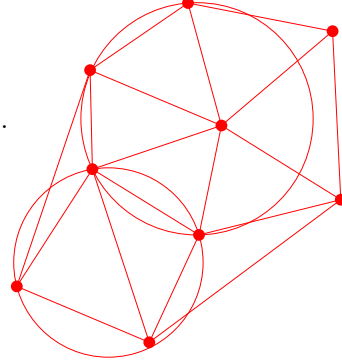# 2 Delaunay triangulation: definitions, motivations, properties, classical algorithms.

2.1 Drawing

Draw the Delaunay triangulation of the attached point set.

2.1 Correction:

2.2 Nearest neighbors

S a set of n points.

Denote $NN(q)$ the nearest neighbor of q in $S \setminus \{q\}$ (q may belong to S or not).

For a point q, cut the plane in four quadrants using the two lines through q of slopes 1 and -1, and call these quadrants North, West, South, and East. Denote $NN_N(q)$, $NN_W(q)$, $NN_S(q)$, and $NN_E(q)$ the nearest neighbor of q in $S \setminus \{q\}$ in each quadrant.

2.2.1 Nearest neighbor

For $q \in S$, is $NN(q)$ a neighbor of q in the Delaunay triangulation of S ? Prove it or draw a counter example.

If the answer is no, then prove that there exist w a Delaunay neigbor of q such that the distance $\|qw\|$ is not larger than $\alpha \cdot \|qNN(q)\|$ for some constant α. Give the best value for α.

2.2.2 Oriented nearest neighbor

For $q \in S$, is $NN_N(q)$ a neighbor of q in the Delaunay triangulation of S ? Prove it or draw a counter example.

If the answer is no, then prove that there exist w a Delaunay neigbor of q in the half-plane above q such that the distance $\|qw\|$ is not larger than $\alpha \cdot \|qNN_N(q)\|$ for some constant α. Give the best value for α.

2.2.3 Nearest neighbor path

Let T be a triangulation of S, p be a point (not in S) and $q \in S$. Define the sequence $(q_i)_{i \in \mathbb{N}}$ by $q_0 = q$ and q_{i+1} the closest point of p amongst the neighbors of q_i in the triangulation T, and q_i itself (assume no ties).

What are the possible behavior of this sequence ? Does it have a limit ? If yes, what is this limit ?
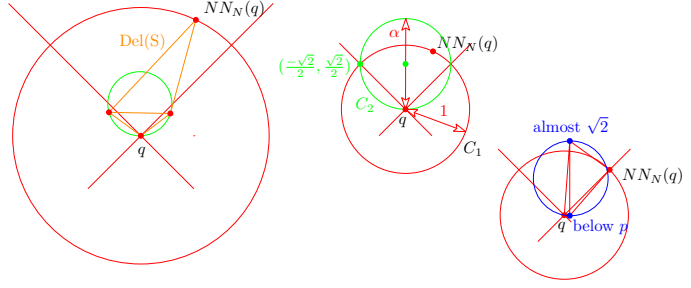
If T is not any triangulation but the Delaunay triangulation, what are the possible behavior of this sequence ? Does it have a limit ? If yes, what is this limit ?

2.2 Correction:

2.2.1 Nearest neighbor

The circle of center q passing through $NN(q)$ enclose no other point of S than q (since $NN(q)$ is the closest neighbor). The disk of diameter $qNN(q)$ is enclosed in the previous one and has q and $NN(q)$ on its boundary, thus this disk witnesses that $qNN(q)$ is an edge of the Delaunay triangulation.

2.2.2 Oriented nearest neighbor

$NN_N(q)$ is not always a neighbor of q in the Delaunay triangulation of S, see above figure.

Consider the interections points of the circle C_1 of center q trough $NN(q)$ and the North-West ray and the North-East ray. Then consider the circle C_2 through q and these two points. C_2 must enclose $NN(q)$ since it enclose the part of C_1 in the North quadrant where $NN(q)$ lies. Thus C_2 is not empty and must contain a Delaunay neighbor of q. Looking at the largest distance between a point inside C_2 and q we get $\alpha = \sqrt{2}$.

The left picture provide an example proving α is tight.

2.2.3 Nearest neighbor path
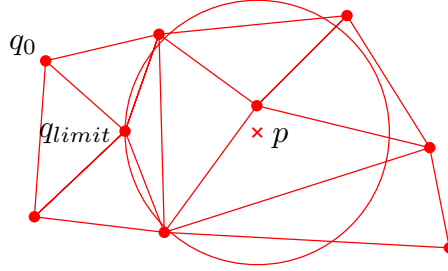
The sequence is stationnary. Actually, the distance $\|pq_i\|$ is positive and decreasing and, since S is finite, can take a finite number of values. Thus it has a limit. Since we assume that no two points of S are at the same distance of p, the sequence is stationnary at some point q_{limit}. See example above.

If the triangulation is the Delaunay triangulation, then the limit is $NN(q)$ because if the circle C of center p through q_{limit} contains a point w, then we can consider the maximal empty disk passing through q_{limit} tangent to C inside C. This circle pass through a point w' of S ($w' = w$ or another point inside C better than w) witnessing an edge $q_{limit}w'$ of the Delaunay triangulation proving that q_{limit} has a neighbor closest to p establishing a contradiction.

2.3 Diameter

Let S be a set of points in the plane. The diameter of S is the pair of point in S^2 that realizes the largest distance (assume no degenracies).

The collision problem is: "given a two set of n real numbers in some interval, determine if if the intersection of the two sets is non empty"

Theorem: The collision problem has an $\Omega(n \log n)$ lower bound in the real-RAM model.

2.3.1 Diameter lower bound

Prove that the diameter problem has an $\Omega(n \log n)$ lower bound in the real-RAM model.

Hint: design a stupid algorithm for the collision problem for a set numbers in $[0, \frac{\pi}{2}]$.

2.3 Correction:

Stupid algorithm for the collision problem:

Let α_i and β_i be two sets of n numbers in $[0, \frac{\pi}{2}]$.

Create $2n$ points $p_i = (\cos \alpha_i, \sin \alpha_i)$ and $q_i = (-\cos \beta_i, -\sin \beta_i)$.

Solve the diameter problem on this set of points.

If the diameter $p_i q_j$ has length 2 answer that $\alpha_i = \beta_j$ as a witness of non empty intersection.

If the length is strictly less than 2, then anser "empty intersection".

The complexity of this algorithm is linear plus the complexity of solving the diameter problem, thus the diameter problem cannot be solved faster than $n \log n$ without contradicting the lower bound on the collision problem.