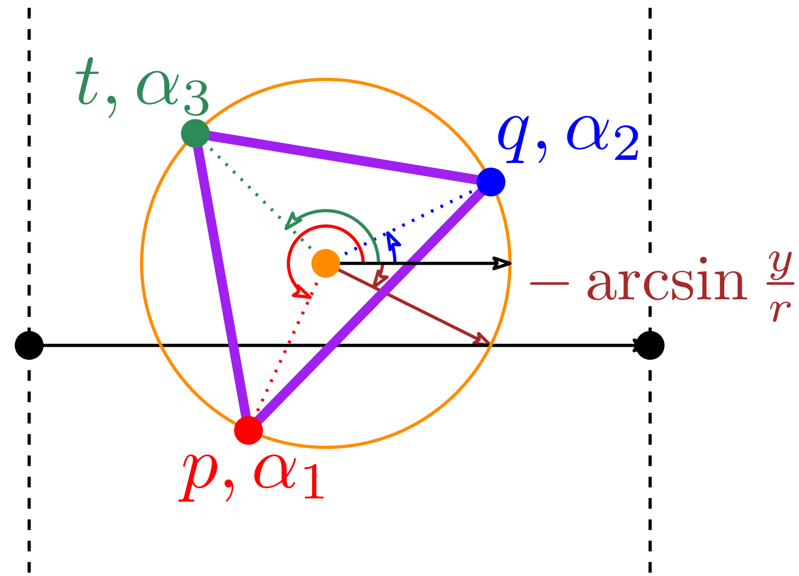
Straight walk analysis



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$$\cdot\, r^3$$

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$$\cdot\, r^3 2area(\alpha_1\alpha_2\alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}y\mathrm{d}r$$

$$\simeq n^3 \int_0^\infty \int_{-1}^{1} \int_{\pi+\arcsin h}^{2\pi-\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} e^{-n\pi r^2}$$

$rh = y$

$$\cdot\, r^3 2area(\alpha_1\alpha_2\alpha_3)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1 r\mathrm{d}h\mathrm{d}r$$

Straight walk analysis

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$$\times \int_{-1}^1 \int_{\pi+\arcsin h}^{2\pi-\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} 2area(\alpha_1\alpha_2\alpha_3) \mathrm{d}\alpha_3 \mathrm{d}\alpha_2 \mathrm{d}\alpha_1 r \mathrm{d}h$$

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ask Maple !

$$\times \int_{-1}^{1} \int_{\pi+\arcsin h}^{2\pi-\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} 2 area(\alpha_1\alpha_2\alpha_3) \mathrm{d}\alpha_3 \mathrm{d}\alpha_2 \mathrm{d}\alpha_1 r \mathrm{d}h$$

$$\simeq n^3 \int_0^\infty e^{-n\pi r^2} r^3 \, \frac{512}{9} r \, \mathrm{d}r$$

Straight walk analysis

X a Poisson point process of density n

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$$\times\int_{-1}^{1}\int_{\pi+\arcsin h}^{2\pi-\arcsin h}\int_{-\arcsin h}^{\pi+\arcsin h}\int_{-\arcsin h}^{\pi+\arcsin h}2area(\alpha_1\alpha_2\alpha_3)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1 r\mathrm{d}h$$

$$\simeq n^3\int_0^\infty e^{-n\pi r^2}r^3\ \frac{512}{9}r\ \mathrm{d}r$$

$$=\frac{512}{9}n^3\frac{3}{8\pi^2n^2\sqrt{n}}=\frac{64}{3\pi^2}\sqrt{n}\simeq 2.16\sqrt{n}$$

Sample of other probabilistic results

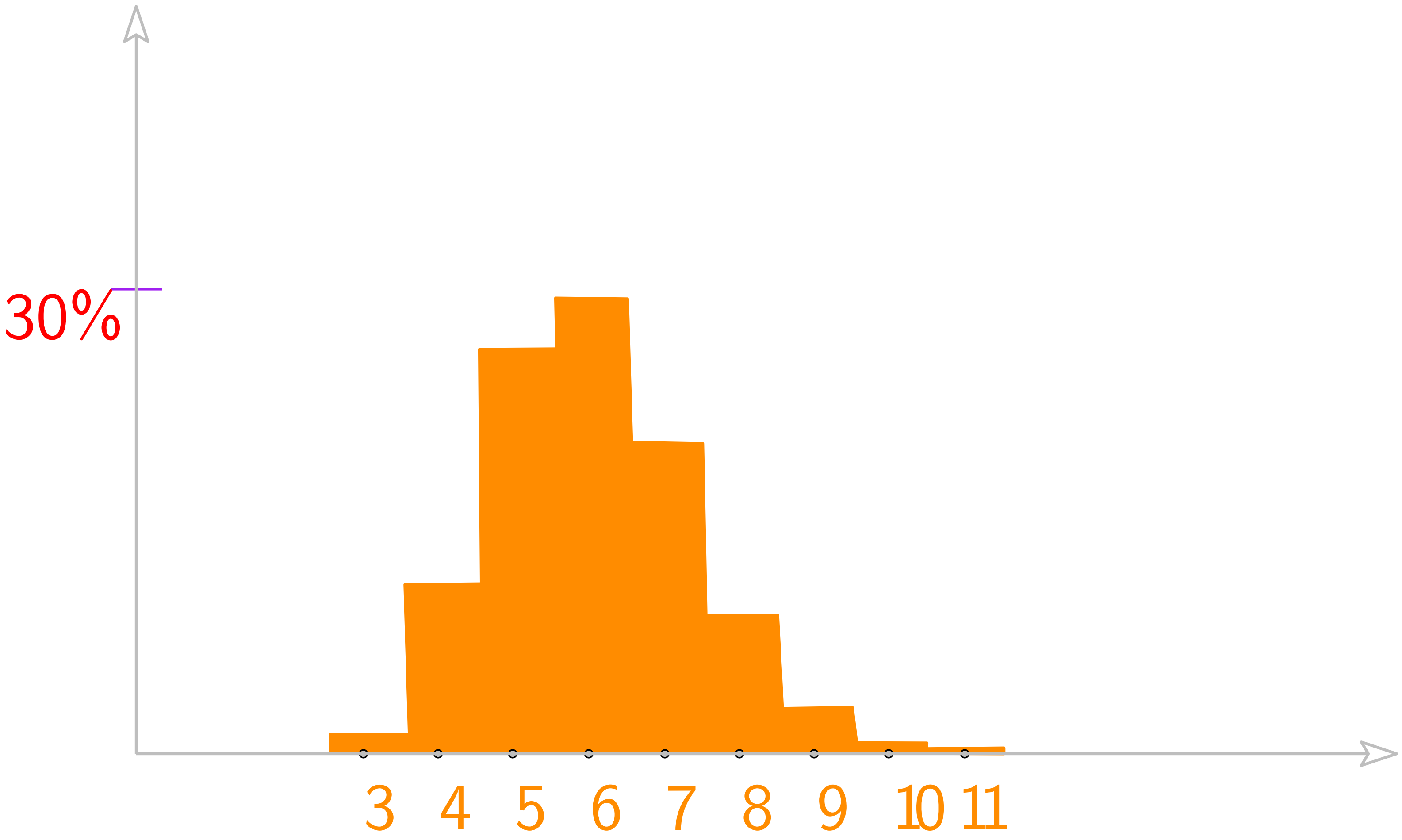
Expected degree

2D $\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = 6$

Expected degree

2D

$$\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = 6$$



Expected degree

2D $\qquad \mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = 6$

3D $\qquad \mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = \frac{48\pi^2}{35} + 2 \simeq 15.535$

Expected degree

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3D on a cylinder $\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = \Theta(\log n)$

Expected degree

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3D on a cylinder $\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = \Theta(\log n)$

3D on a surface $O(1) \leq \mathbb{E}\left[(\mathrm{d}^\circ(p)\right] \leq O(\log n)$

generic

`conjecture`

Expected maximum degree

Poisson distribution intensity 1, window $[0, \sqrt{n}]^2$

$$\mathbb{E}\left[\max(\mathrm{d}^\circ(p)\right] = \Theta\left(\frac{\log n}{\log\log n}\right)$$

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no boundaries!

Expected maximum degree

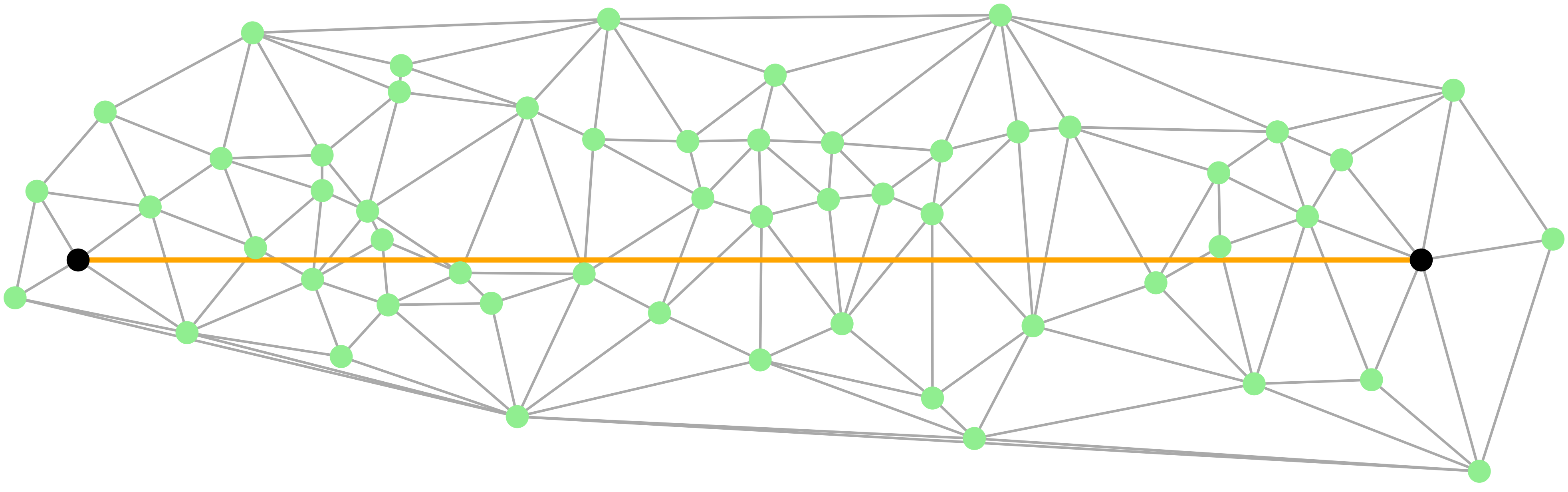
Poisson distribution intensity 1, window $[0, \sqrt{n}]^2$

$$\mathbb{E}\left[\max(\mathrm{d}^\circ(p)\right] = \Theta\left(\frac{\log n}{\log\log n}\right)$$

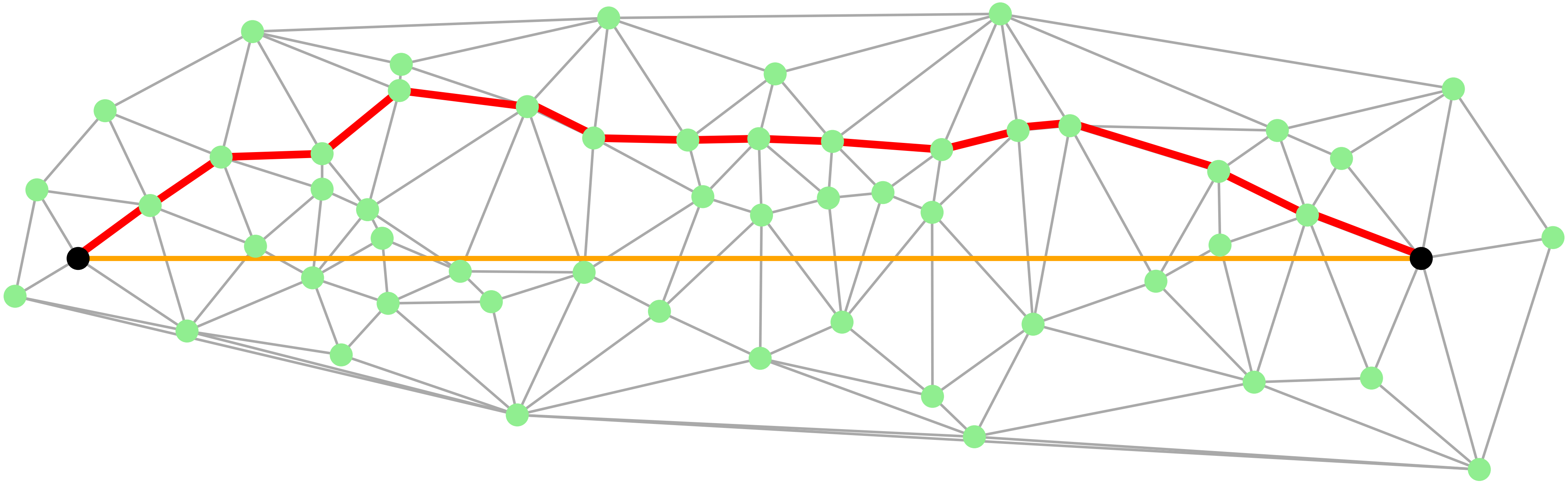
Poisson distribution intensity n, bounded domain

$$\mathbb{E}\left[\max(\mathrm{d}^\circ(p)\right] = O\left(\log^{2+\epsilon} n\right)$$

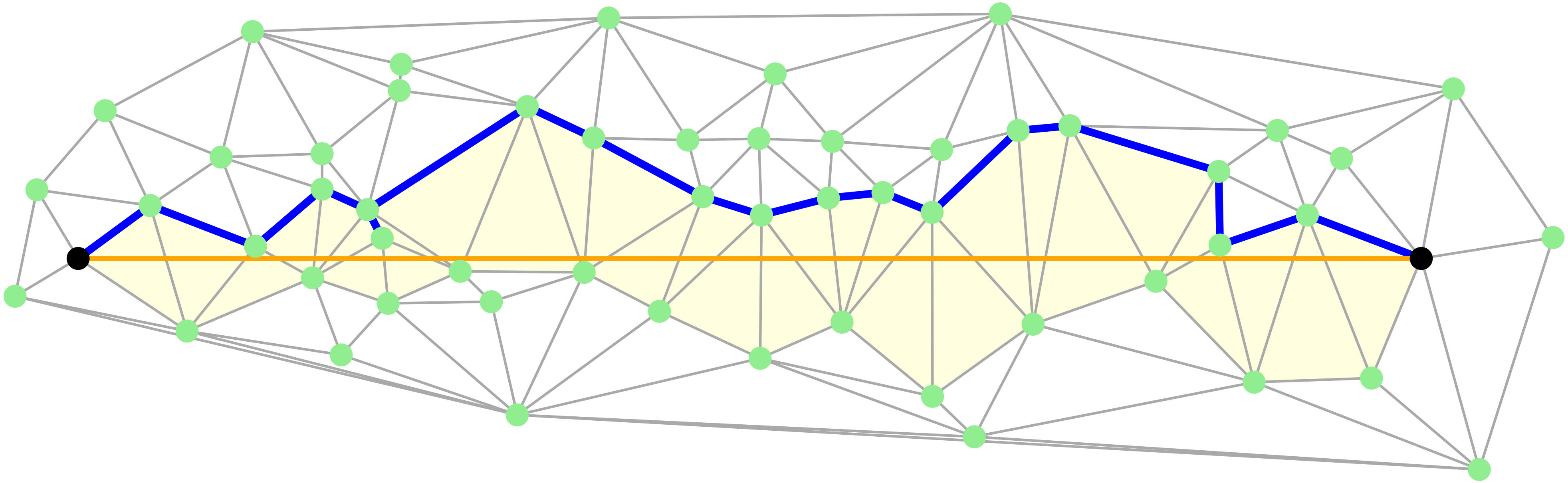
Walk between vertices



Walk between vertices

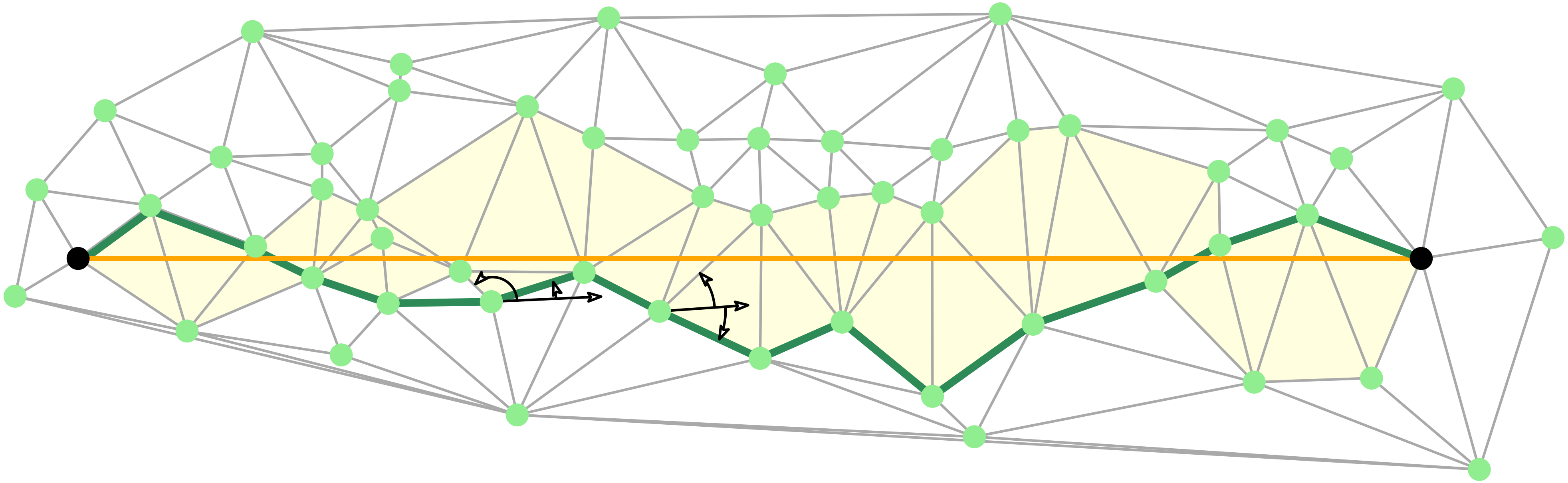


Walk between vertices



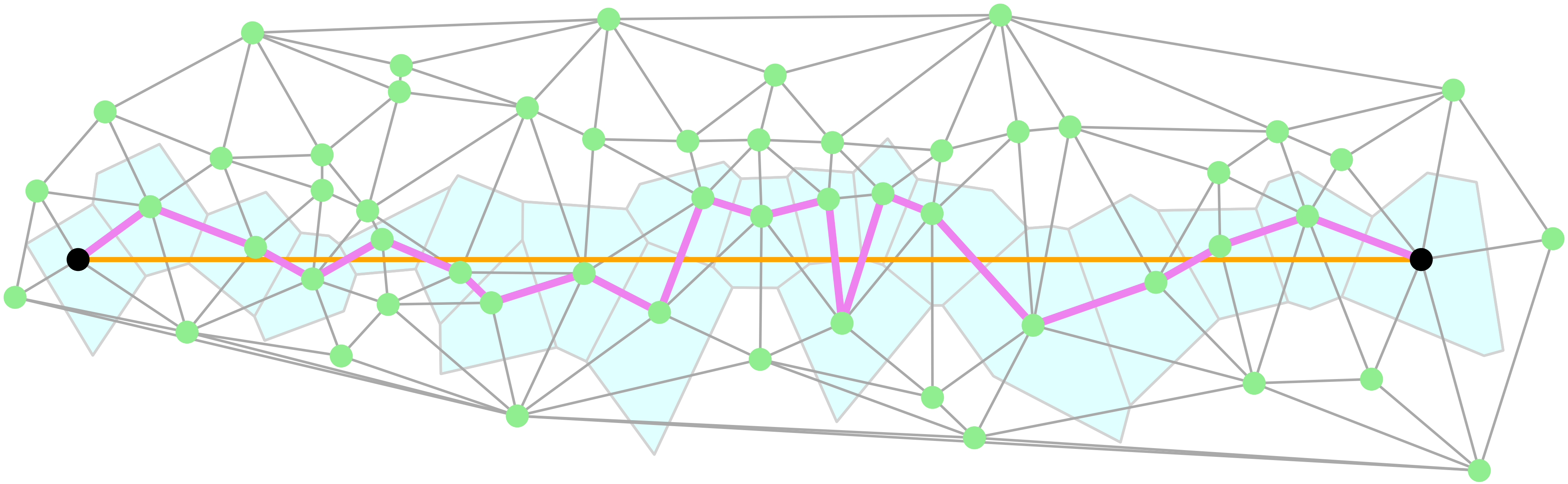
Upper path

Walk between vertices



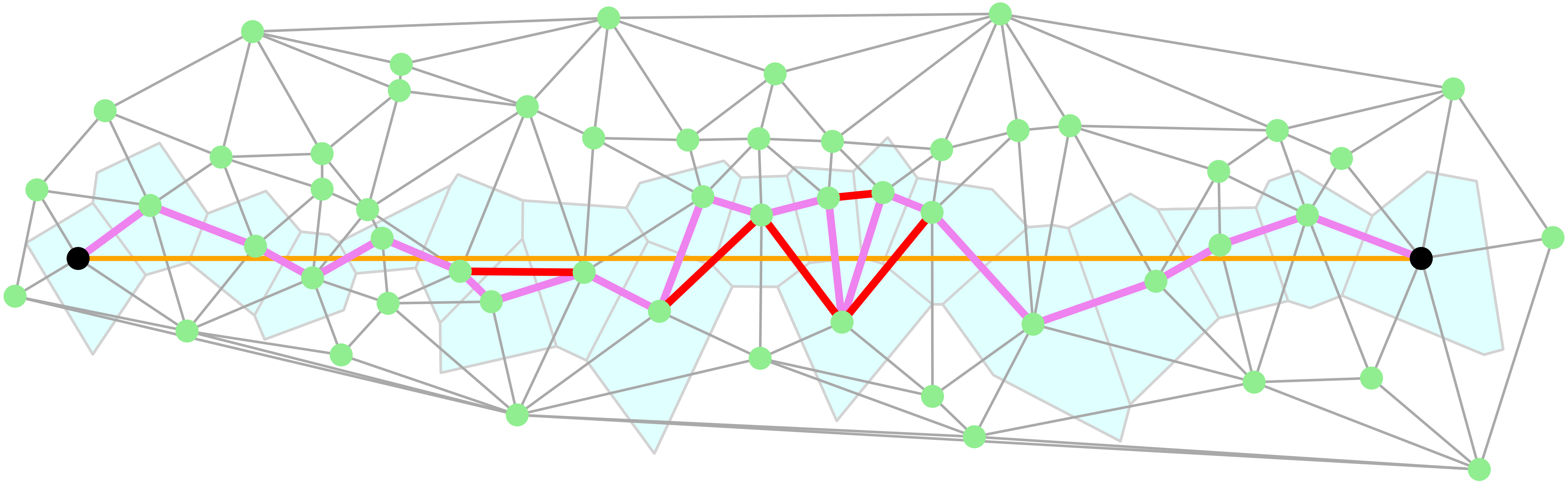
Compass walk

Walk between vertices



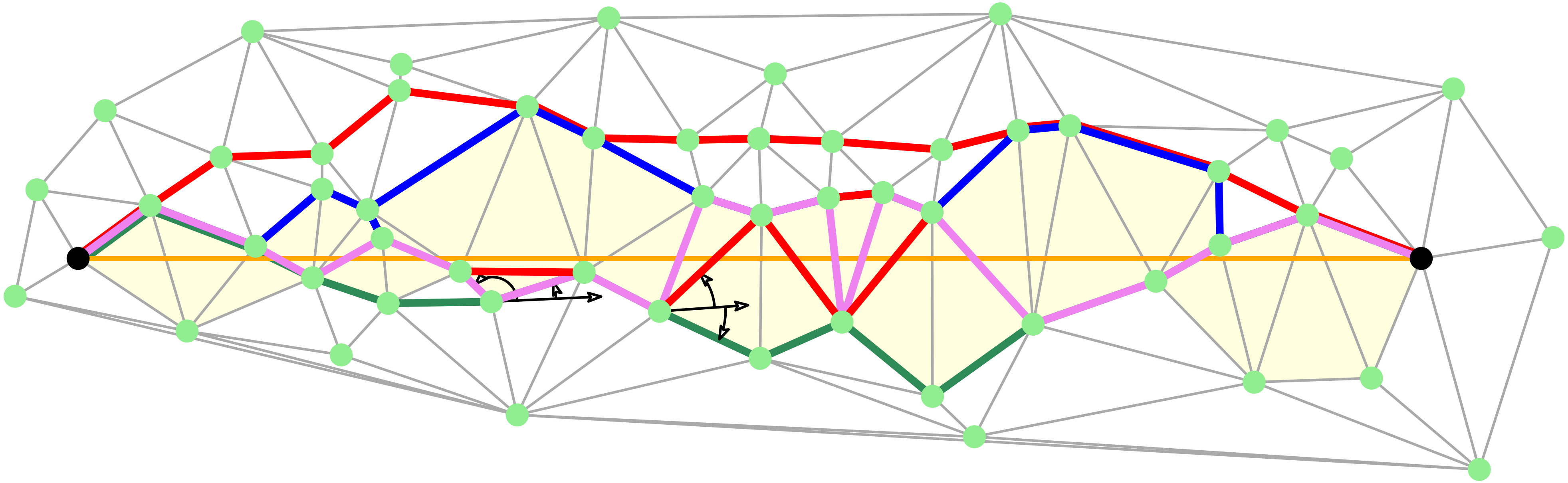
Voronoi path

Walk between vertices



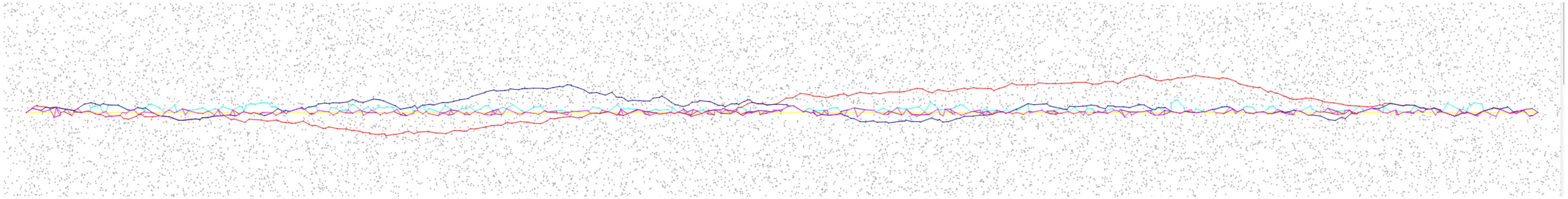
Voronoi path with shortcuts

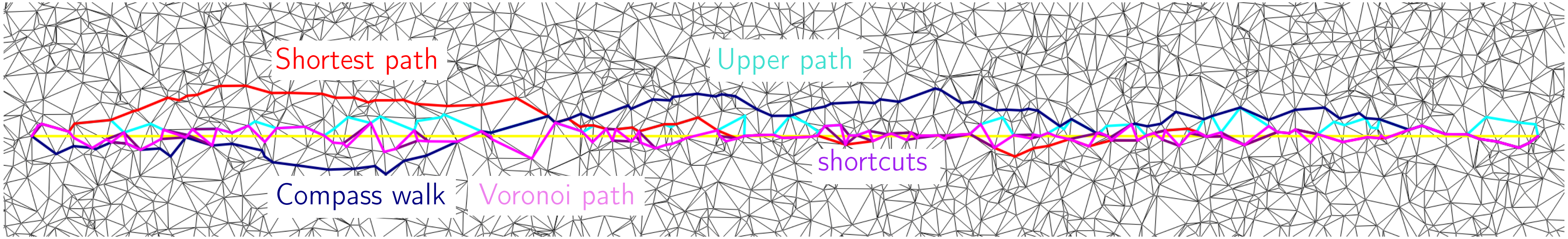
Walk between vertices



Shortest path
Upper path
Compass walk
Voronoi path with shortcuts

Walk between vertices





Expected length (experiments)

Euclidean length	1
Shortest path	1.04
Compass walk	1.07
Shortened V. path	1.16
Upper path	1.18
Voronoi path	1.27

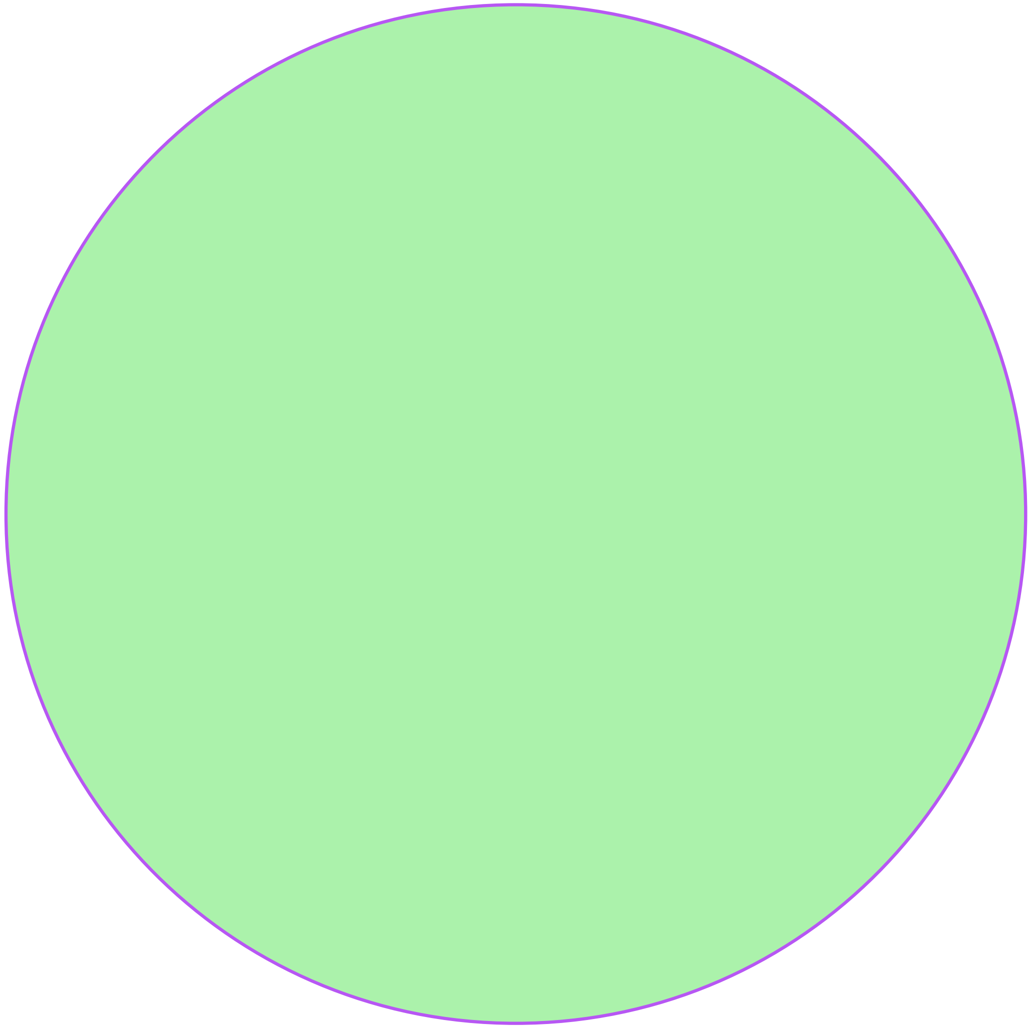
Walk between vertices

Expected length (experiments)		theory
Euclidean length	1	
Shortest path	1.04	$\geq 1 + 10^{-11}$
Compass walk	1.07	numerical integration
Shortened V. path	1.16	1.16
Upper path	1.18	$\frac{35}{3\pi^2} \simeq 1.18$
Voronoi path	1.27	$\frac{4}{\pi} \simeq 1.27$ [Baccelli et al., 2000]

Smoothed analysis of convex hull

Smoothed analysis of convex hull

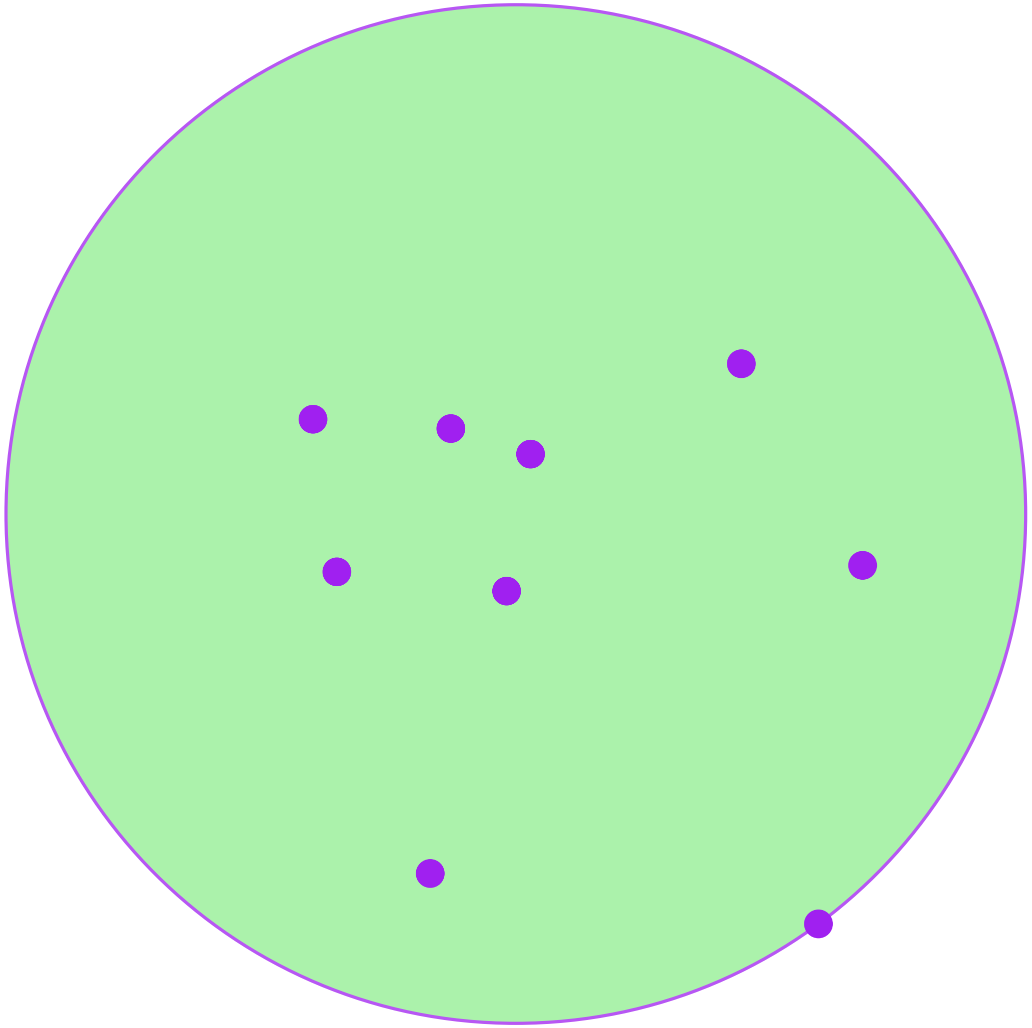
K unit ball of $\mathbb{R}^d$



Smoothed analysis of convex hull

K unit ball of $\mathbb{R}^d$

initial point set

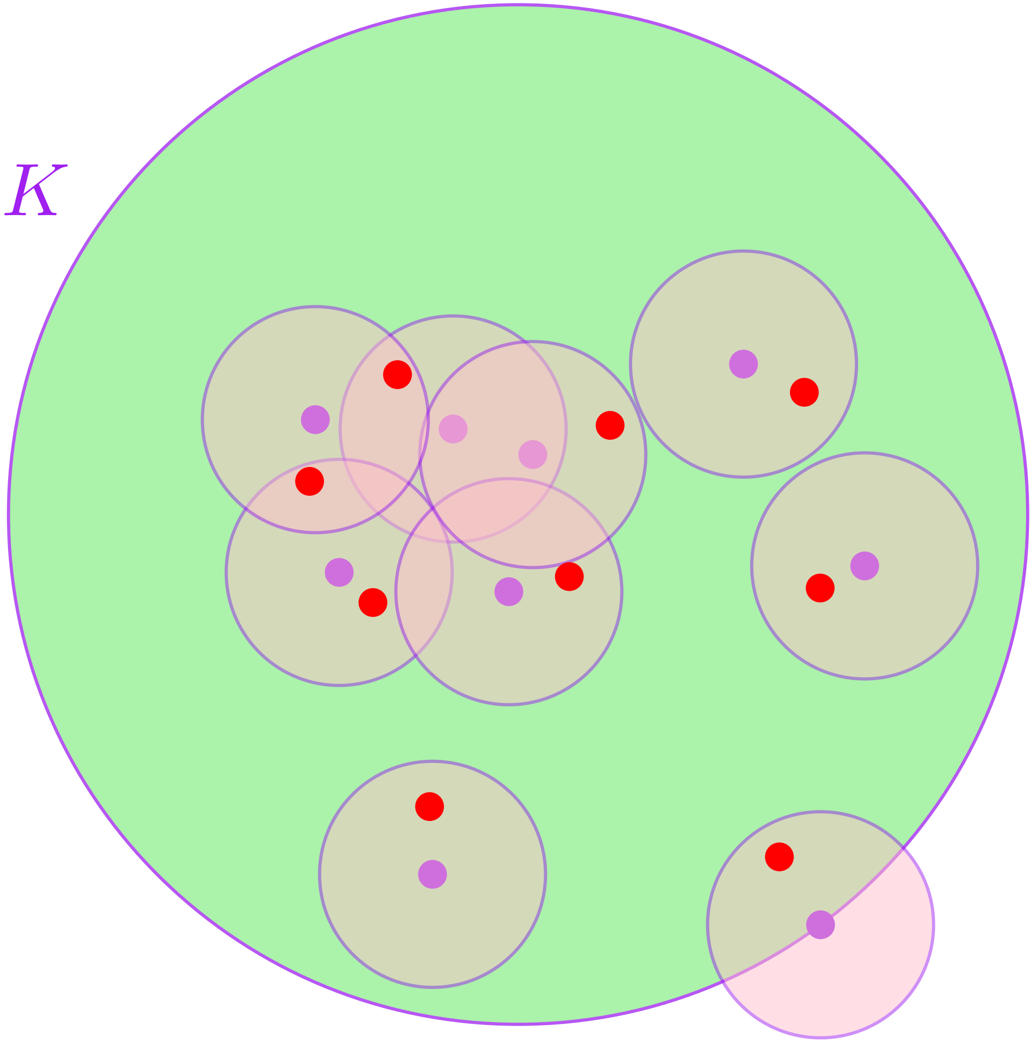


Smoothed analysis of convex hull

K unit ball of $\mathbb{R}^d$

initial point set

Add noise, uniform in δK



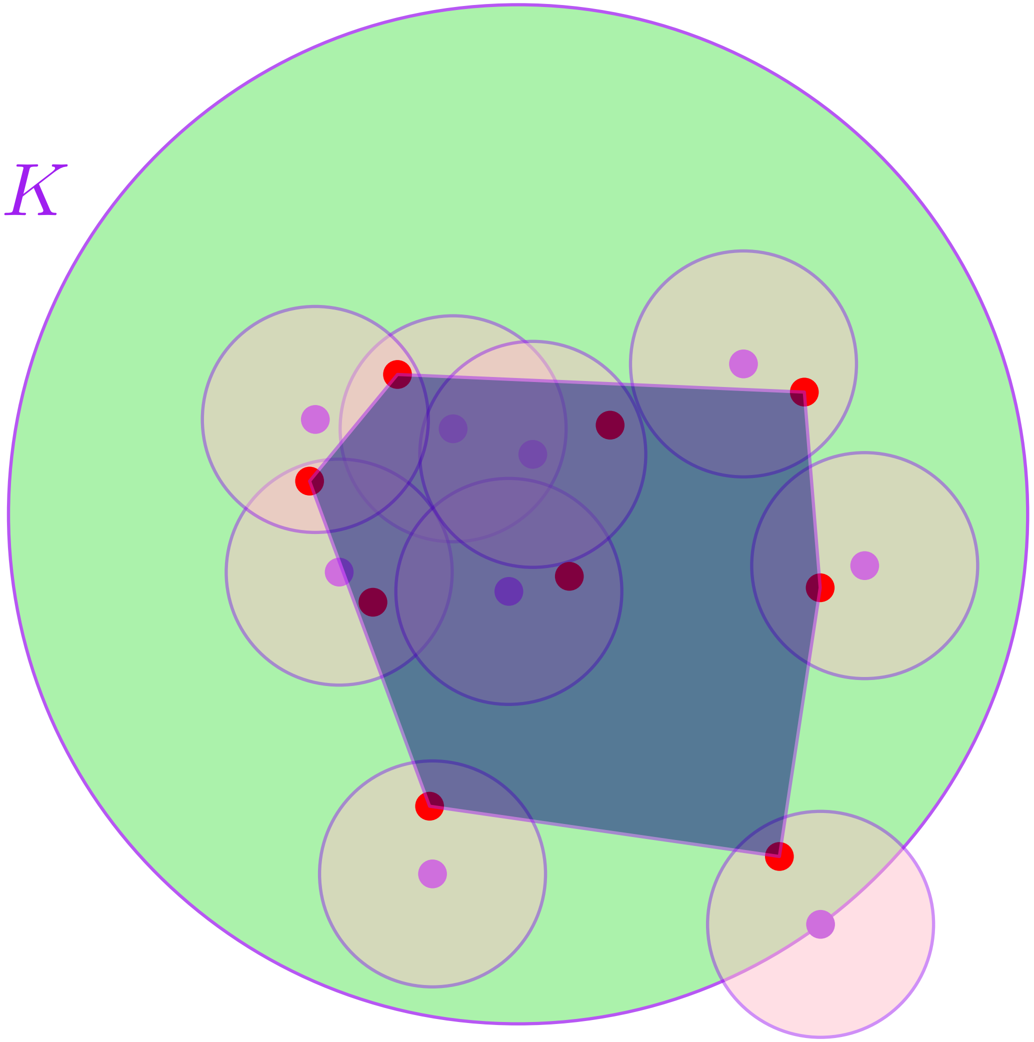
Smoothed analysis of convex hull

K unit ball of $\mathbb{R}^d$

initial point set

Add noise, uniform in δK

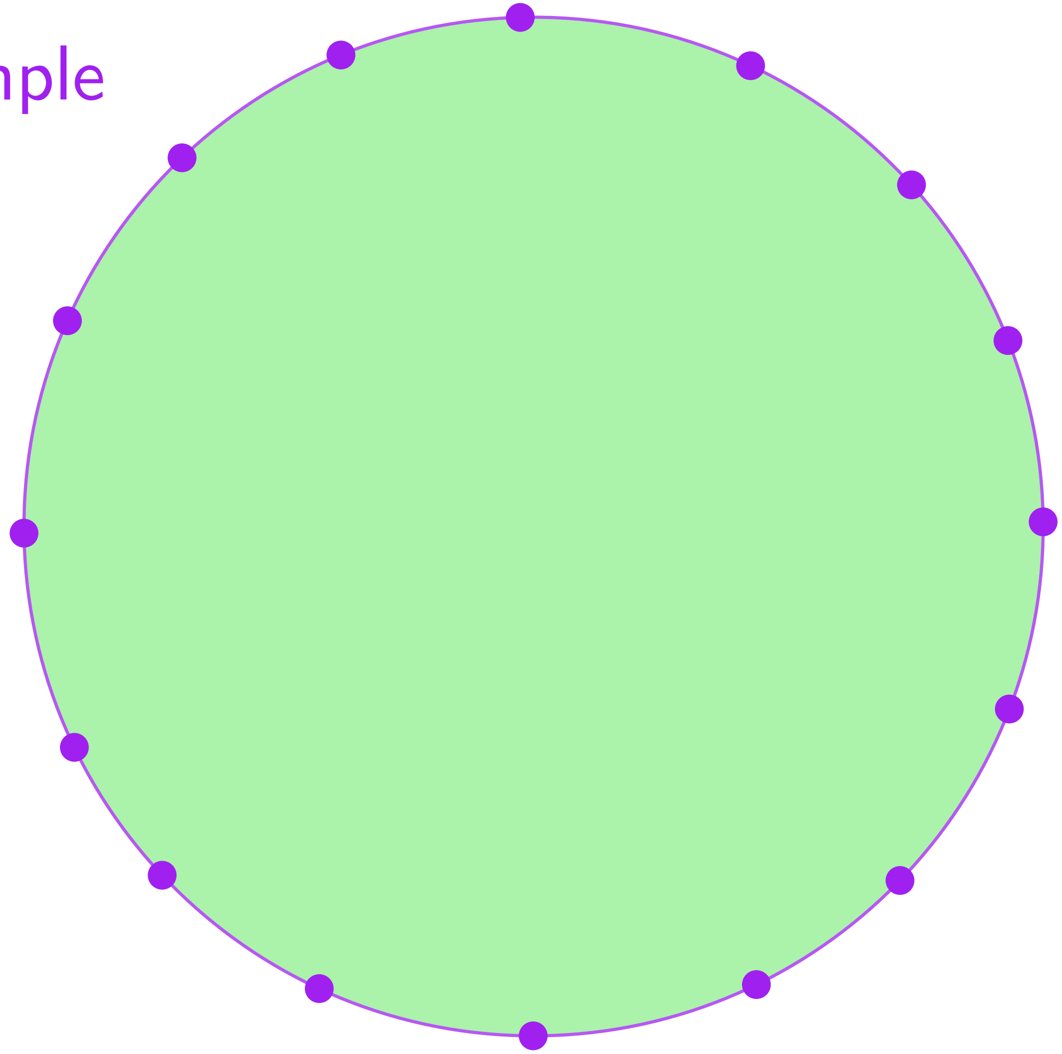
Convex hull



Smoothed analysis of convex hull

K unit ball of $\mathbb{R}^d$

special case: (ϵ, κ) sample

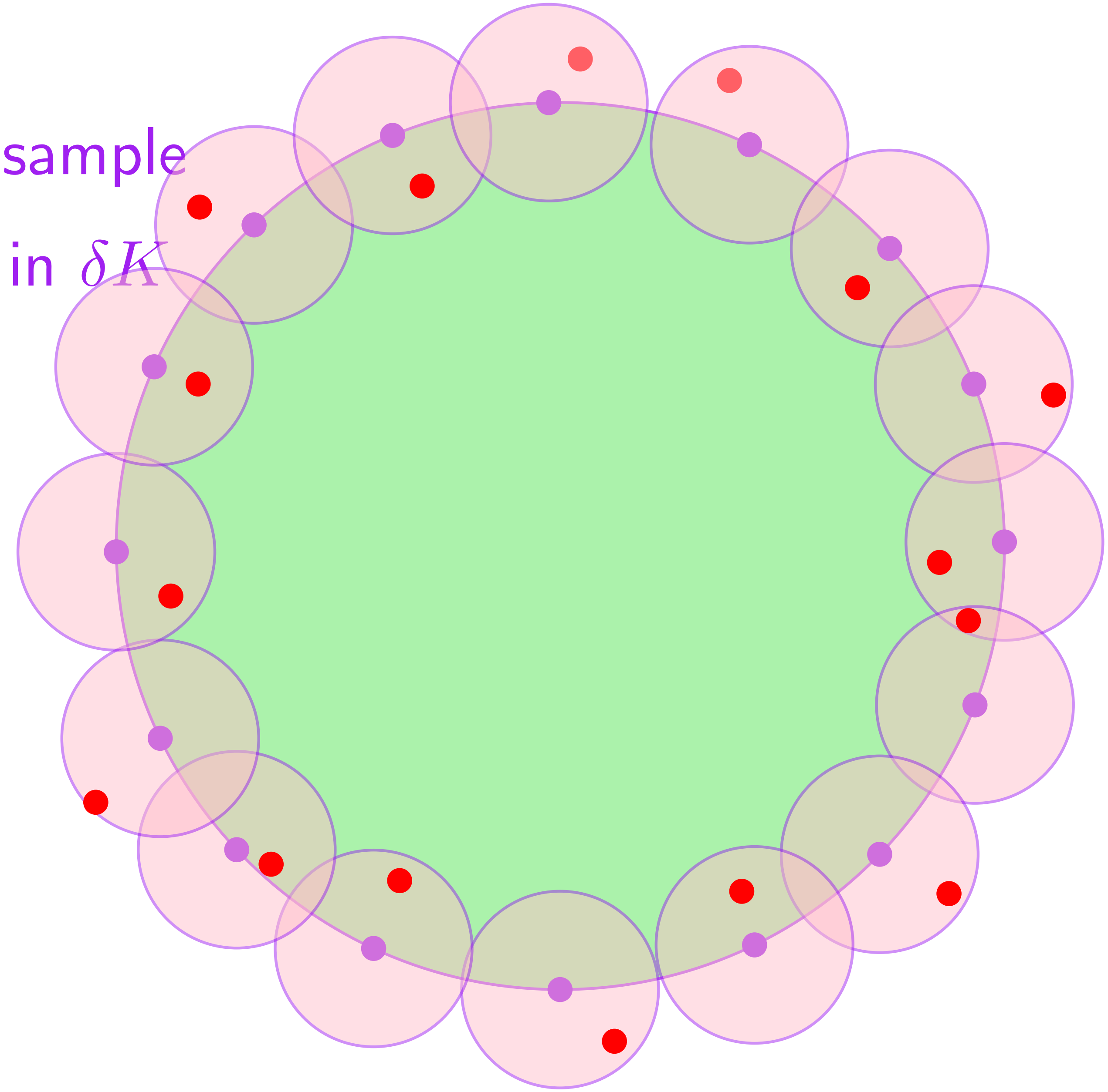


Smoothed analysis of convex hull

K unit ball of $\mathbb{R}^d$

special case: (ϵ, κ) sample

Add noise, uniform in δK



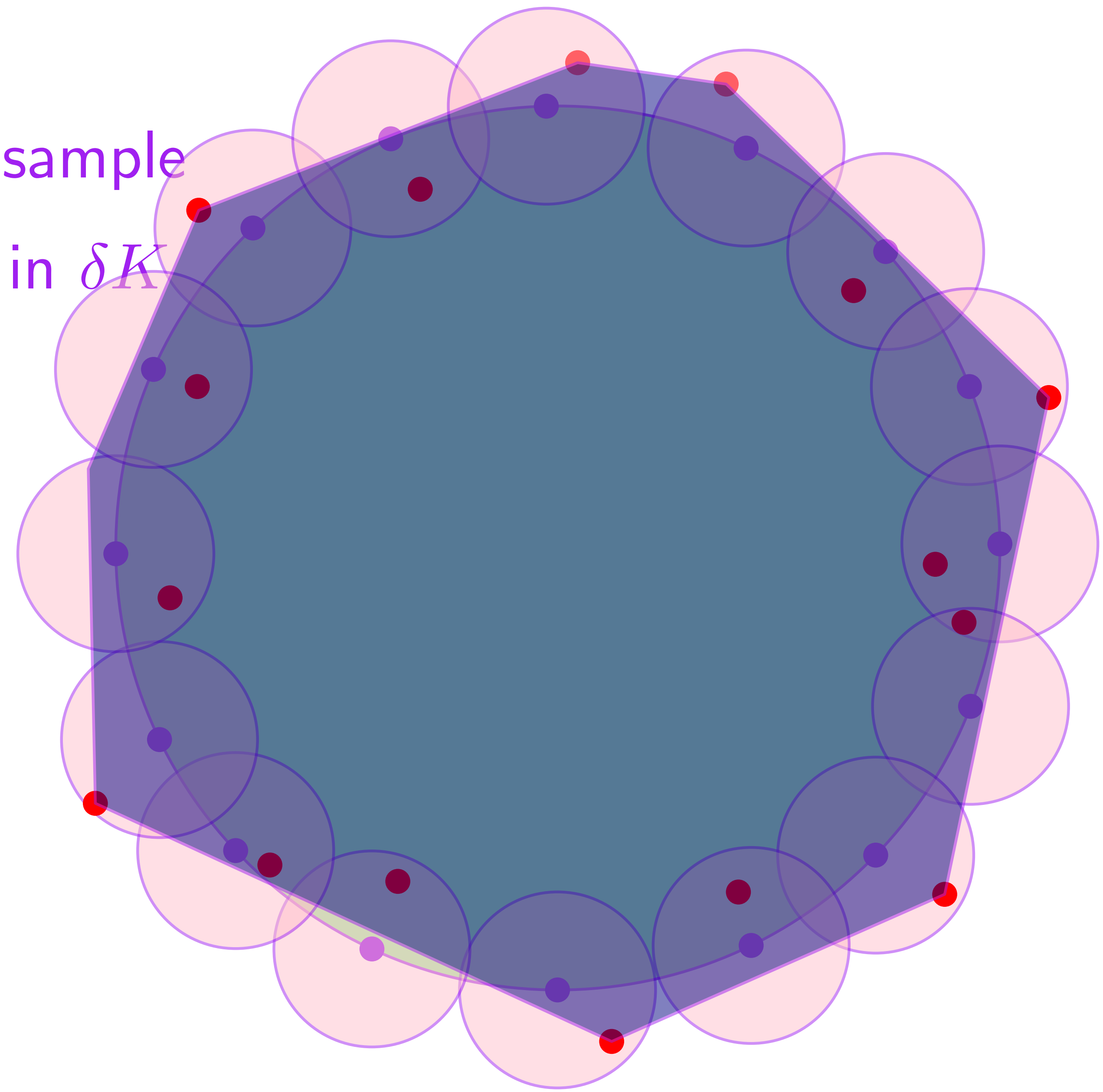
Smoothed analysis of convex hull

K unit ball of $\mathbb{R}^d$

special case: (ϵ, κ) sample

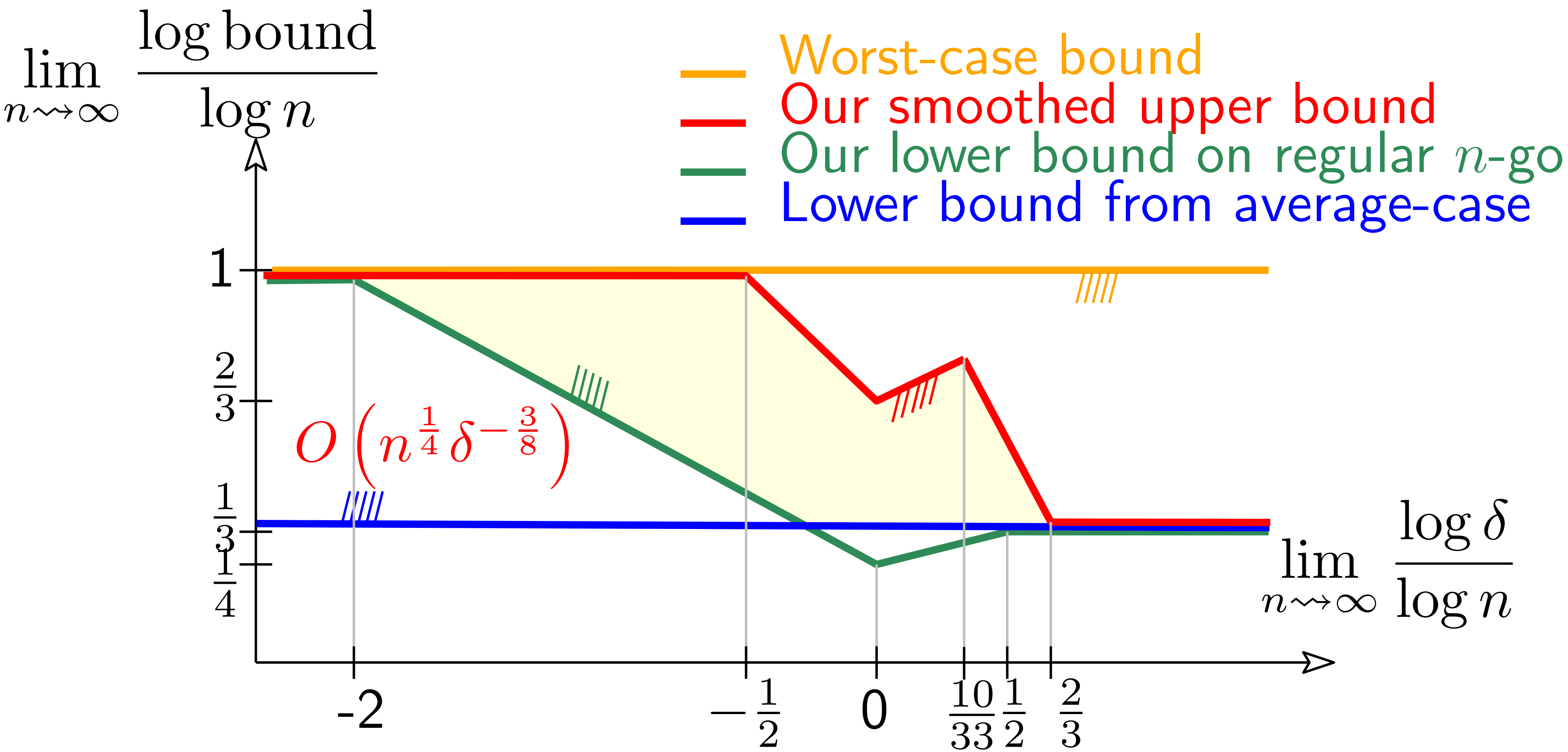
Add noise, uniform in δK

Convex hull



Smoothed analysis of convex hull

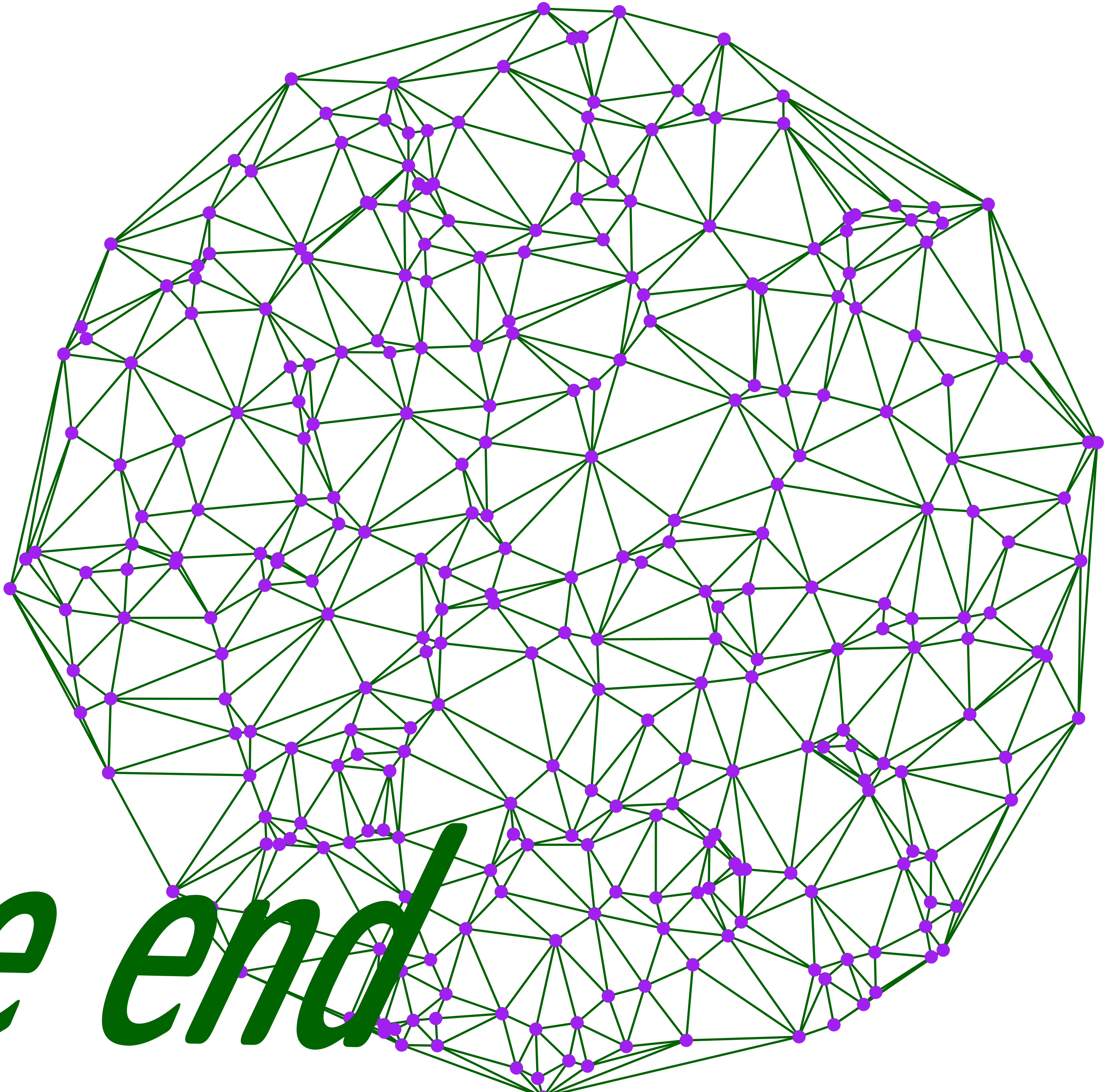
Dimension 2



Smoothed analysis of convex hull

Open problems

- Tighter analysis for CH
- Delaunay size in 3D
- Delaunay walk in 2D
- • • •



The end