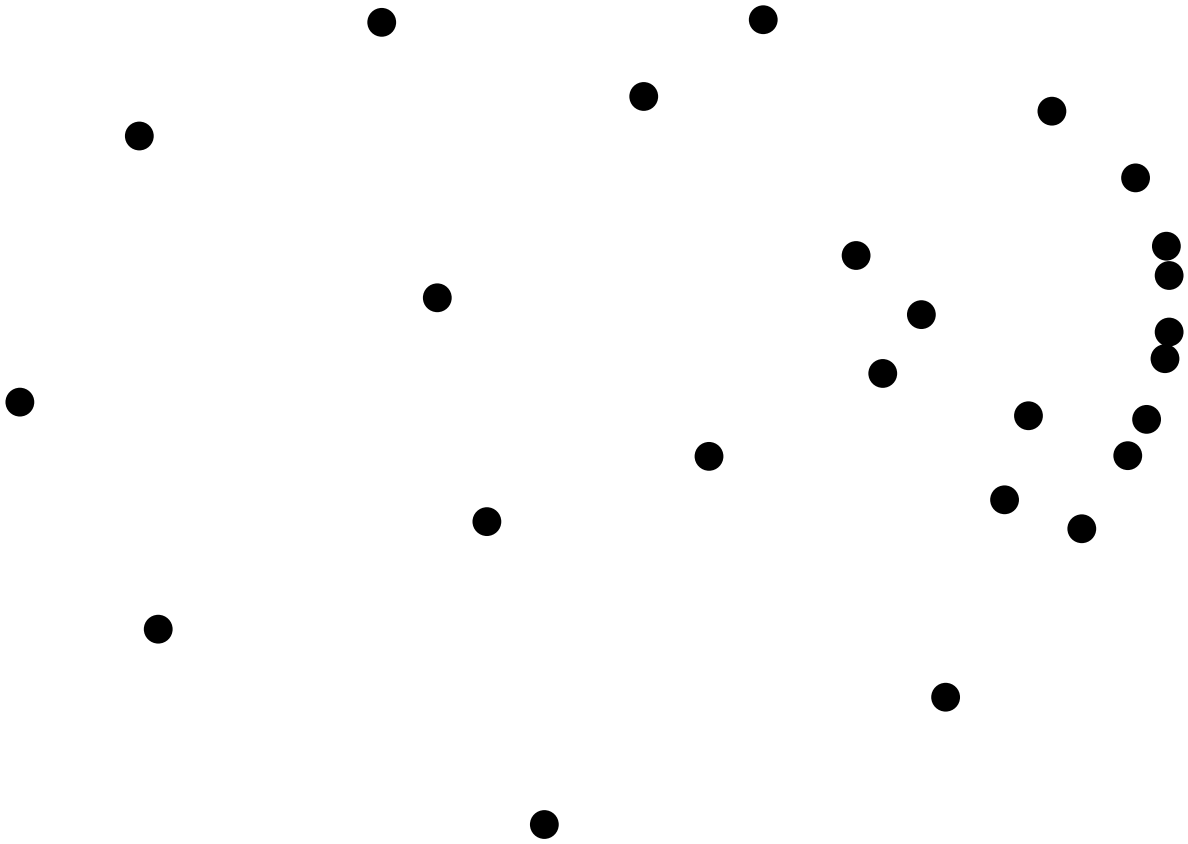
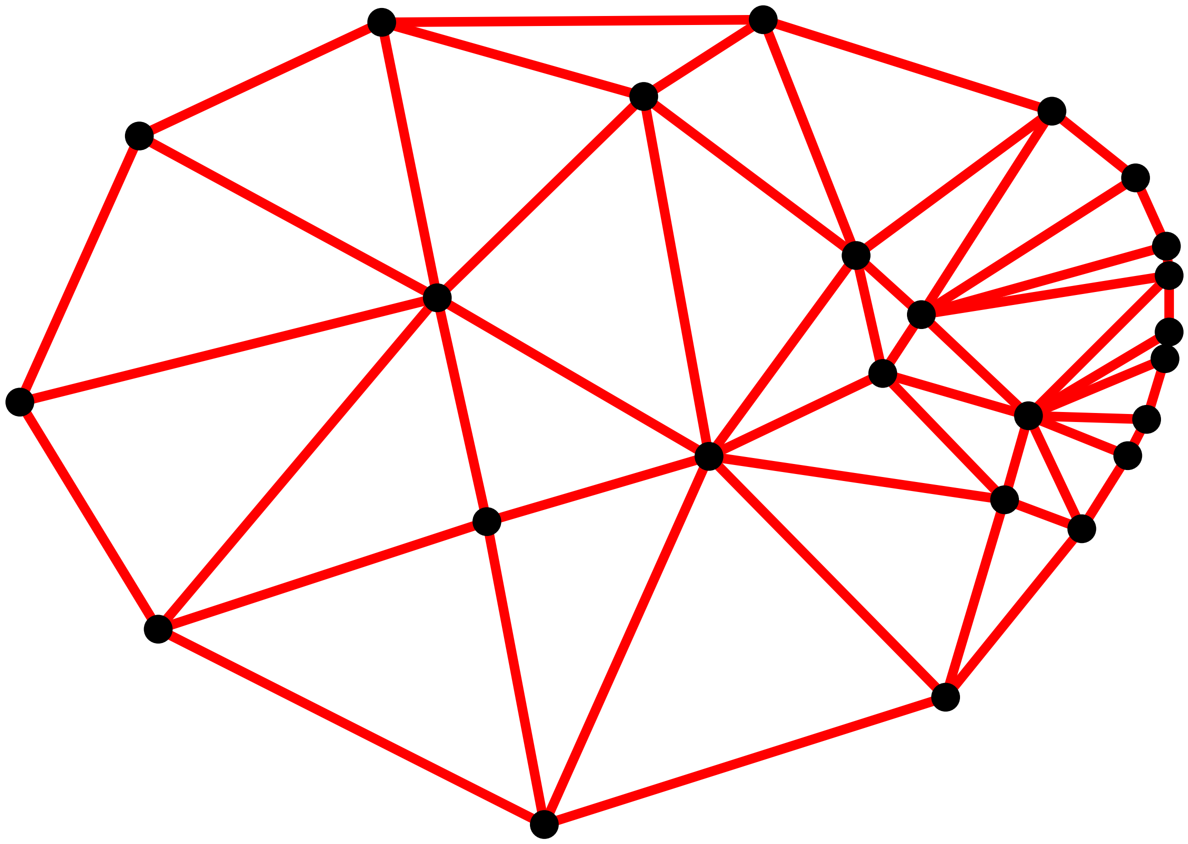
Delaunay Triangulation

Delaunay Triangulation



Delaunay Triangulation



Delaunay Triangulation: pencils of circles

Imagine moving circles

Delaunay Triangulation: pencils of circles

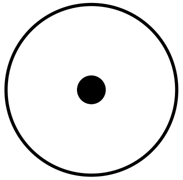
Imagine moving circles

fixed center

increasing radius

Delaunay Triangulation: pencils of circles

Imagine moving circles

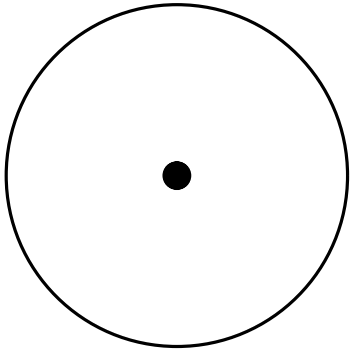


fixed center

increasing radius

Delaunay Triangulation: pencils of circles

Imagine moving circles

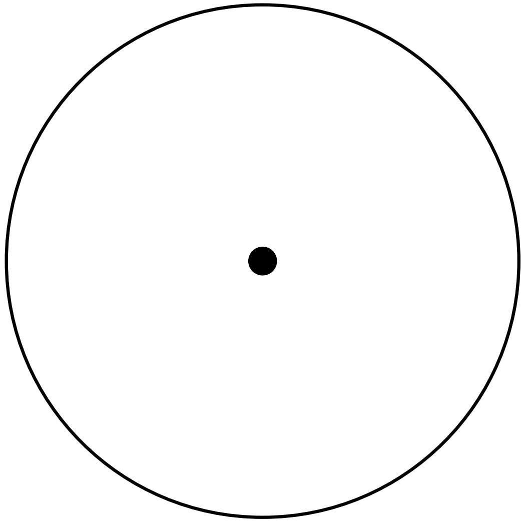


fixed center

increasing radius

Delaunay Triangulation: pencils of circles

Imagine moving circles

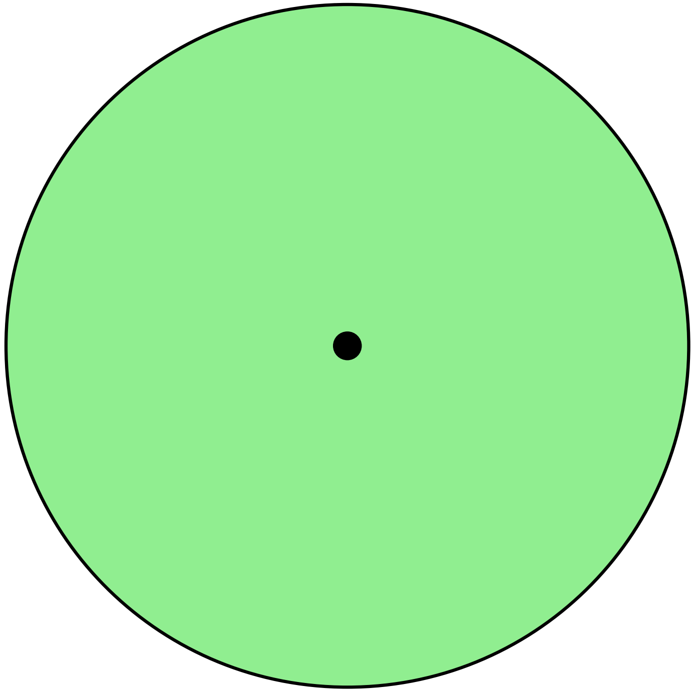


fixed center

increasing radius

Delaunay Triangulation: pencils of circles

Imagine moving circles



fixed center

increasing radius

Cocentric pencil

Delaunay Triangulation: pencils of circles

Imagine moving circles

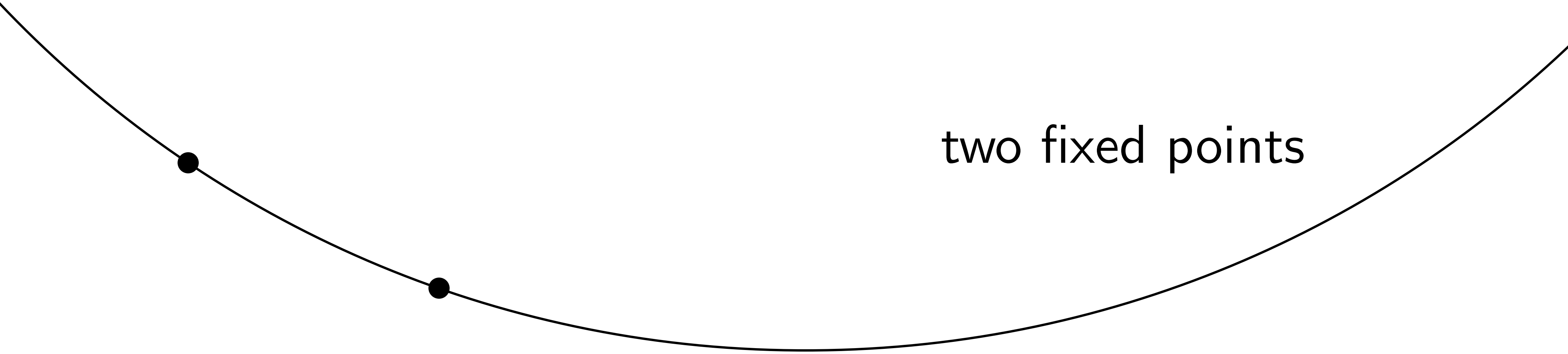
Delaunay Triangulation: pencils of circles

Imagine moving circles

two fixed points

Delaunay Triangulation: pencils of circles

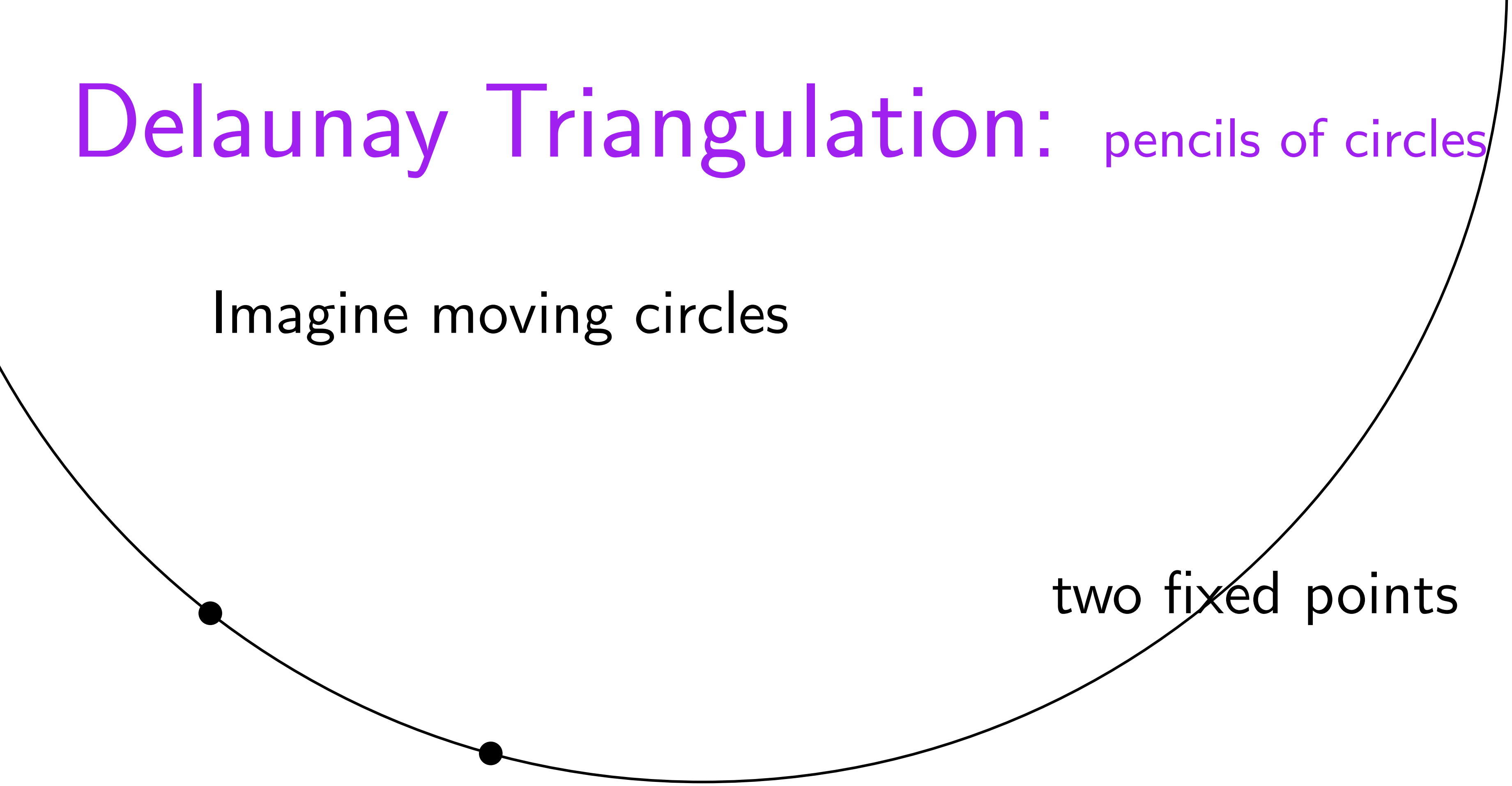
Imagine moving circles



two fixed points

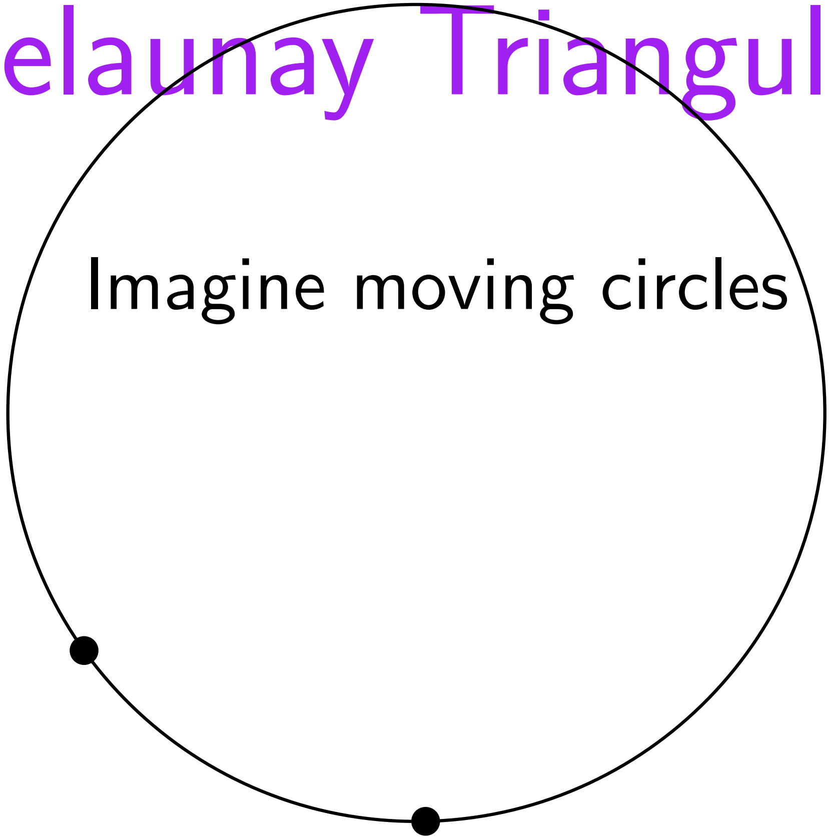
Delaunay Triangulation: pencils of circles

Imagine moving circles



two fixed points

Delaunay Triangulation: pencils of circles

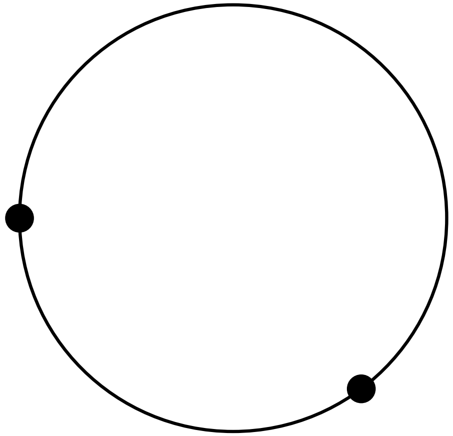


Imagine moving circles

two fixed points

Delaunay Triangulation: pencils of circles

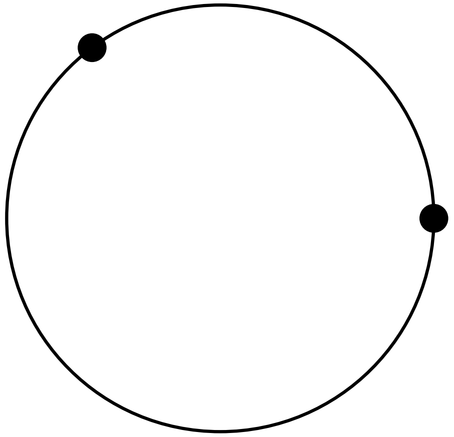
Imagine moving circles



two fixed points

Delaunay Triangulation: pencils of circles

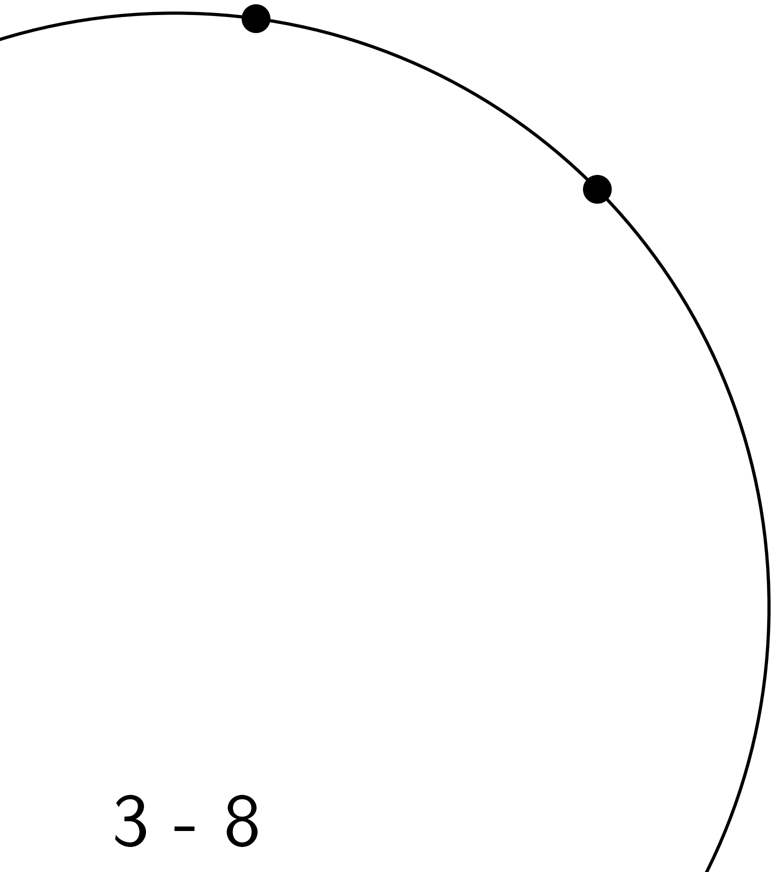
Imagine moving circles



two fixed points

Delaunay Triangulation: pencils of circles

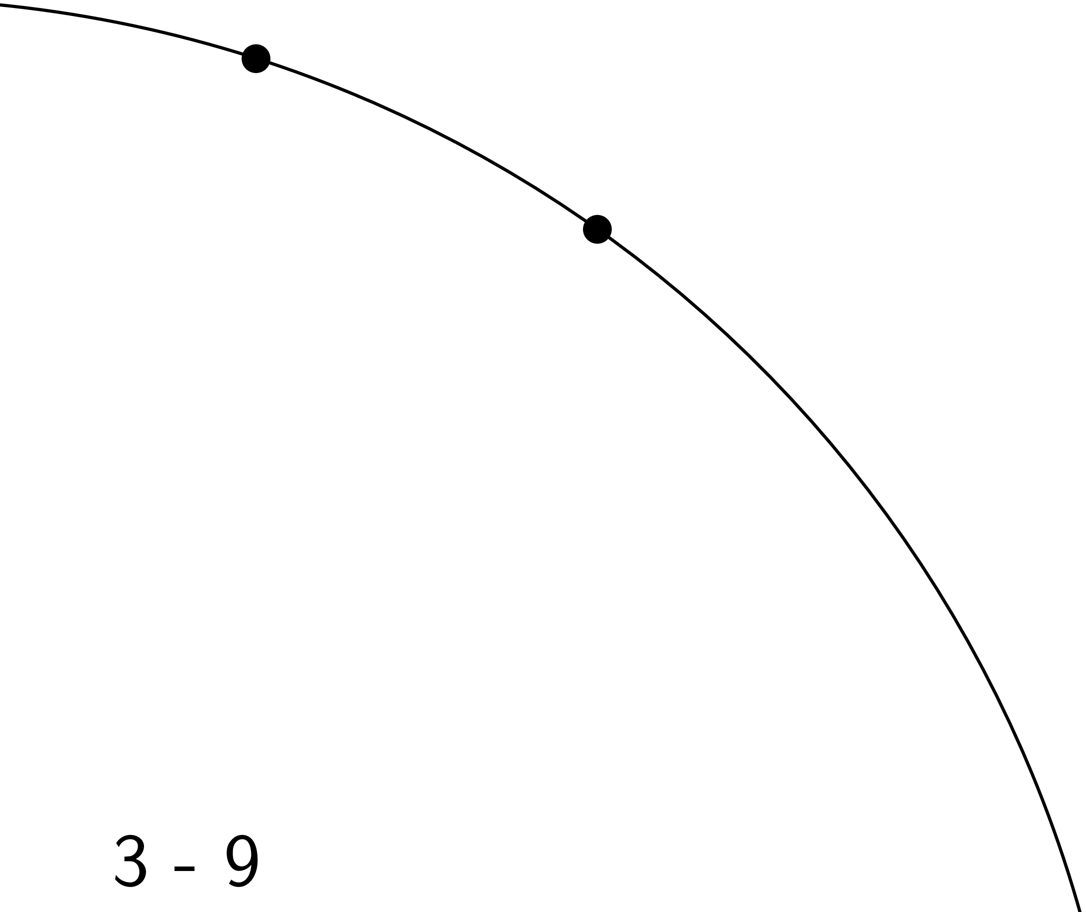
Imagine moving circles



two fixed points

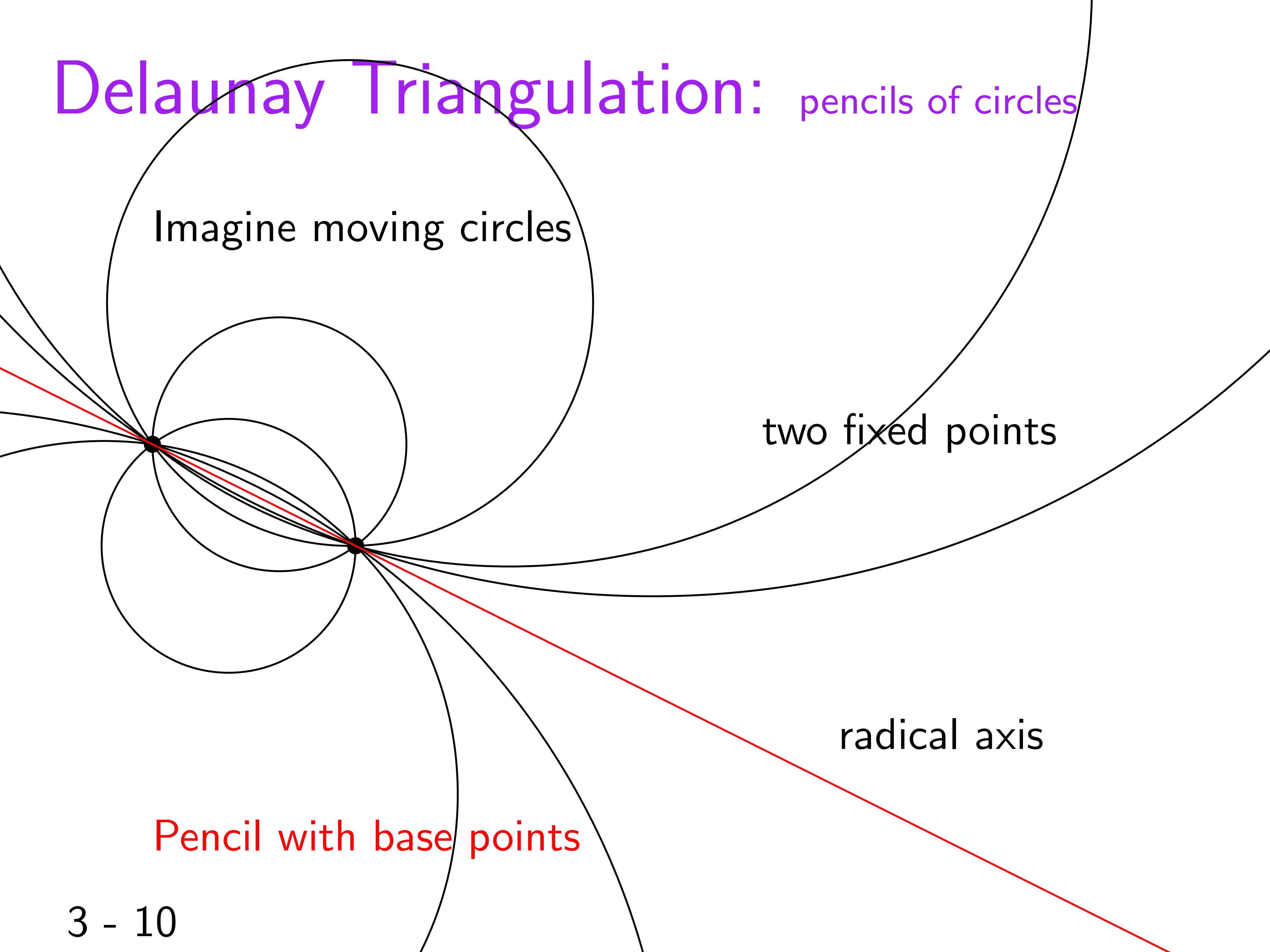
Delaunay Triangulation: pencils of circles

Imagine moving circles



two fixed points

Delaunay Triangulation: pencils of circles



Imagine moving circles

two fixed points

radical axis

Pencil with base points

Delaunay Triangulation: pencils of circles

Imagine moving circles

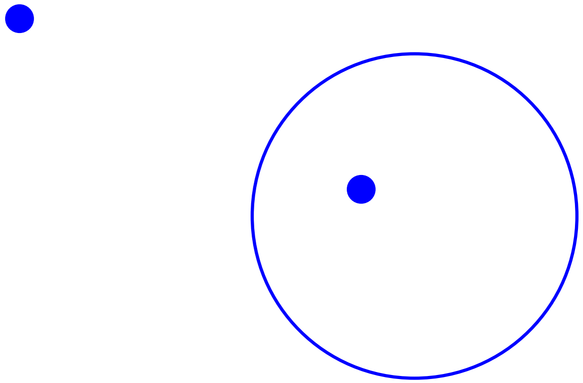
Delaunay Triangulation: pencils of circles

Imagine moving circles

two fixed points

Delaunay Triangulation: pencils of circles

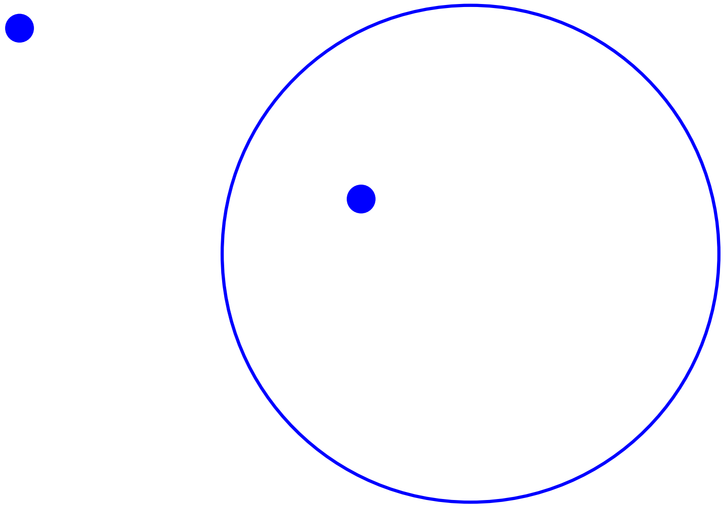
Imagine moving circles



two fixed points

Delaunay Triangulation: pencils of circles

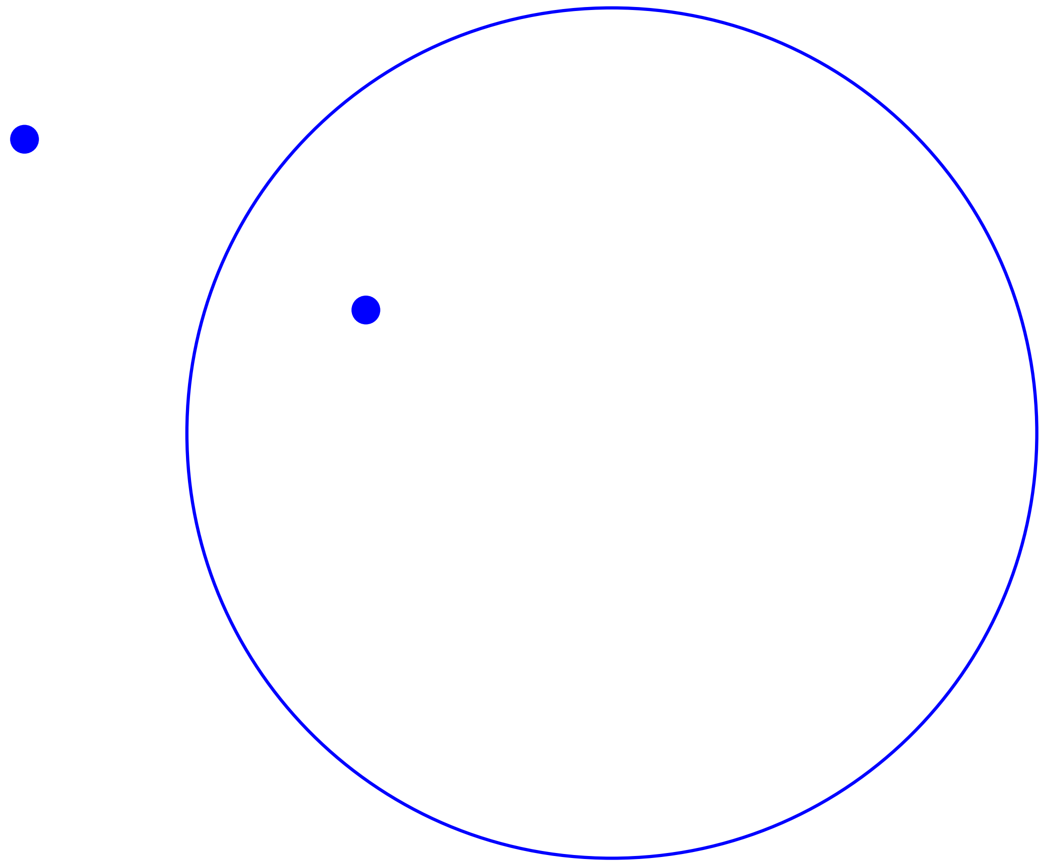
Imagine moving circles



two fixed points

Delaunay Triangulation: pencils of circles

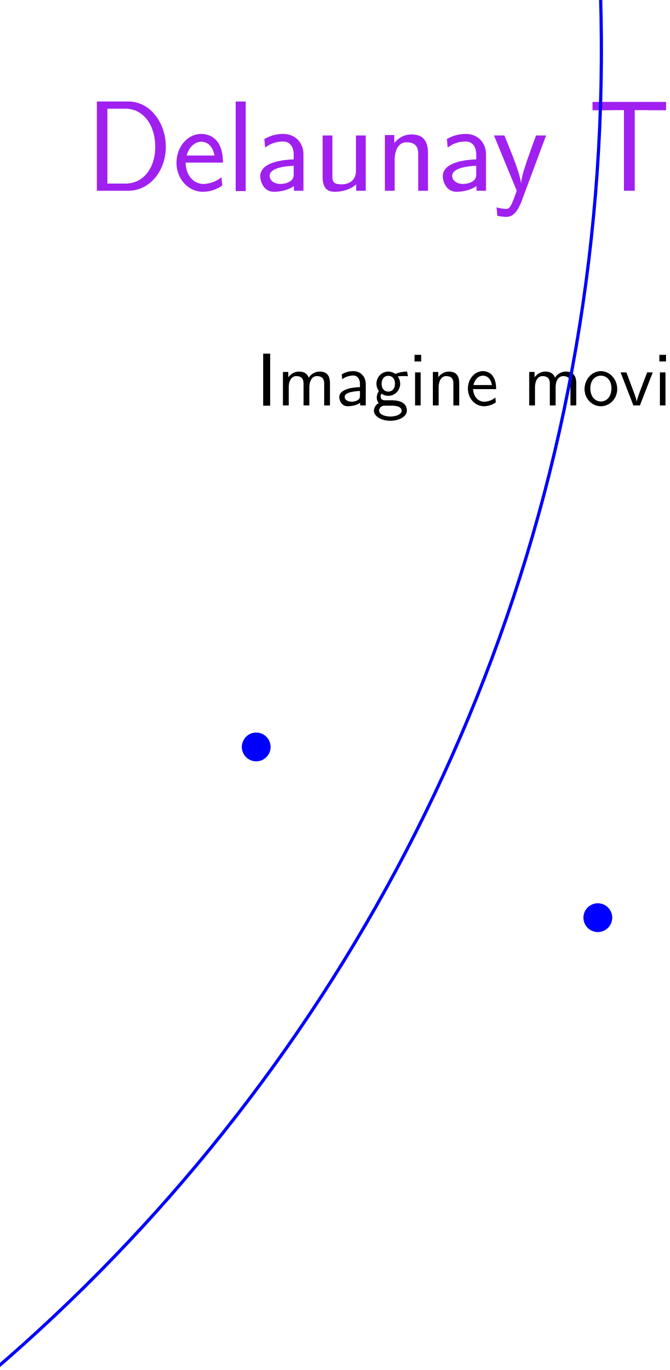
Imagine moving circles



two fixed points

Delaunay Triangulation: pencils of circles

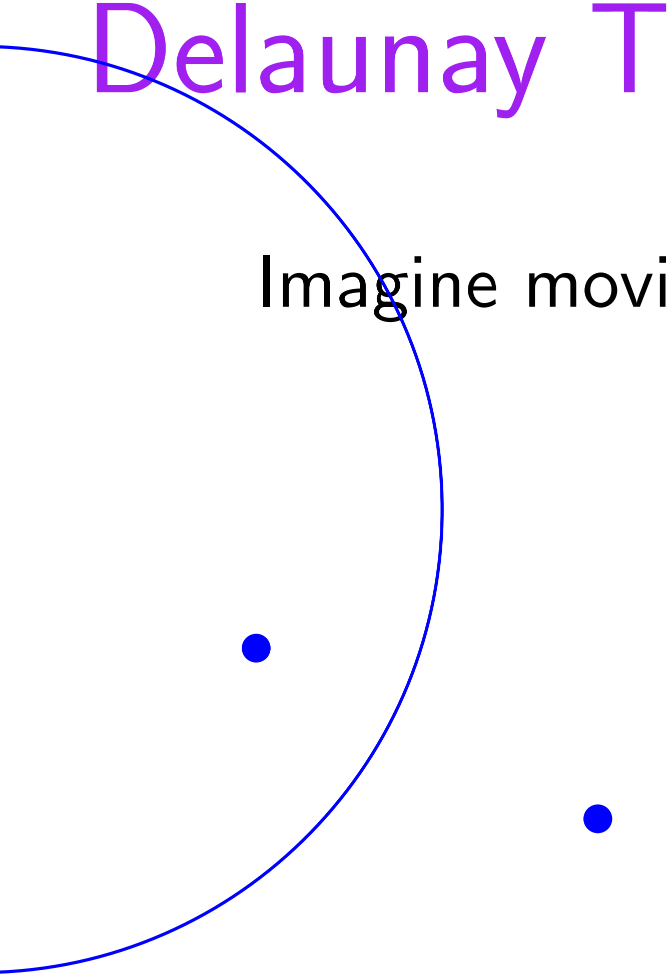
Imagine moving circles



two fixed points

Delaunay Triangulation: pencils of circles

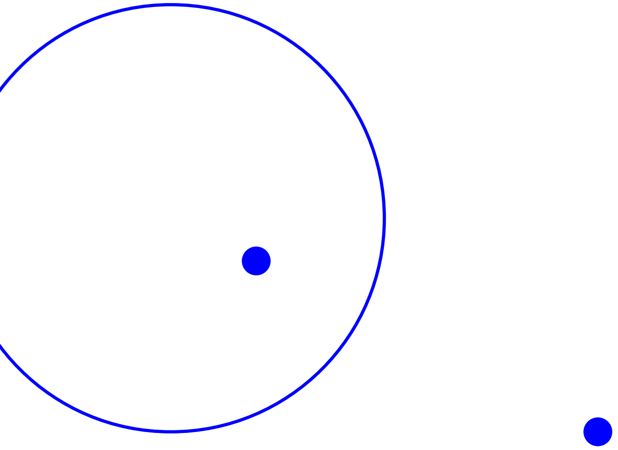
Imagine moving circles



two fixed points

Delaunay Triangulation: pencils of circles

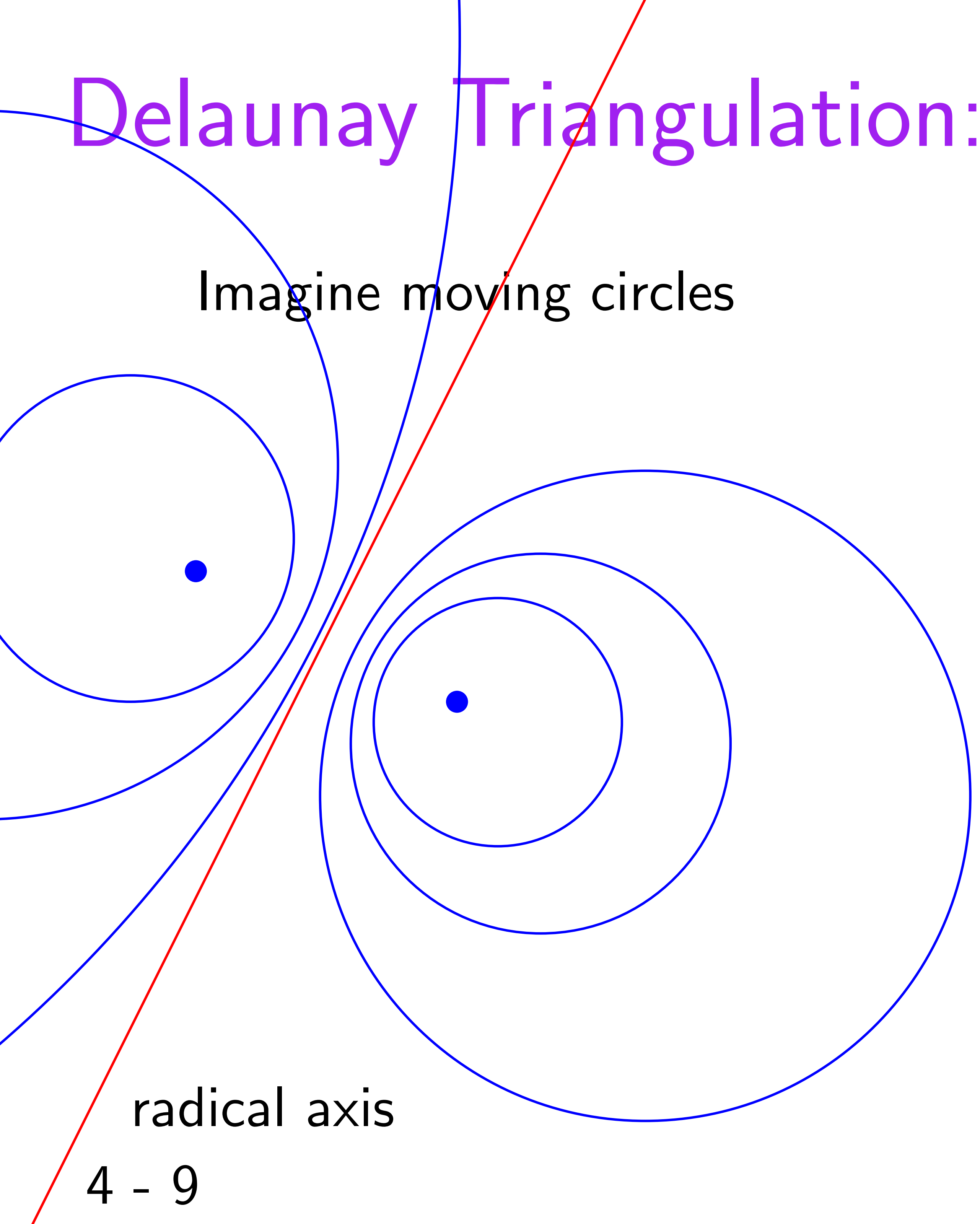
Imagine moving circles



two fixed points

Delaunay Triangulation: pencils of circles

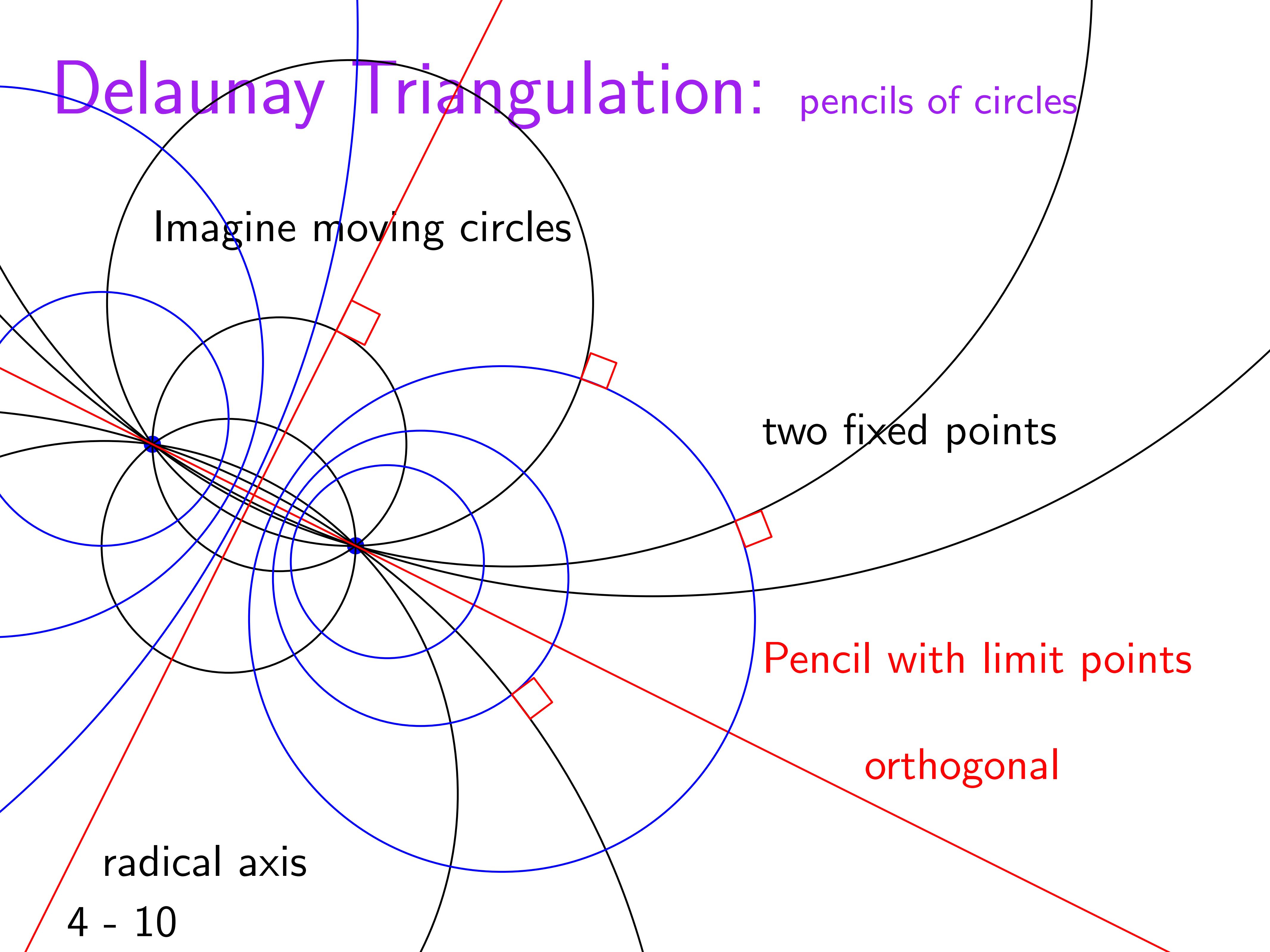
Imagine moving circles



two fixed points

Pencil with limit points

Delaunay Triangulation: pencils of circles



Imagine moving circles

two fixed points

Pencil with limit points

orthogonal

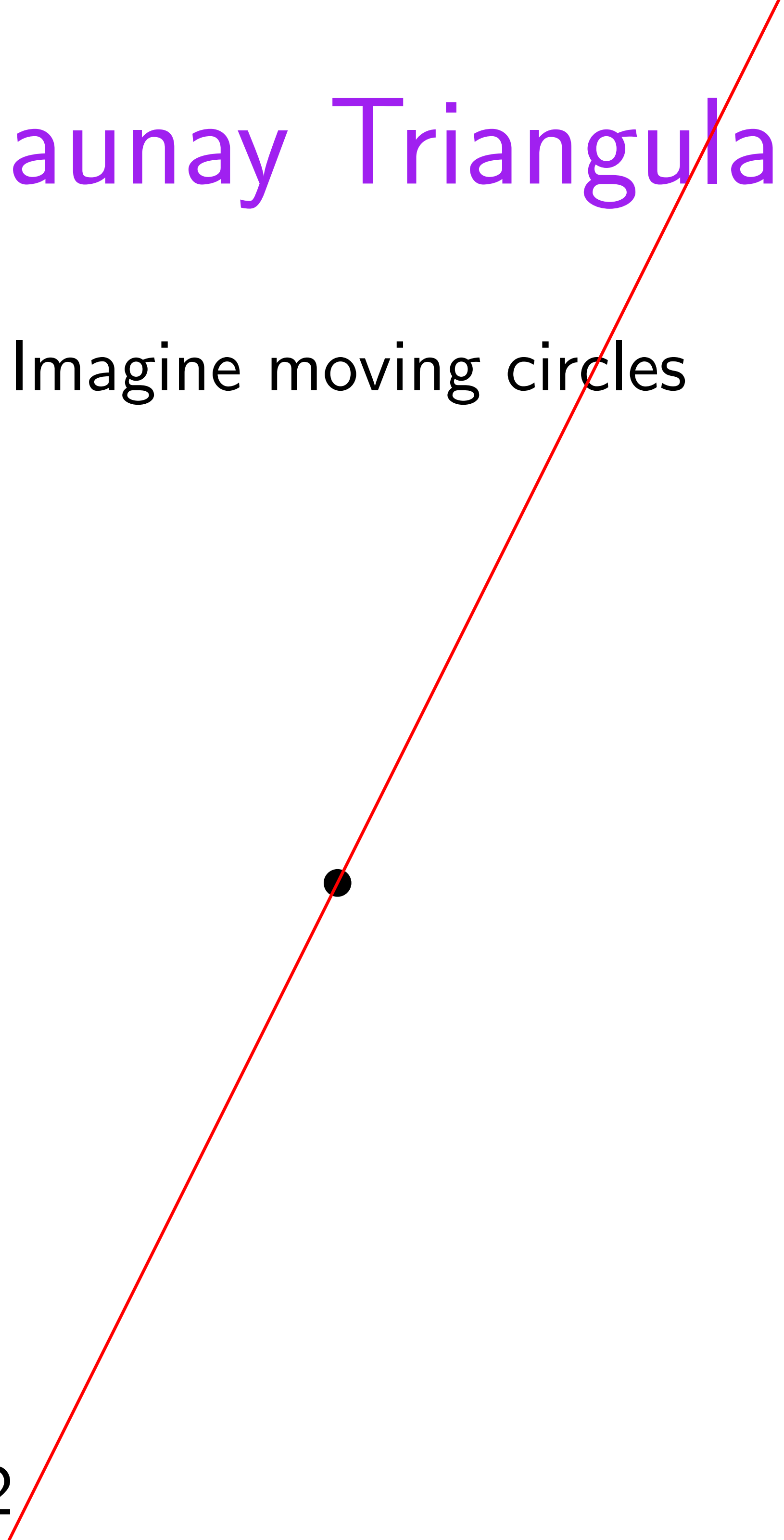
radical axis

Delaunay Triangulation: pencils of circles

Imagine moving circles

Delaunay Triangulation: pencils of circles

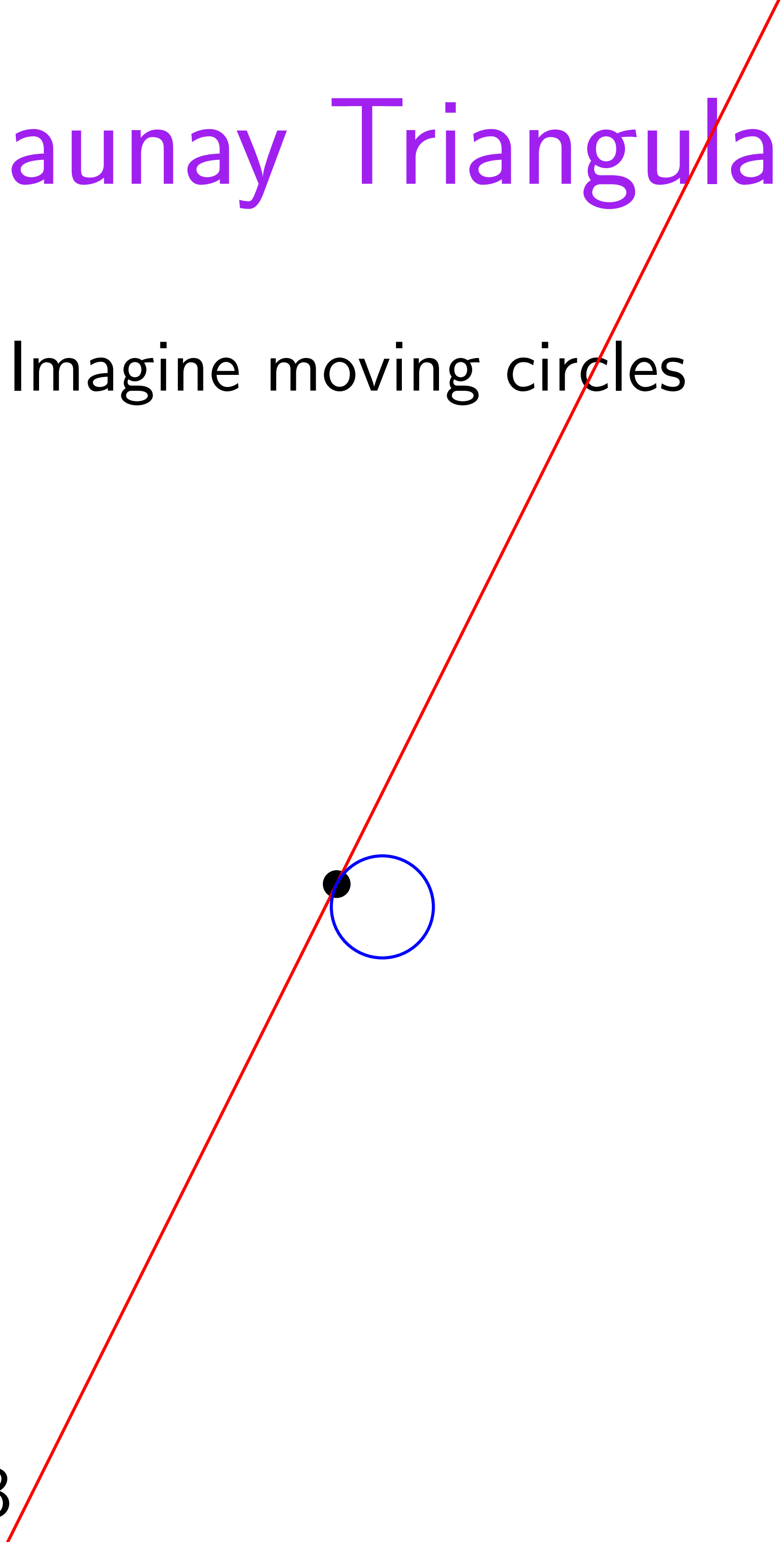
Imagine moving circles



a point on a line

Delaunay Triangulation: pencils of circles

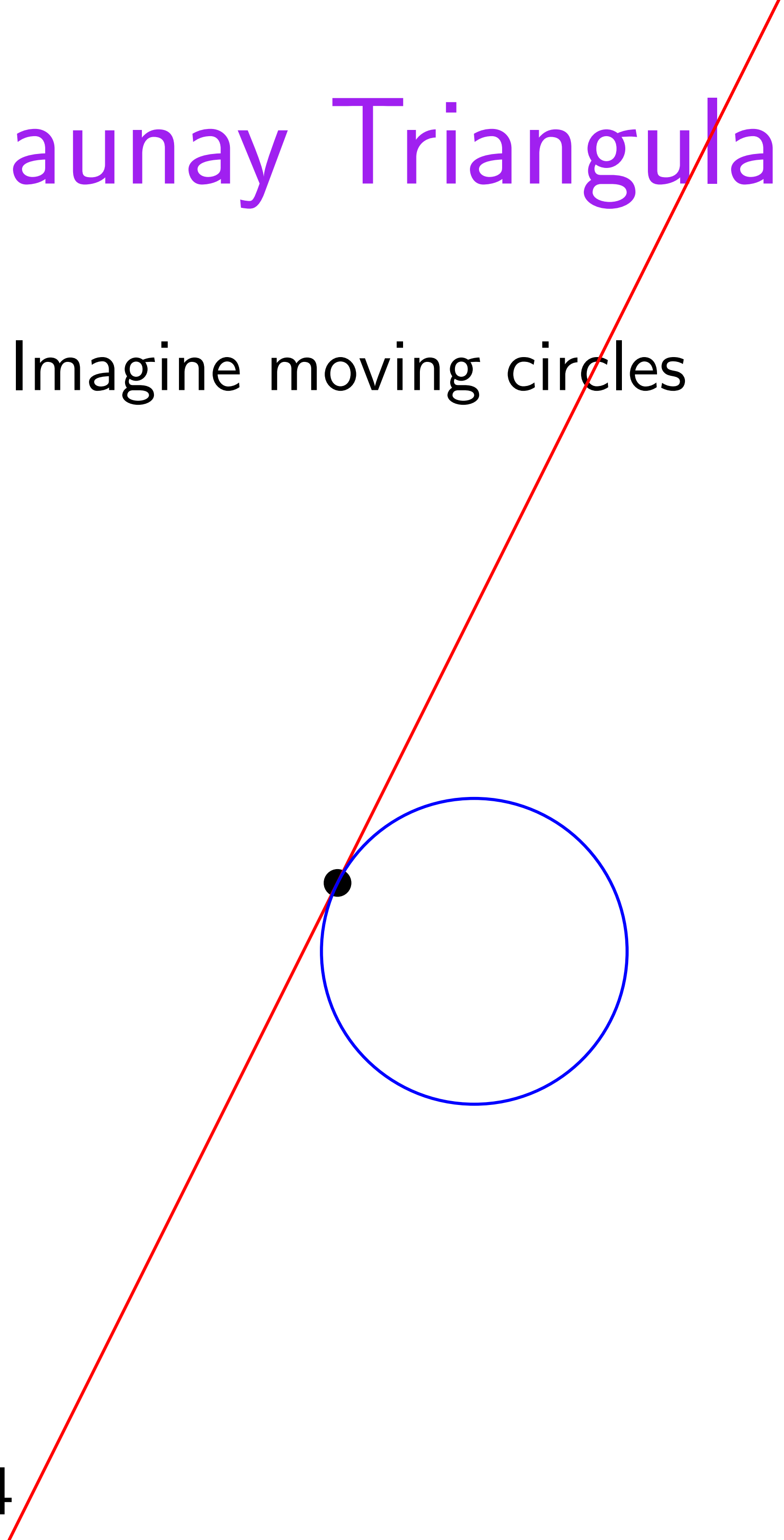
Imagine moving circles



a point on a line

Delaunay Triangulation: pencils of circles

Imagine moving circles

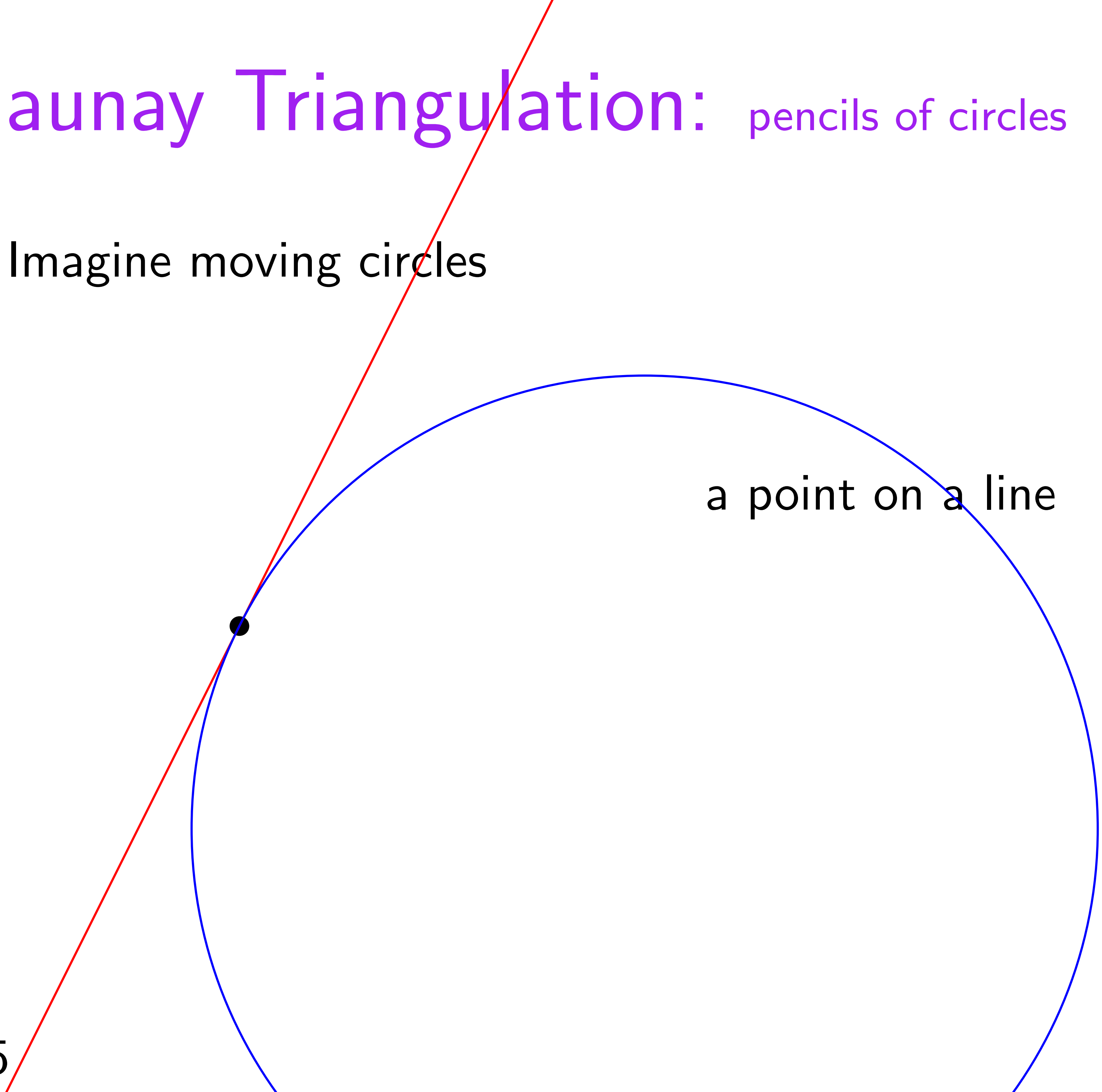


a point on a line

Delaunay Triangulation: pencils of circles

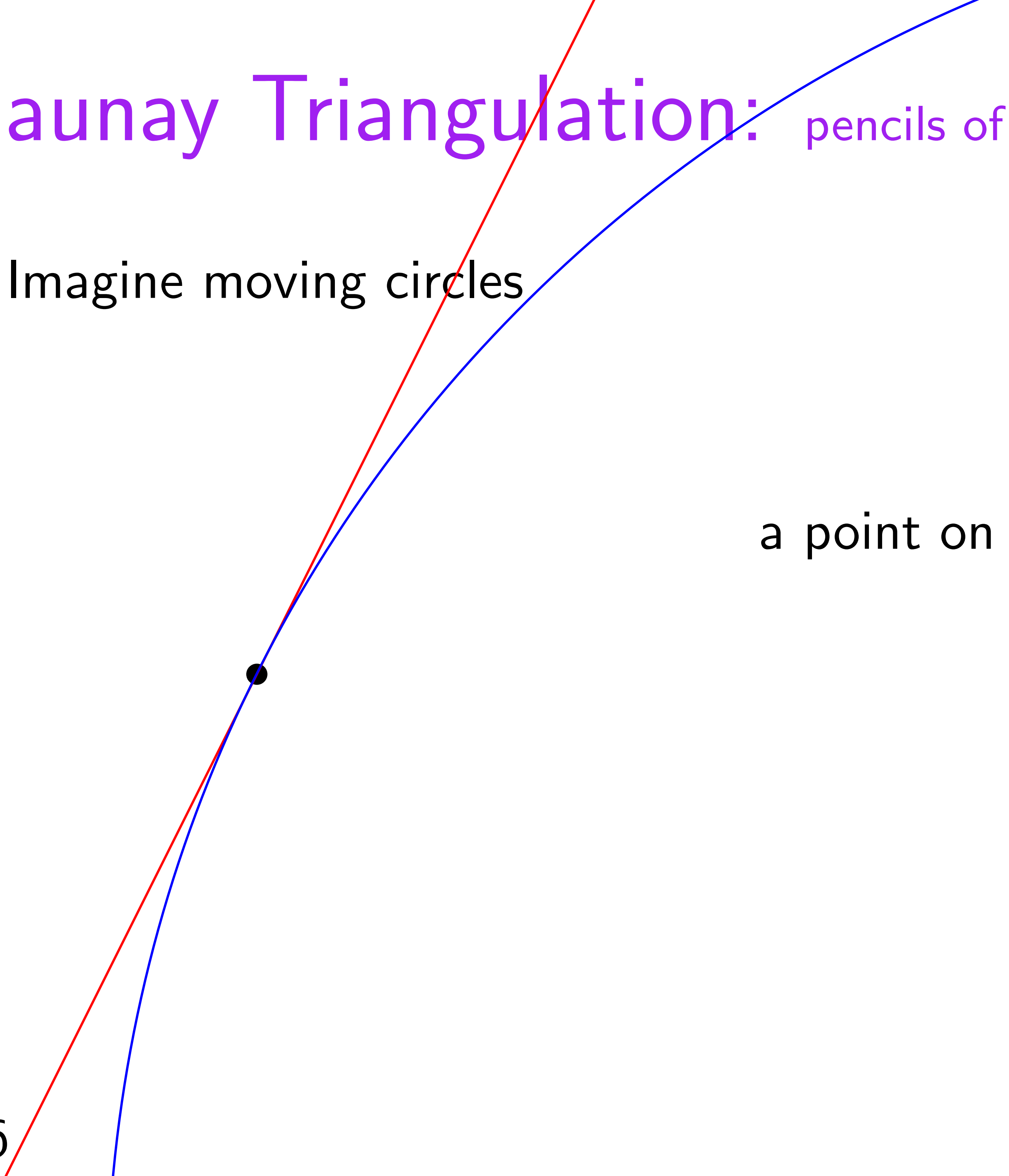
Imagine moving circles

a point on a line



Delaunay Triangulation: pencils of circles

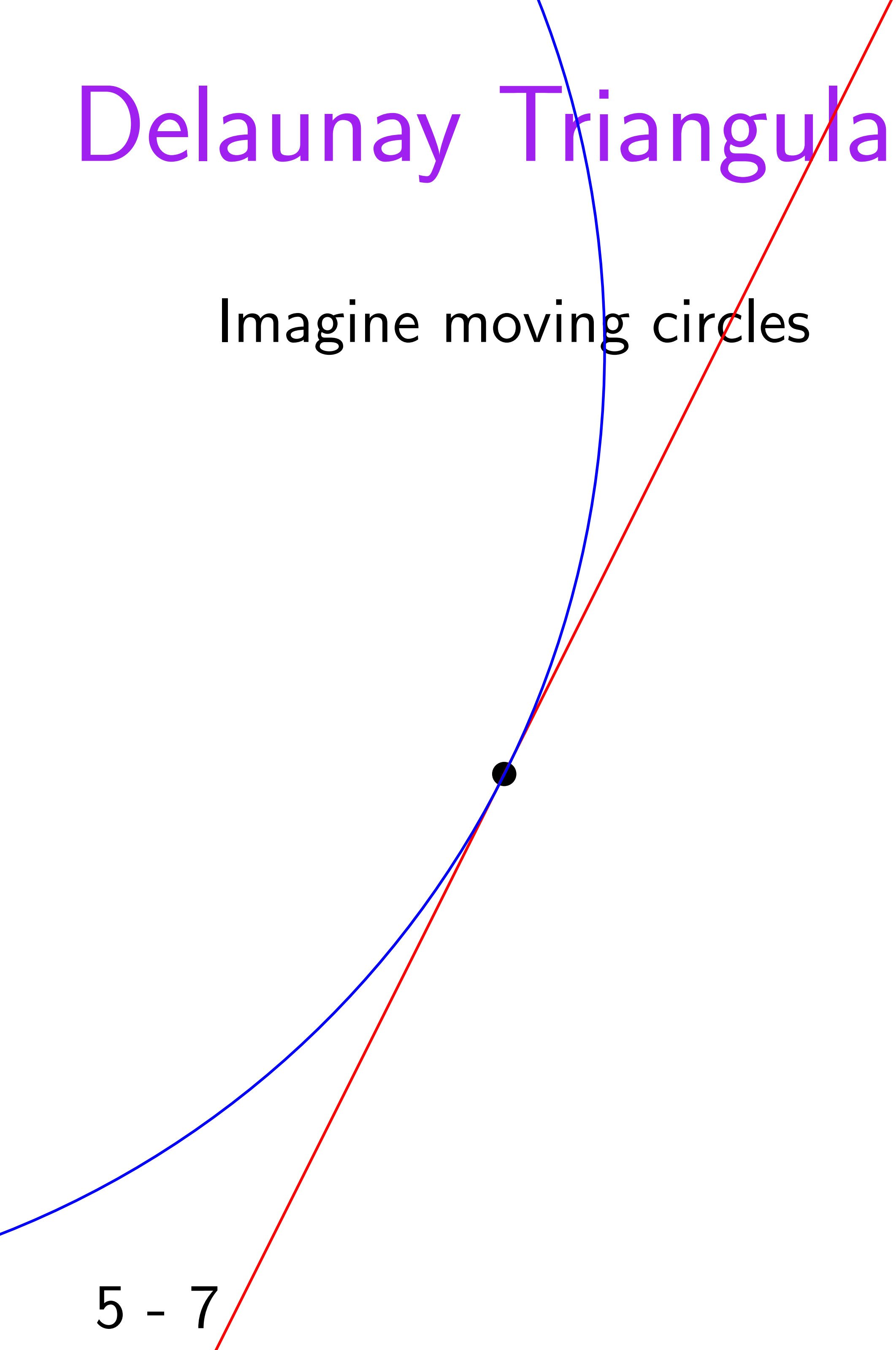
Imagine moving circles



a point on a line

Delaunay Triangulation: pencils of circles

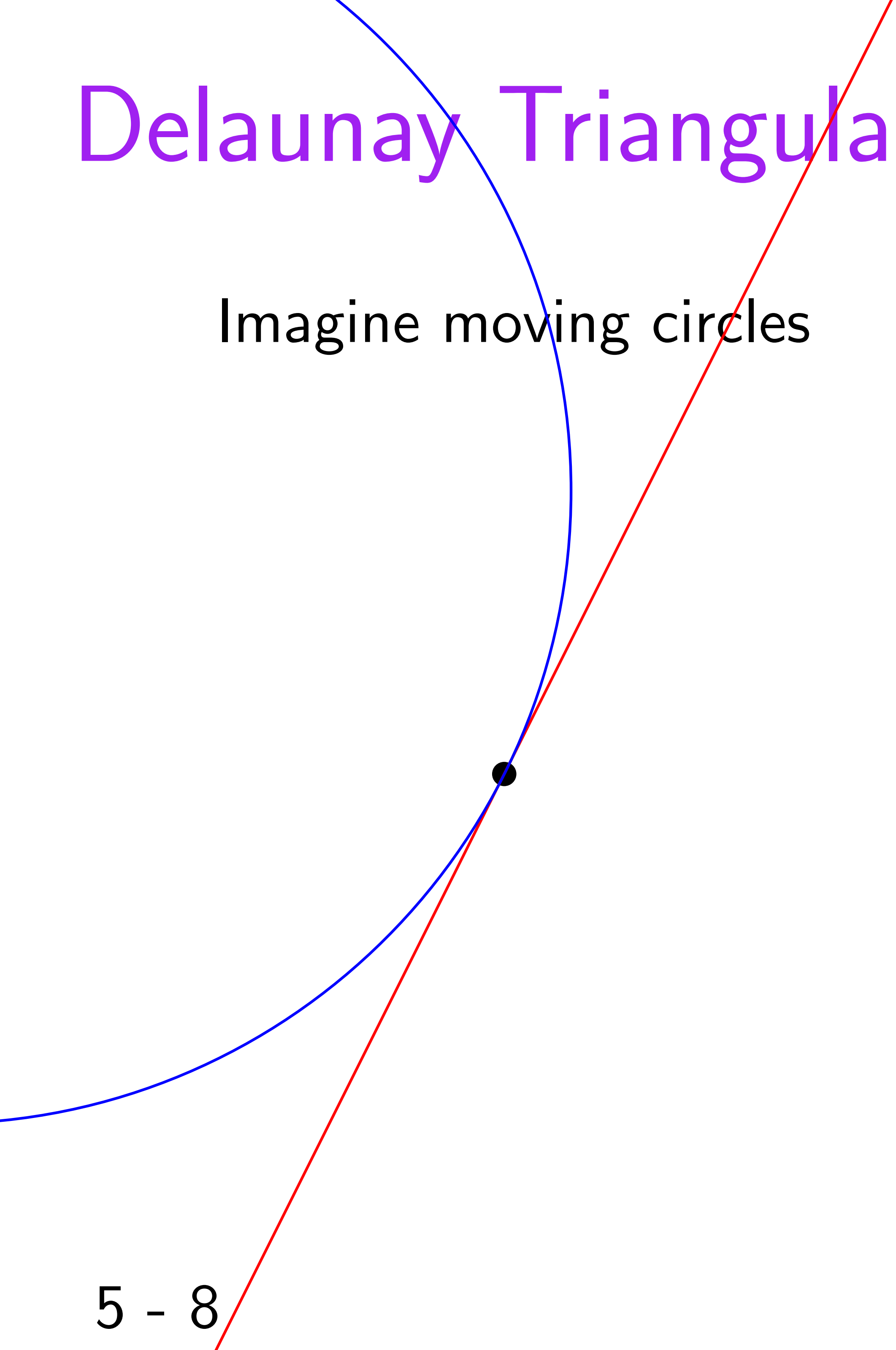
Imagine moving circles



a point on a line

Delaunay Triangulation: pencils of circles

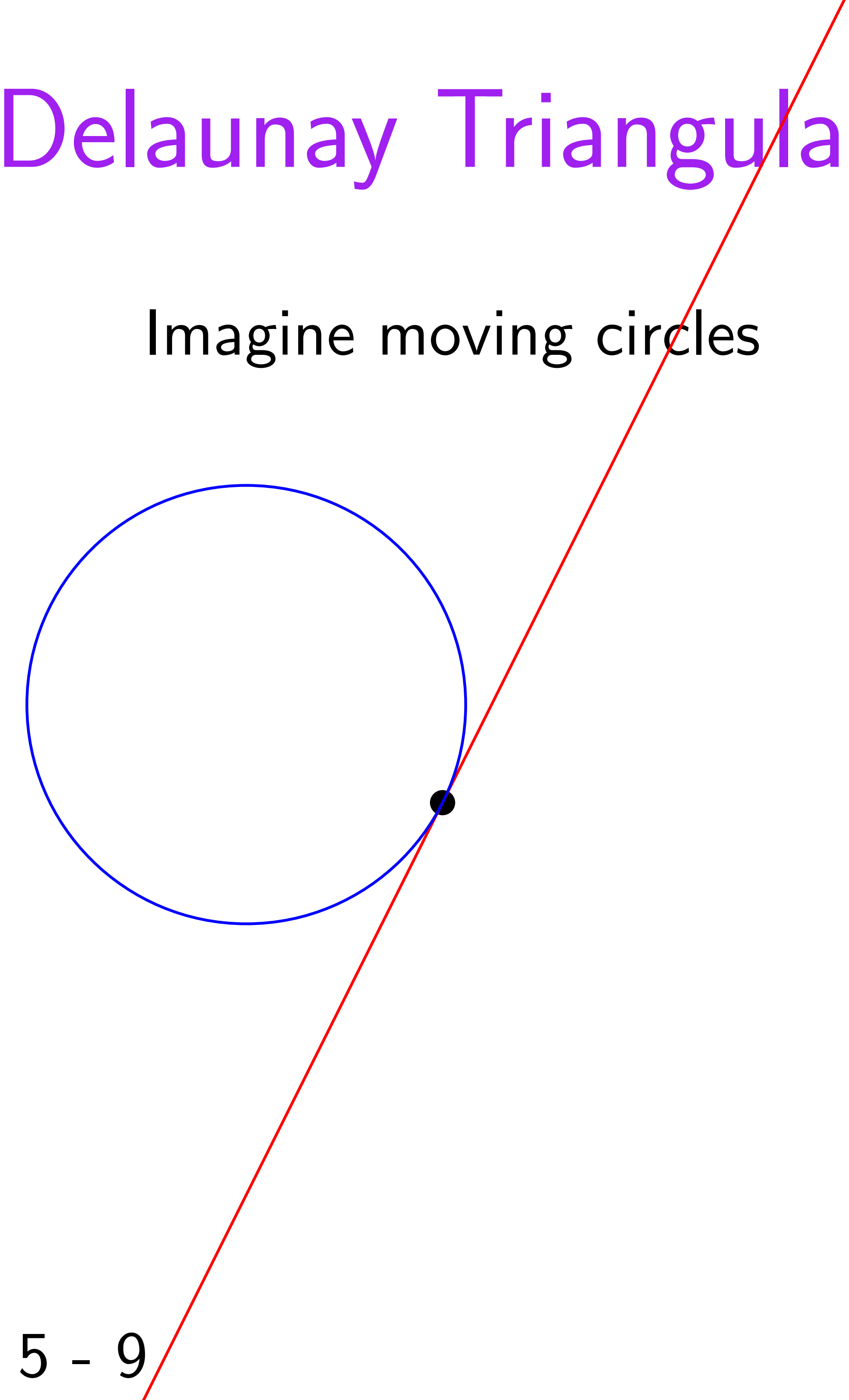
Imagine moving circles



a point on a line

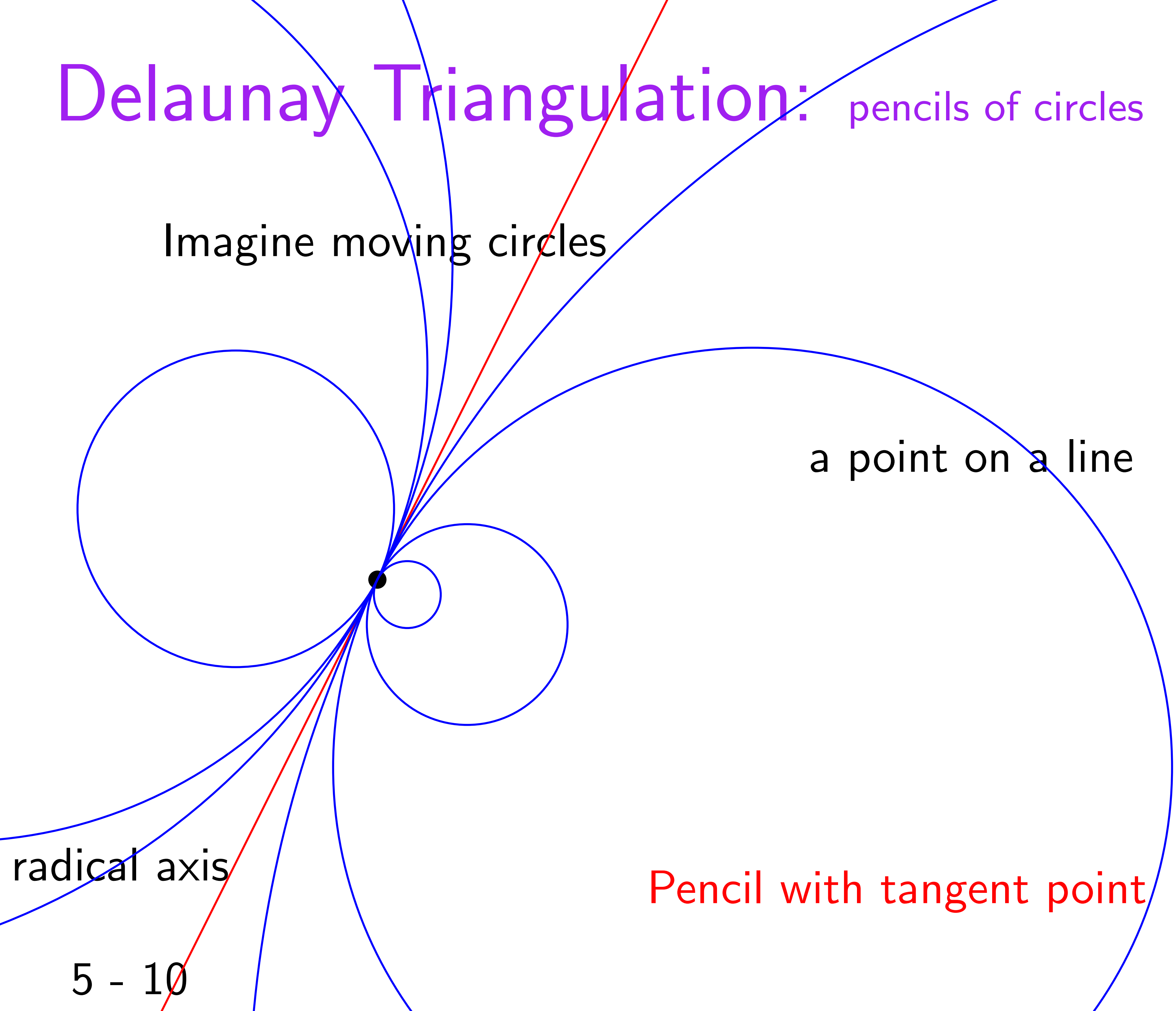
Delaunay Triangulation: pencils of circles

Imagine moving circles



a point on a line

Delaunay Triangulation: pencils of circles

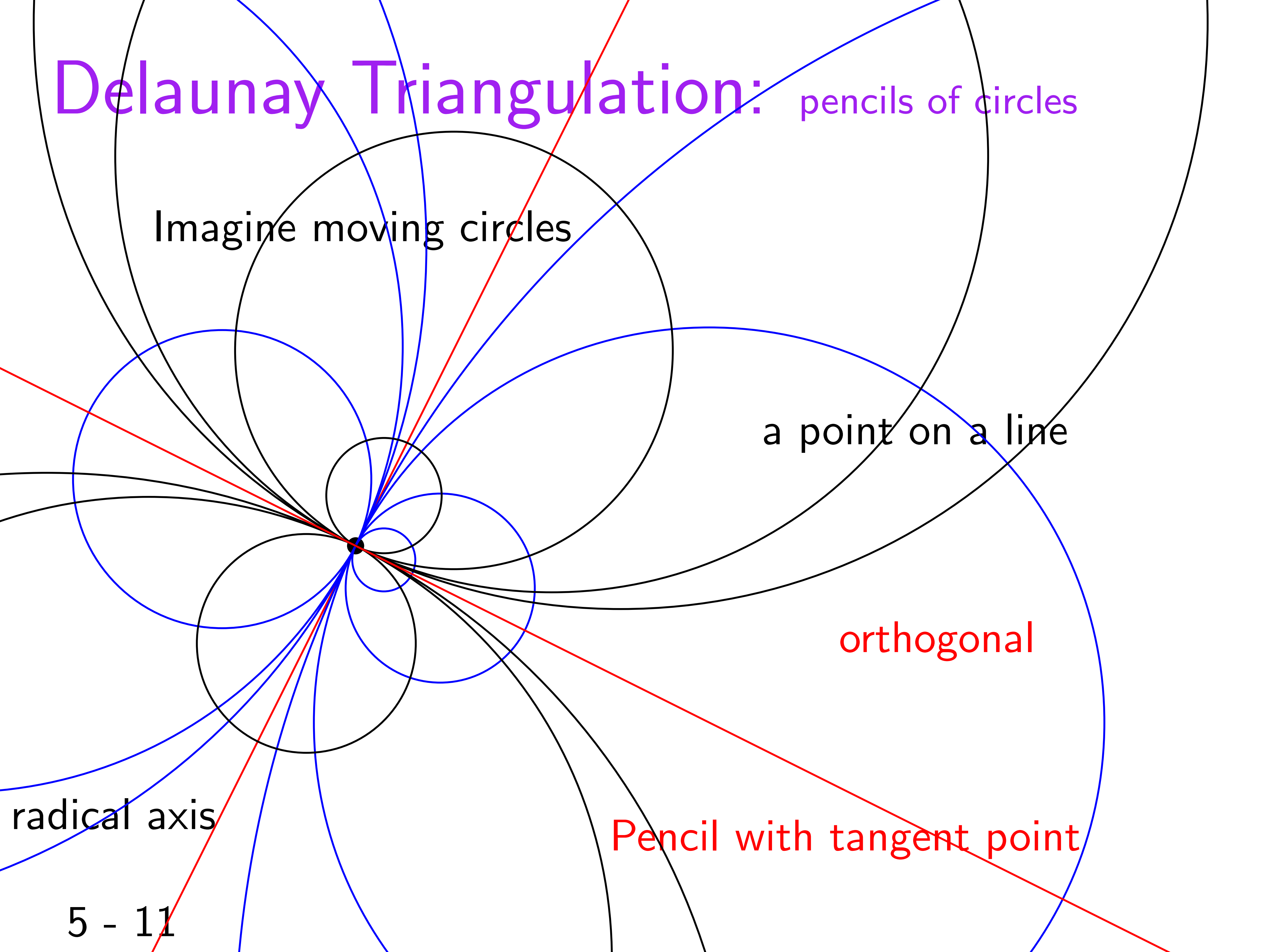


Imagine moving circles

a point on a line

Pencil with tangent point

Delaunay Triangulation: pencils of circles



Imagine moving circles

a point on a line

orthogonal

radical axis

Pencil with tangent point

Delaunay Triangulation: pencils of circles

Circle equation

$$x^2 + y^2 - 2ax - 2by + c = 0$$

Delaunay Triangulation: pencils of circles

Circle equation

$$x^2 + y^2 - 2ax - 2by + c = 0$$

Another circle equation

$$x^2 + y^2 - 2a'x - 2b'y + c' = 0$$

Delaunay Triangulation: pencils of circles

Circle equation

$$x^2 + y^2 - 2ax - 2by + c = 0$$

Another circle equation

$$x^2 + y^2 - 2a'x - 2b'y + c' = 0$$

Pencil of circles

$$\begin{aligned}&\lambda \cdot (x^2 + y^2 - 2ax - 2by + c)\\ &\quad + (1 - \lambda) \cdot (x^2 + y^2 - 2a'x - 2b'y + c') = 0\end{aligned}$$

Delaunay Triangulation: pencils of circles

Circle equation

$$x^2 + y^2 - 2ax - 2by + c = 0$$

Another circle equation

$$x^2 + y^2 - 2a'x - 2b'y + c' = 0$$

Pencil of circles

$$\lambda \cdot (x^2 + y^2 - 2ax - 2by + c) \\ +(1 - \lambda) \cdot (x^2 + y^2 - 2a'x - 2b'y + c') = 0$$

A special "circle: the radical axis

$$\lambda = \infty$$

Delaunay Triangulation: pencils of circles

Power of a point w.r.t a circle

$$x^2 + y^2 - 2ax - 2by + c$$

Delaunay Triangulation: pencils of circles

Power of a point w.r.t a circle

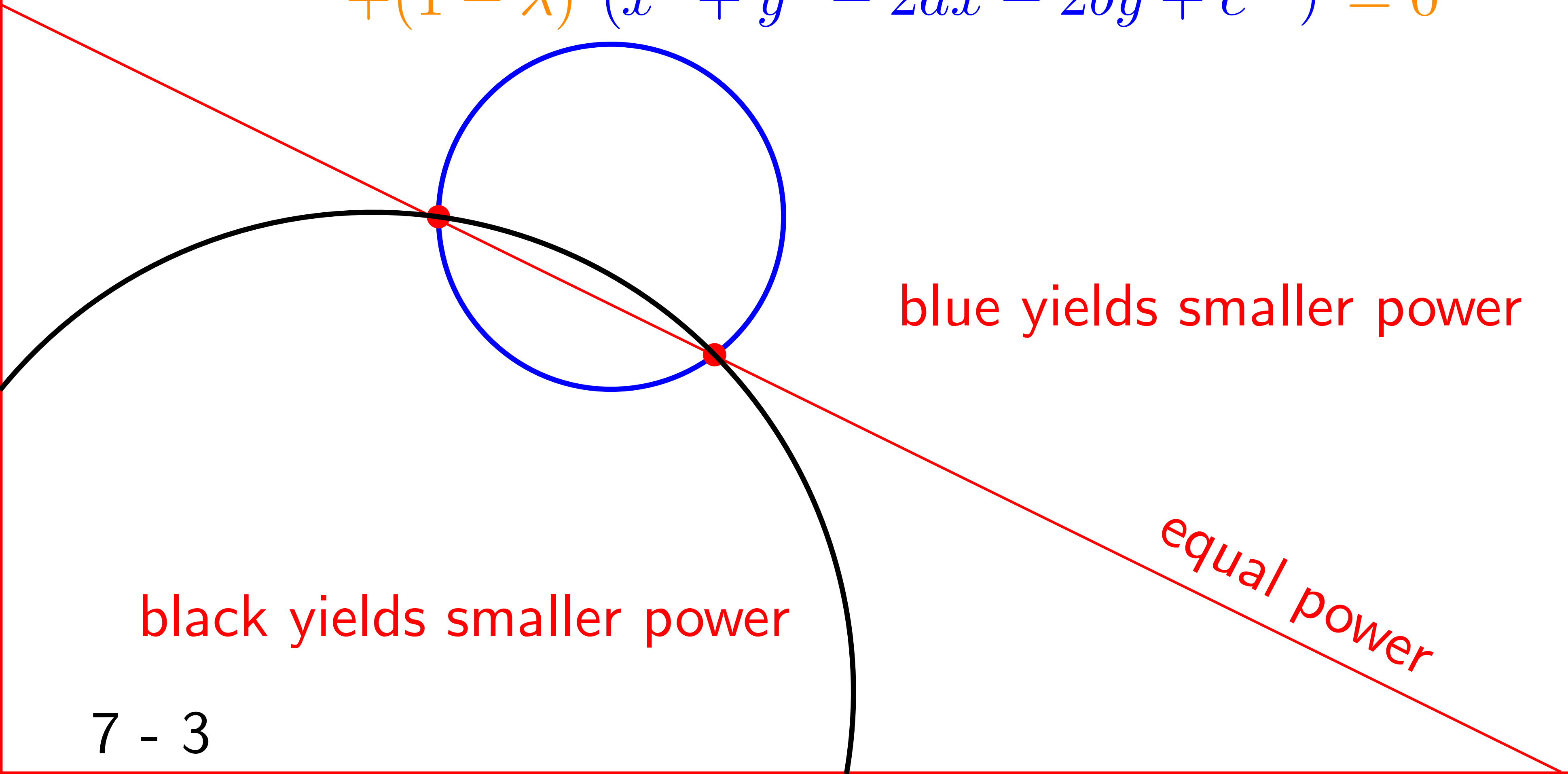
$$x^2 + y^2 - 2ax - 2by + c$$

$= 0$	on the circle
< 0	inside the circle
> 0	outside the circle

Delaunay Triangulation: pencils of circles

Power of a point w.r.t a circle

$$\begin{aligned} &\lambda\ \ (x^2+y^2-2a'x-2b'y+c') \\ +&(1-\lambda)\ (x^2+y^2-2ax-2by+c\ \ ) = 0 \end{aligned}$$



blue yields smaller power

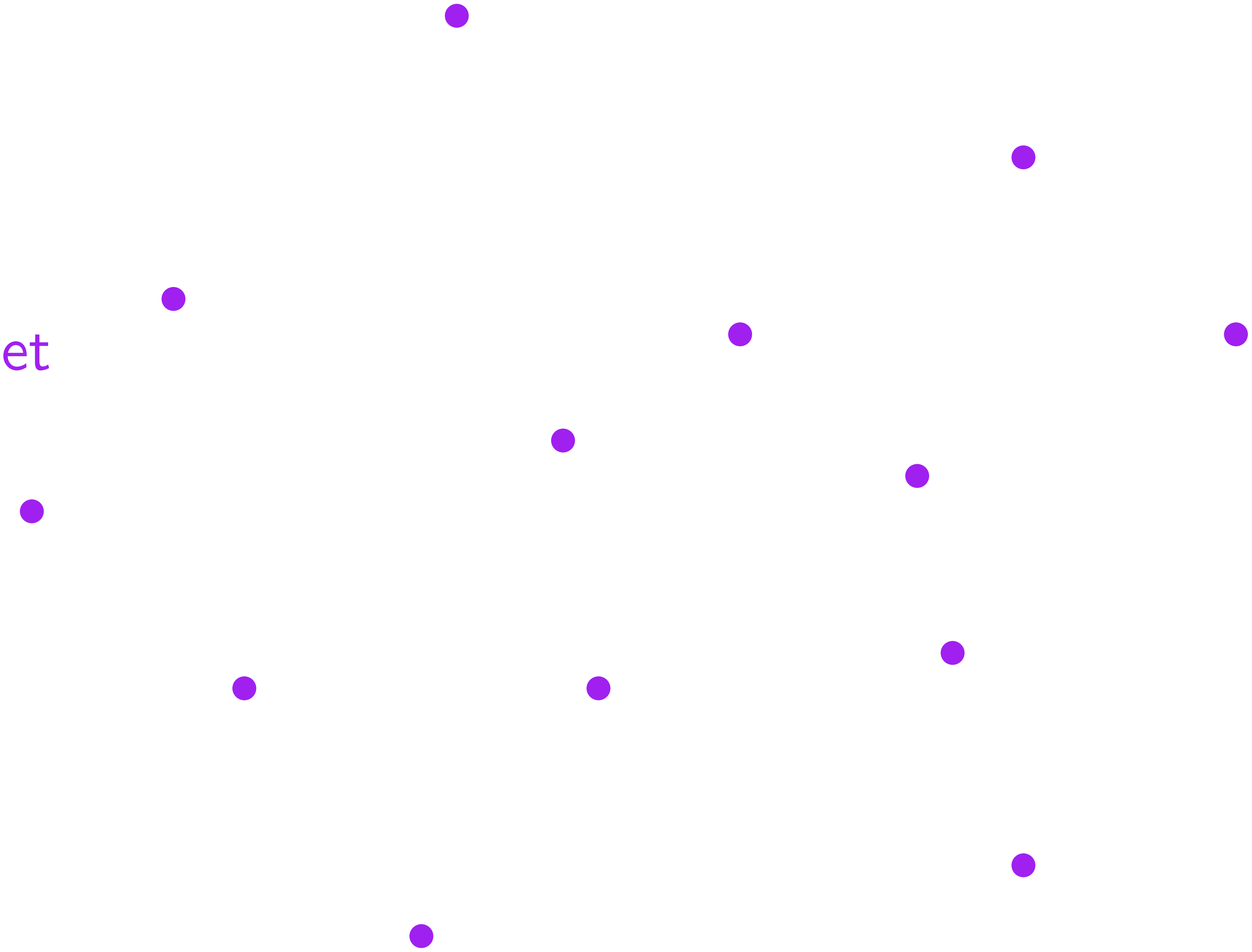
equal power

black yields smaller power

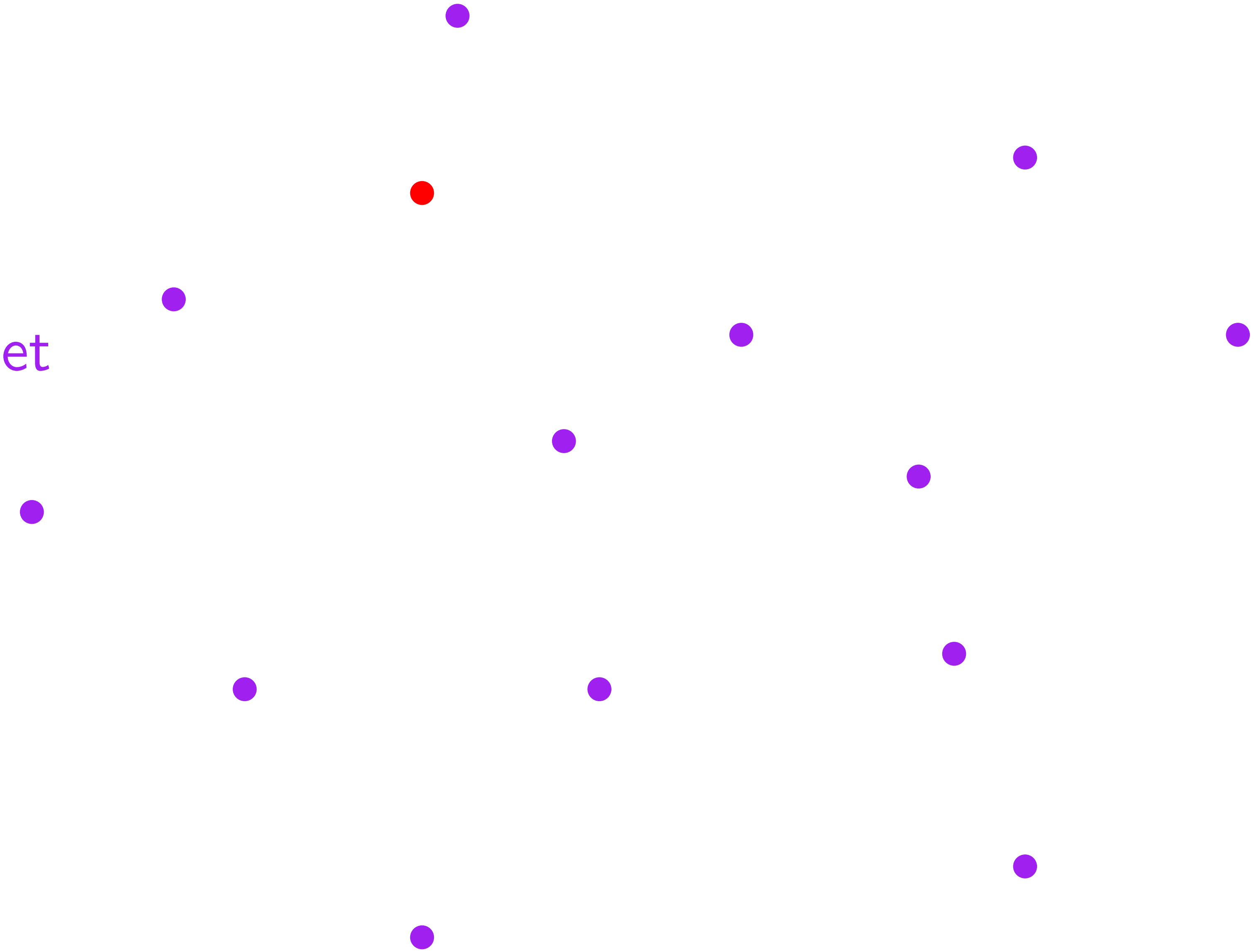
Delaunay Triangulation: definition, empty circle property

Delaunay Triangulation: definition, empty circle property

Point set



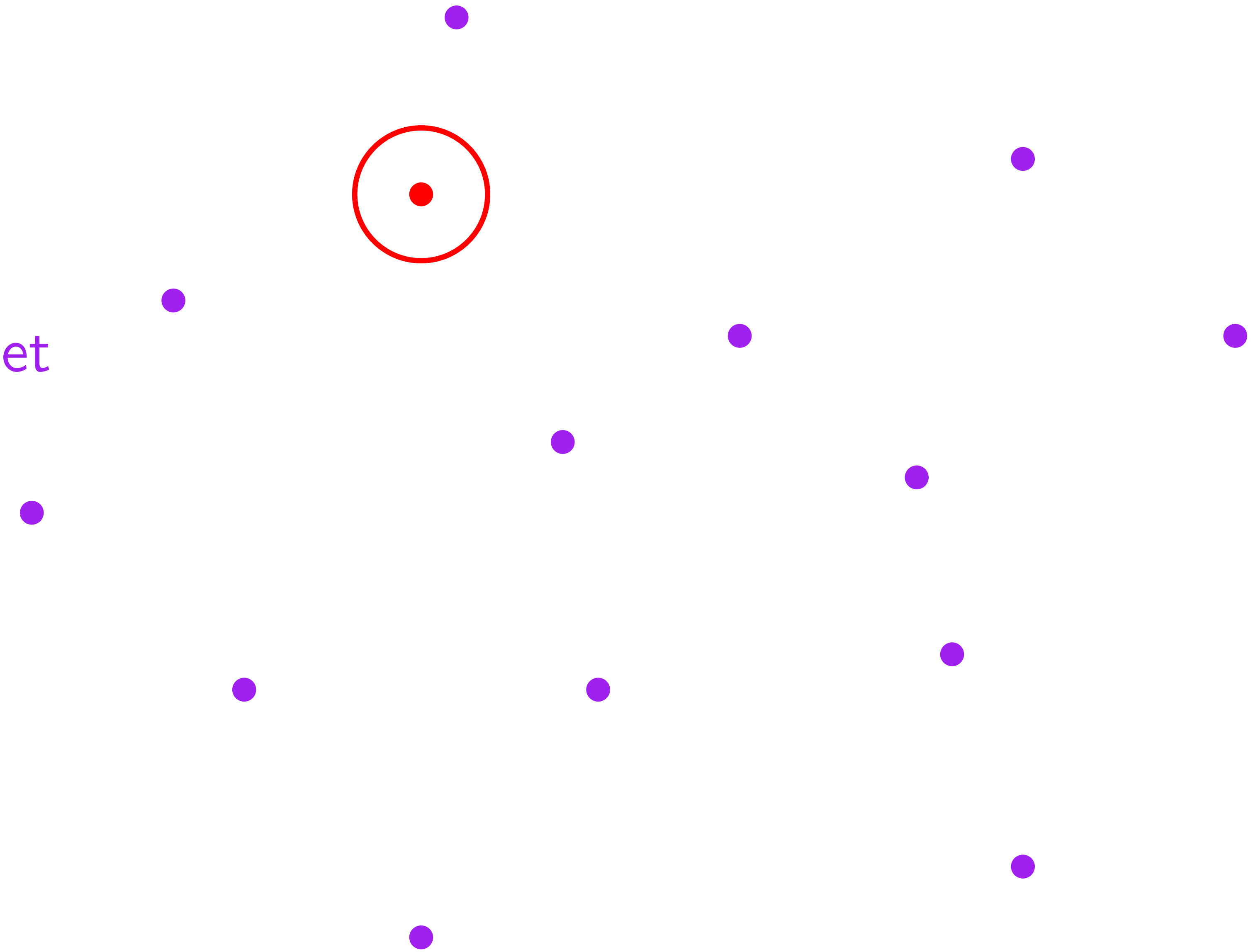
Delaunay Triangulation: definition, empty circle property



Point set

Query

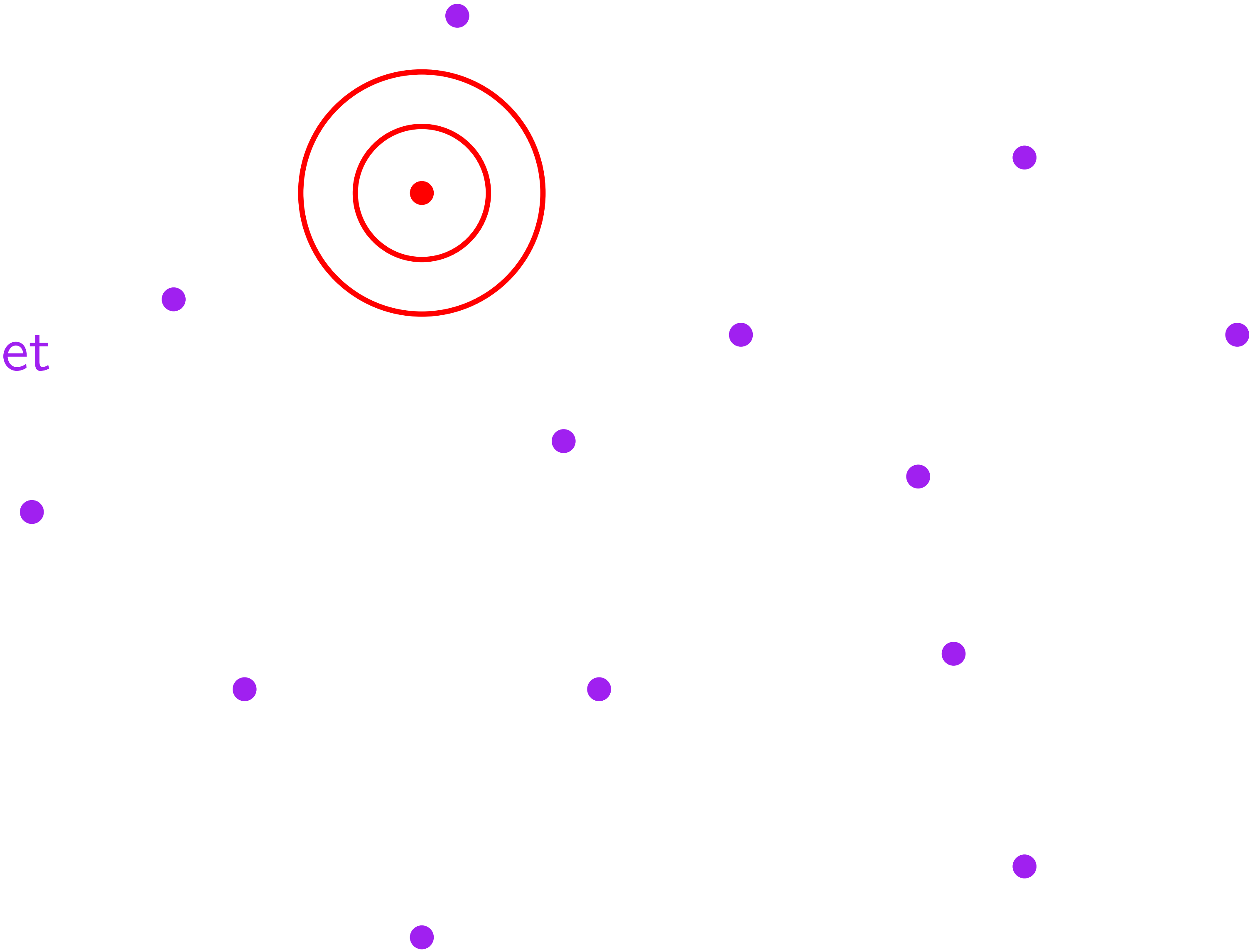
Delaunay Triangulation: definition, empty circle property



Point set

Query

Delaunay Triangulation: definition, empty circle property



Point set

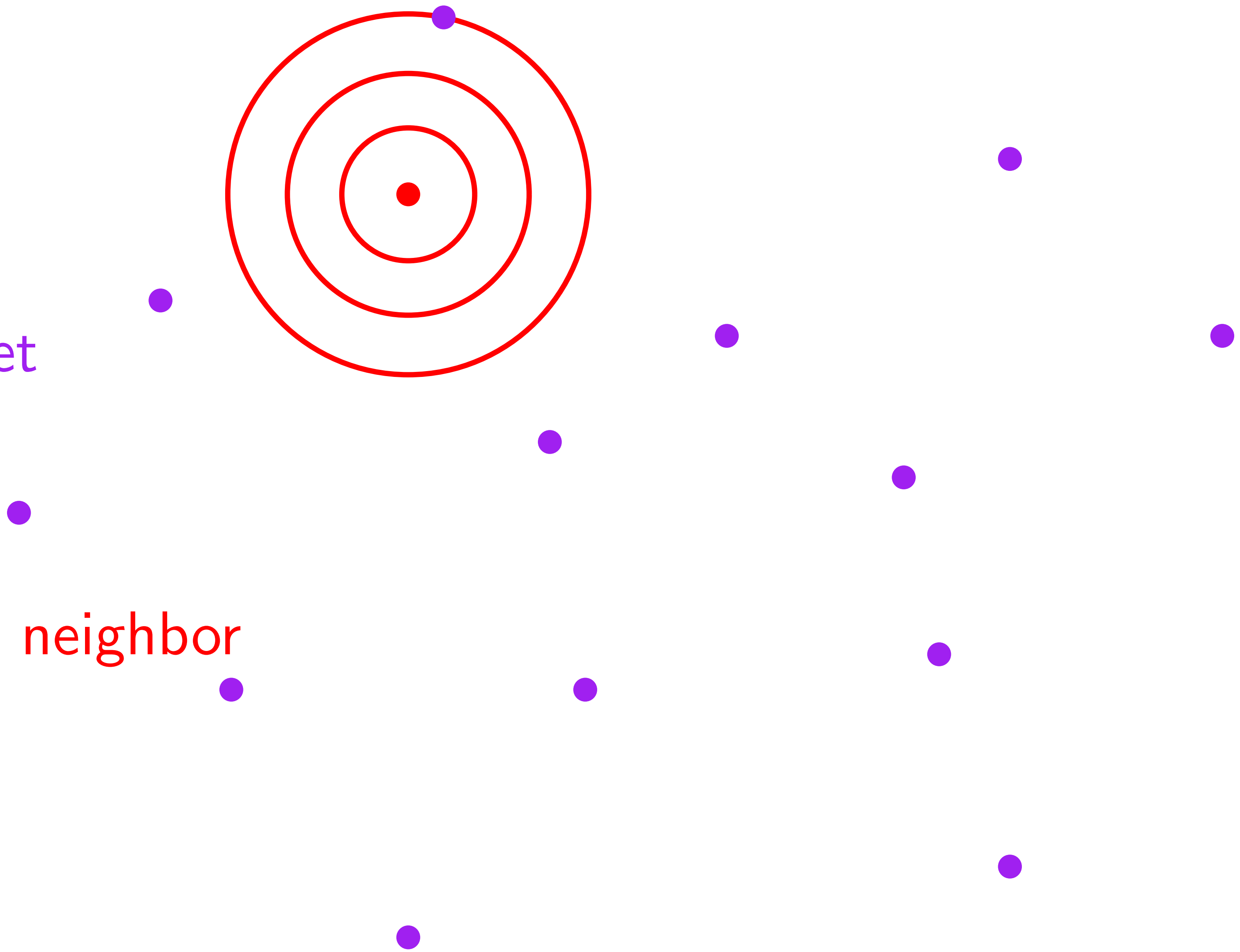
Query

Delaunay Triangulation: definition, empty circle property

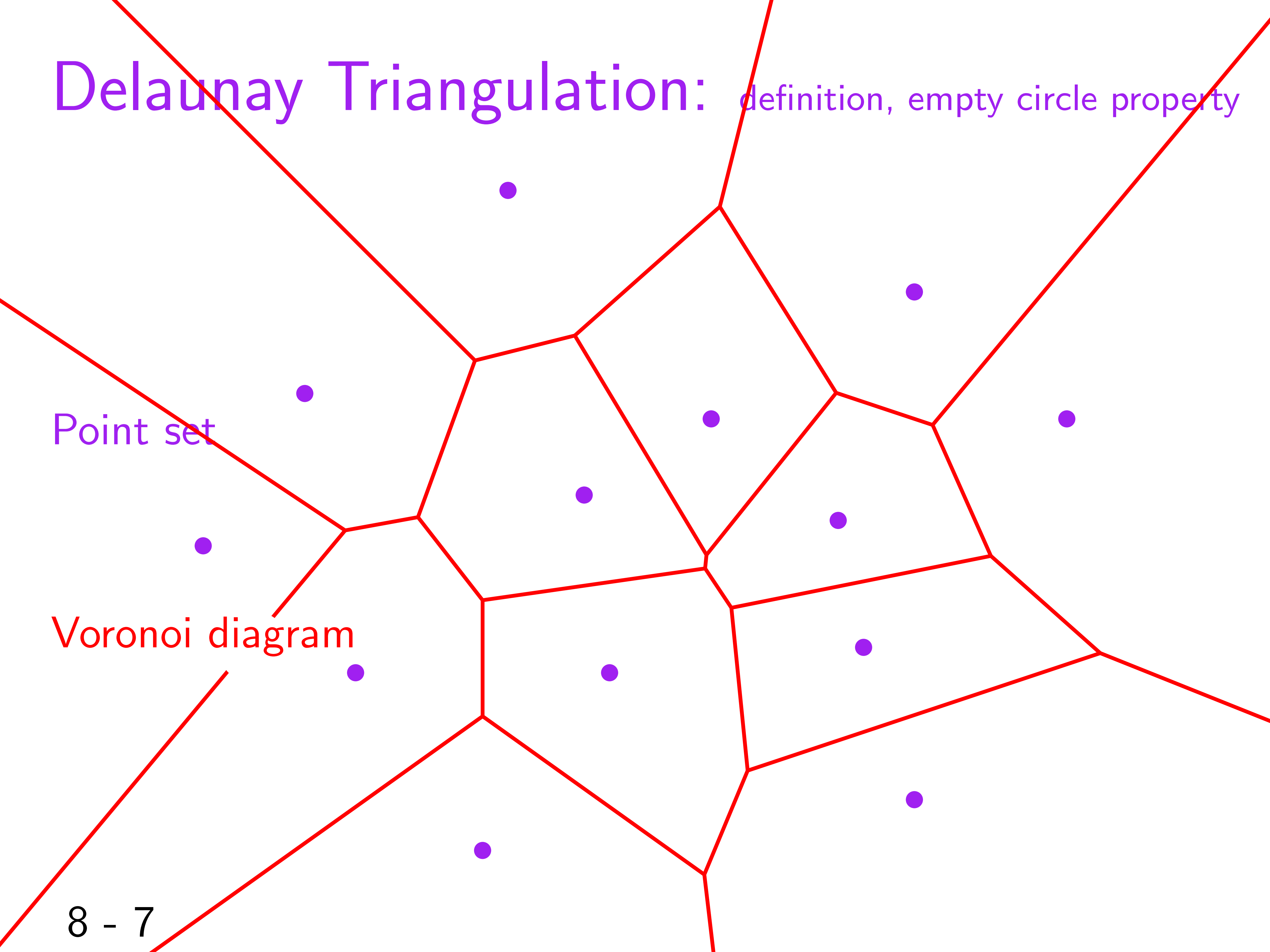
Point set

Query

Nearest neighbor



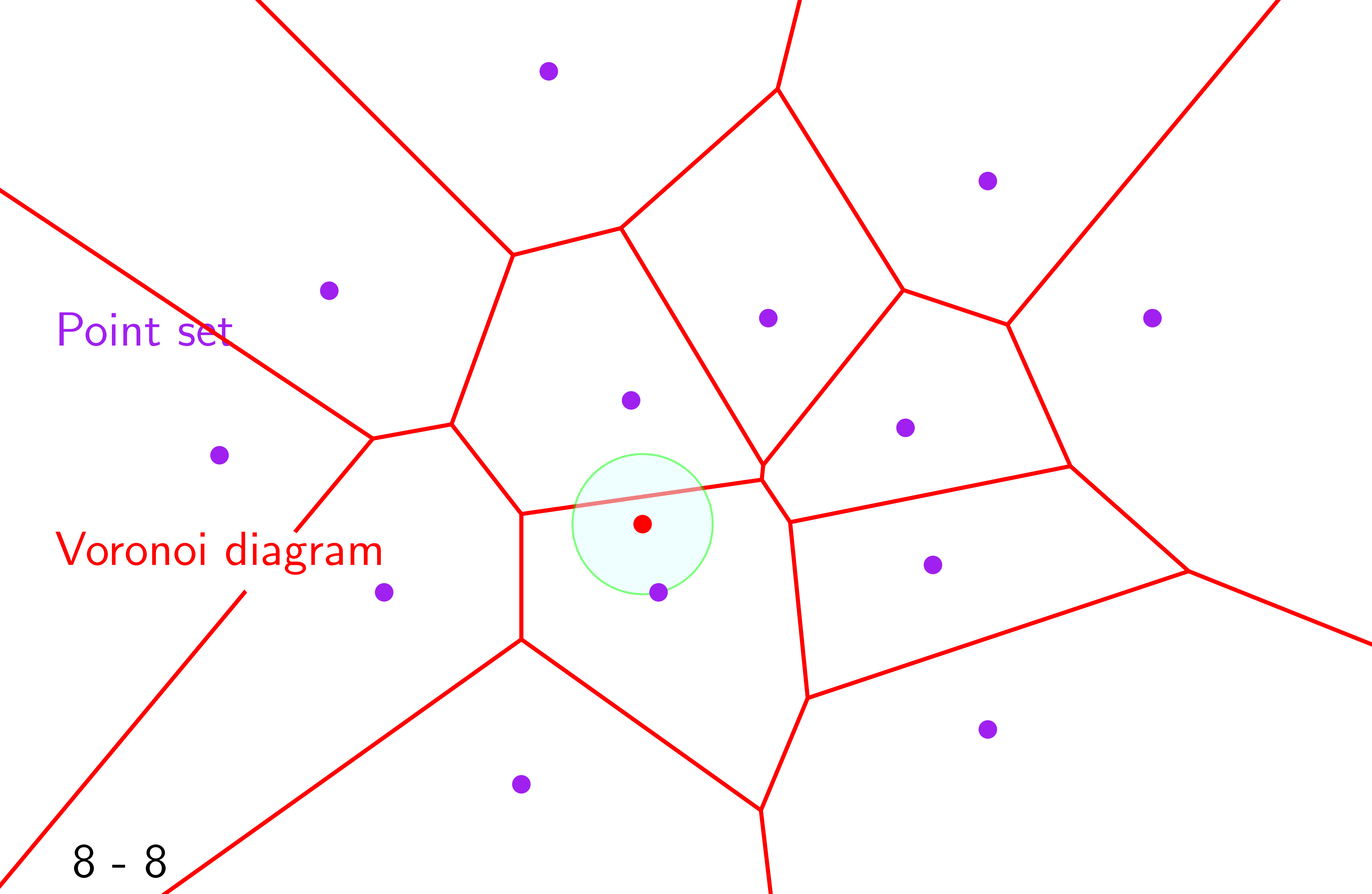
Delaunay Triangulation: definition, empty circle property



Point set

Voronoi diagram

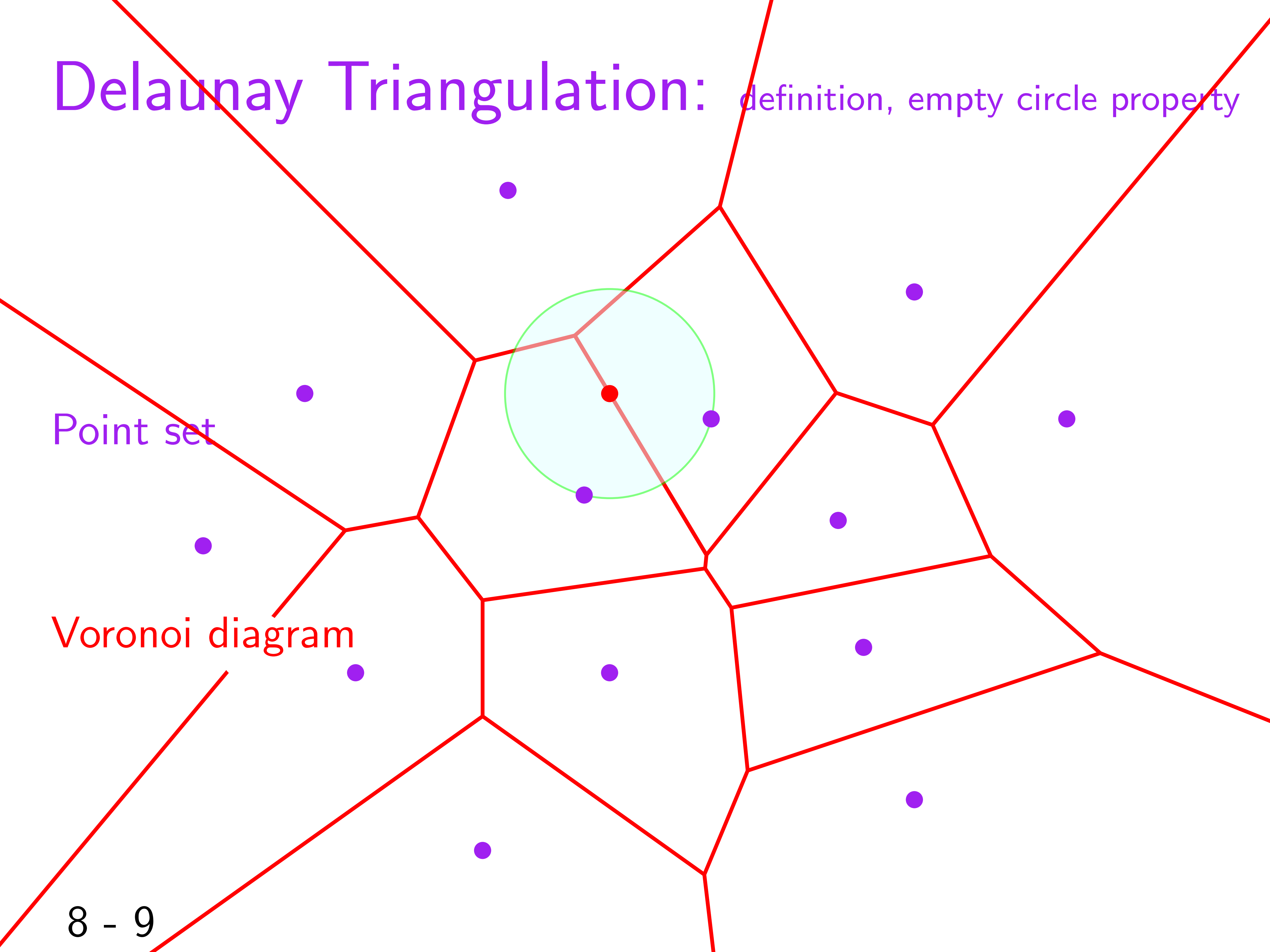
Delaunay Triangulation: definition, empty circle property



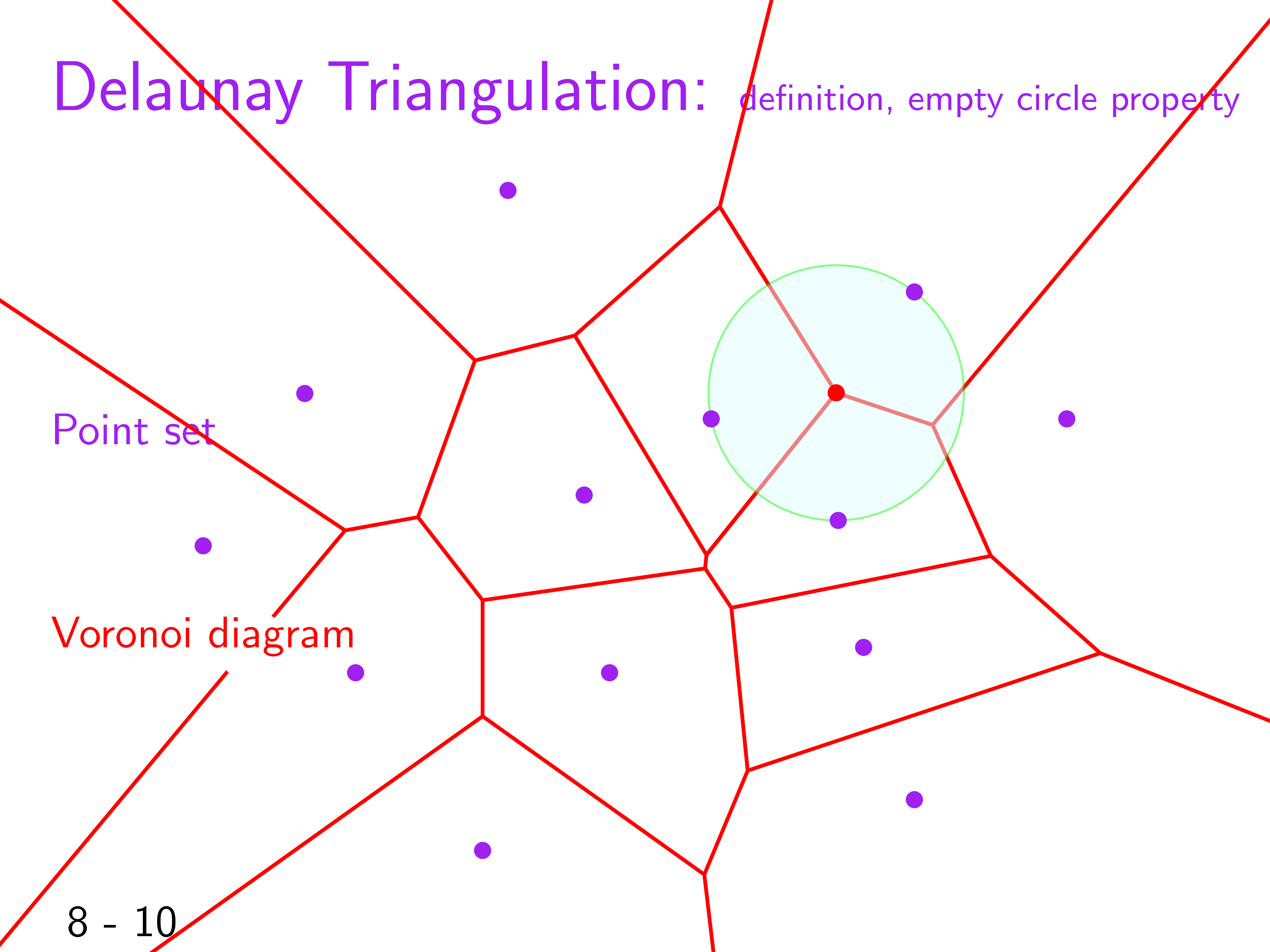
Point set

Voronoi diagram

Delaunay Triangulation: definition, empty circle property



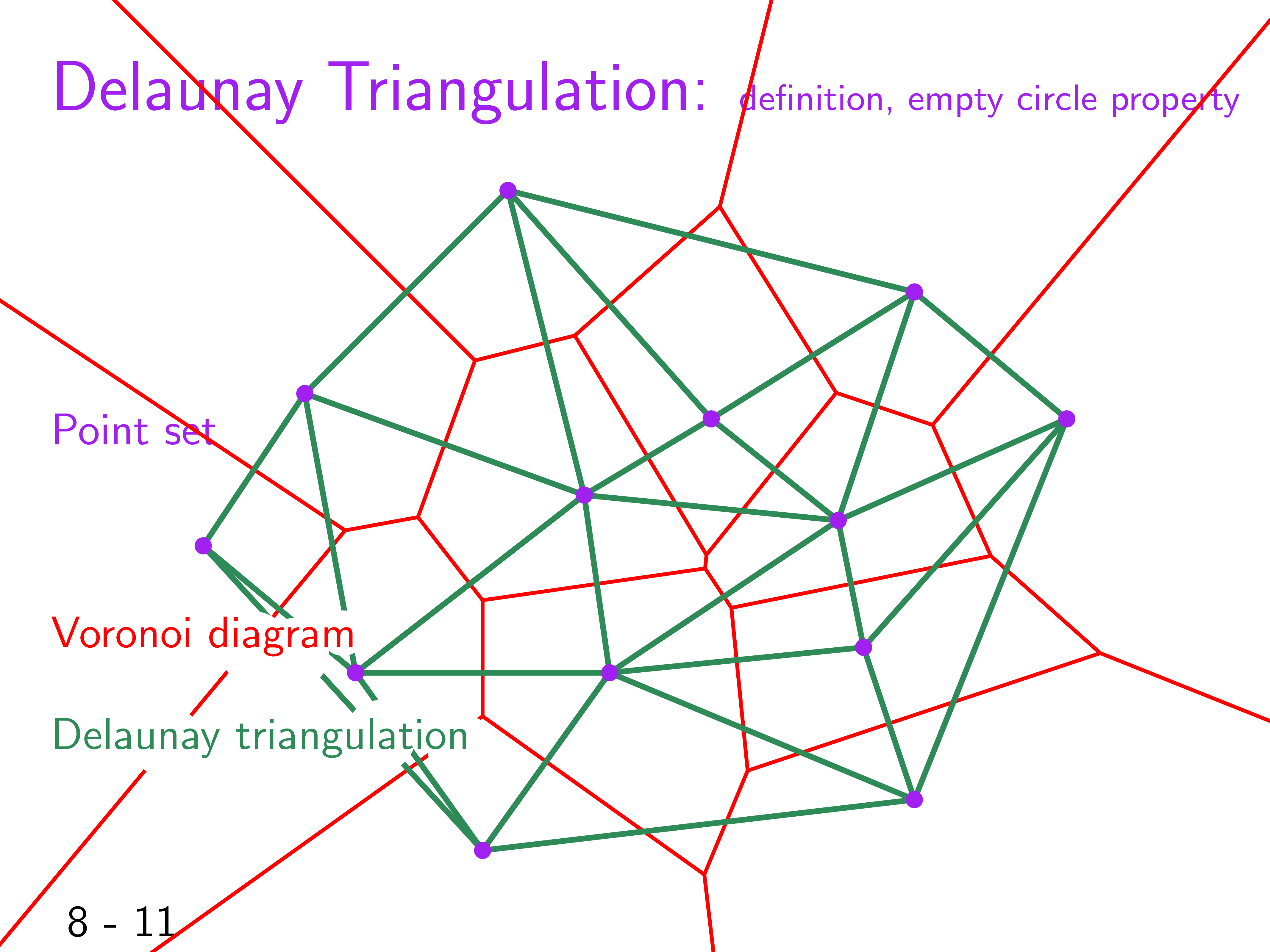
Delaunay Triangulation: definition, empty circle property



Point set

Voronoi diagram

Delaunay Triangulation: definition, empty circle property

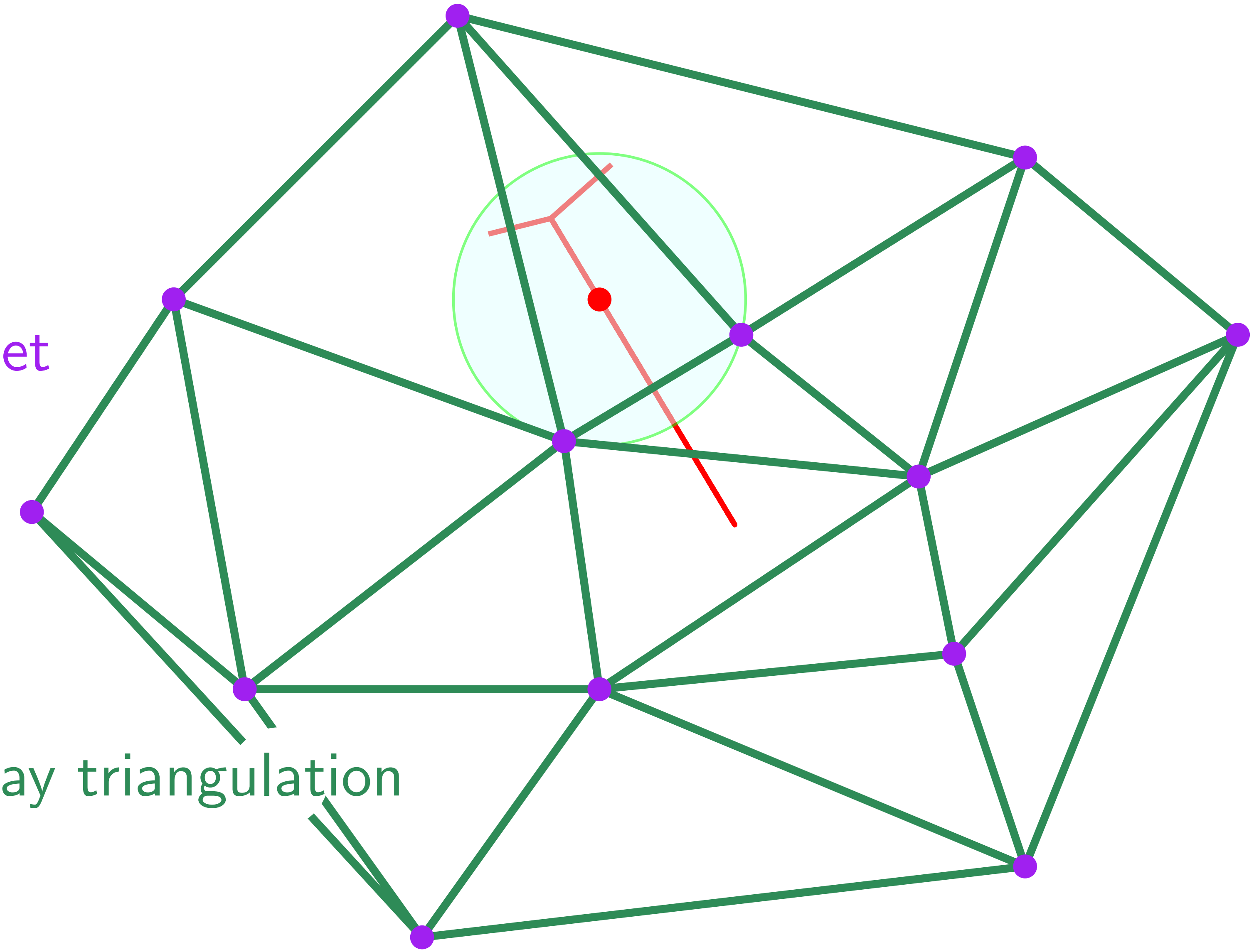


Point set

Voronoi diagram

Delaunay triangulation

Delaunay Triangulation: definition, empty circle property

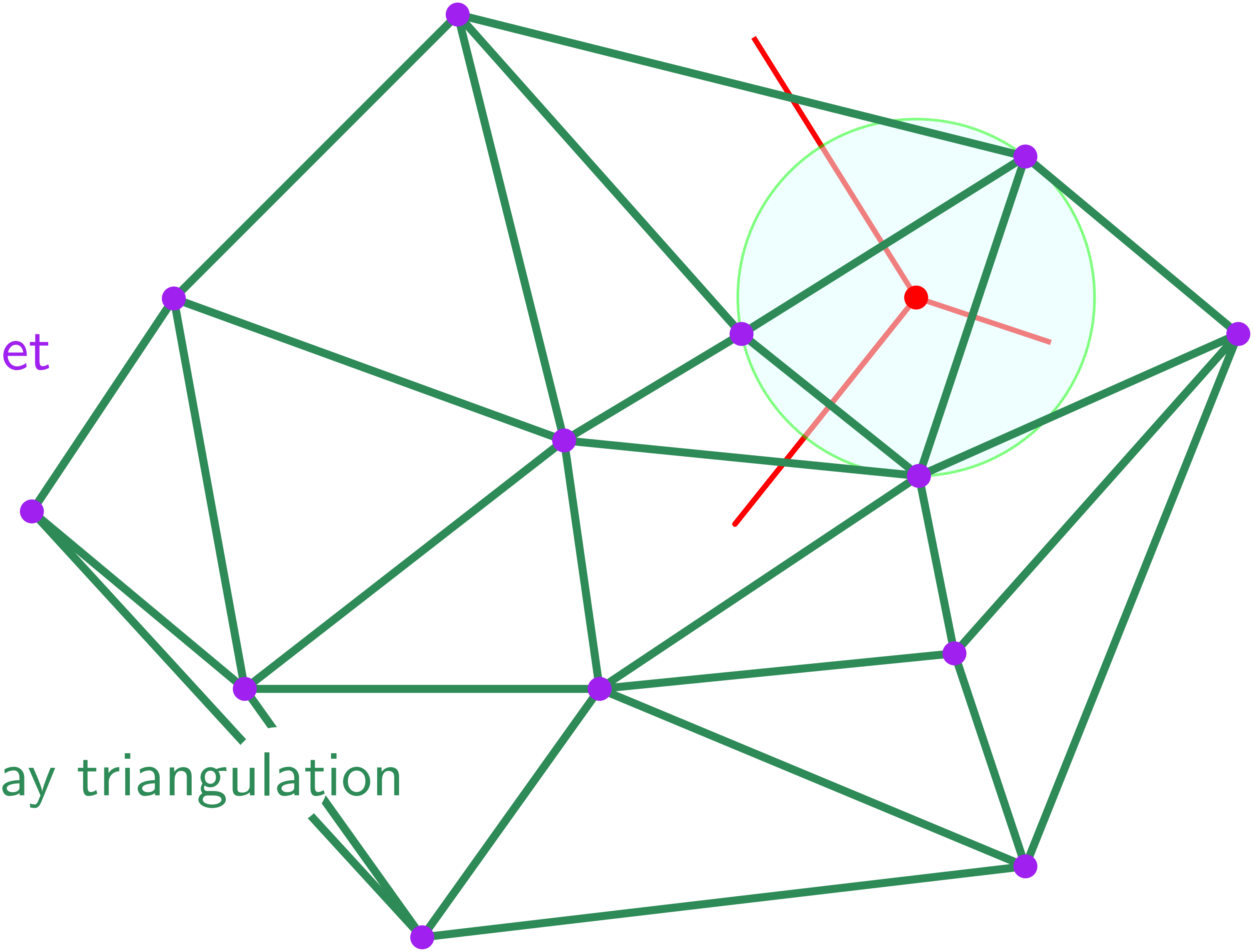


Point set

Delaunay triangulation

Empty circle property

Delaunay Triangulation: definition, empty circle property



Point set

Delaunay triangulation

Empty circle property

Delaunay Triangulation:

Teaser reconstruction lecture

Input: a set of points on an unknown curve

Delaunay Triangulation:

Teaser reconstruction lecture

Input: a set of points on an unknown curve

Output: the curve (the points in order along the curve)

Delaunay Triangulation:

Teaser reconstruction lecture

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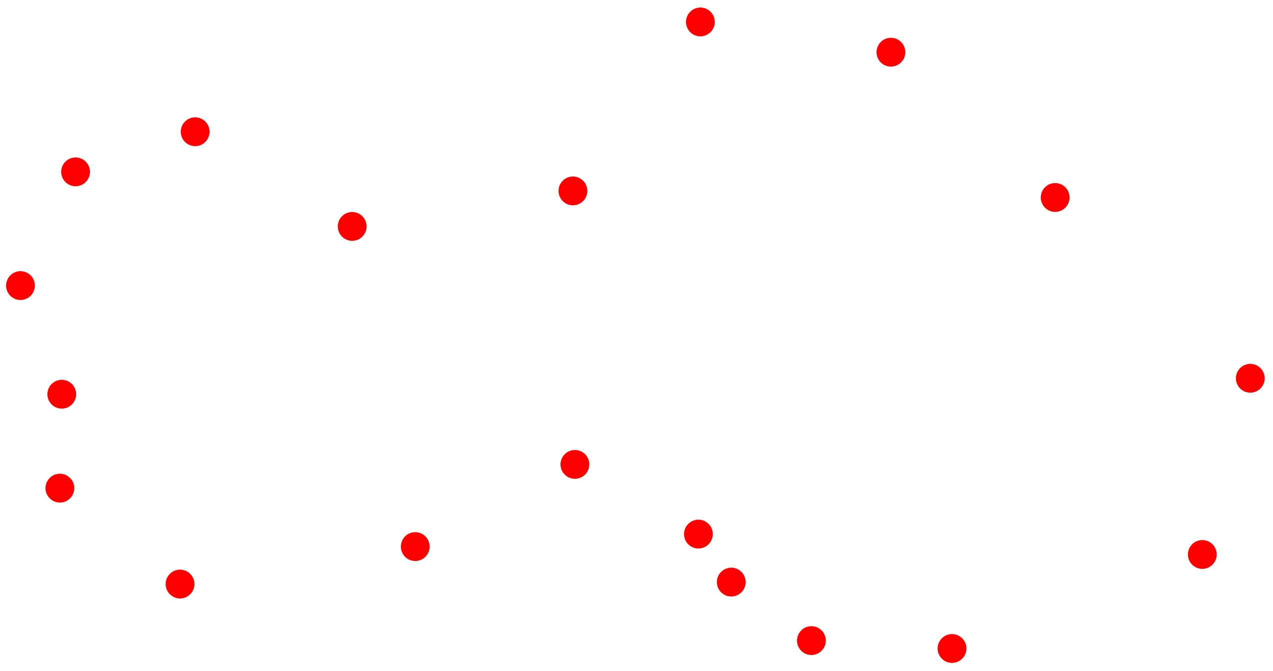

Delaunay Triangulation:

Teaser reconstruction lecture

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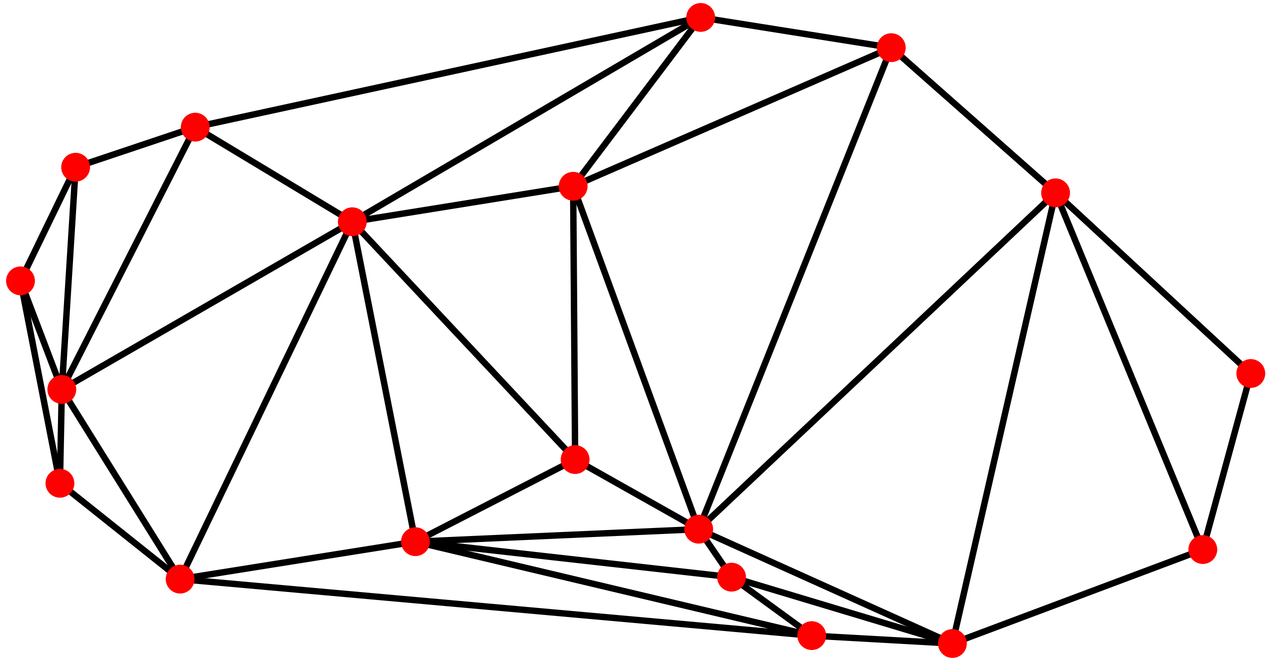

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Teaser reconstruction lecture

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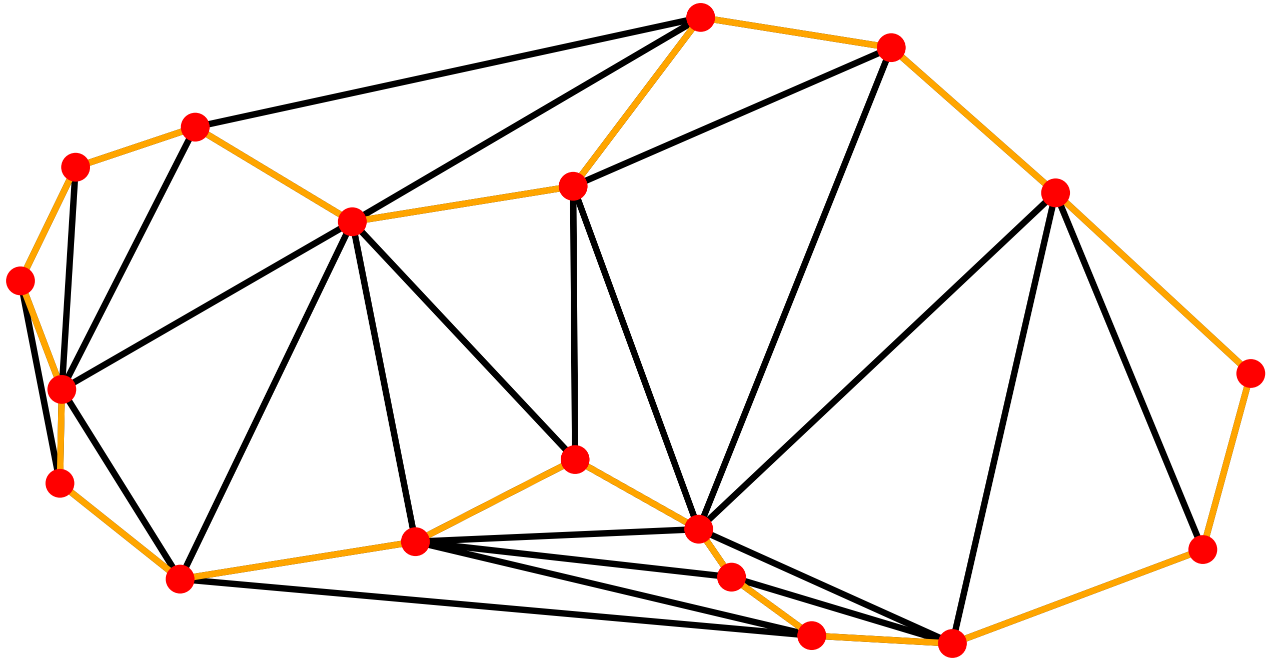

If good sampling, ouput ∈ Delaunay

Delaunay Triangulation:

Teaser reconstruction lecture

Delaunay Triangulation:

Teaser reconstruction lecture

Input: a set of points on an unknown surface

Output: the surface (a triangulation of the points approximating the surface)

Delaunay Triangulation:

Teaser reconstruction lecture

Input: a set of points on an unknown surface

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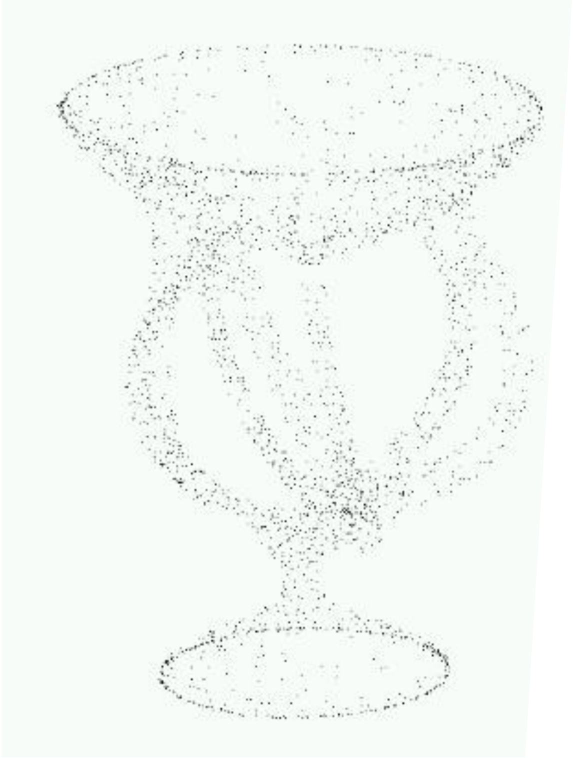
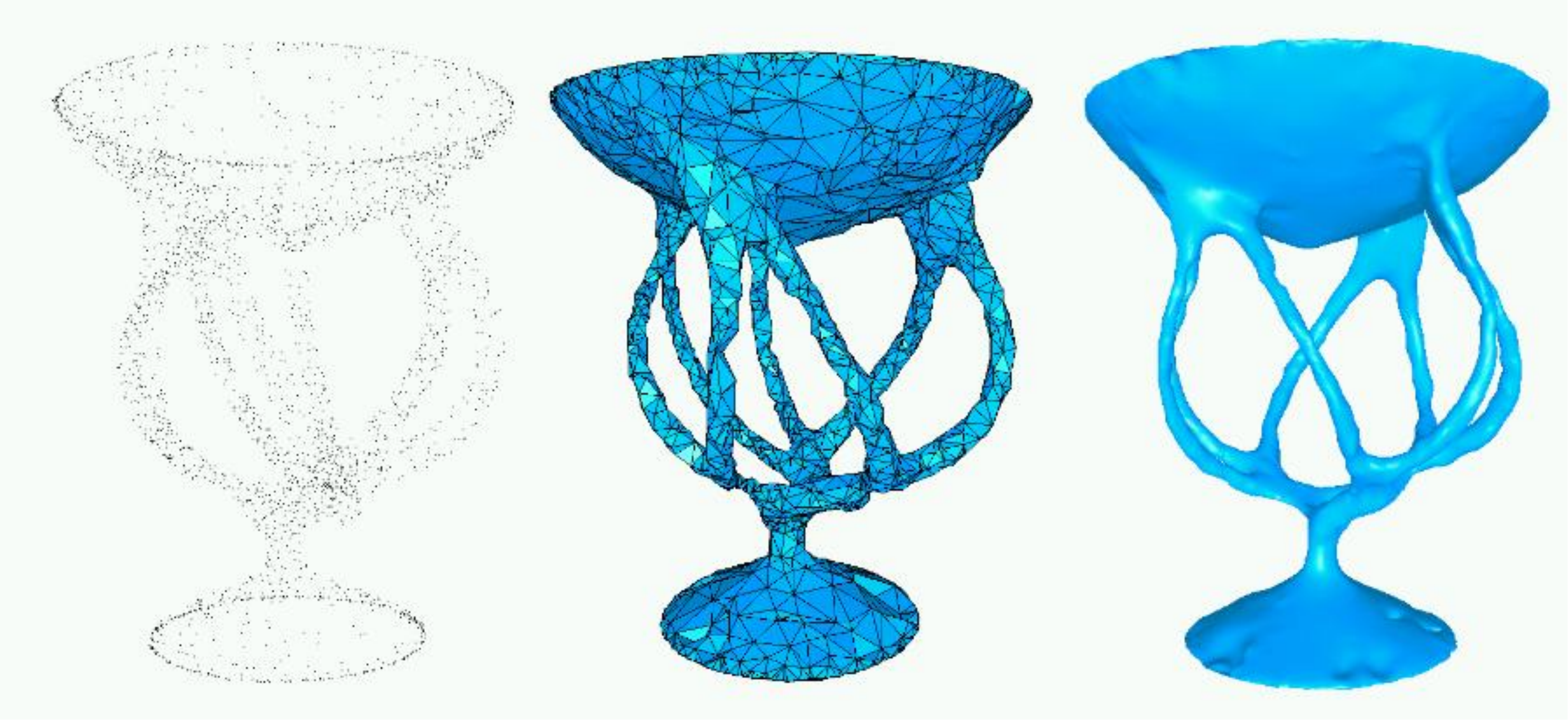


Delaunay Triangulation:

Teaser reconstruction lecture

Input: a set of points on an unknown surface

Output: the surface (a triangulation of the points approximating the surface)



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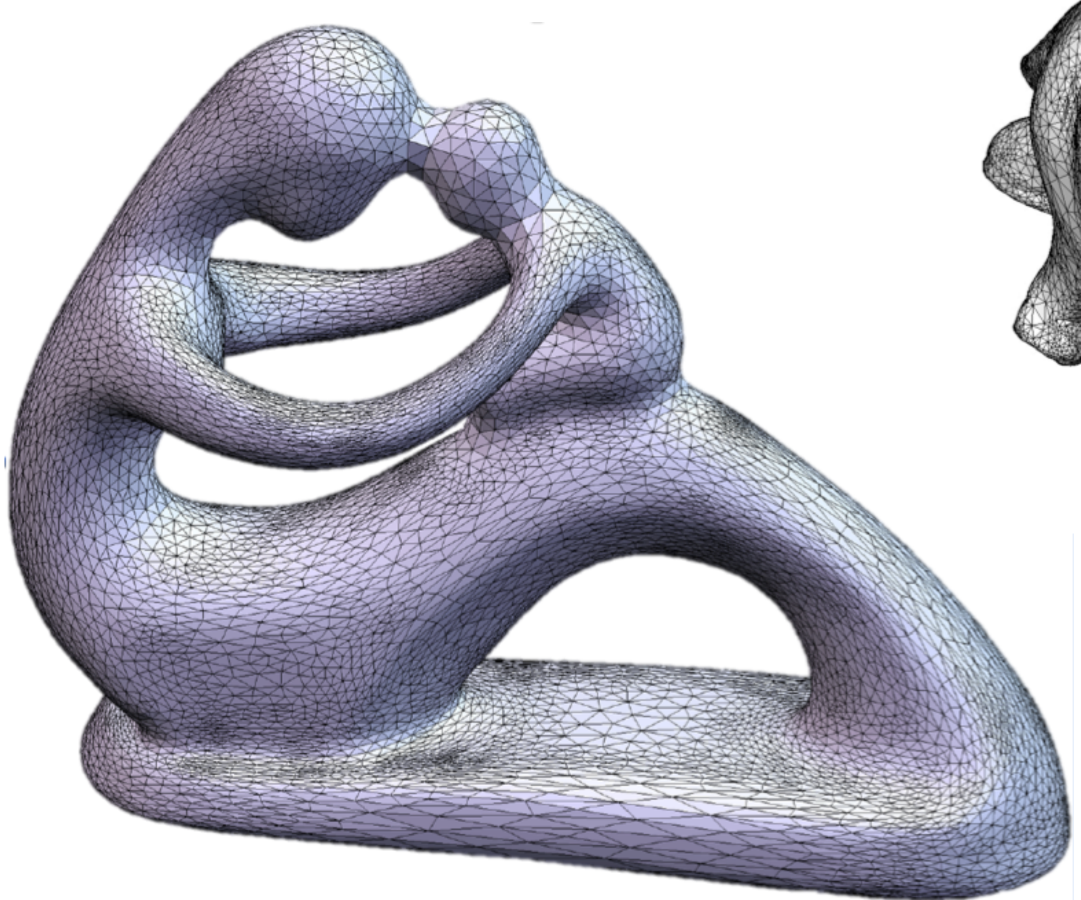
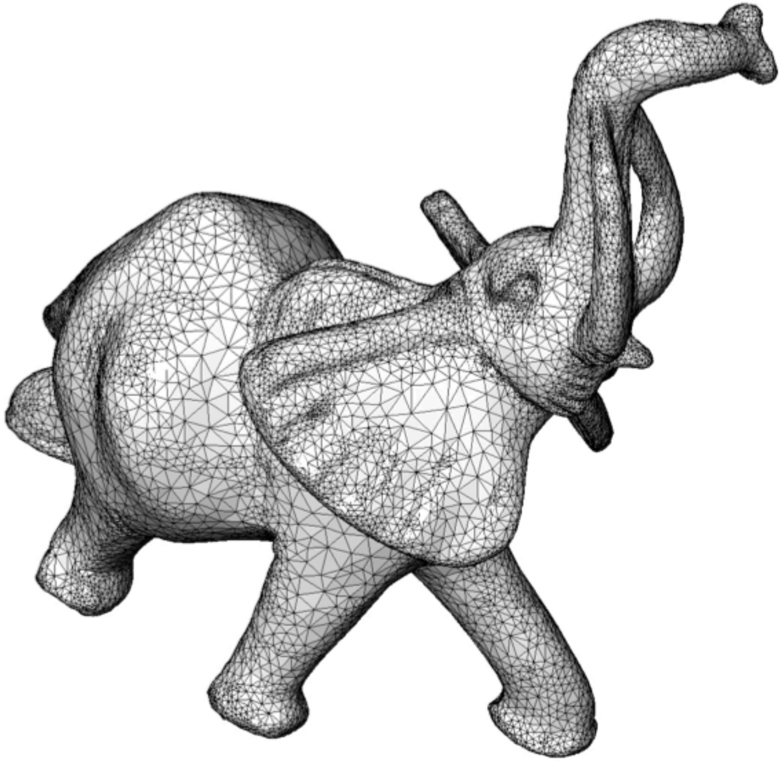
Delaunay Triangulation:

Teaser reconstruction lecture





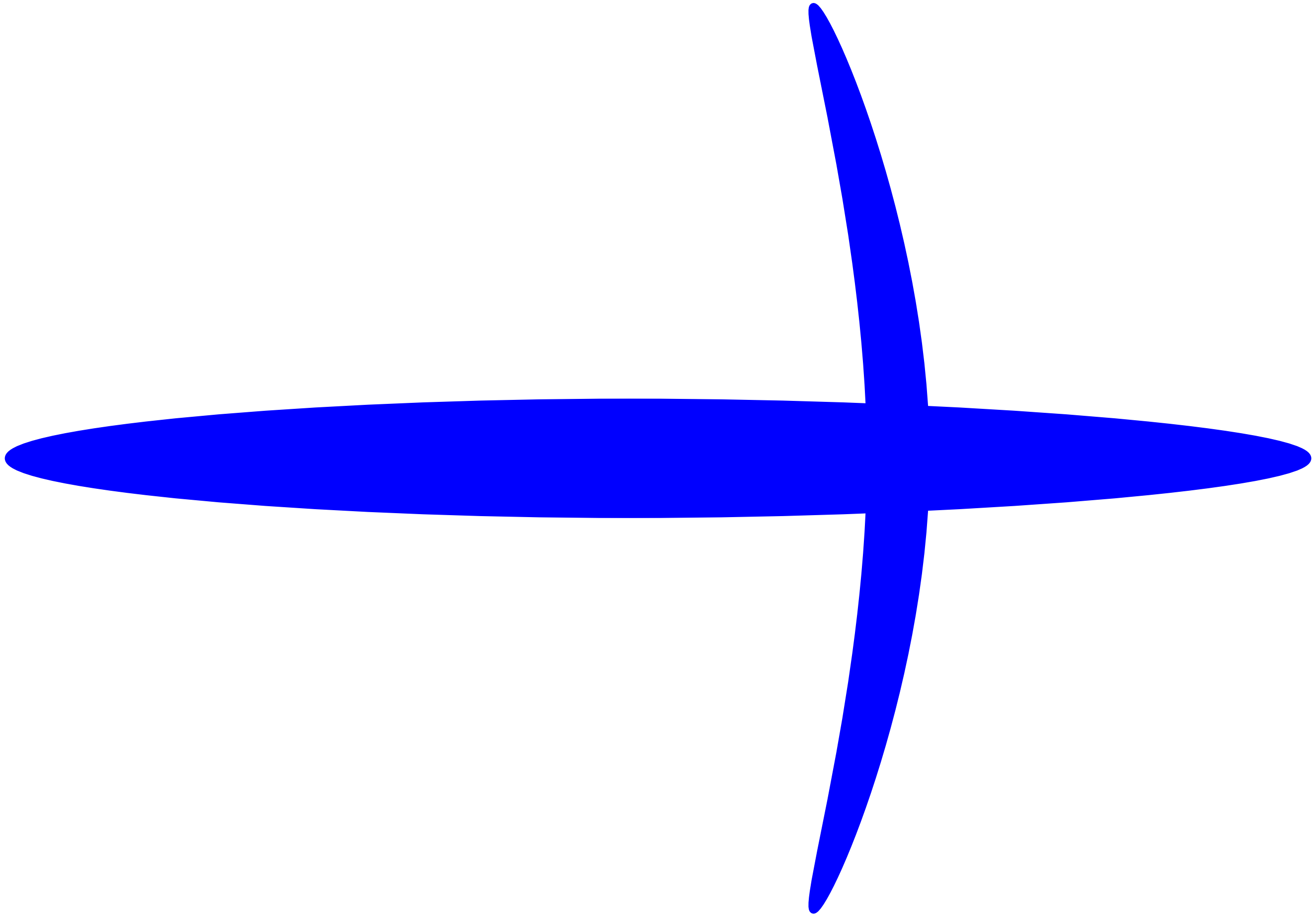
Delaunay Triangulation:

Teaser meshing lecture

Delaunay Triangulation:

Teaser meshing lecture

Shape



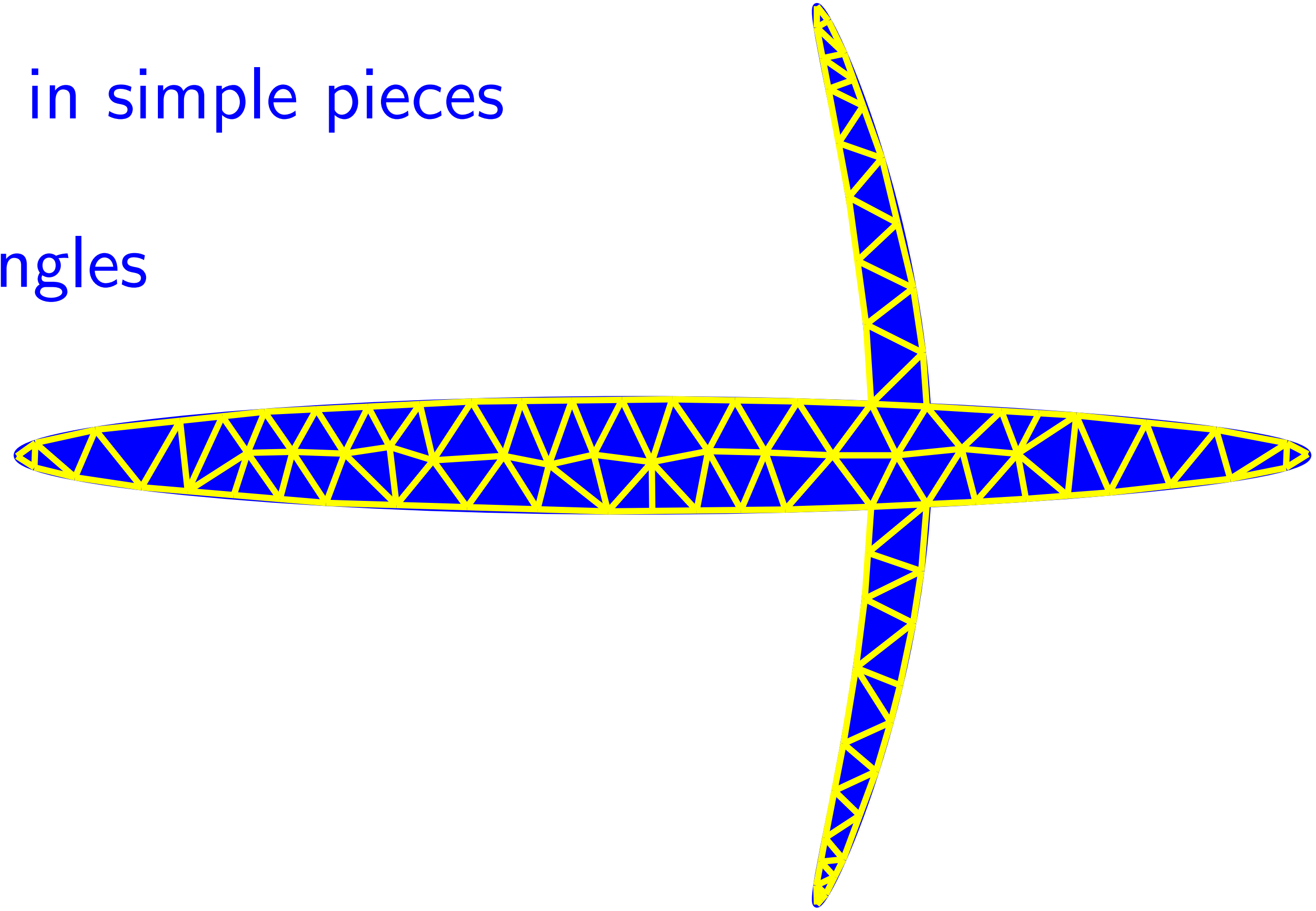
Delaunay Triangulation:

Teaser meshing lecture

Shape

Cut in simple pieces

triangles



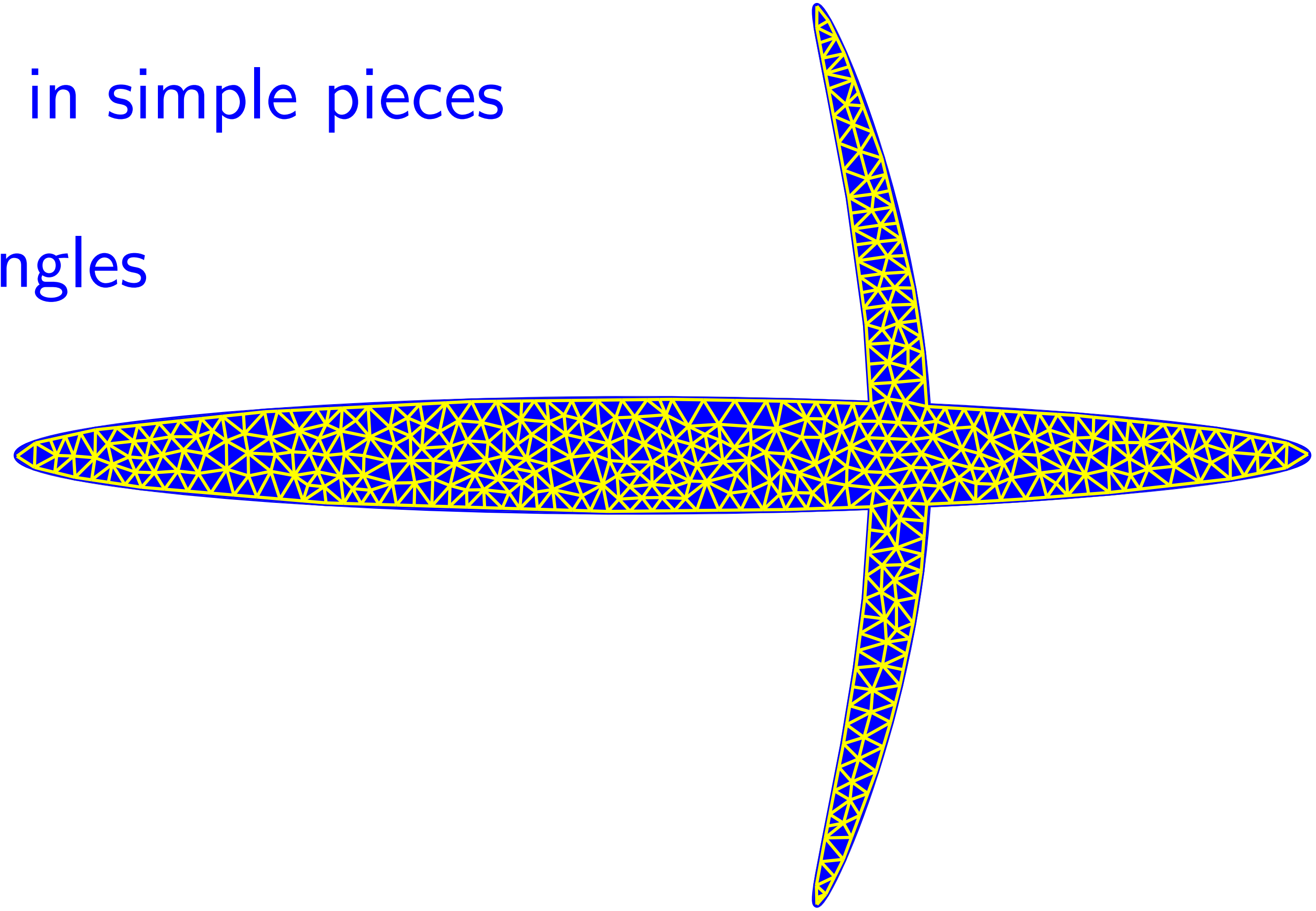
Delaunay Triangulation:

Teaser meshing lecture

Shape

Cut in simple pieces

triangles



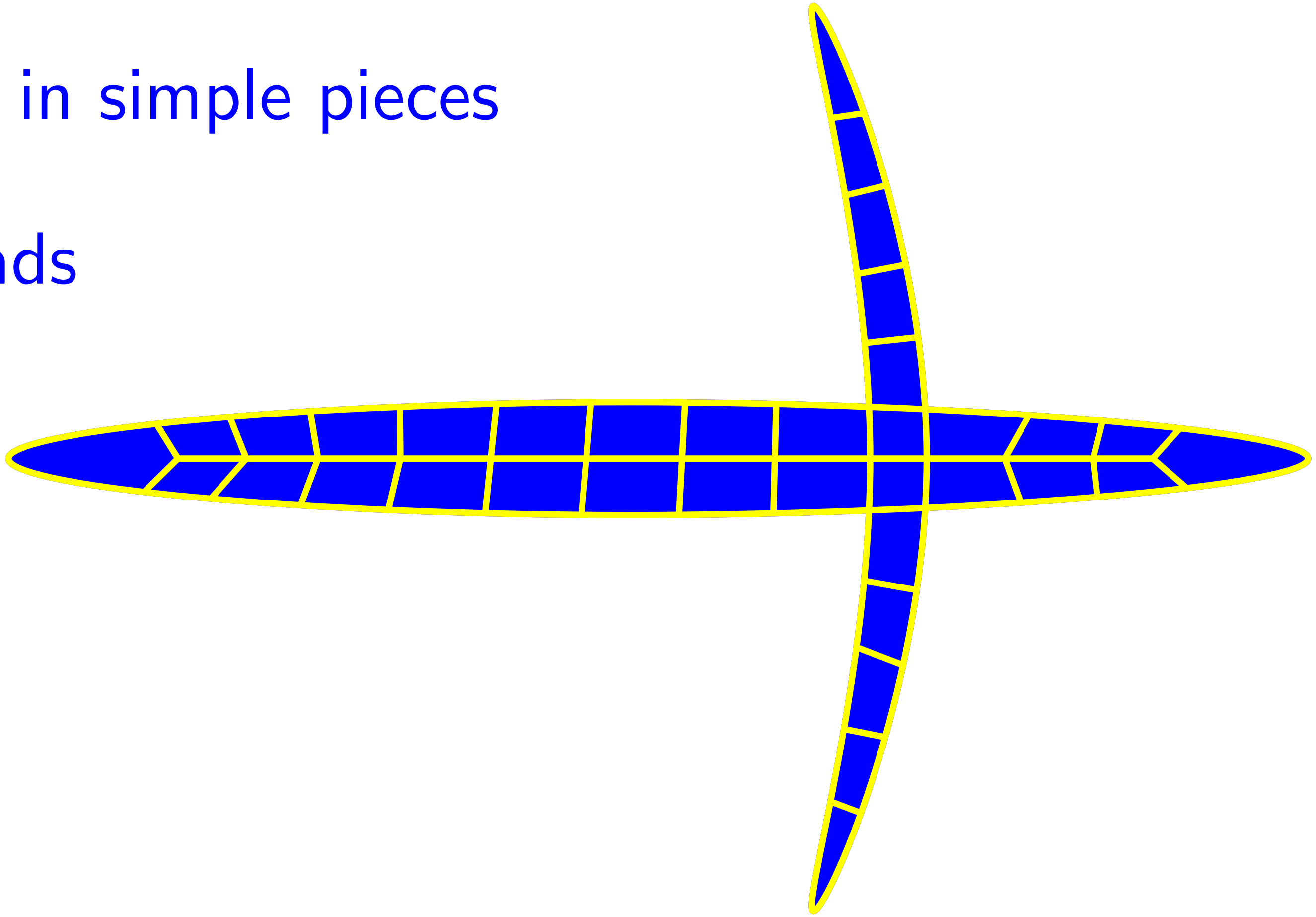
Delaunay Triangulation:

Teaser meshing lecture

Shape

Cut in simple pieces

Quads



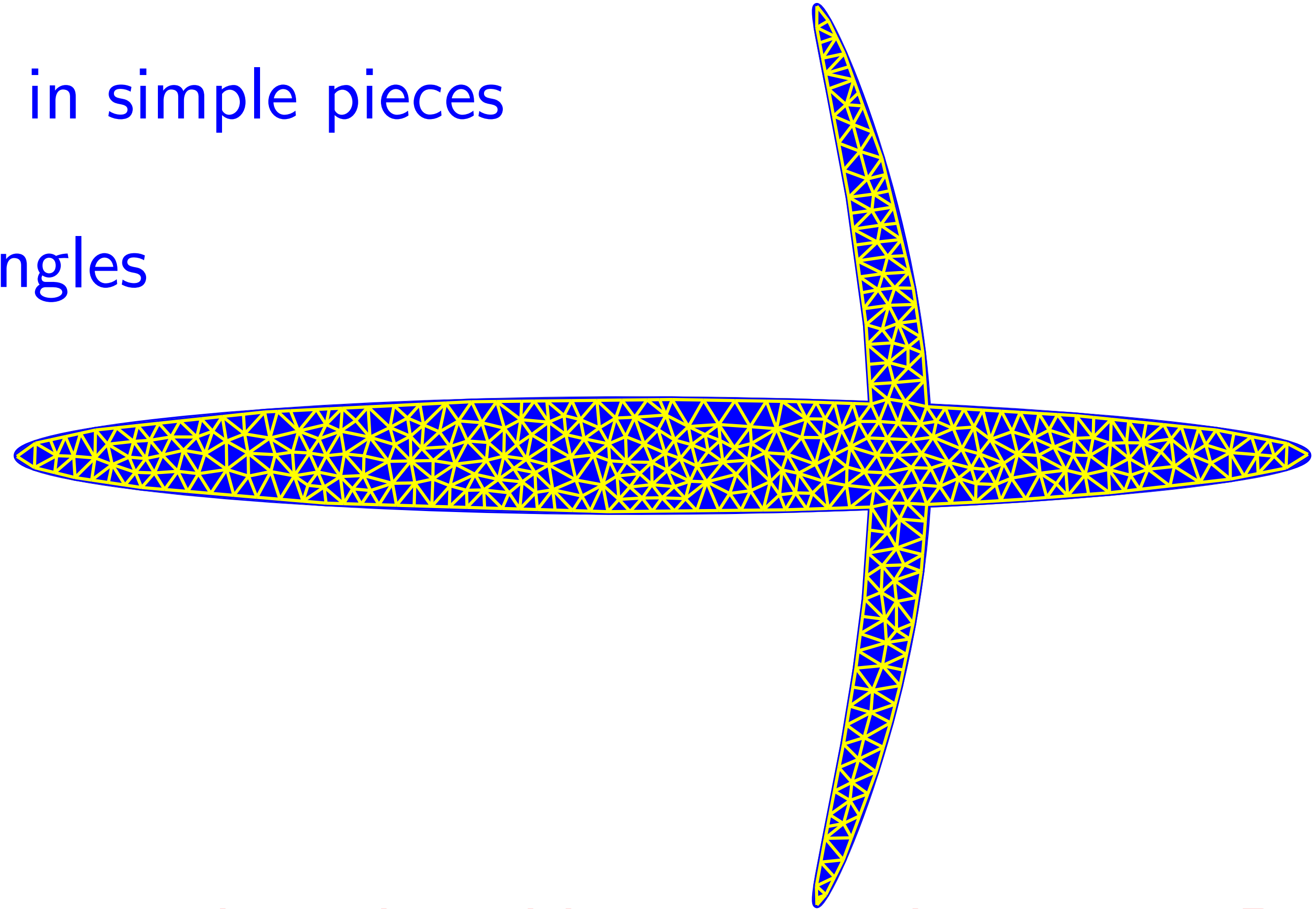
Delaunay Triangulation:

Teaser meshing lecture

Shape

Cut in simple pieces

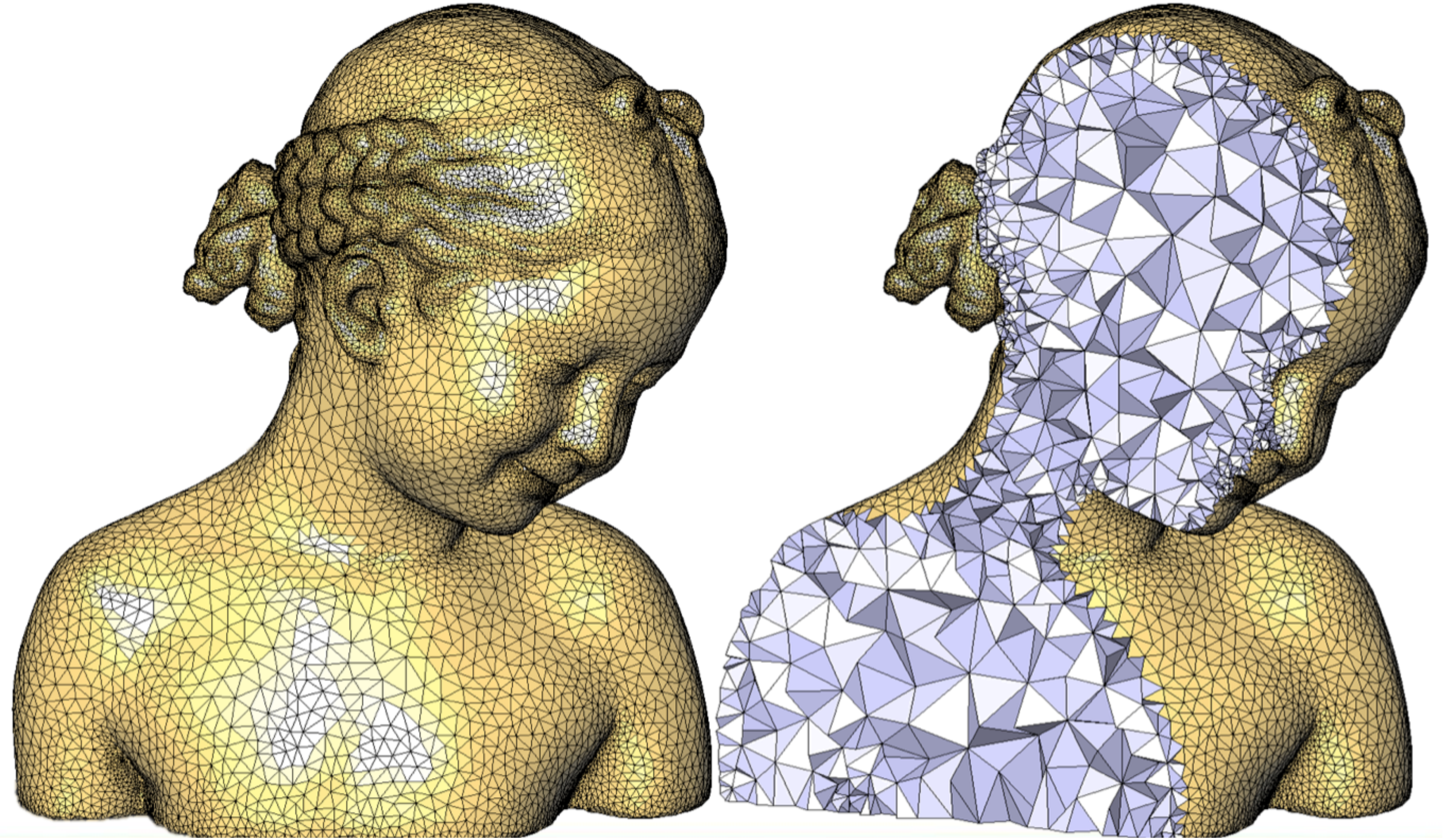
triangles



For unstructured mesh: add points and compute Delaunay

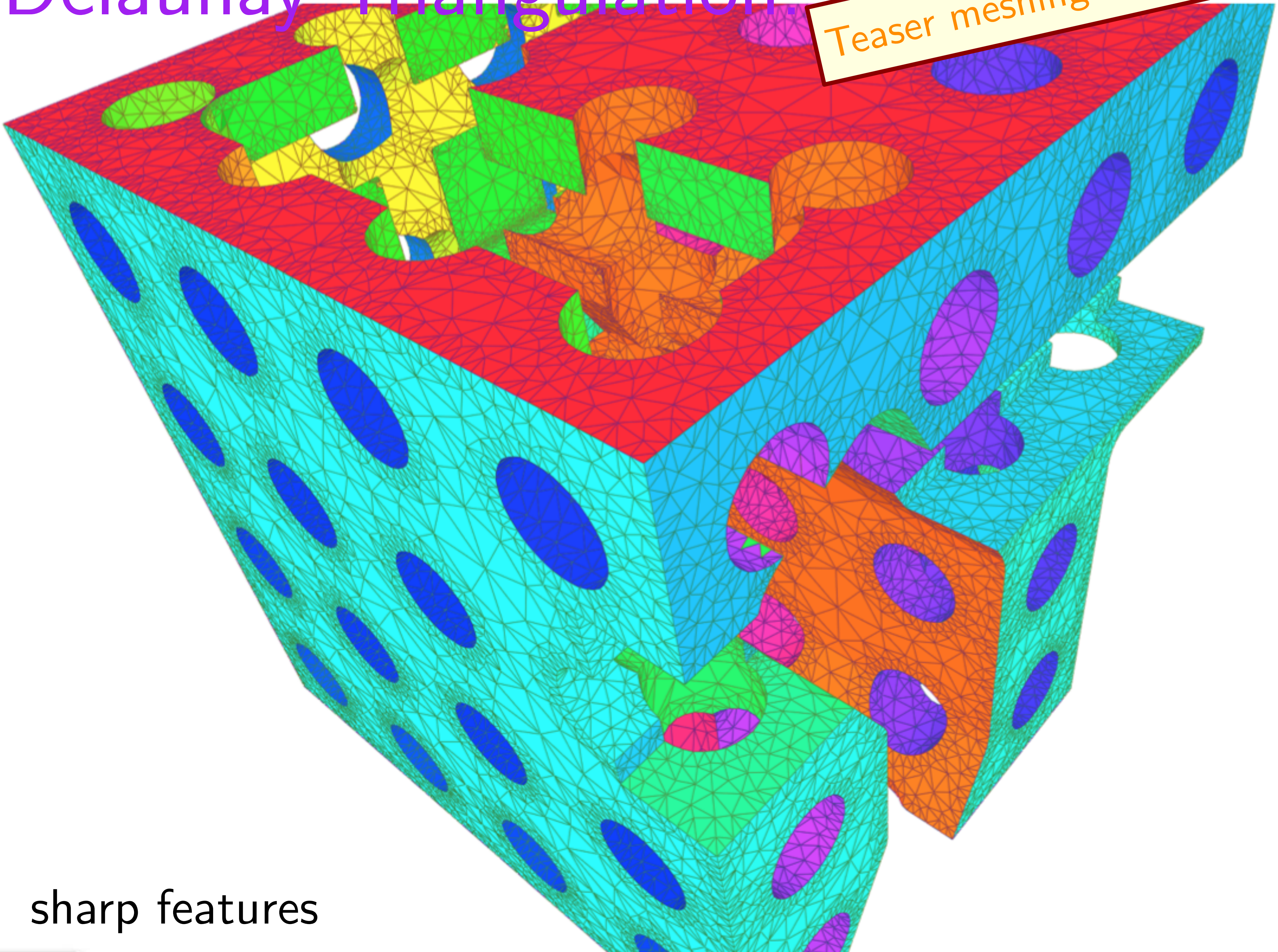
Delaunay Triangulation:

Teaser meshing lecture



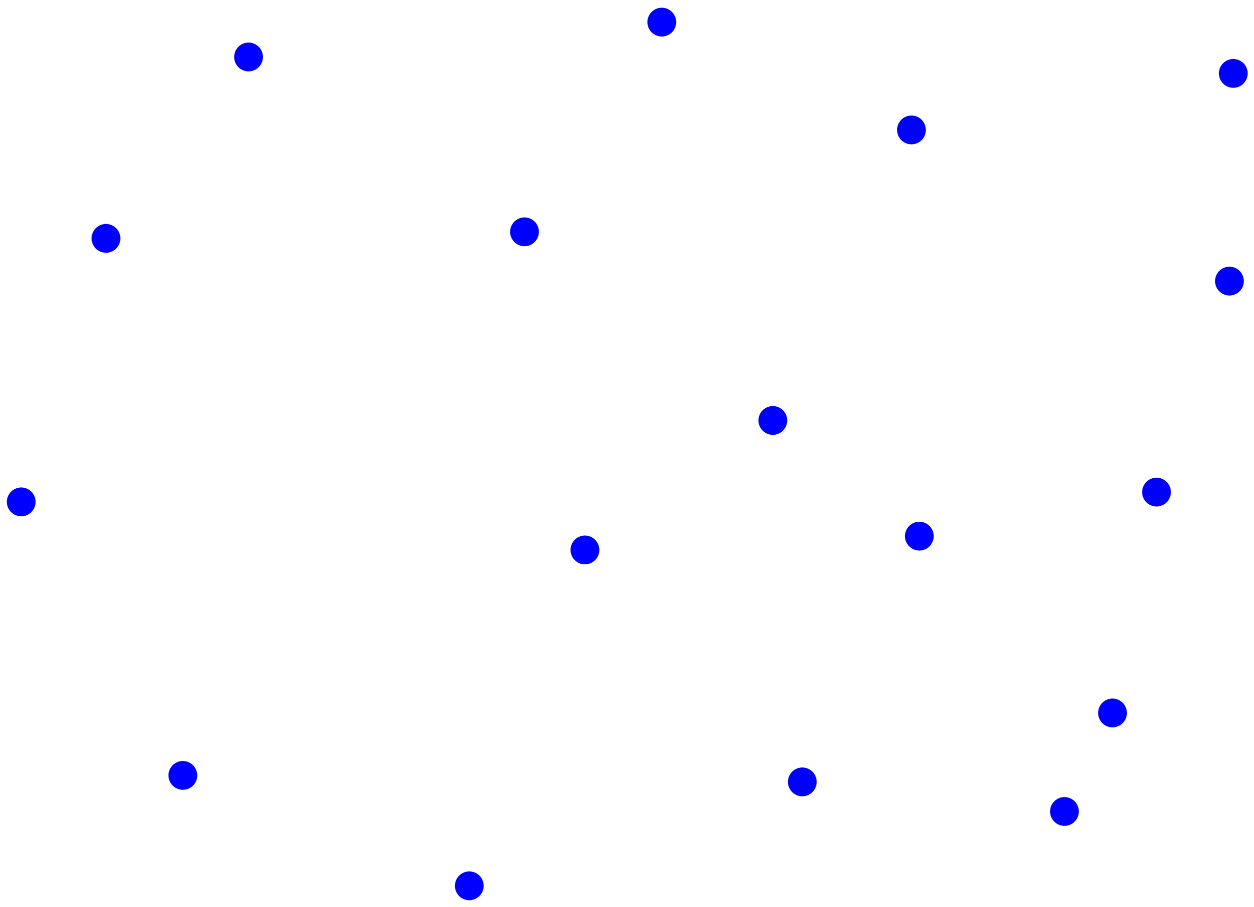
Delaunay Triangulation:

Teaser meshing lecture



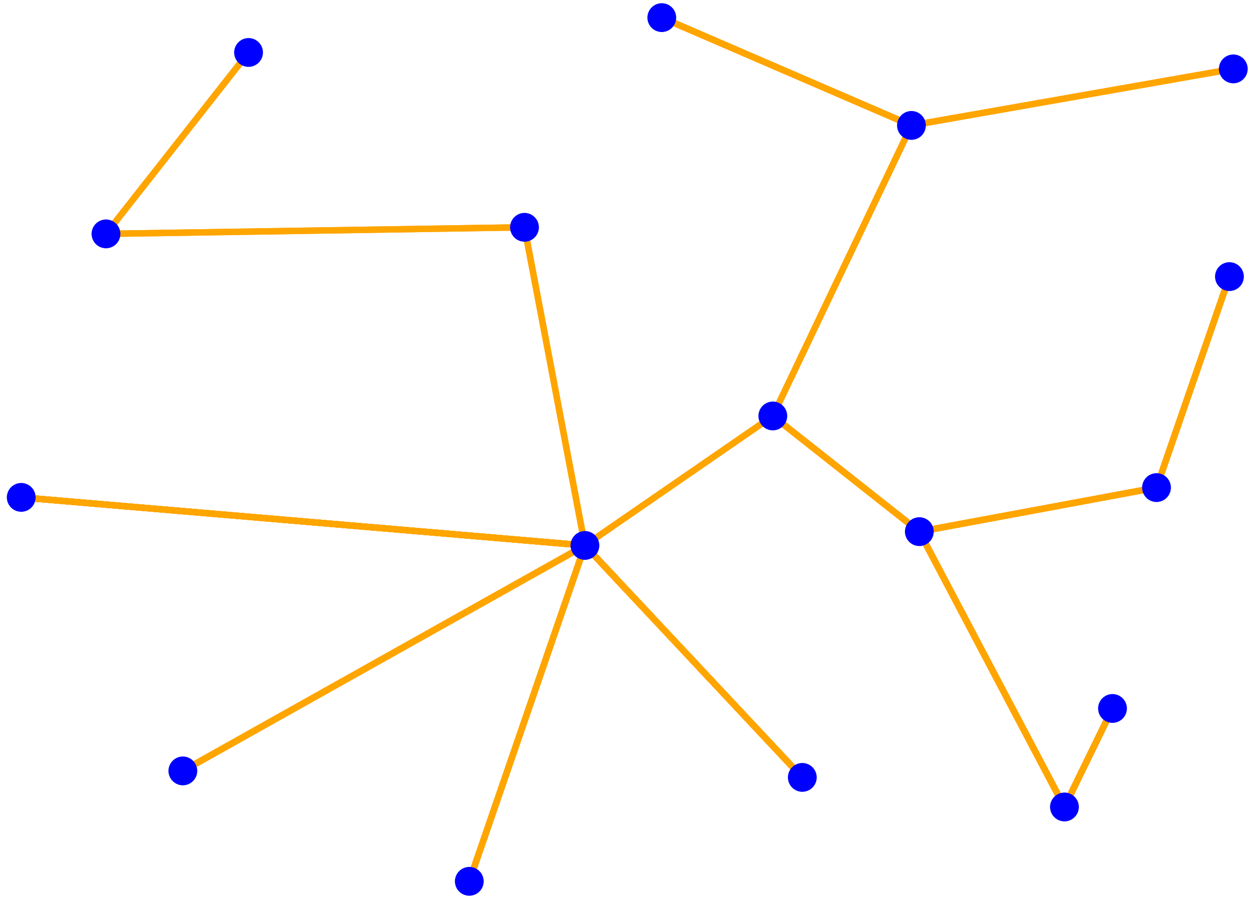
sharp features

Delaunay Triangulation: EMST



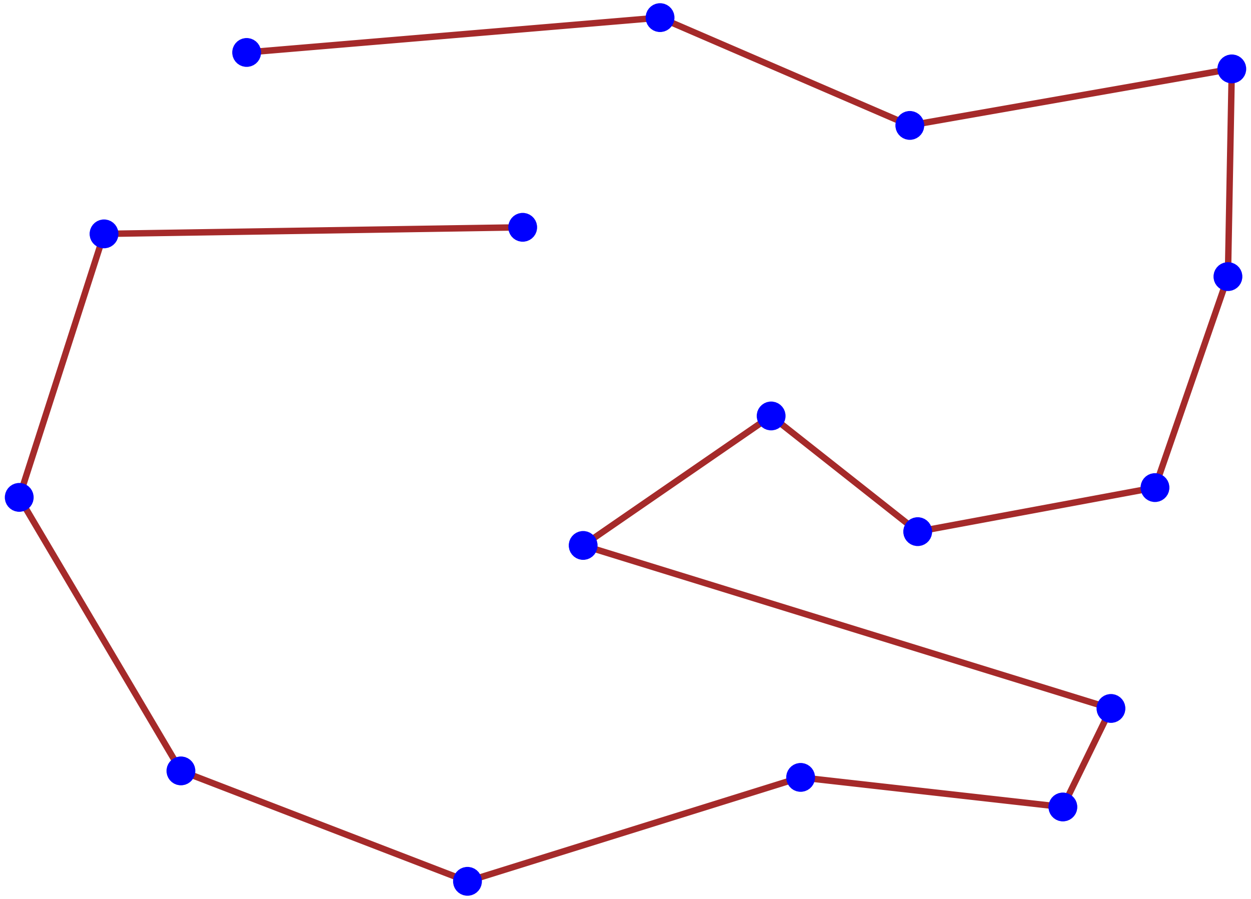
Delaunay Triangulation: EMST

A spanning tree



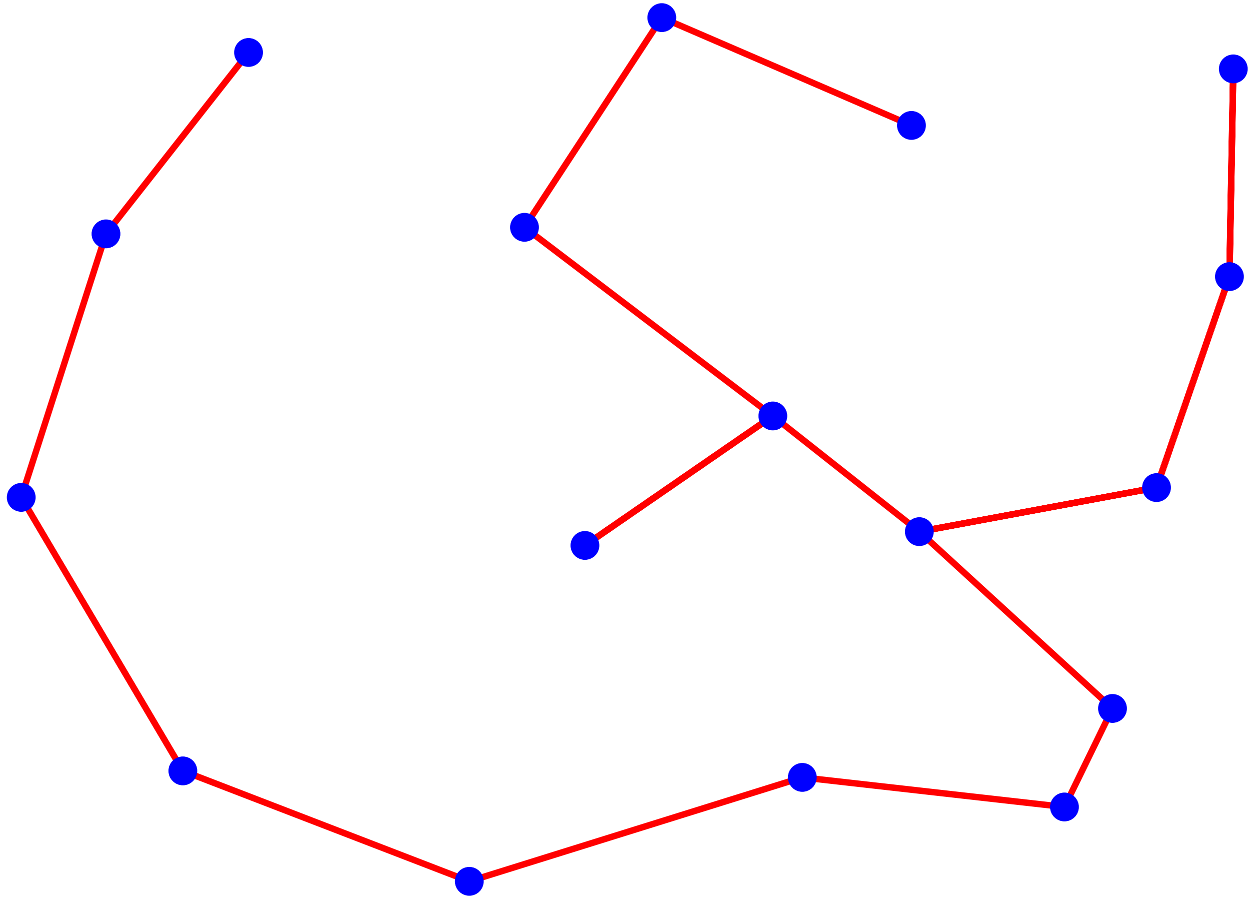
Delaunay Triangulation: EMST

Another spanning tree



Delaunay Triangulation: EMST

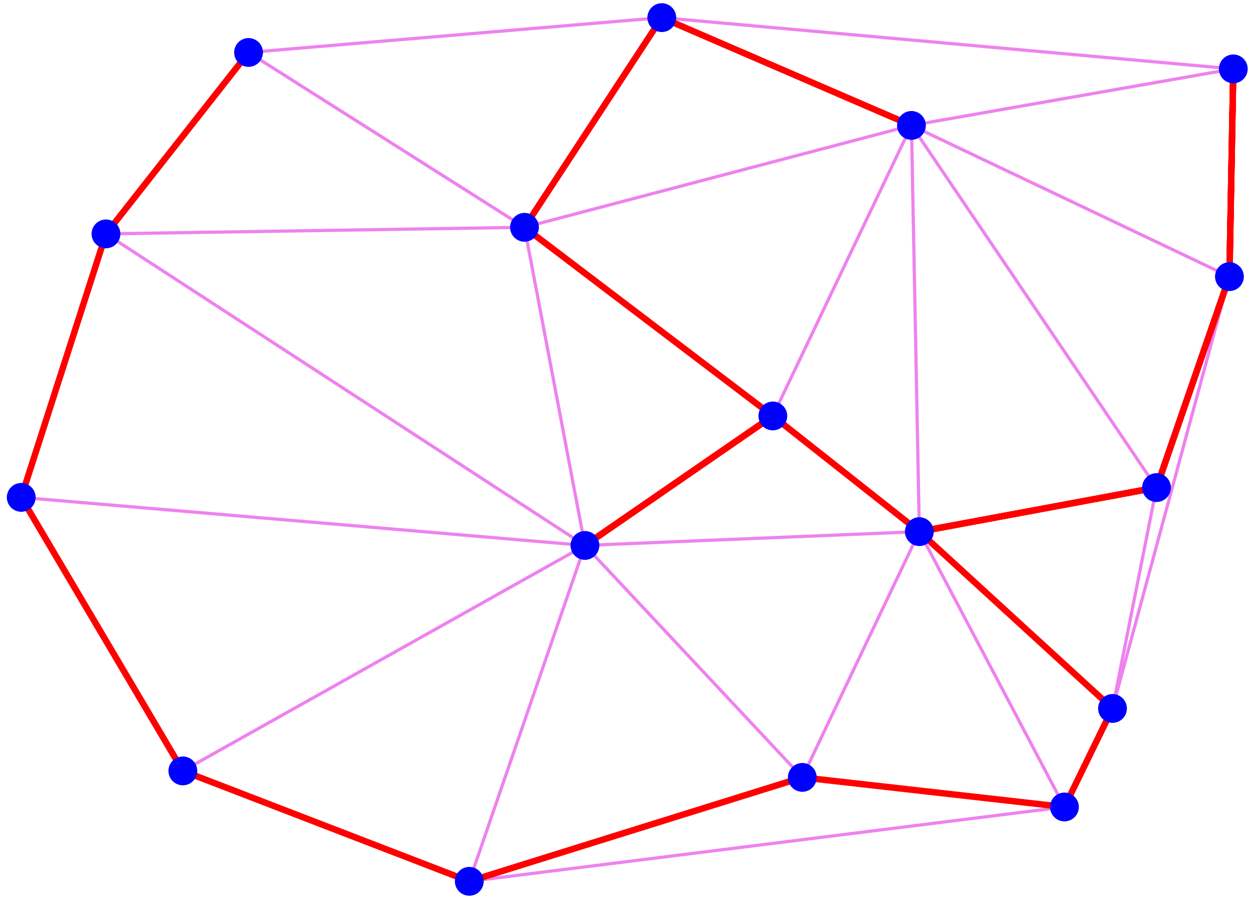
The Euclidean Minimum-length Spanning Tree



Delaunay Triangulation: EMST

The Euclidean Minimum-length Spanning Tree

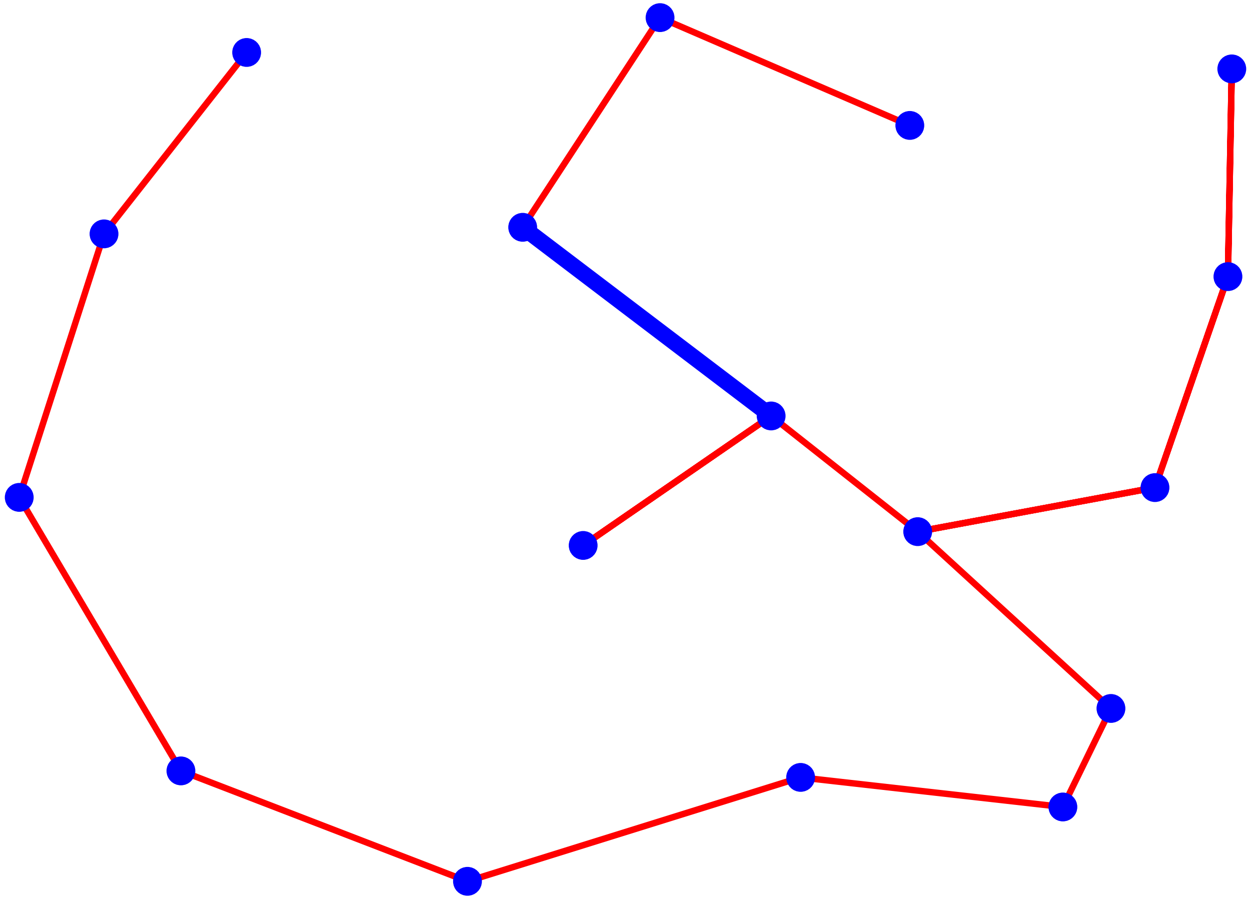
is included in Delaunay



Delaunay Triangulation: EMST

The Euclidean Minimum-length Spanning Tree
is included in Delaunay

Proof:

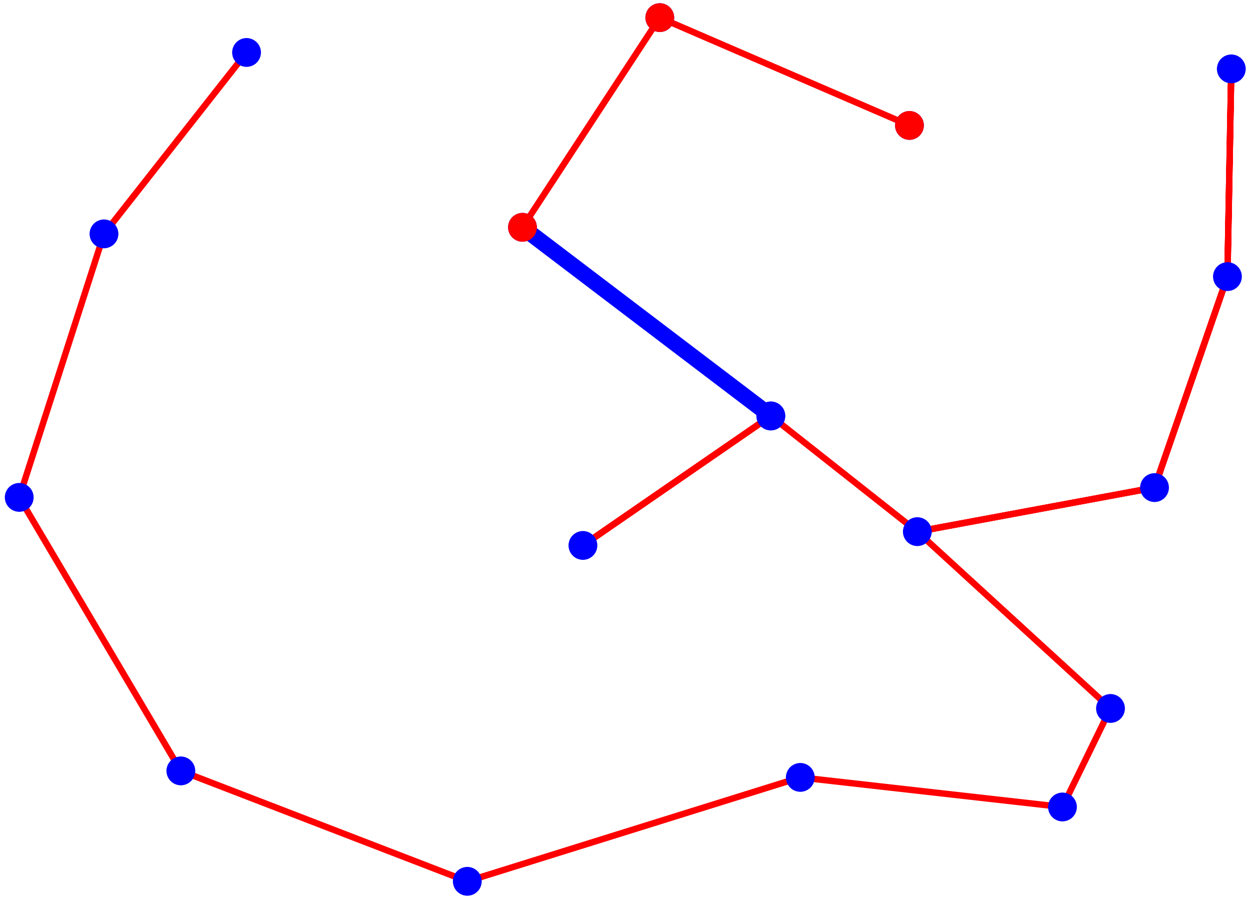


Choose an edge of EMST

Delaunay Triangulation: EMST

The Euclidean Minimum-length Spanning Tree

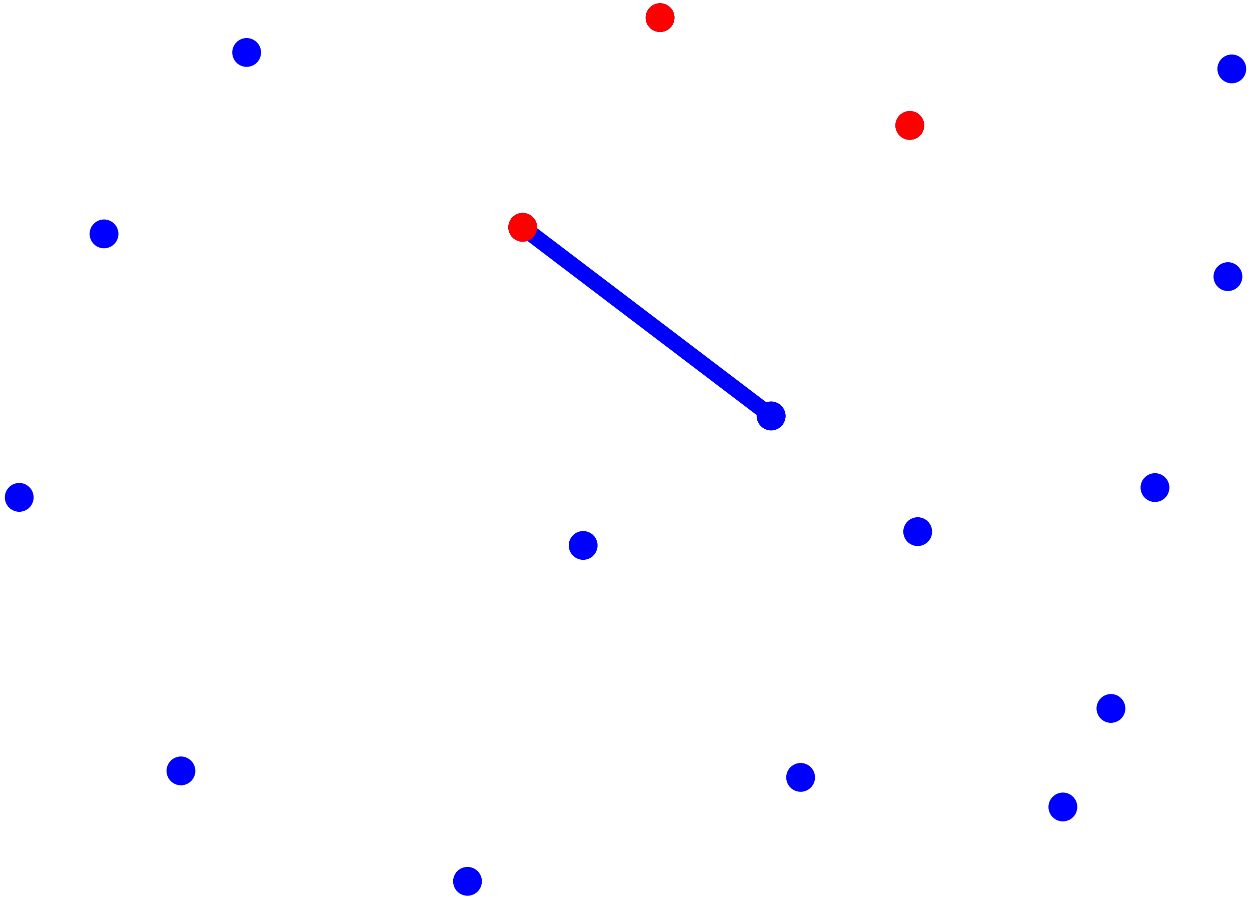
Proof:



Split points

Delaunay Triangulation: EMST

Proof:

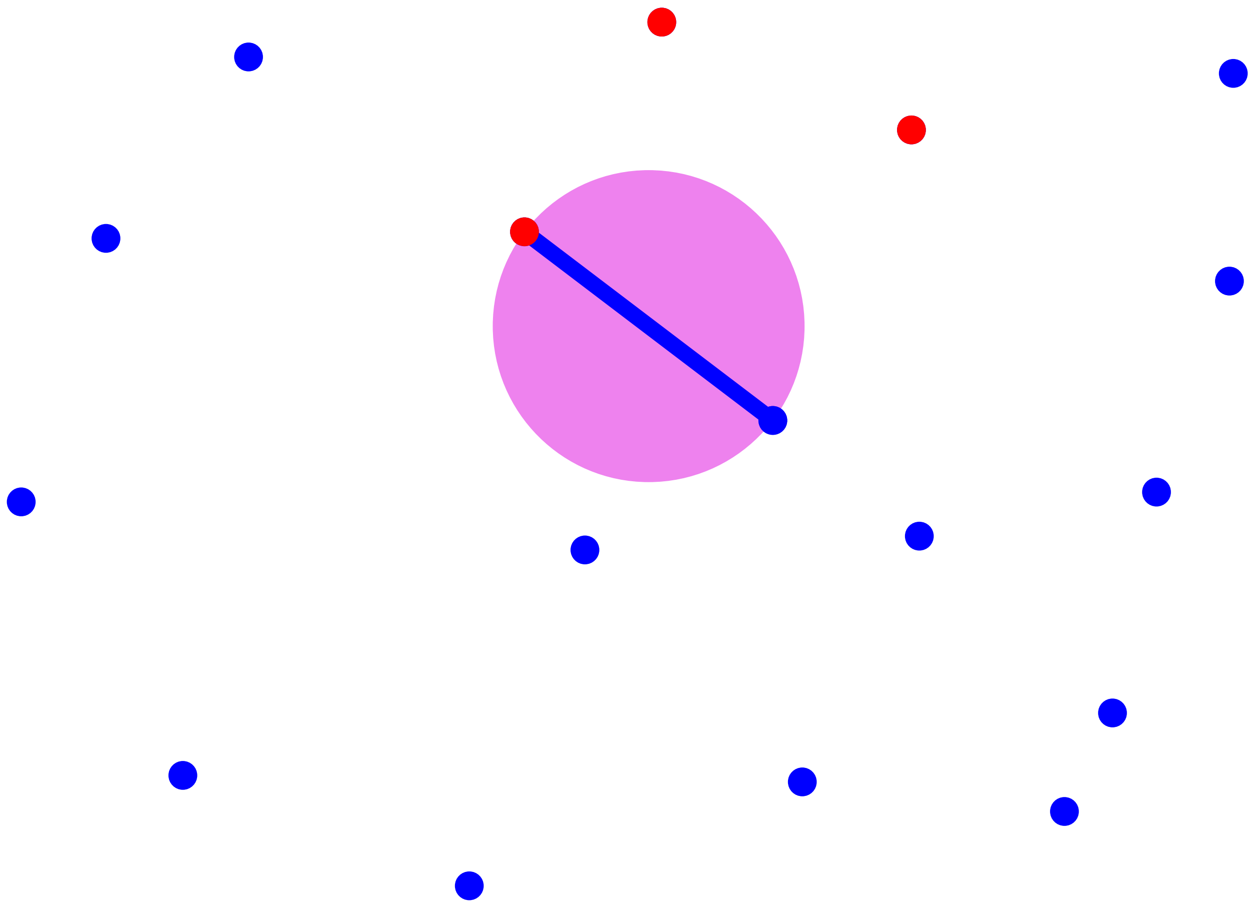


Split points

Delaunay Triangulation: EMST

Is diametral circle empty ?

Proof:



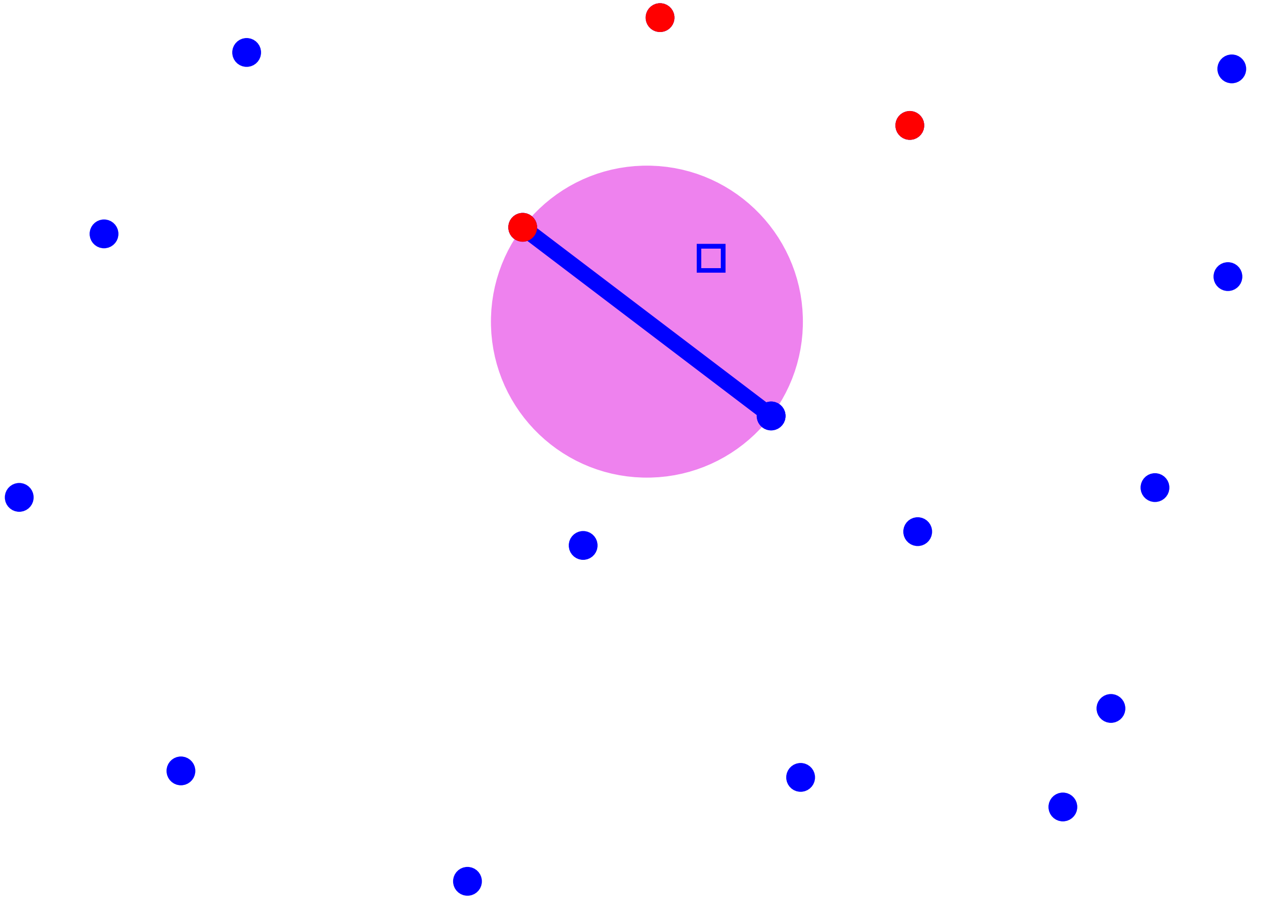
Split points

Delaunay Triangulation: EMST

Is diametral circle empty ?

assume ∃ blue point inside

Proof:



Split points

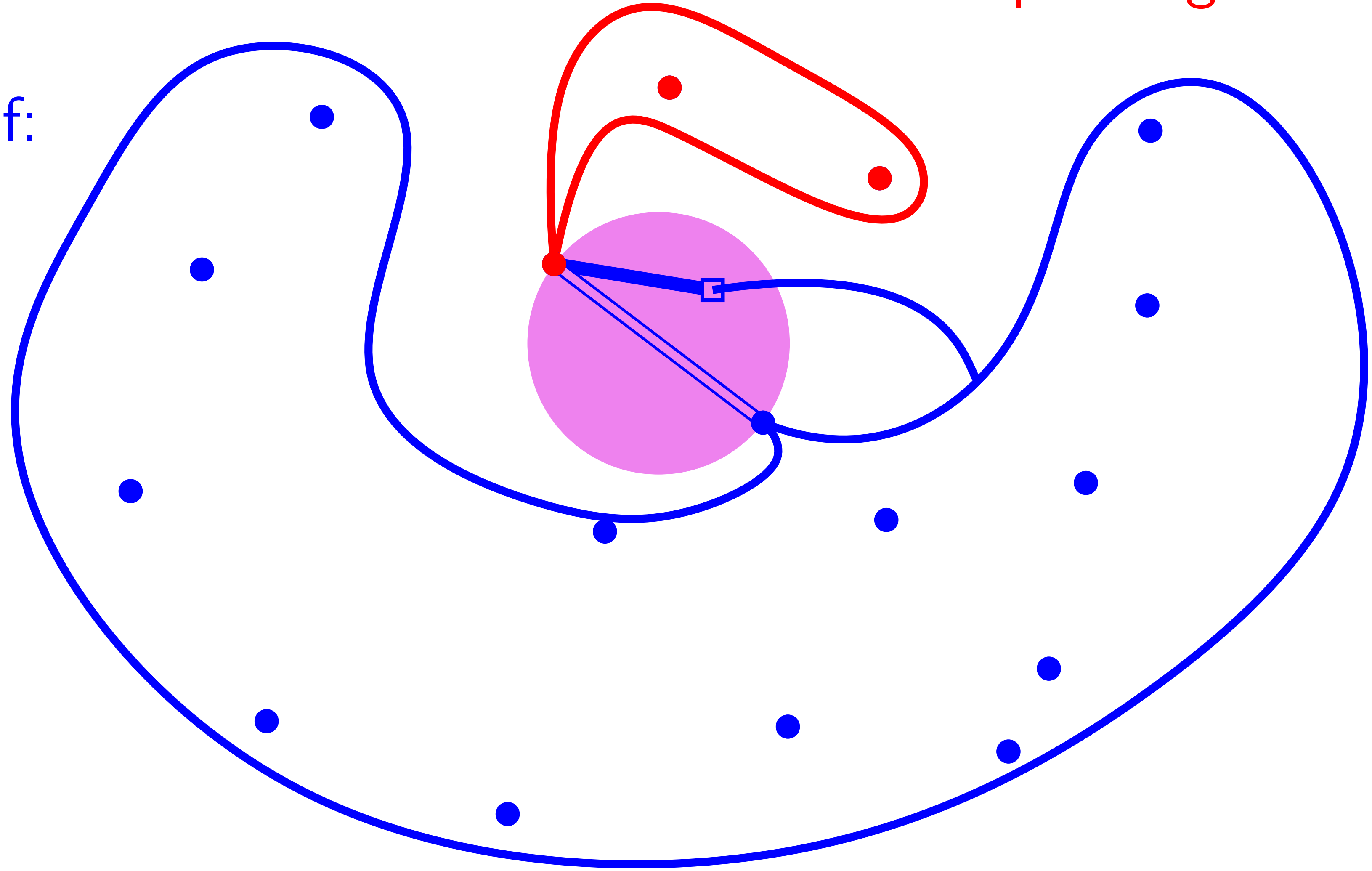
Delaunay Triangulation: EMST

Is diametral circle empty ?

assume ∃ blue point inside

better spanning tree

Proof:



Split points

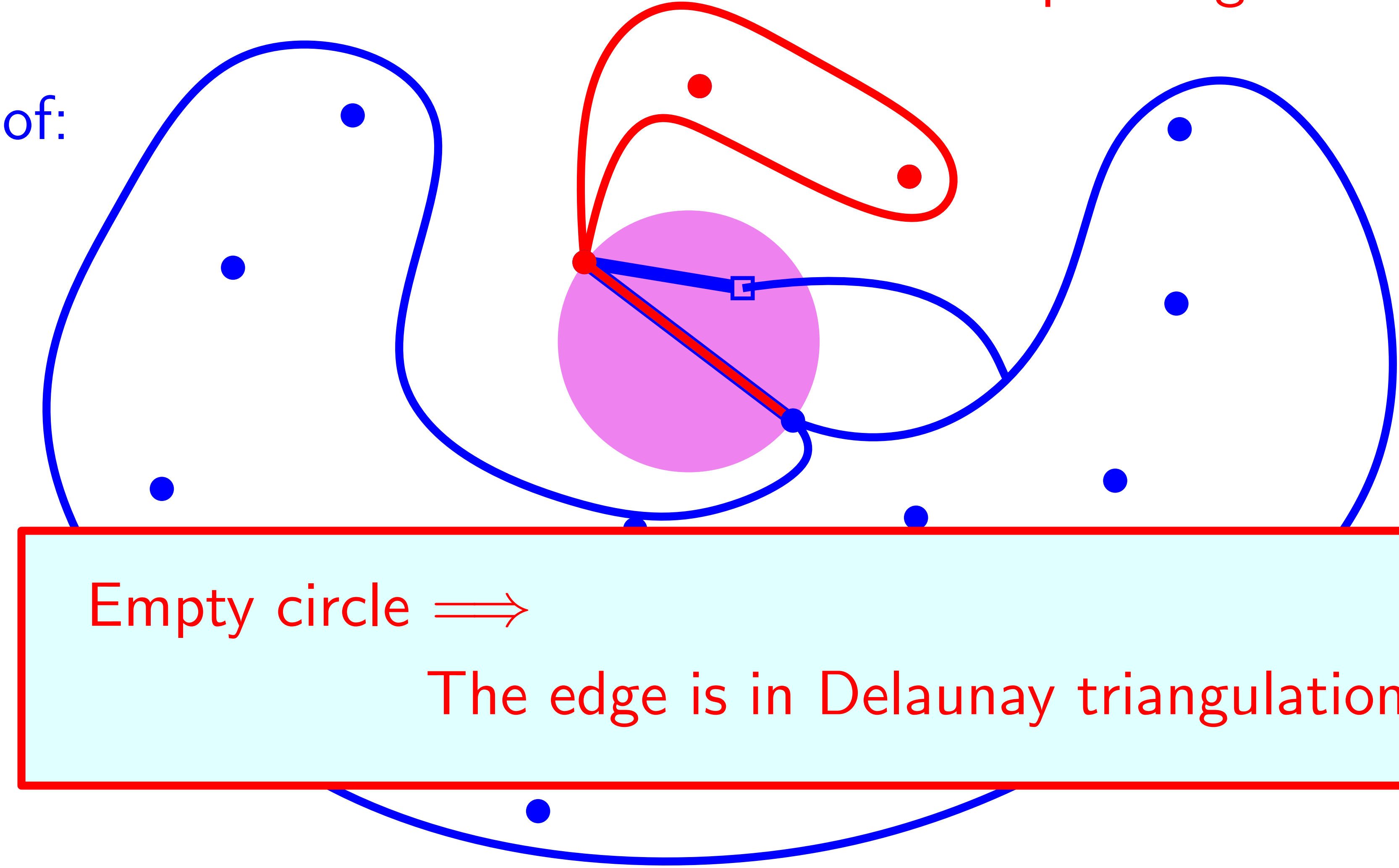
Delaunay Triangulation: EMST

Is diametral circle empty ?

assume ∃ blue point inside

better spanning tree

Proof:



Empty circle ⟹

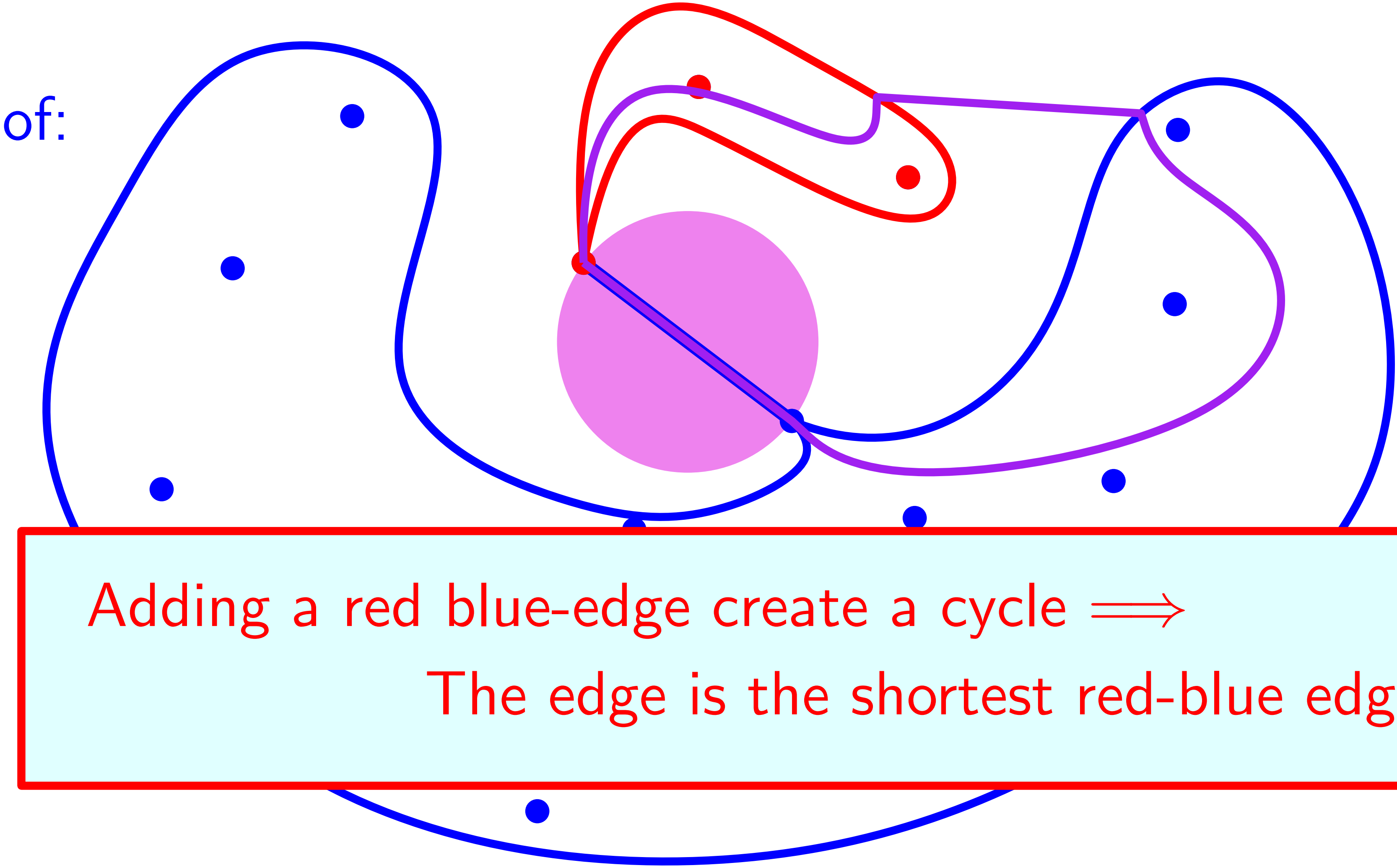
The edge is in Delaunay triangulation

Split points

Delaunay Triangulation: EMST

Is diametral circle empty ?

Proof:



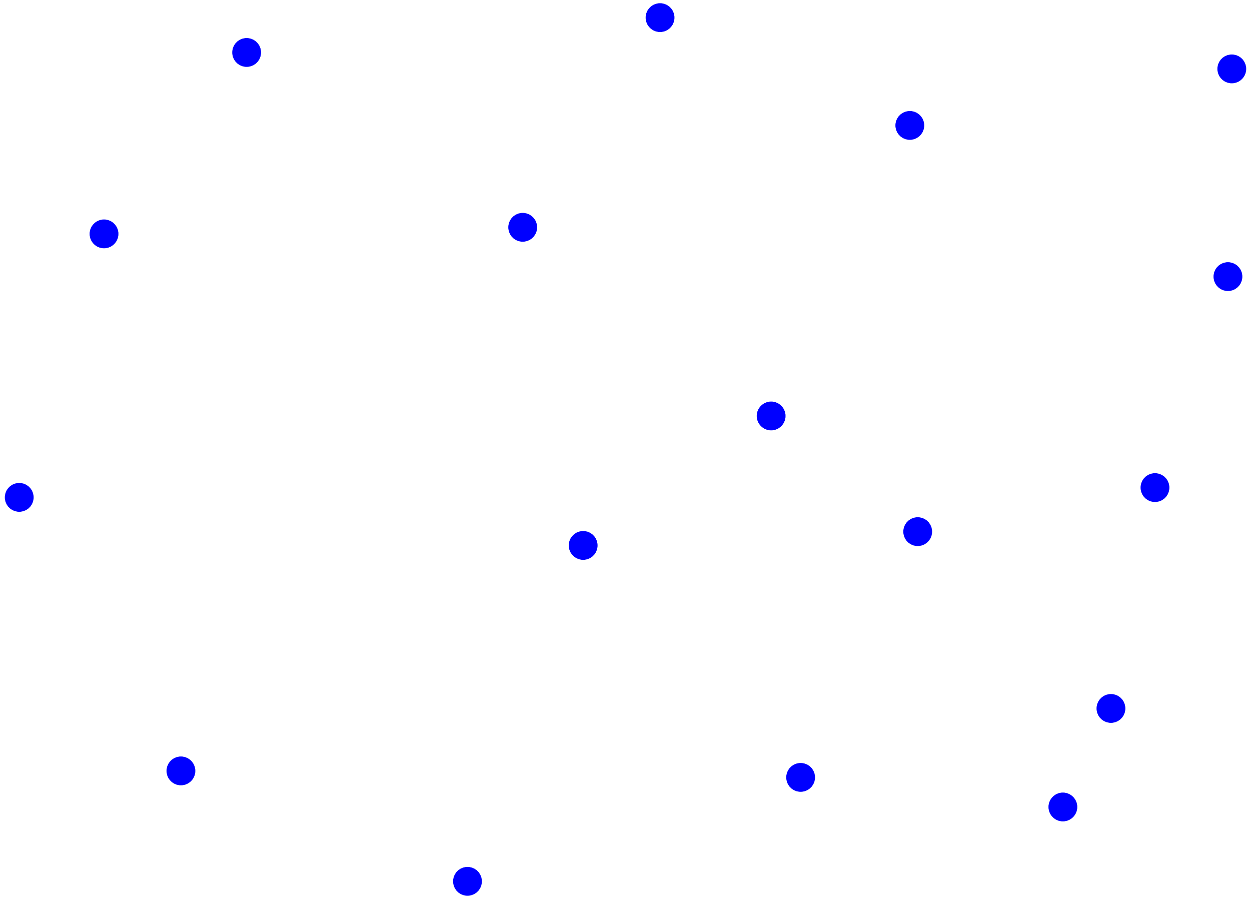
Adding a red blue-edge create a cycle $\Longrightarrow$

The edge is the shortest red-blue edge

Split points

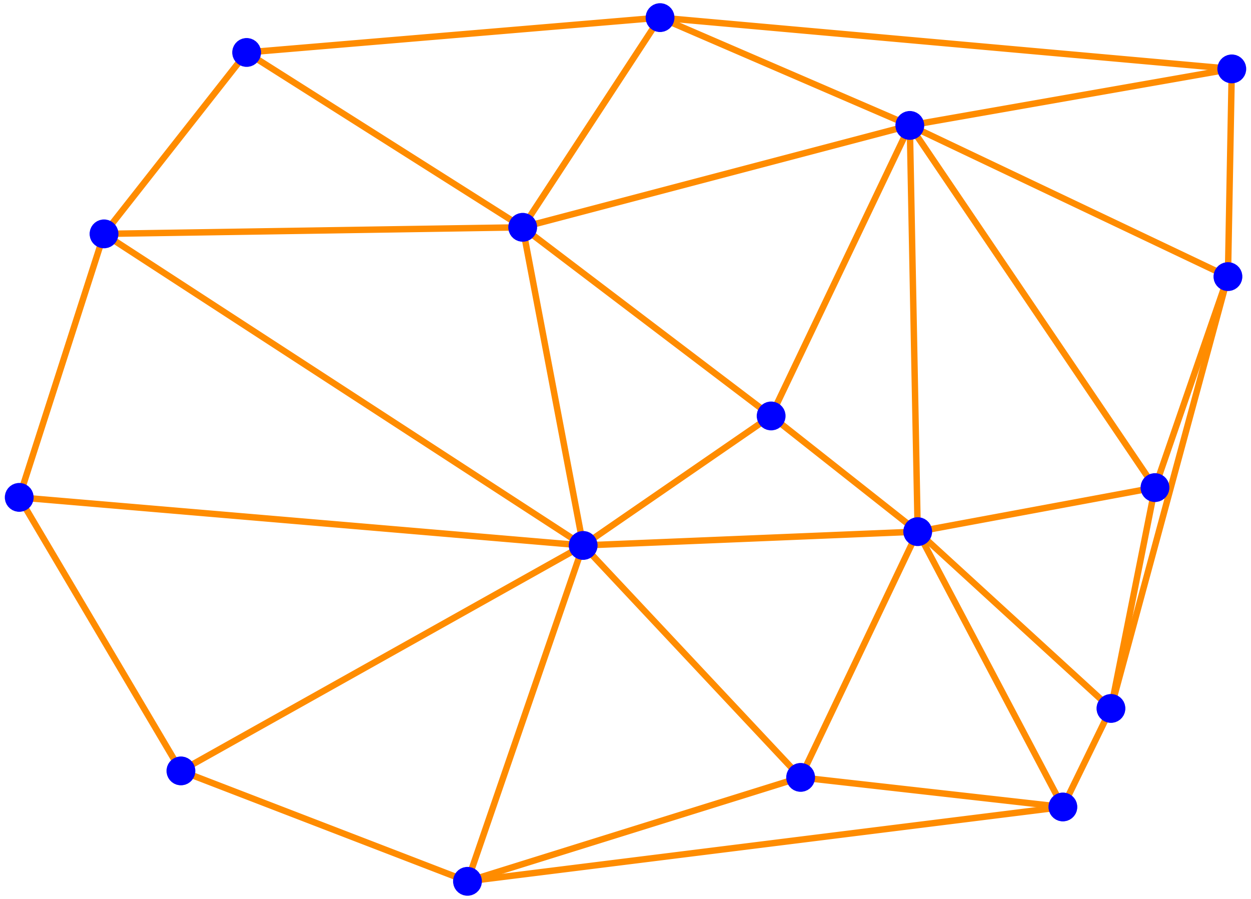
Delaunay Triangulation: EMST

Algorithm



Delaunay Triangulation: EMST

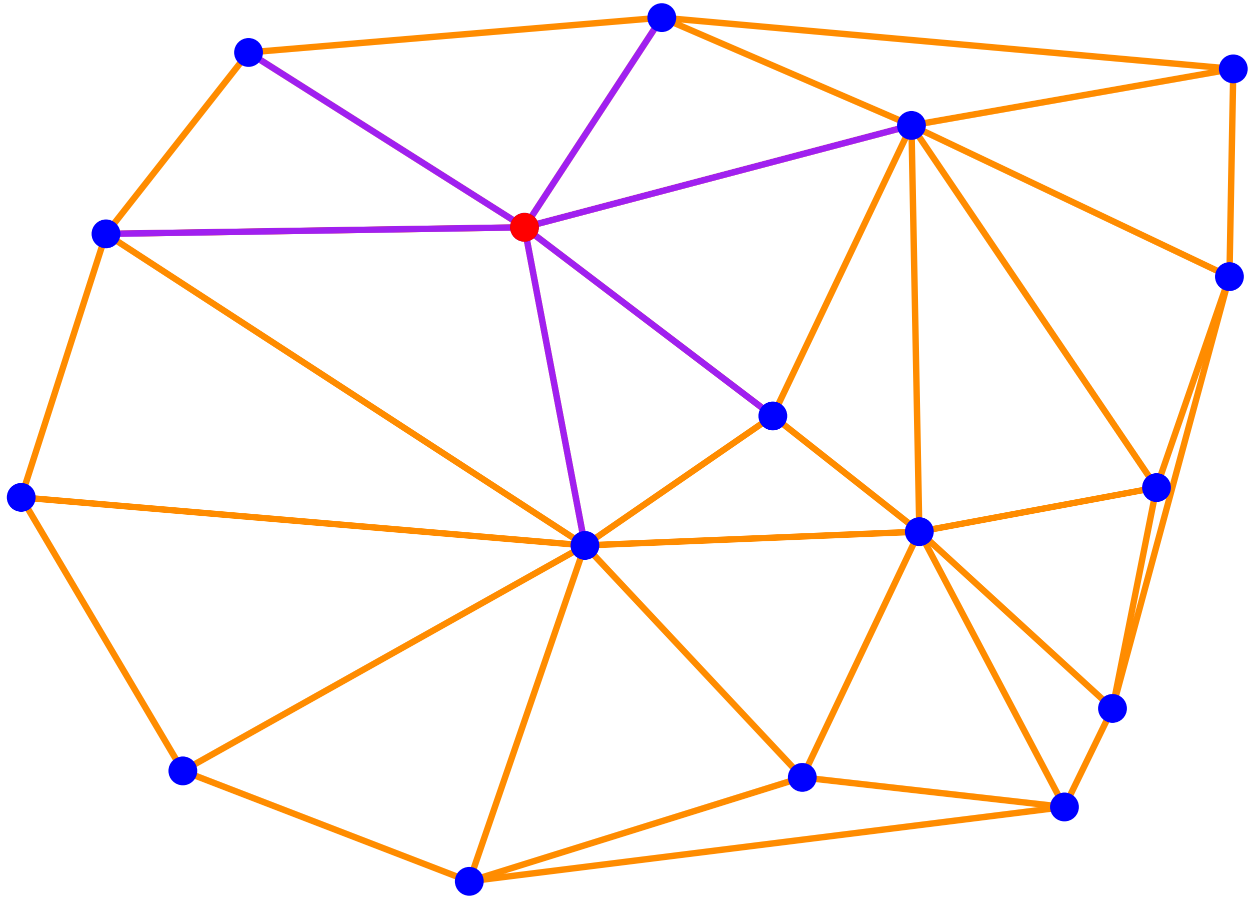
Algorithm



Delaunay Triangulation: EMST

Algorithm

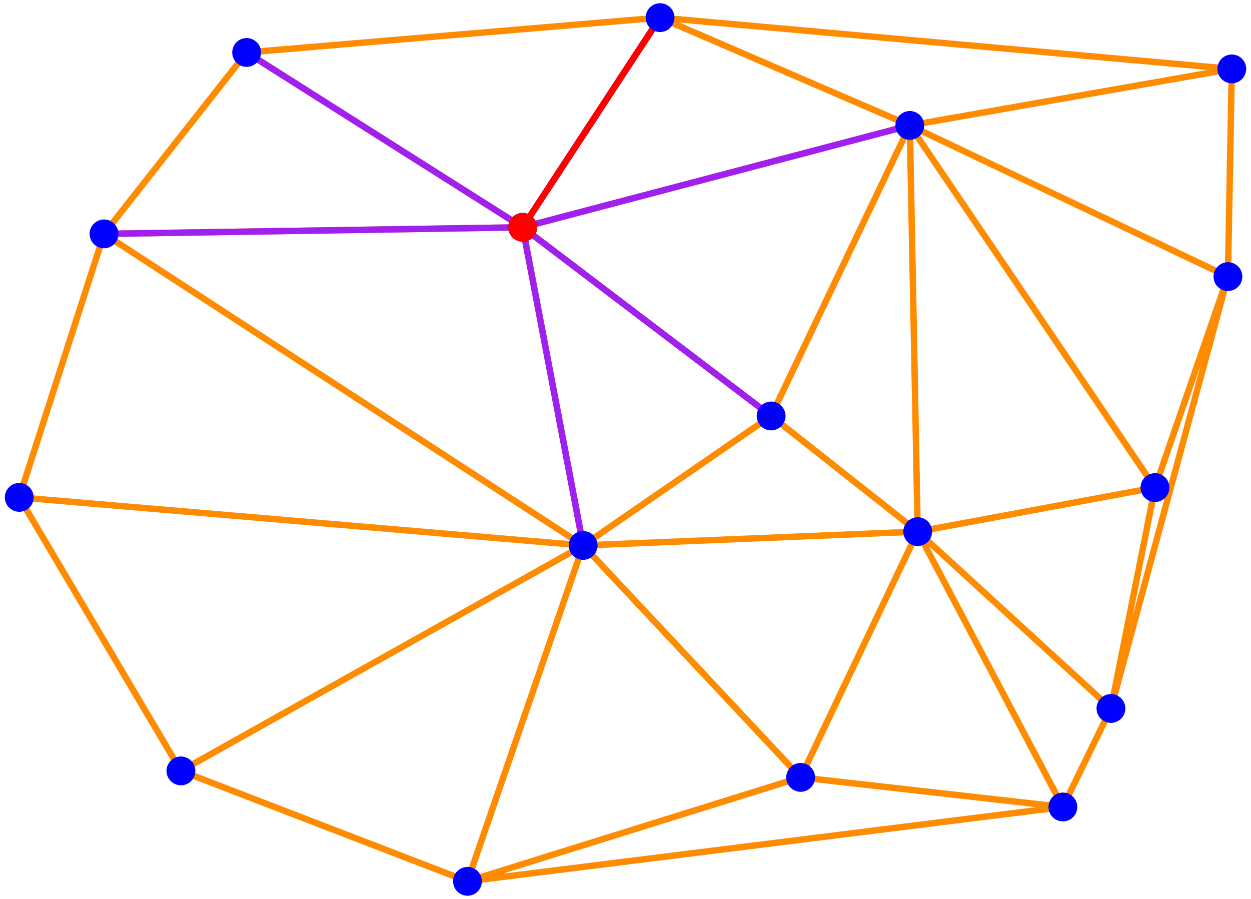
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

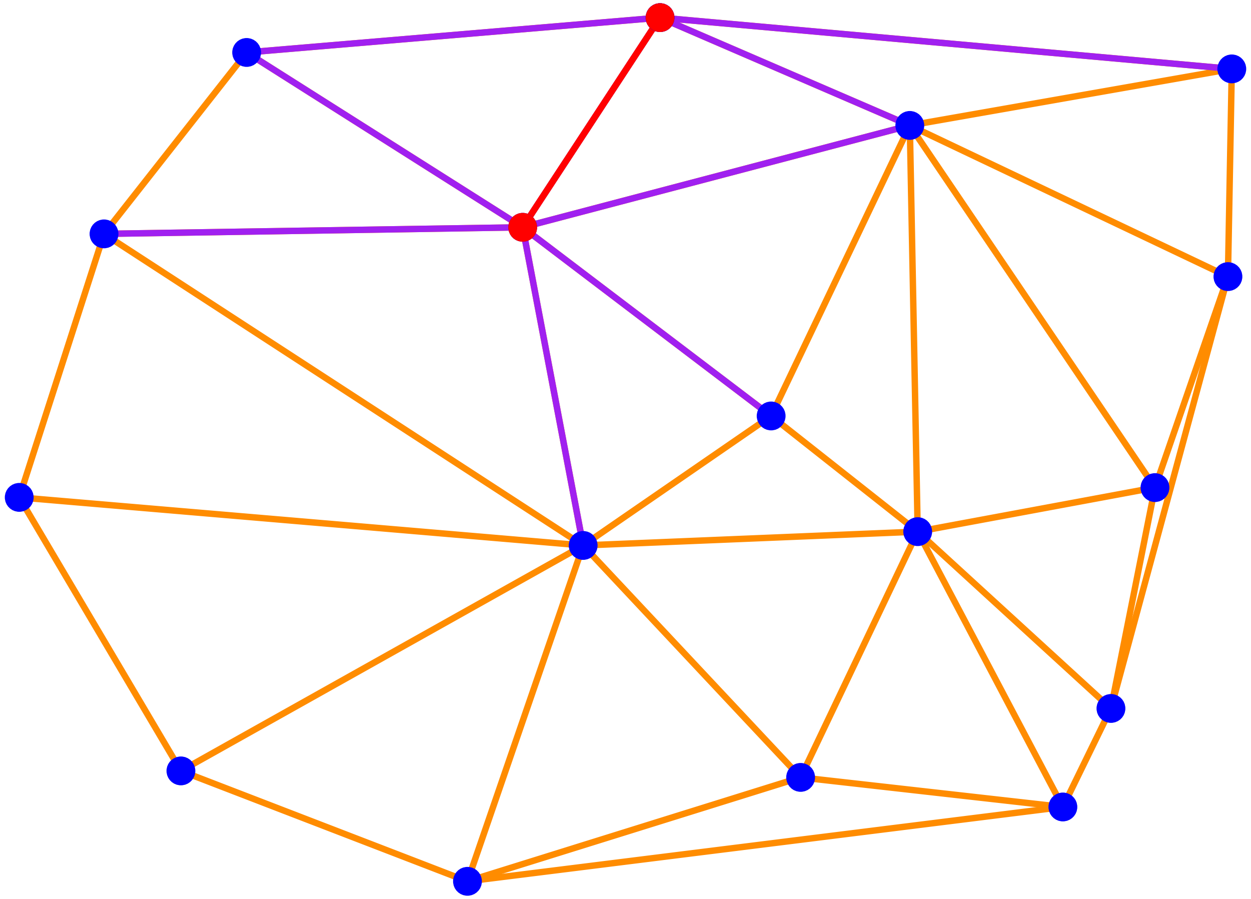
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

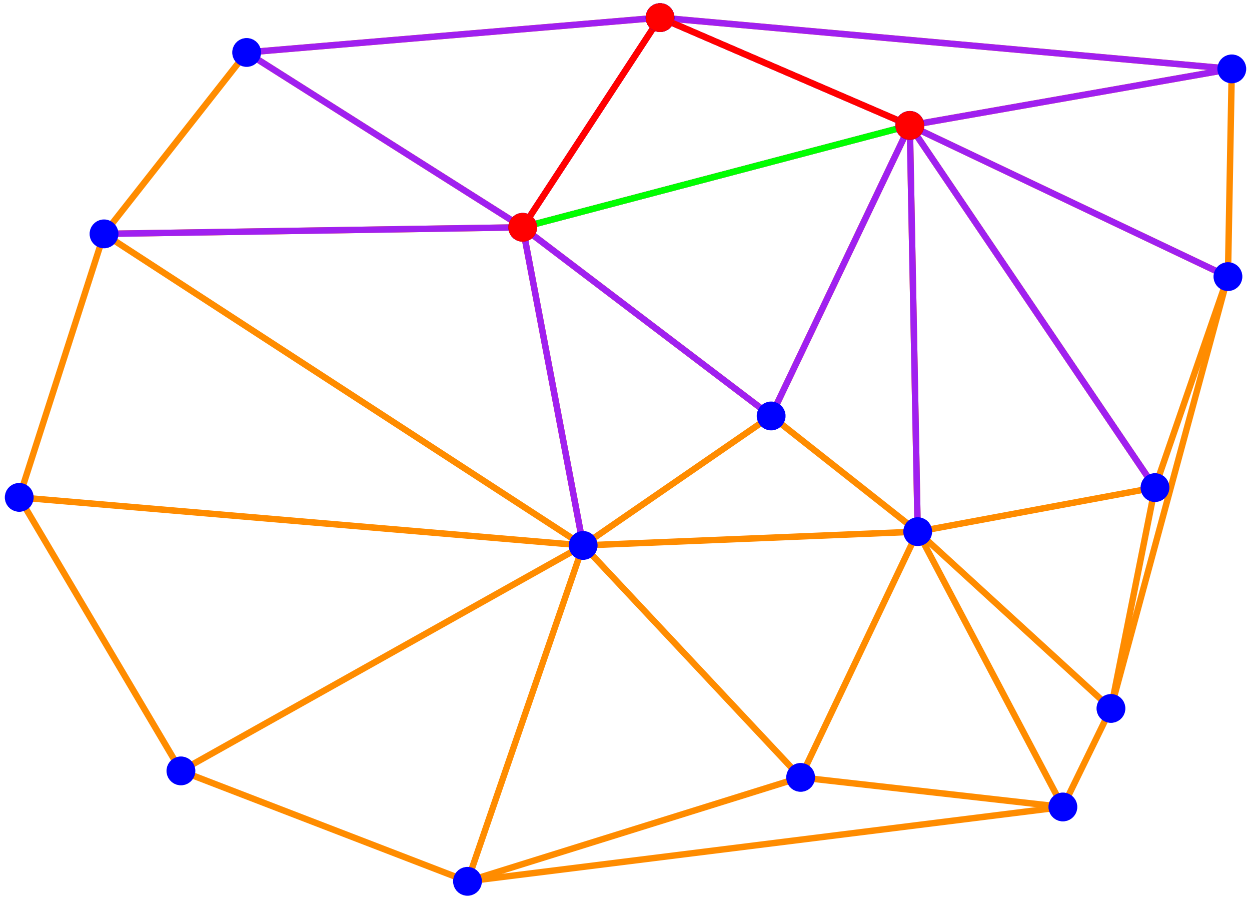
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

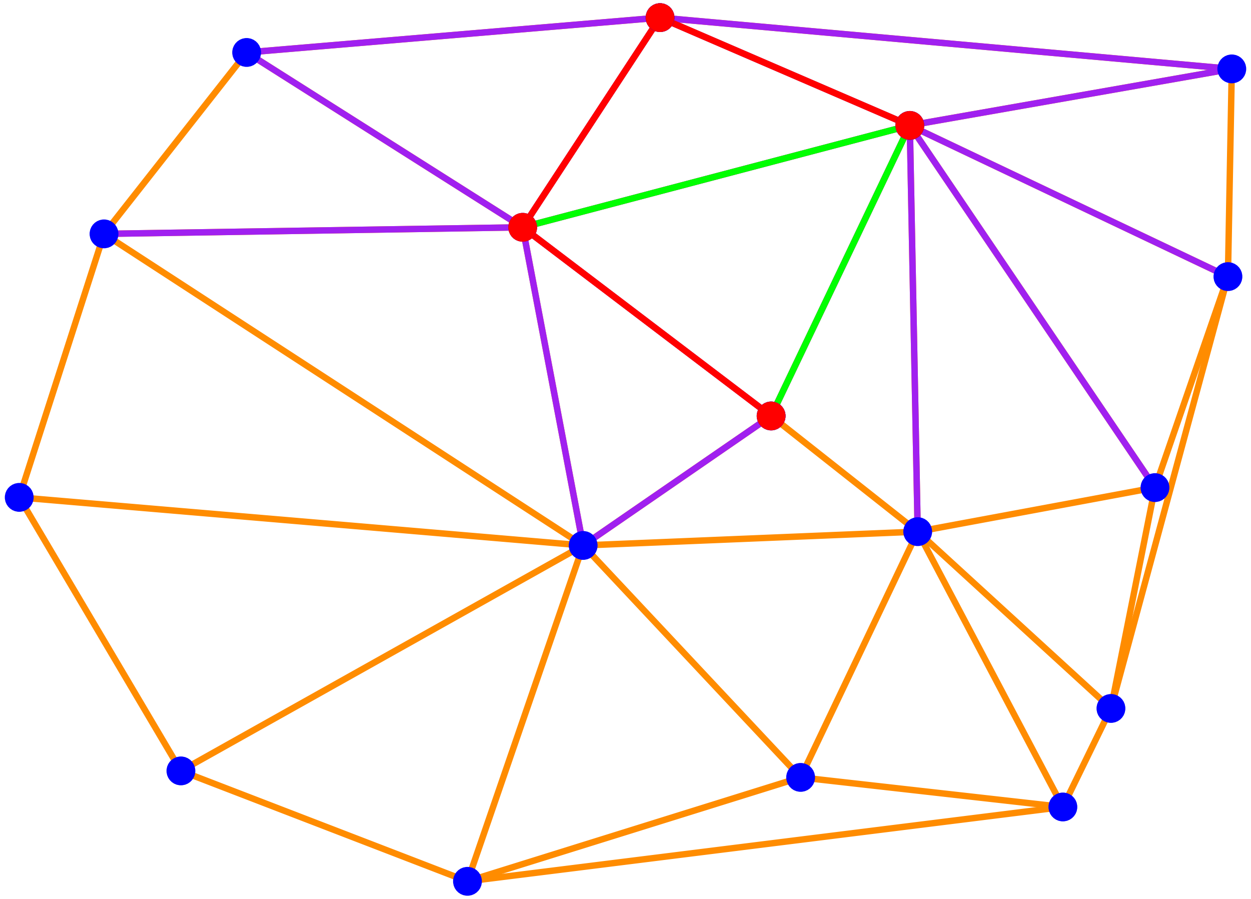
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

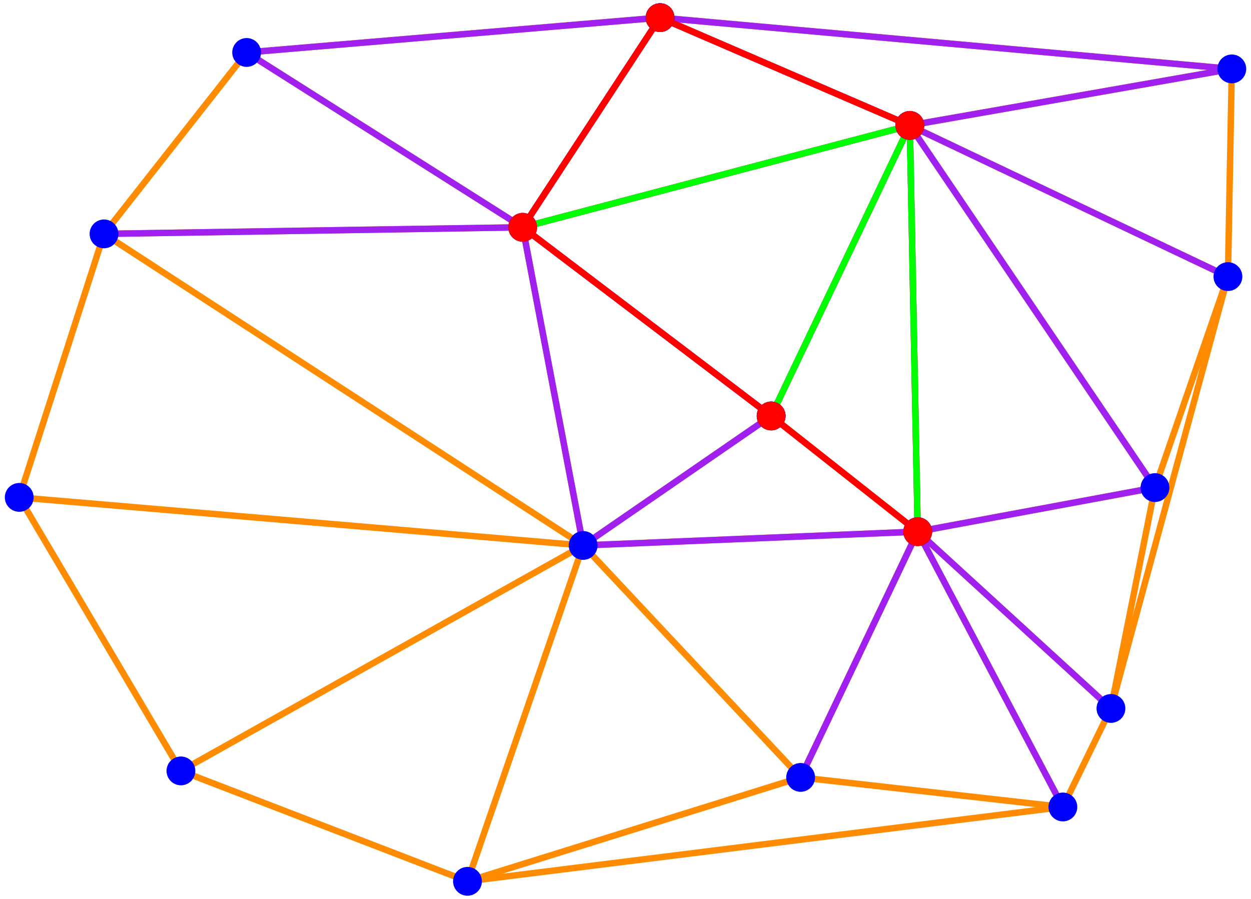
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

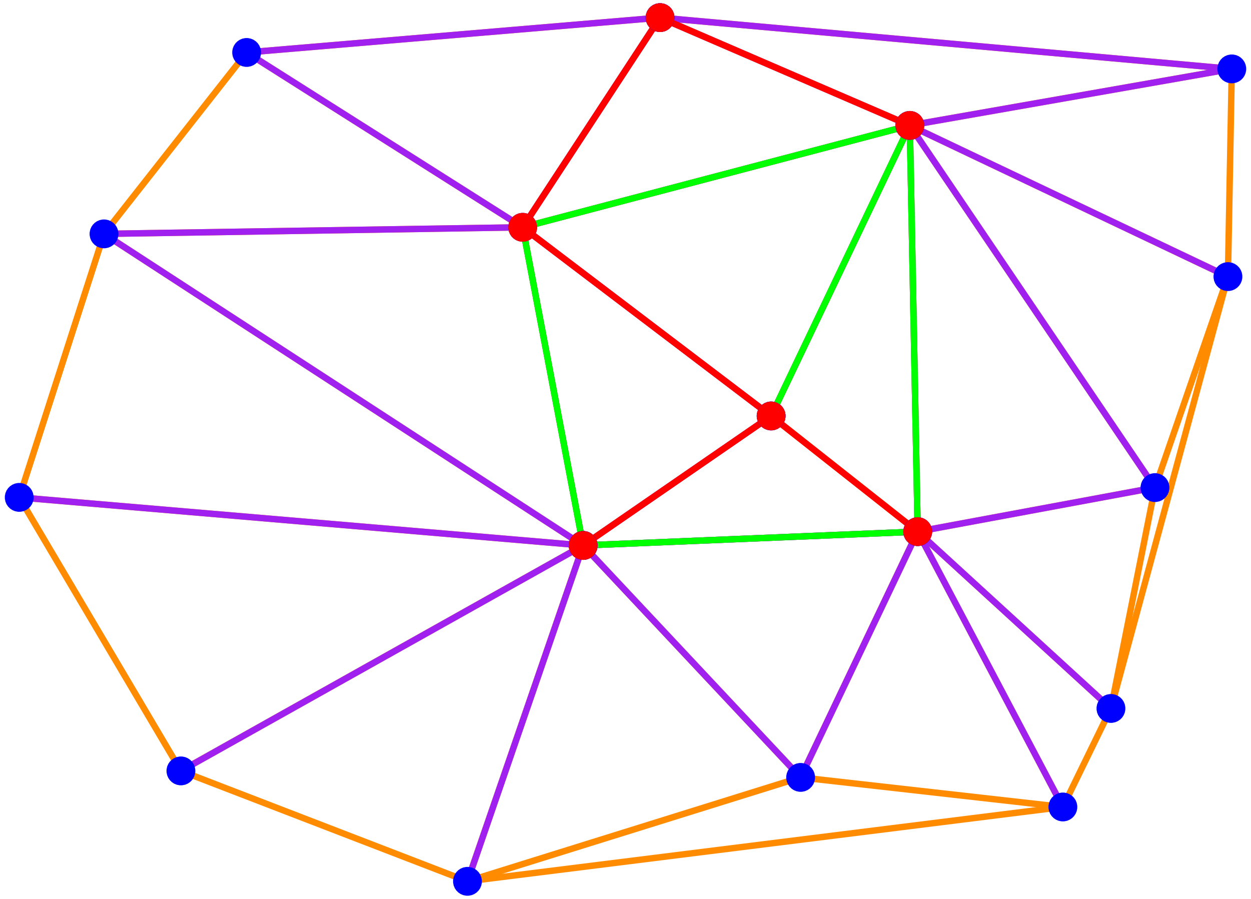
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

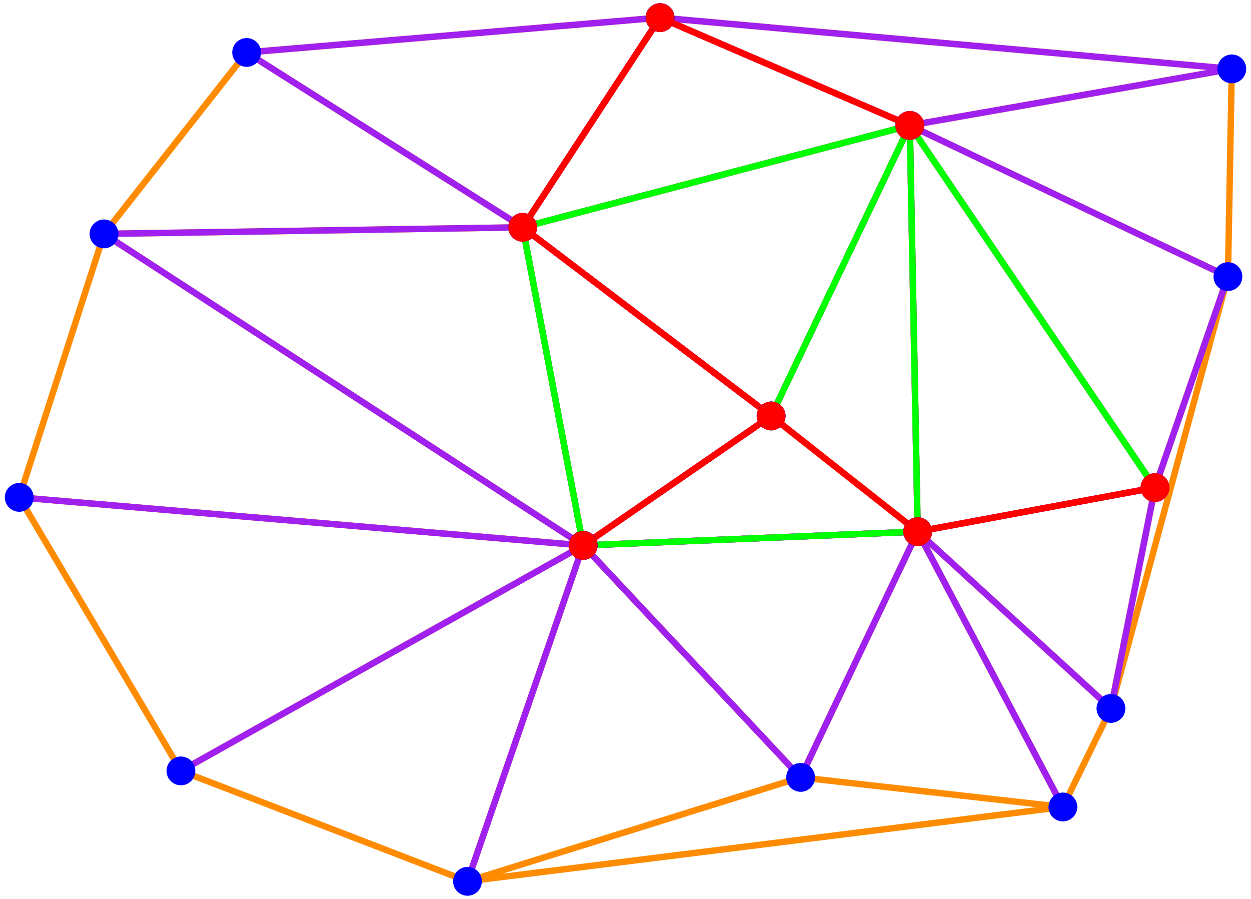
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

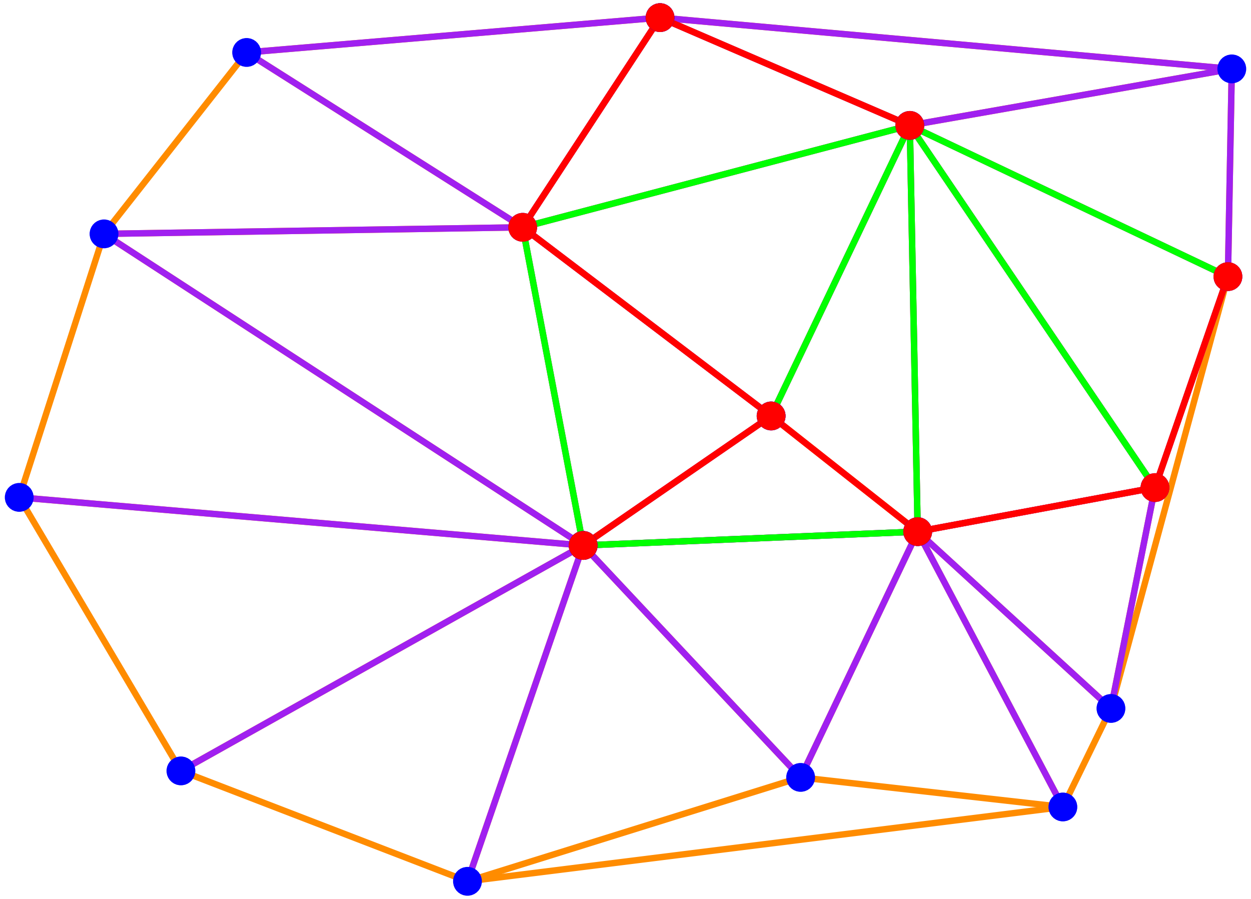
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

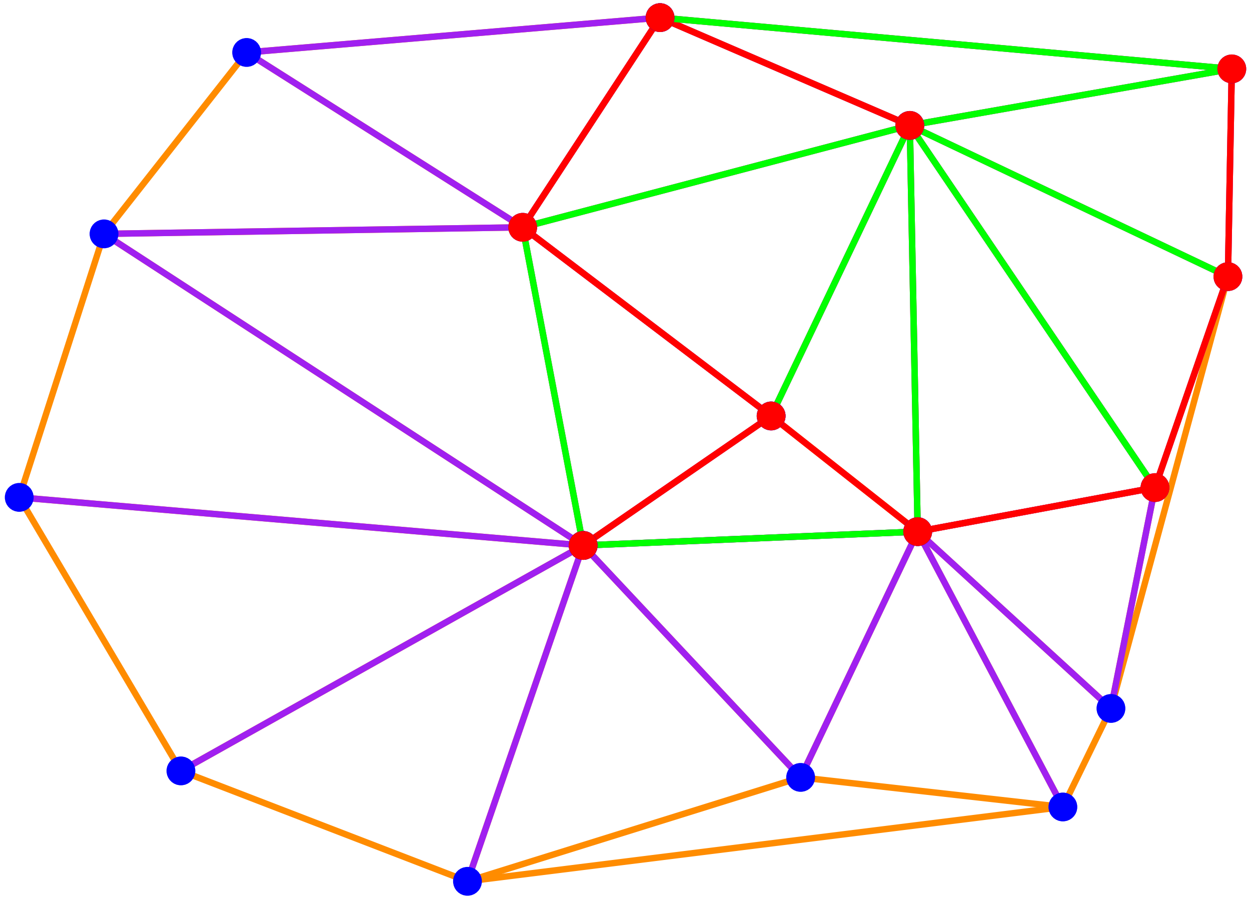
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

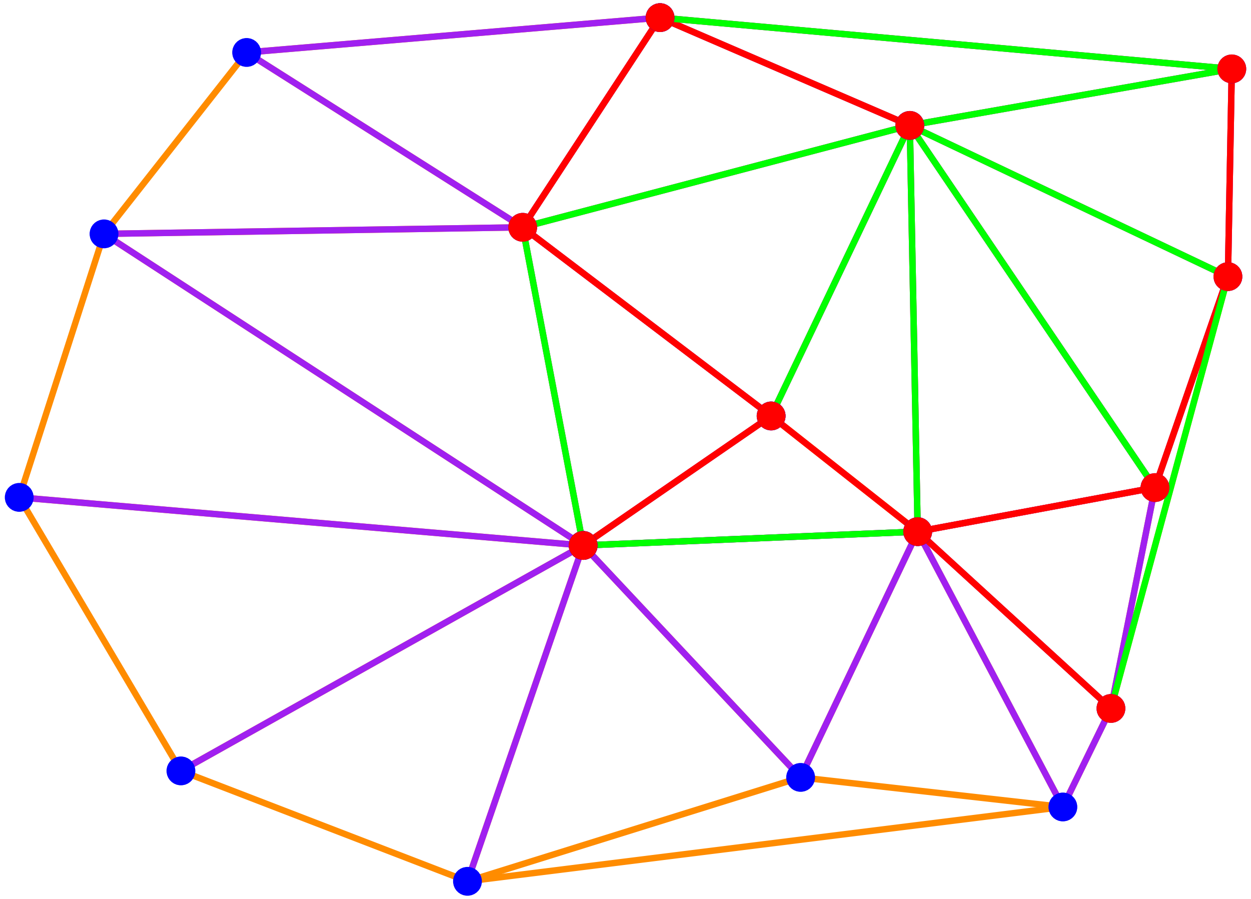
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

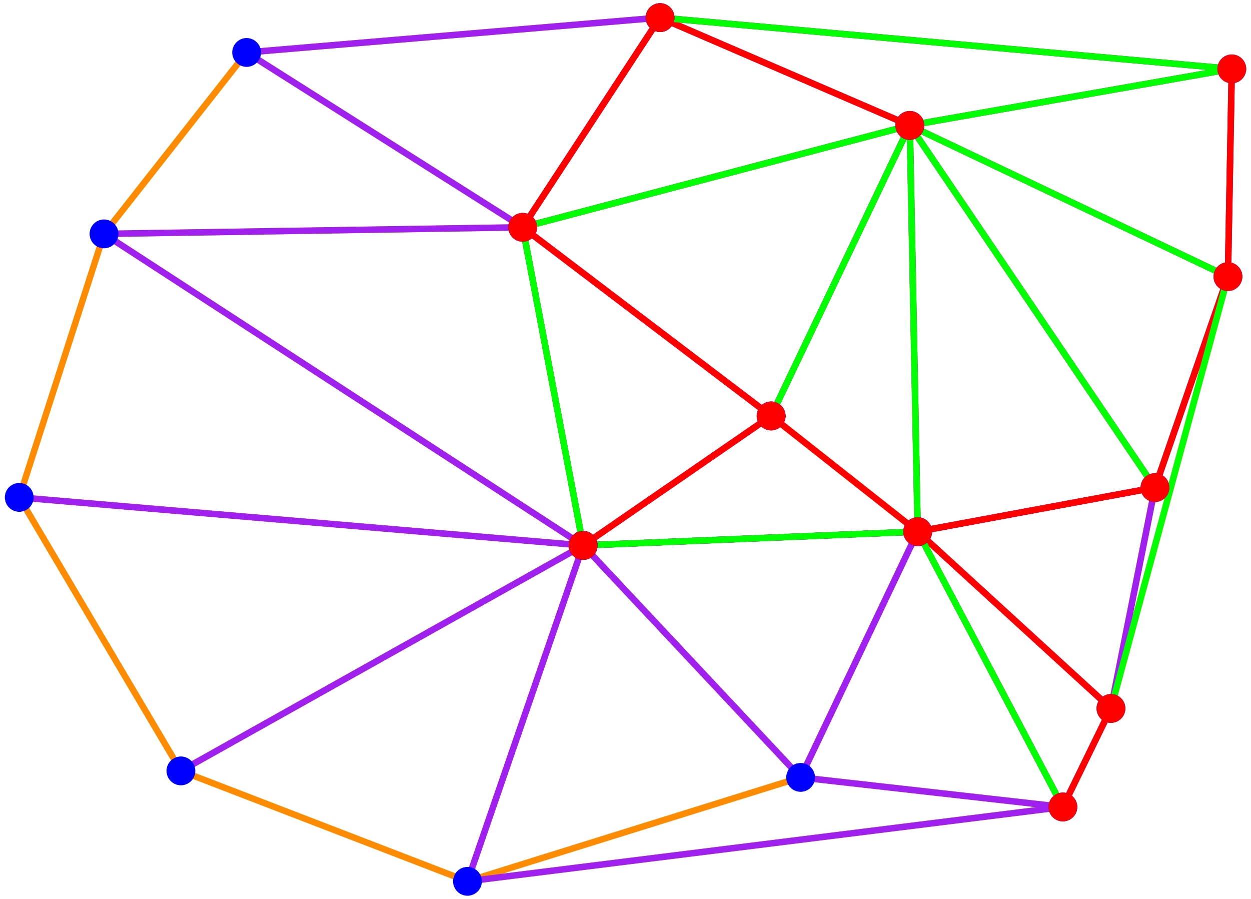
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

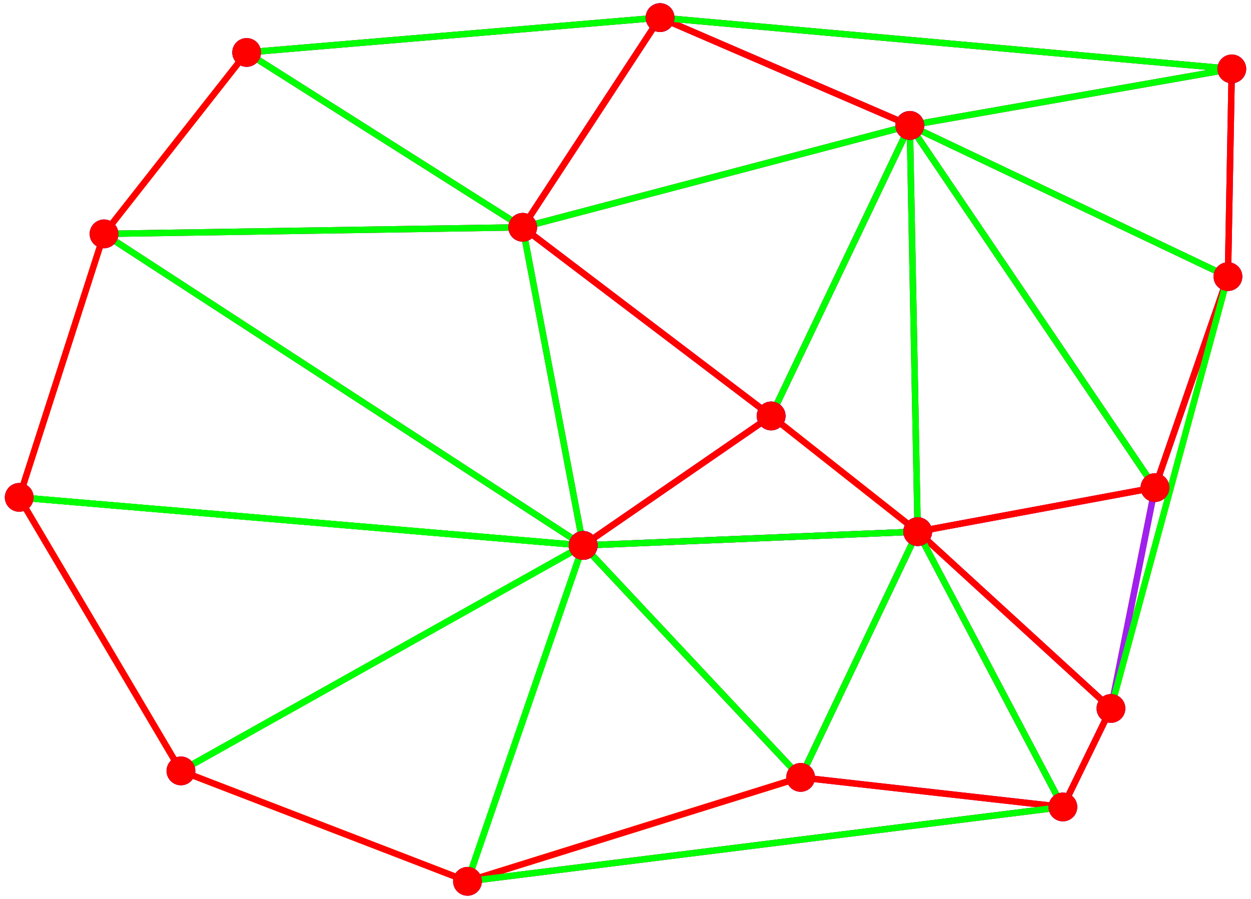
choose shorter purple edge



Delaunay Triangulation: EMST

Algorithm

choose shorter purple edge



$O(n \log n)$ after Delaunay

~~Delaunay~~ Triangulation: size

~~Delaunay~~ Triangulation: size

Convex hull

Three dimensions

Euler relation

Polytope boundary

Vertices Edges Faces

$$n - e + f = 2$$

triangular faces $3f = 2e$

$$f = 2n - 4$$

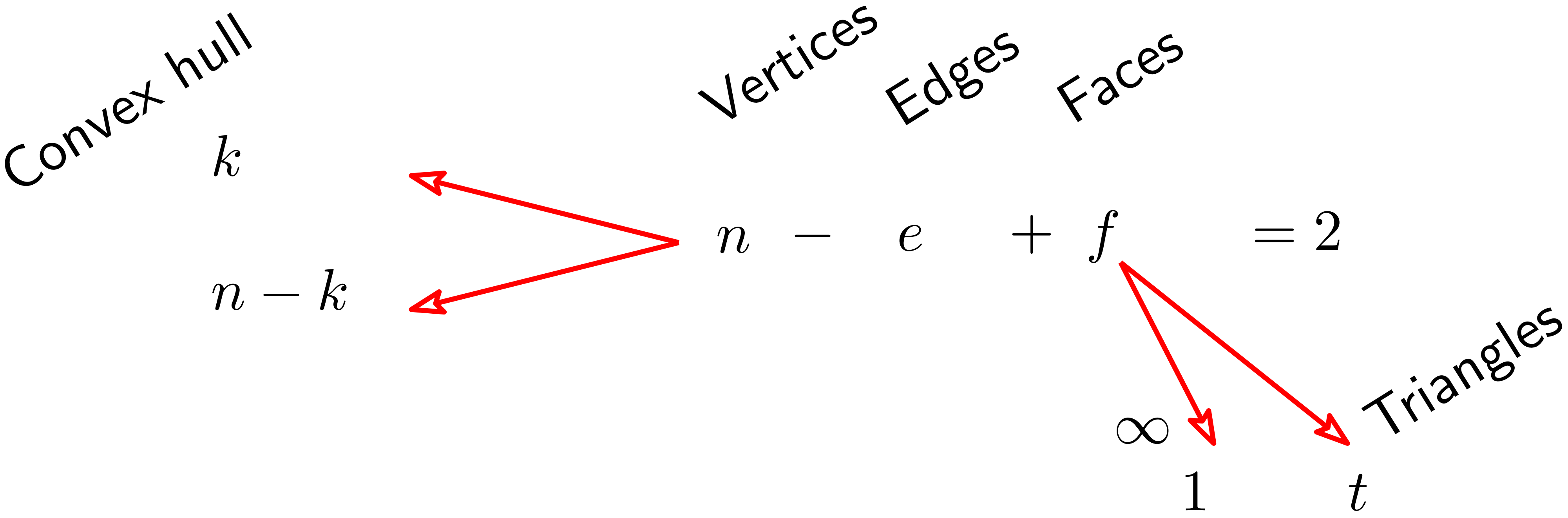
$$e = 3n - 6$$

~~Delaunay~~ Triangulation: size

Vertices Edges Faces

$$n \quad - \quad e \quad + \quad f \qquad = 2$$

~~Delaunay~~ Triangulation: size



Convex hull

k

$n - k$

Vertices

Edges

Faces

$$n - e + f = 2$$

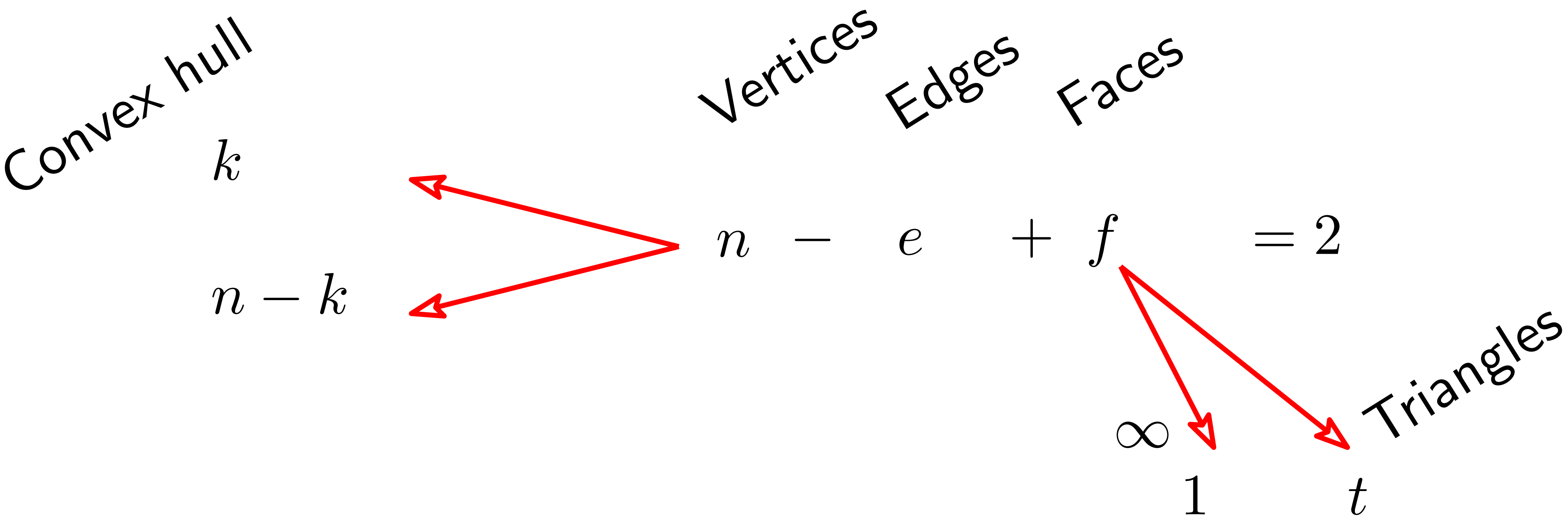
∞

1

t

Triangles

~~Delaunay~~ Triangulation: size



Convex hull

k

$n - k$

Vertices Edges Faces

$n \quad - \quad e \quad + \quad f \quad = 2$

∞

1

t

Triangles

$$n - e + t + 1 = 2$$

~~Delaunay~~ Triangulation: size

Convex hull

k

$n - k$

Vertices Edges Faces

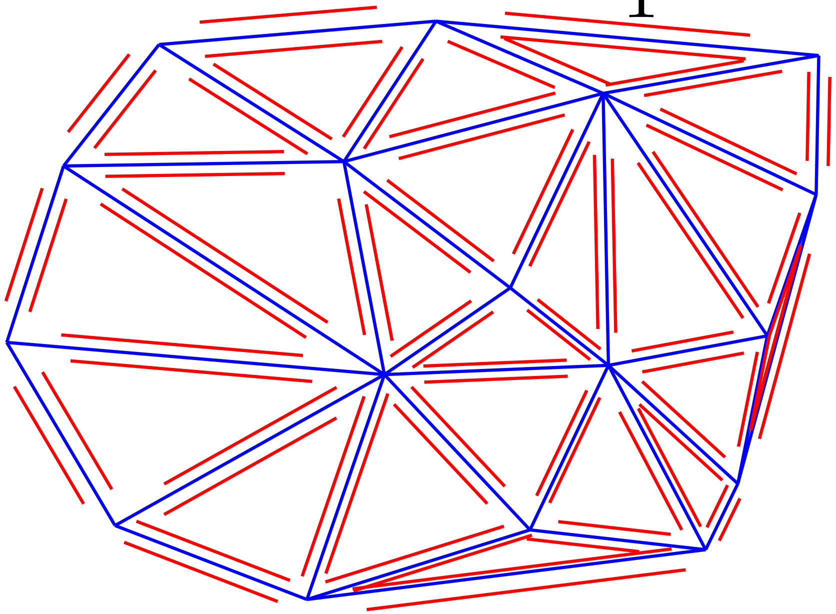
$n \quad - \quad e \quad + \quad f \quad = 2$

∞ 1

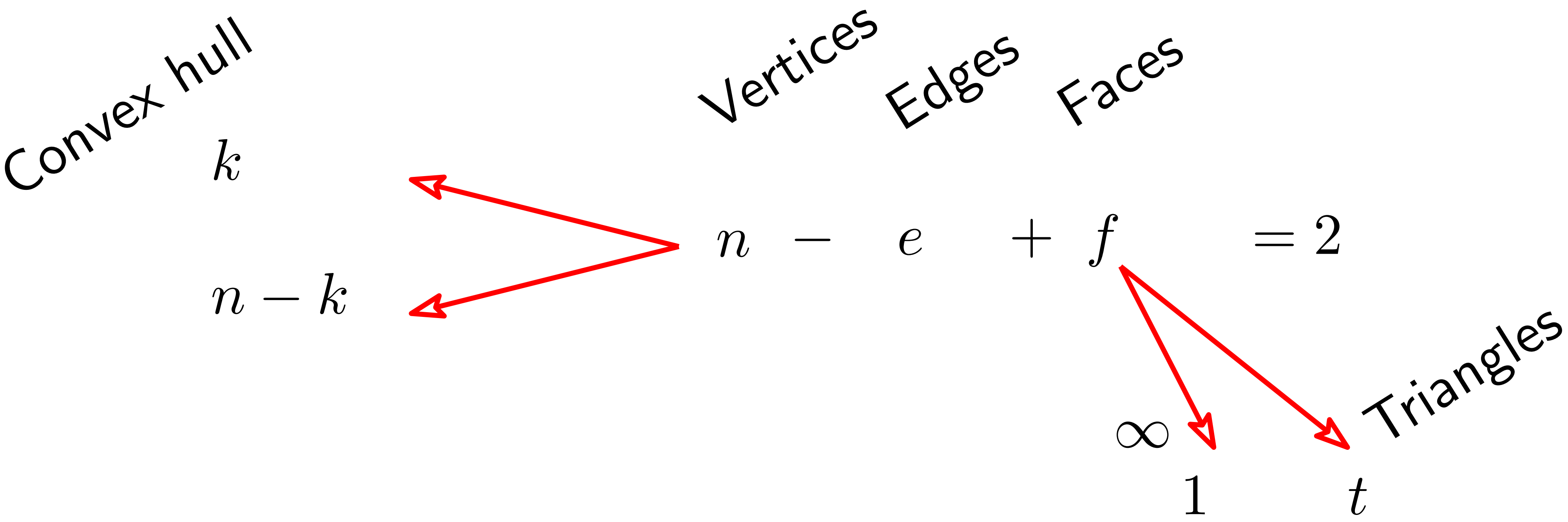
Triangles t

$$n - e + t + 1 = 2$$

$$3t + k = 2e$$



~~Delaunay~~ Triangulation: size



Convex hull

k

$n - k$

Vertices Edges Faces

$n \quad - \quad e \quad + \quad f \quad = 2$

∞

1

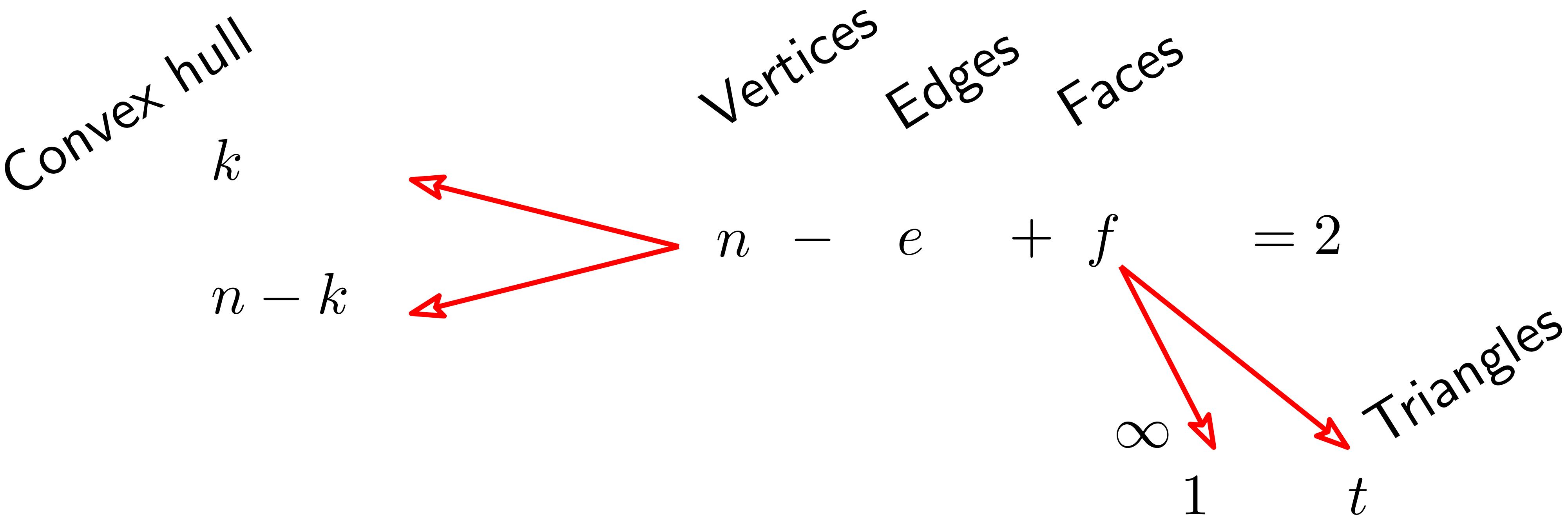
t

Triangles

$$n - e + t + 1 = 2$$

$$3t + k = 2e \qquad\qquad 2n - 3t - k + 2t = 2$$

~~Delaunay~~ Triangulation: size



Convex hull

k

$n - k$

Vertices

Edges

Faces

$n \quad - \quad e \quad + \quad f \quad = 2$

∞

1

t

Triangles

$$n - e + t + 1 = 2$$

$$3t + k = 2e \qquad\qquad 2n - 3t - k + 2t = 2$$

$$t = 2n - k - 2 < 2n$$

$$e = 3n - k - 3 < 3n$$

~~Delaunay~~ Triangulation: size

$$\sum_{p\in S} d^\circ(p) = 2e = 6n - 2k - 6$$

$$\mathbb{E}(d^\circ(p)) = \frac{1}{n}\sum_{p\in S} d^\circ(p) < 6$$

average on the choice of point p in set of points S

$$n - e + t + 1 = 2$$

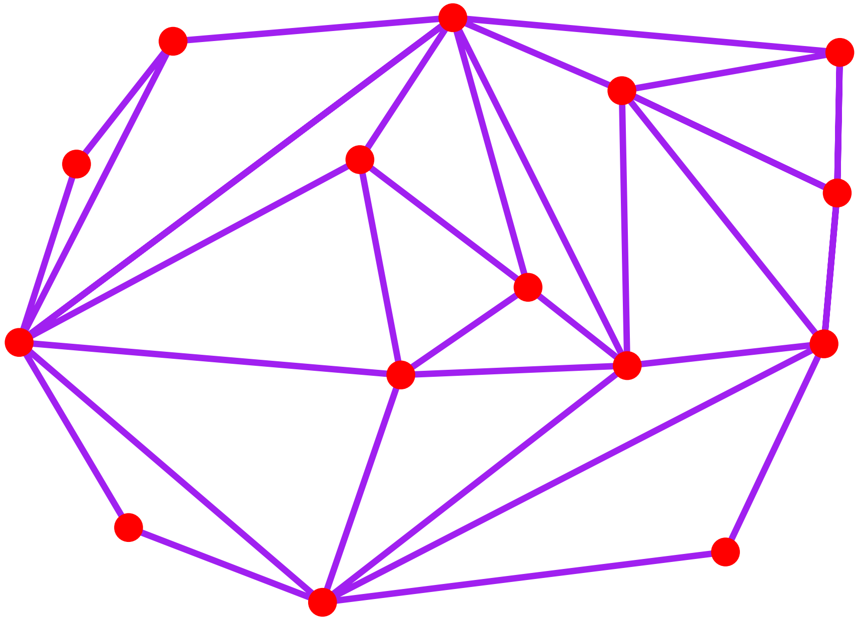
$$3t + k = 2e \qquad 2n - 3t - k + 2t = 2$$

$$t = 2n - k - 2 < 2n$$

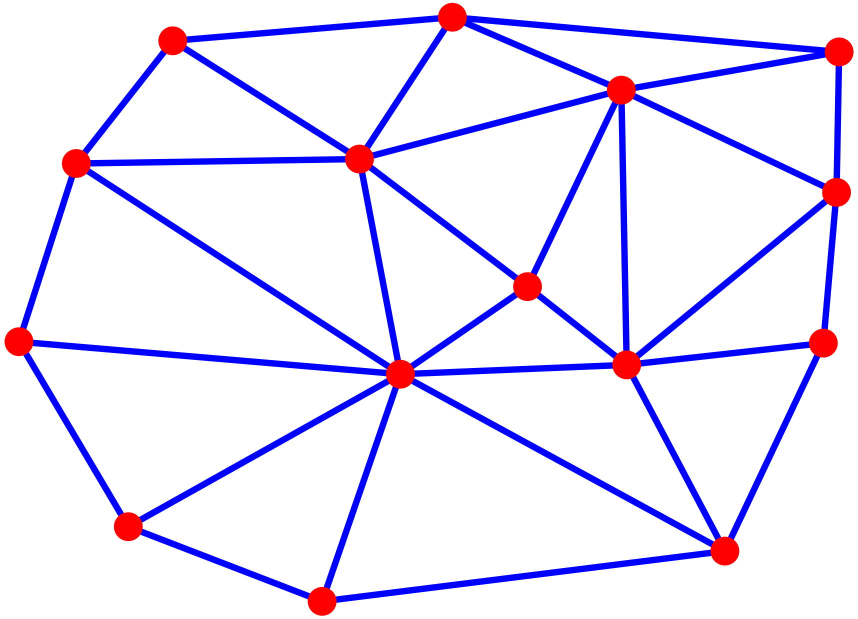
$$e = 3n - k - 3 < 3n$$

Delaunay Triangulation: max-min angle

Delaunay Triangulation: max-min angle

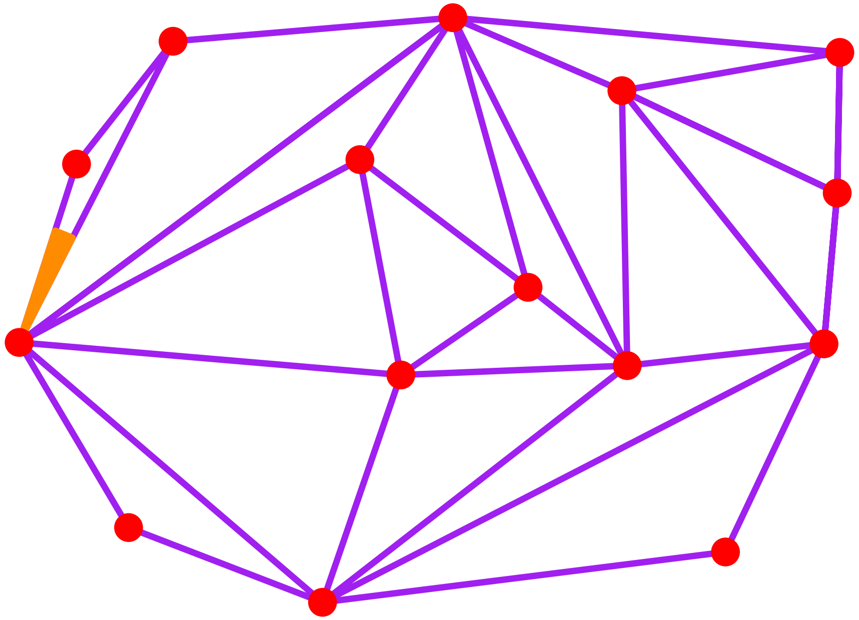


Triangulation

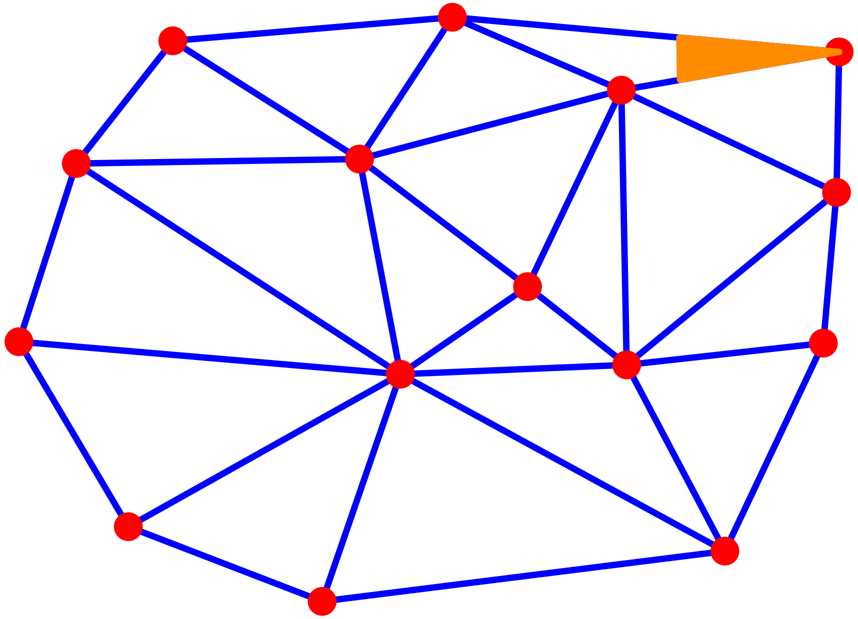


Delaunay

Delaunay Triangulation: max-min angle



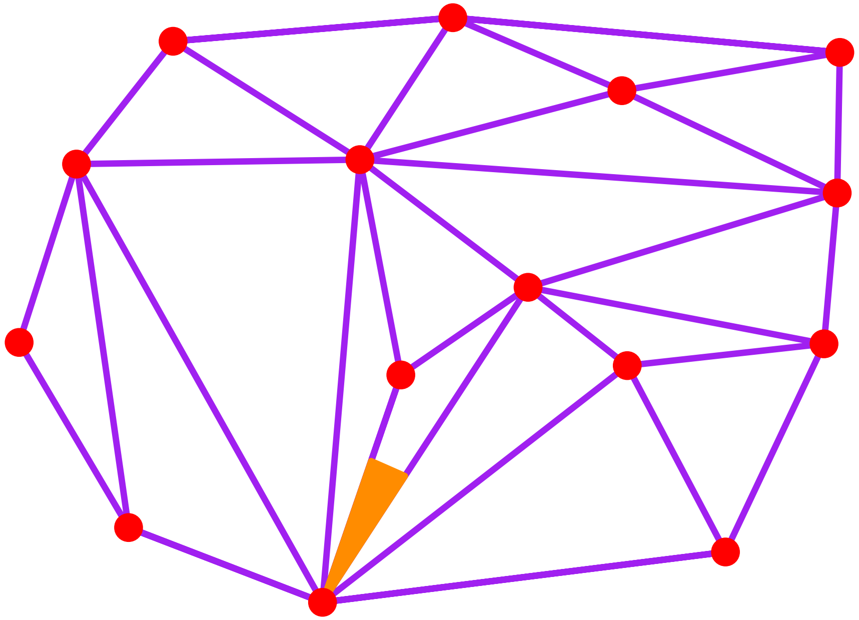
Triangulation



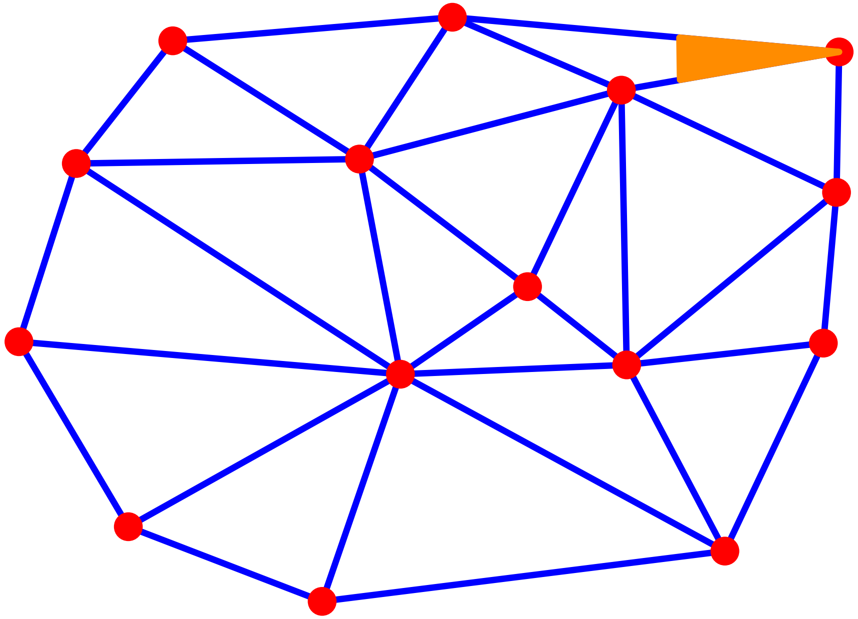
Delaunay

smallest angle

Delaunay Triangulation: max-min angle



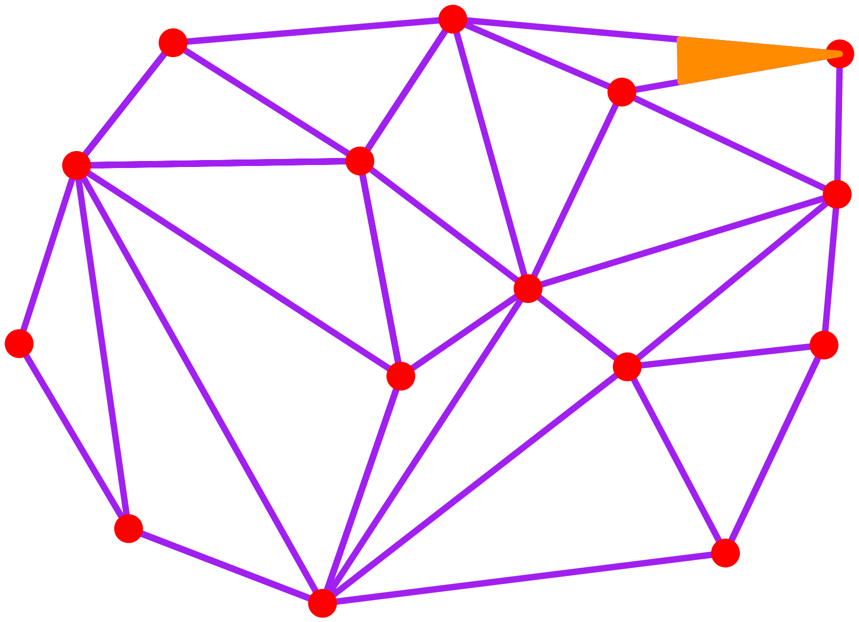
Triangulation



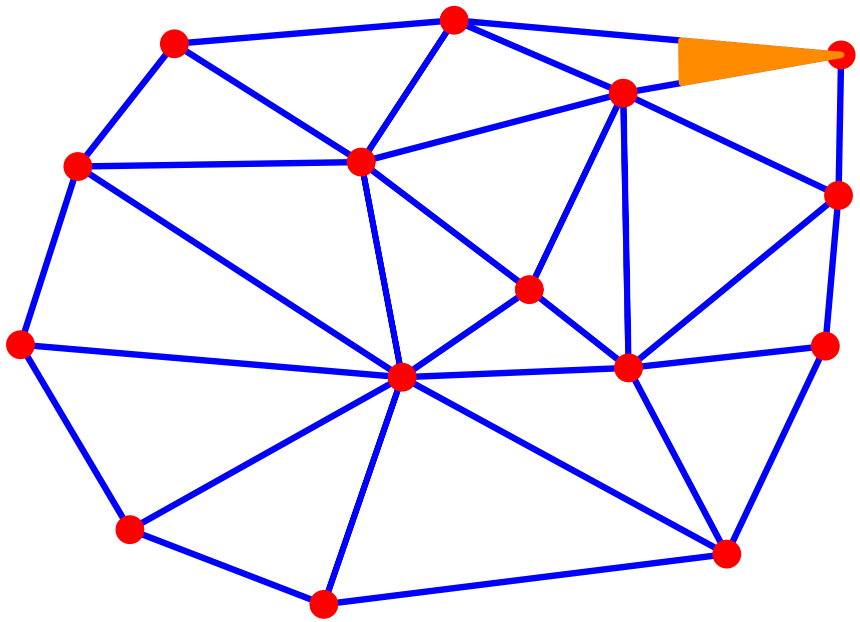
Delaunay

smallest angle

Delaunay Triangulation: max-min angle



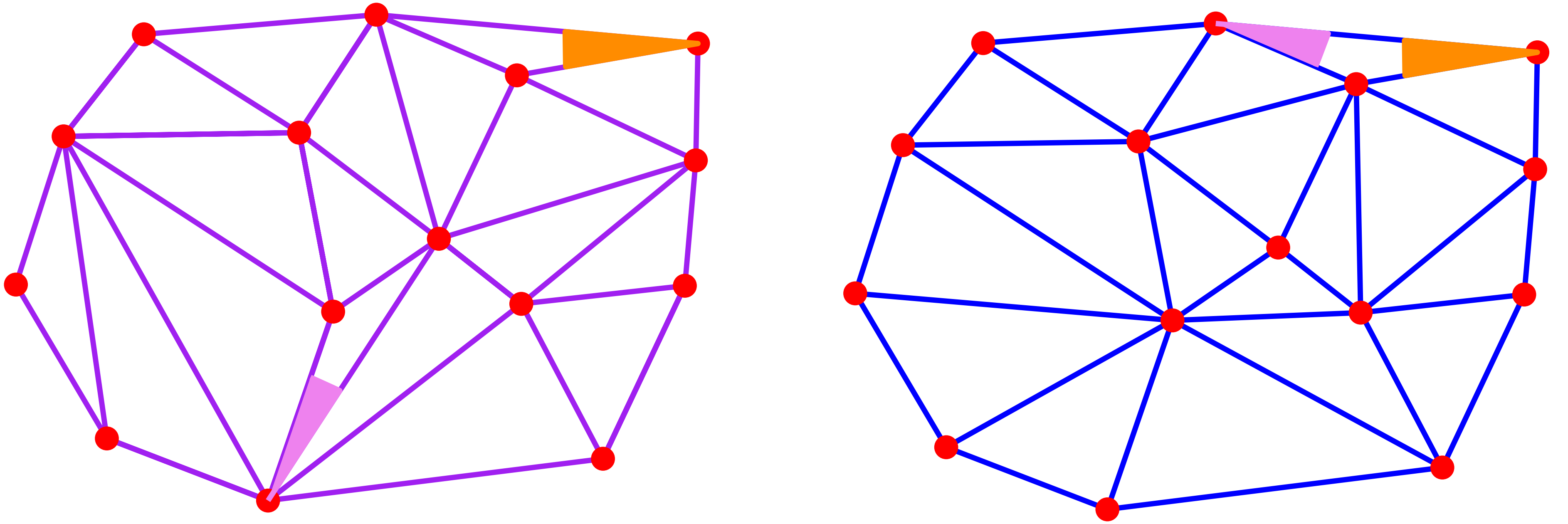
Triangulation



Delaunay

smallest angle

Delaunay Triangulation: max-min angle



Triangulation

Delaunay

smallest angle

second smallest angle

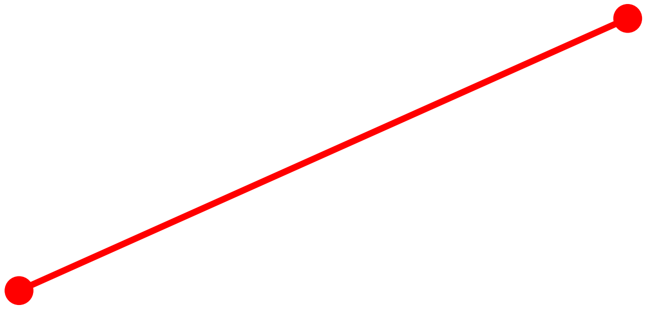
Delaunay Triangulation: max-min angle

Proof

Delaunay Triangulation: max-min angle

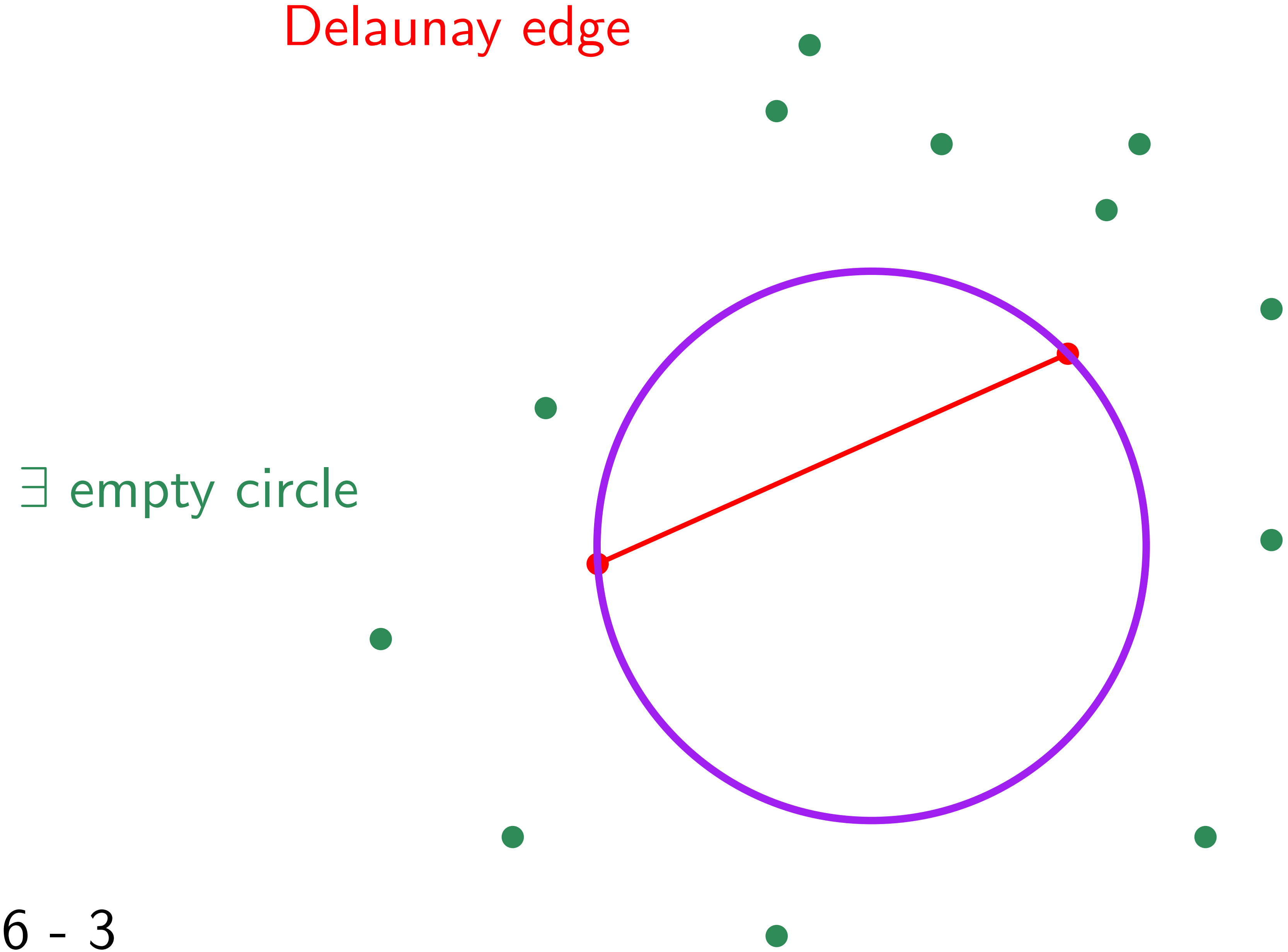
Definition

Delaunay edge



Delaunay Triangulation: max-min angle

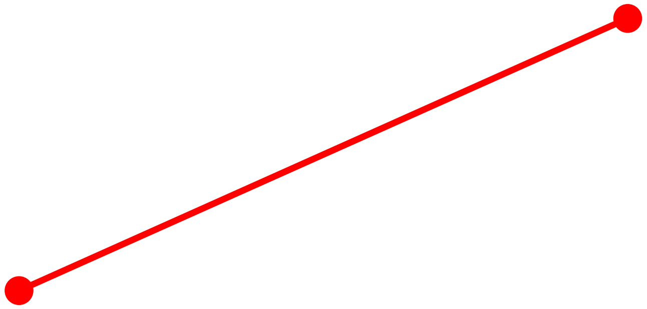
Definition



Delaunay Triangulation: max-min angle

Definition

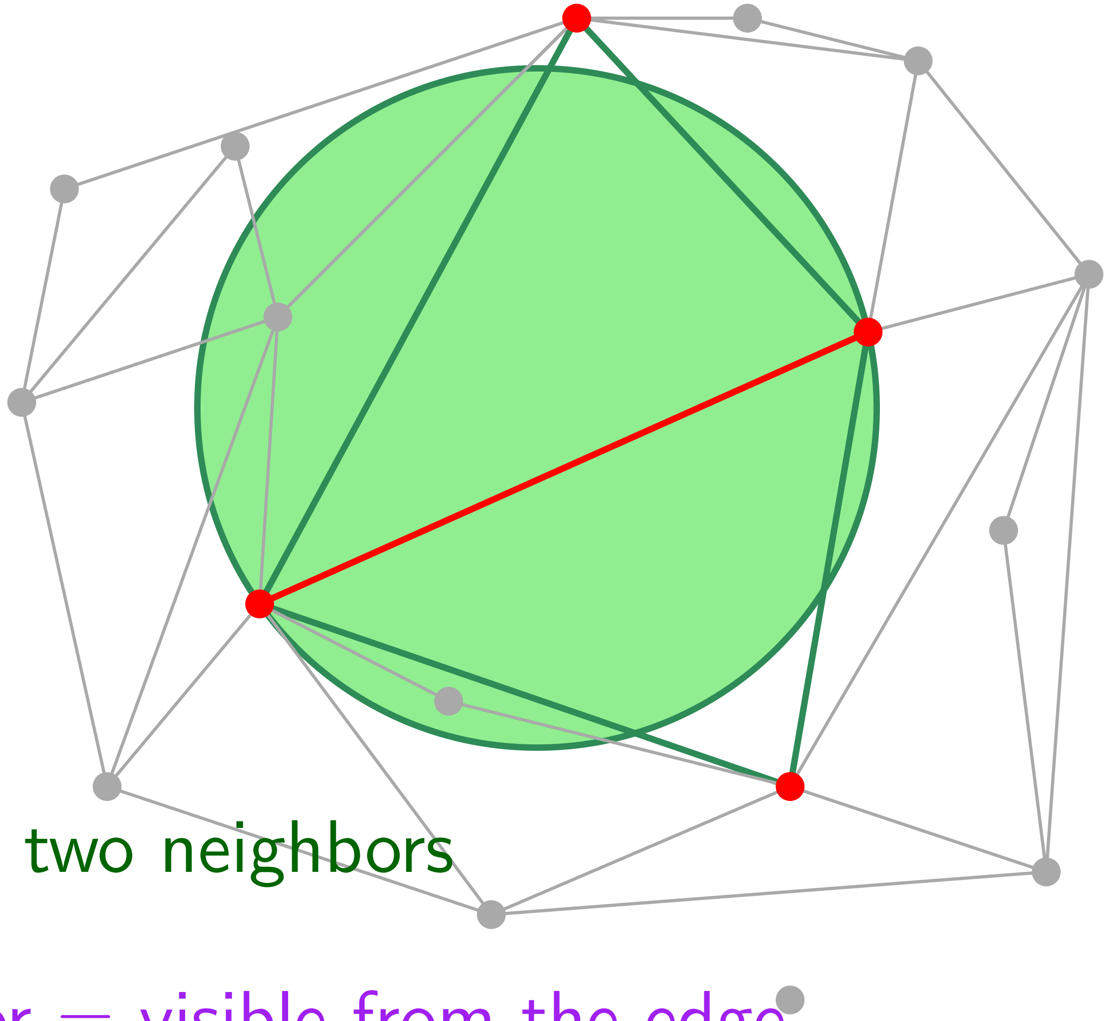
locally Delaunay edge w.r.t. a triangulation



Delaunay Triangulation: max-min angle

Definition

locally Delaunay edge w.r.t. a triangulation



∃ circle

not enclosing the two neighbors

neighbor = visible from the edge

Delaunay Triangulation: max-min angle

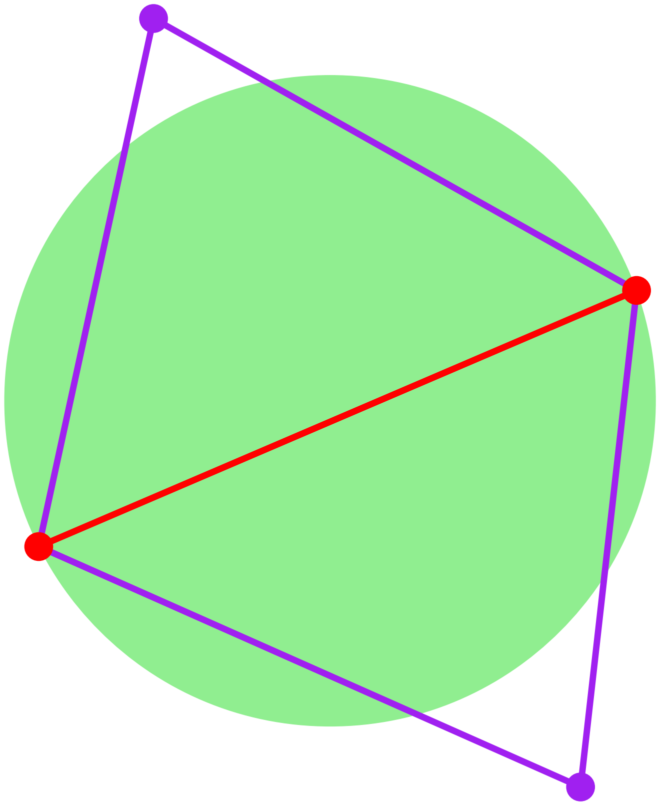
Lemma $(\forall$ edge: locally Delaunay$) \iff$ Delaunay

Delaunay Triangulation: max-min angle

Lemma (∀ edge: locally Delaunay) $\iff$ Delaunay

Proof:

choose an edge

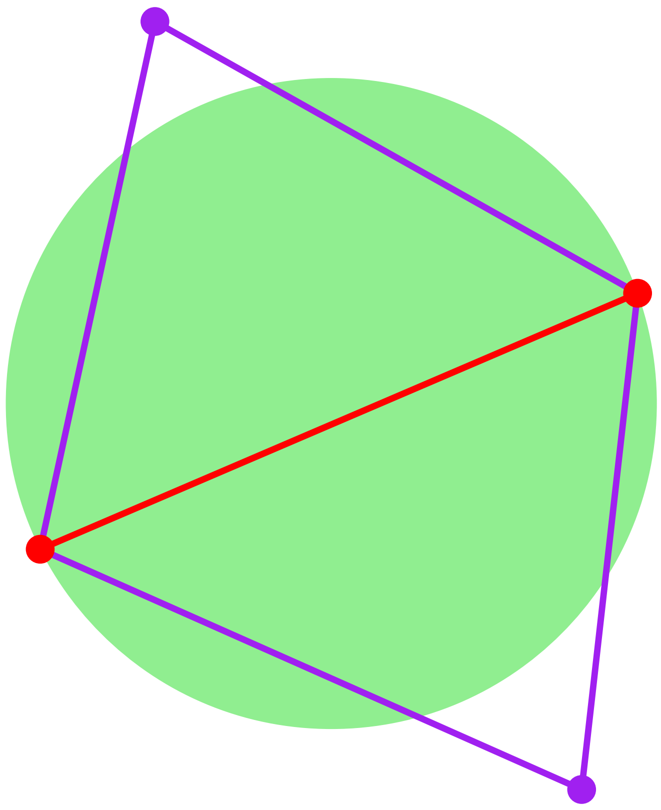


Delaunay Triangulation: max-min angle

Lemma (∀ edge: locally Delaunay) $\Longleftrightarrow$ Delaunay

Proof:

- choose an edge
- edges of the quadrilateral are locally Delaunay



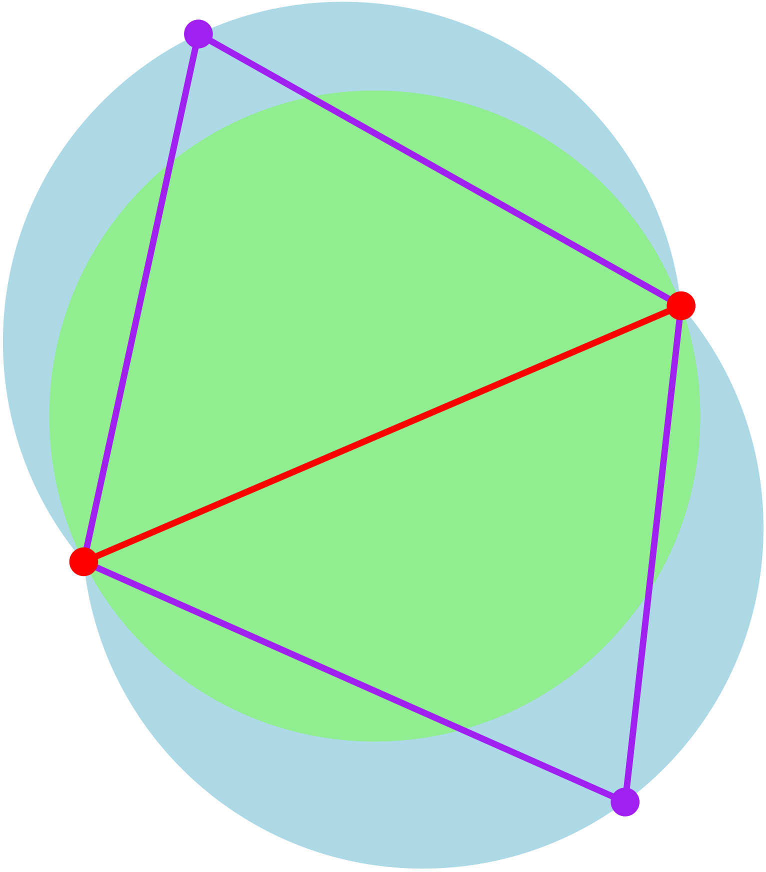
Delaunay Triangulation: max-min angle

Lemma (∀ edge: locally Delaunay) $\iff$ Delaunay

Proof:

- choose an edge
- edges of the quadrilateral

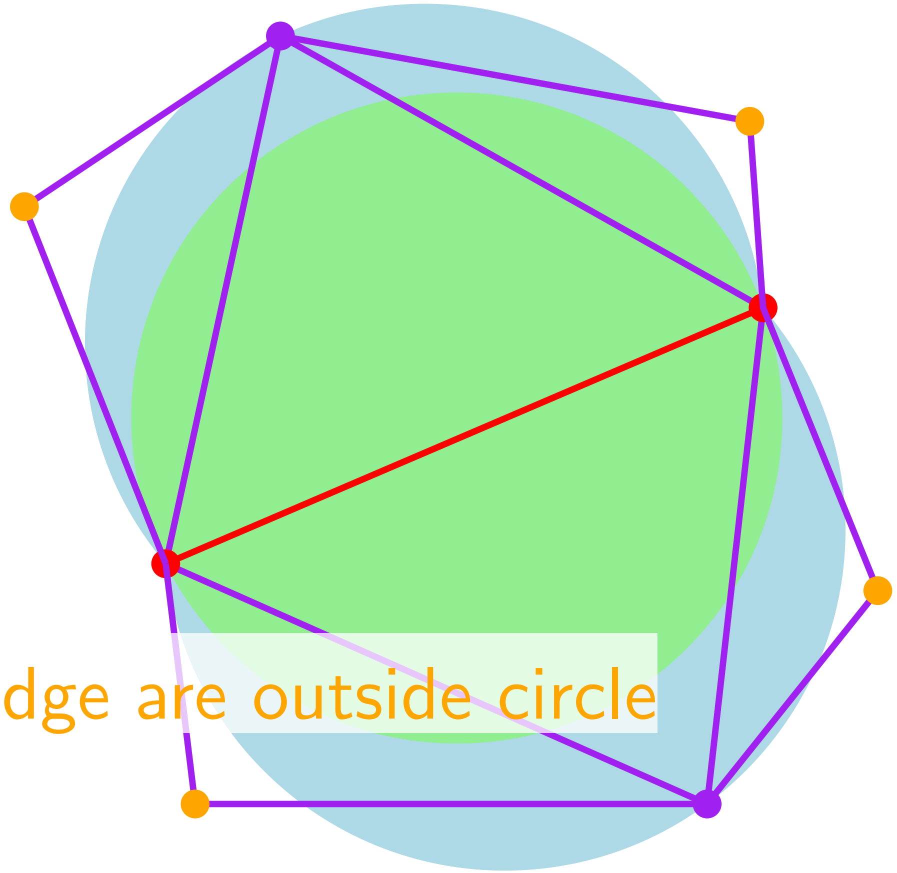
  are locally Delaunay



Delaunay Triangulation: max-min angle

Lemma (∀ edge: locally Delaunay) ⟺ Delaunay

Proof:

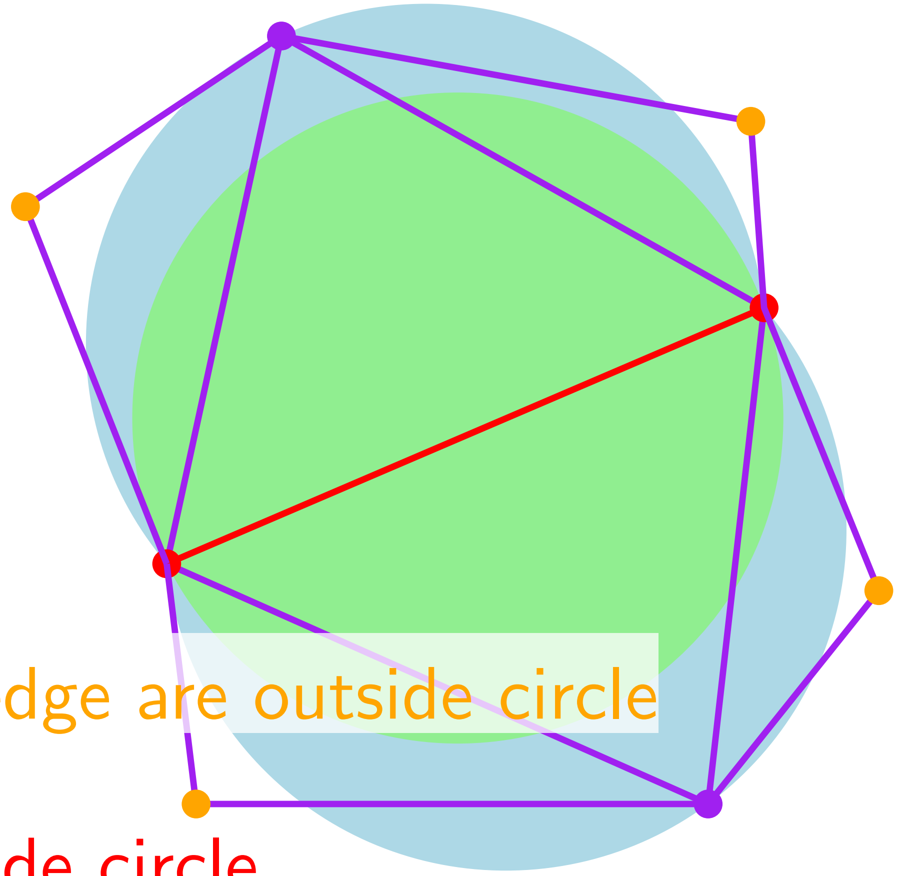


- choose an edge
- edges of the quadrilateral are locally Delaunay
- Vertices visible through one edge are outside circle

Delaunay Triangulation: max-min angle

Lemma (∀ edge: locally Delaunay) $\Longleftrightarrow$ Delaunay

Proof:



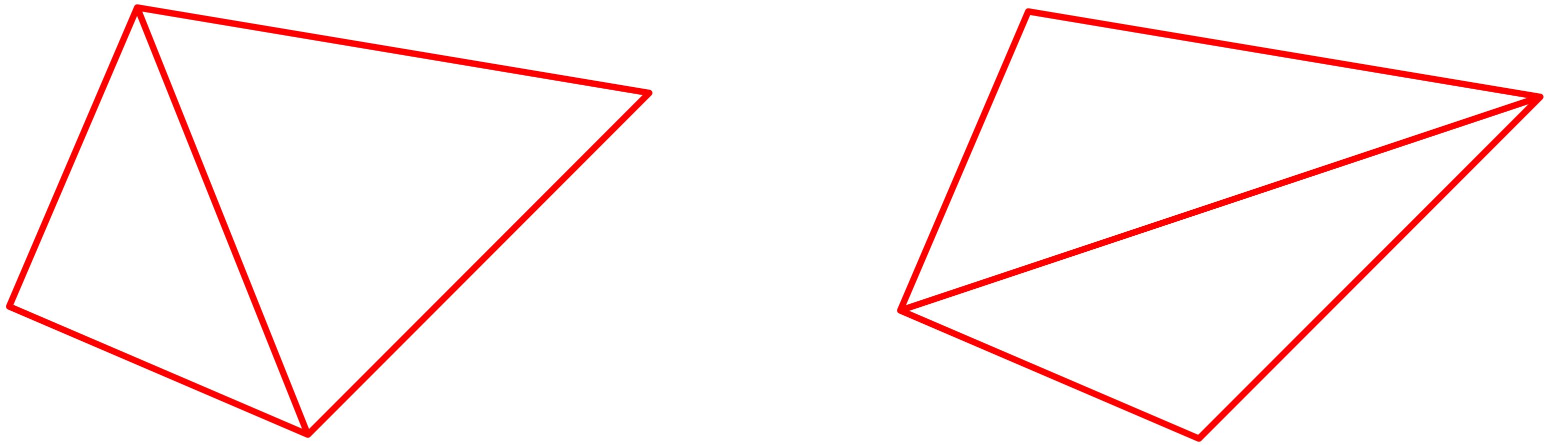
- choose an edge
- edges of the quadrilateral are locally Delaunay
- Vertices visible through one edge are outside circle
- Induction → all vertices outside circle

Delaunay Triangulation: max-min angle

Lemma For four points in convex position

Delaunay $\Longleftrightarrow$ maximize the smallest angle

Two possible triangulation

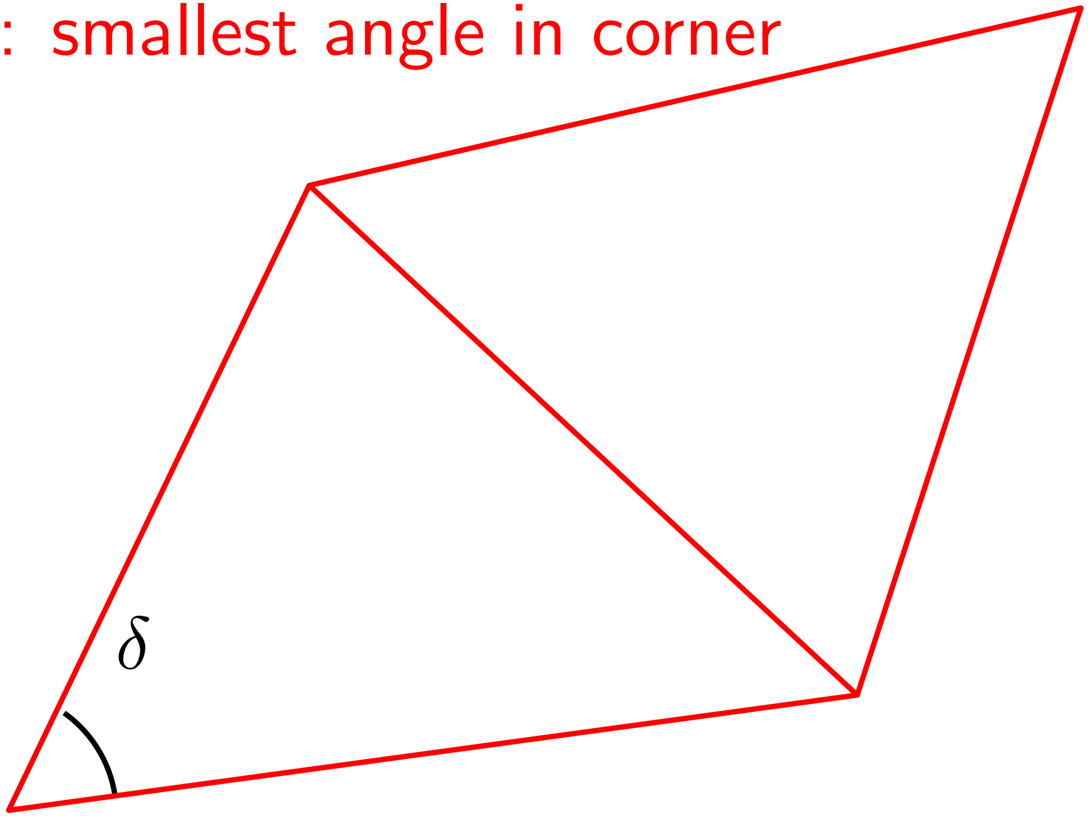


Delaunay Triangulation: max-min angle

Lemma For four points in convex position

Delaunay $\Longleftrightarrow$ maximize the smallest angle

Case 1: smallest angle in corner



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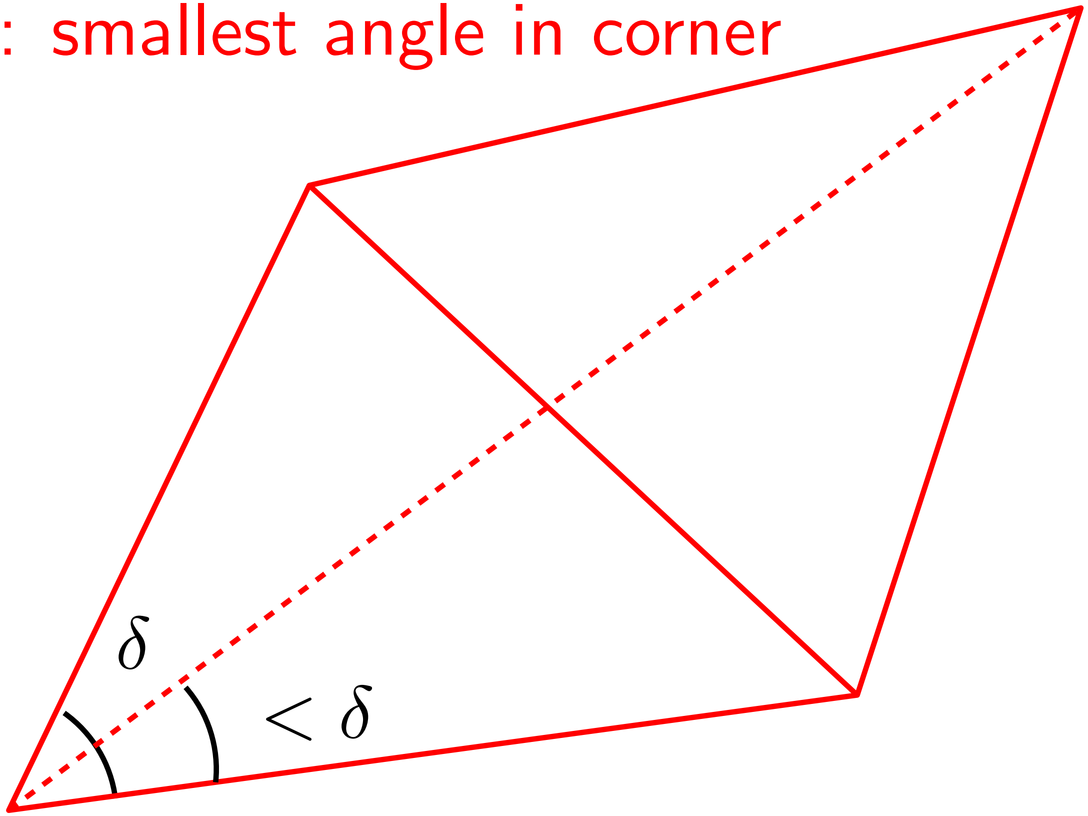

$\exists$ a smaller angle $\in$ other triangulation

Delaunay Triangulation: max-min angle

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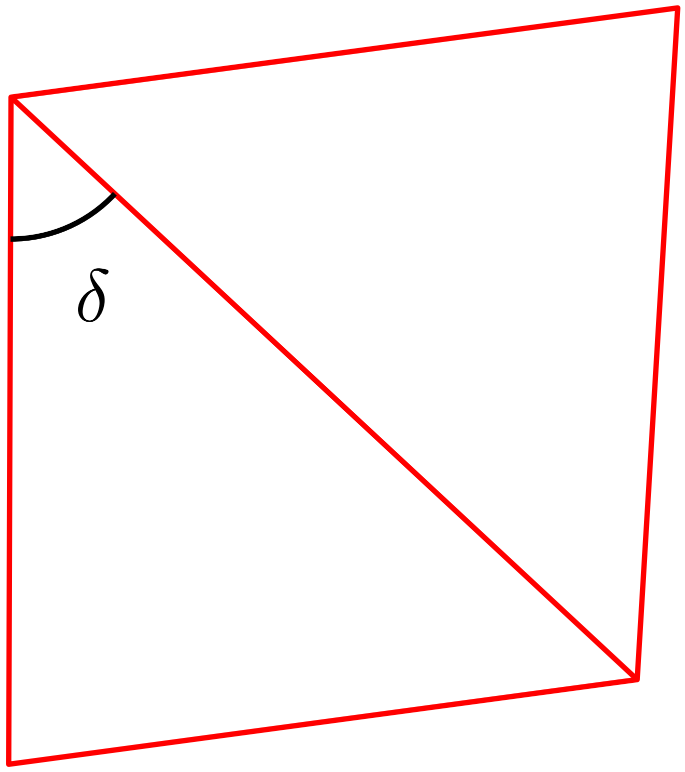

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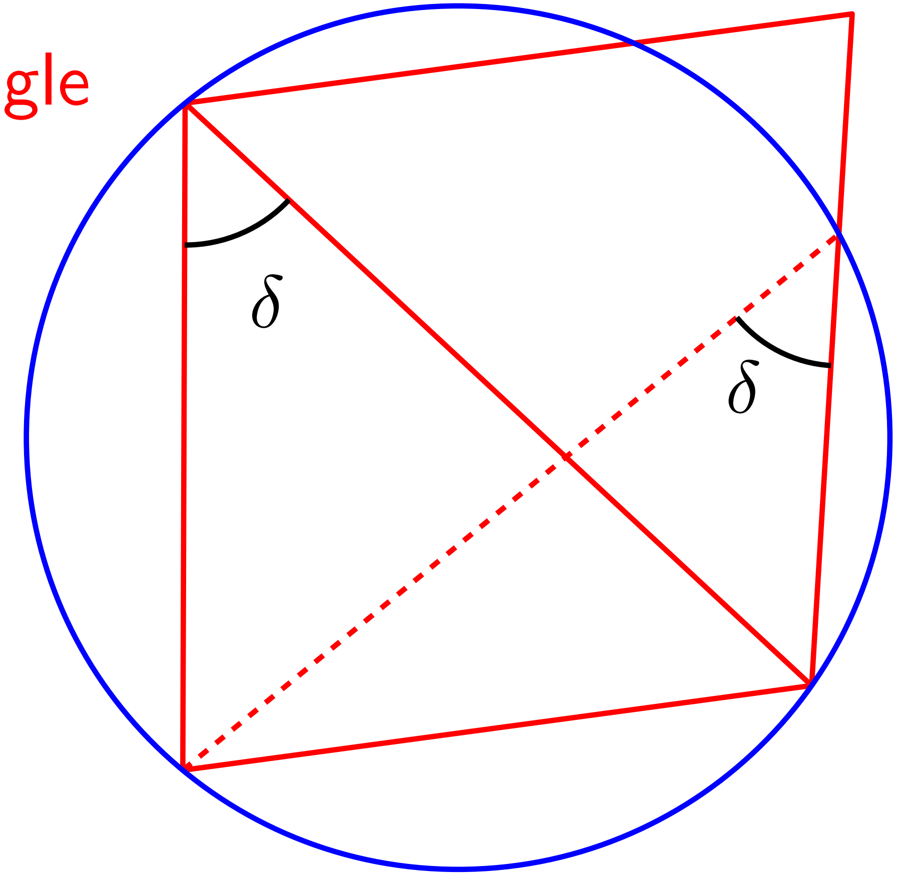

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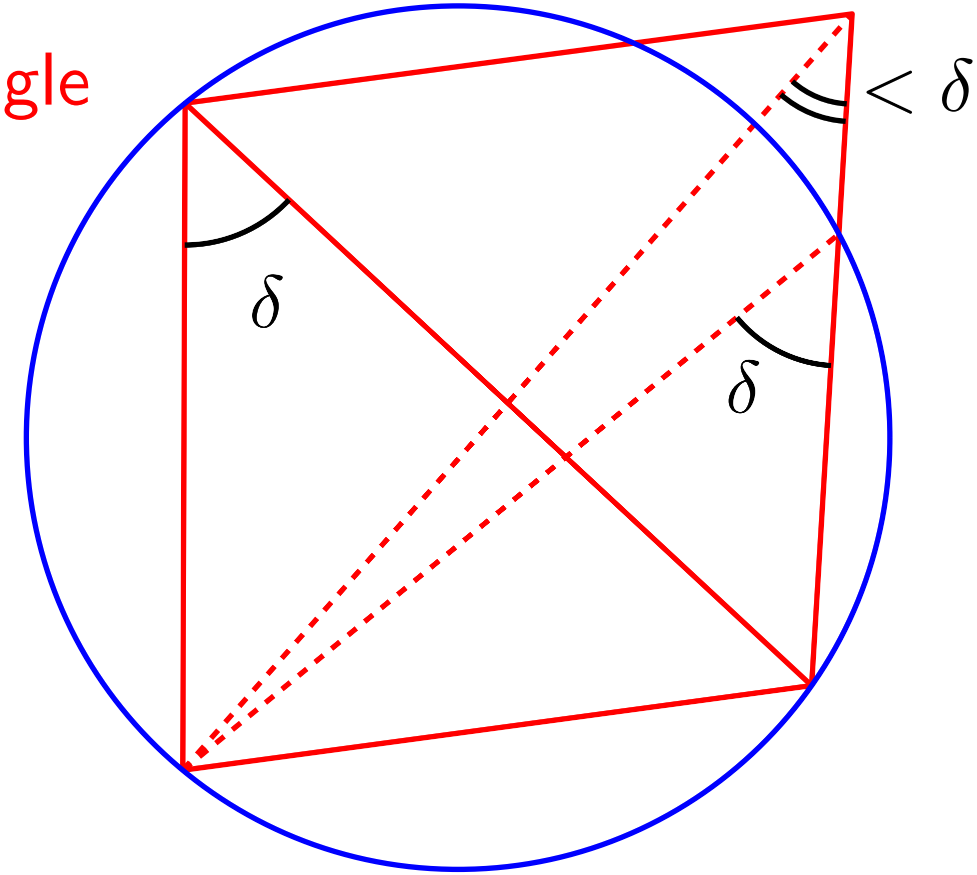

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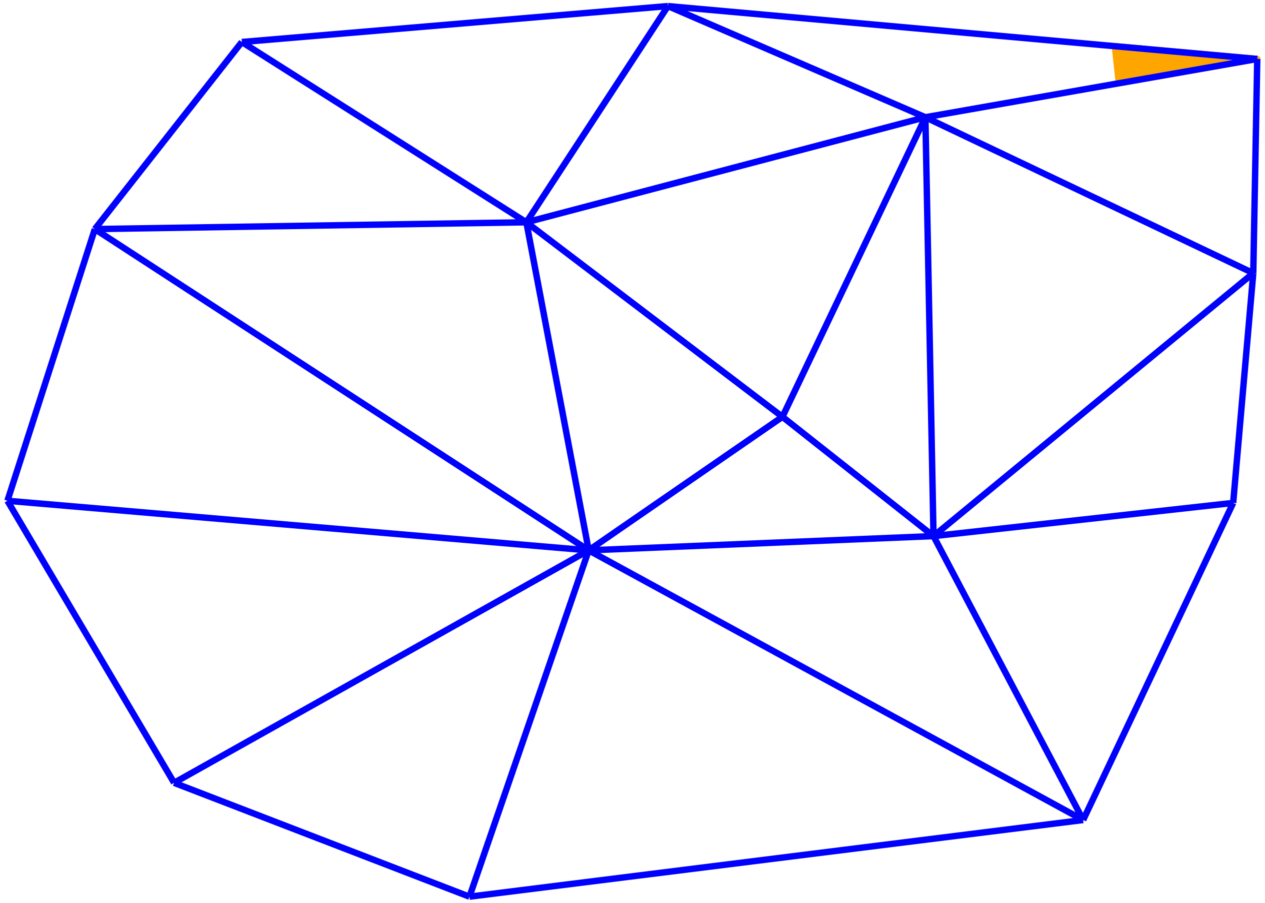
Delaunay Triangulation: max-min angle

Map: Triangulations $\longrightarrow \mathbb{R}^{6n-3k-4}$ smallest angle α_1

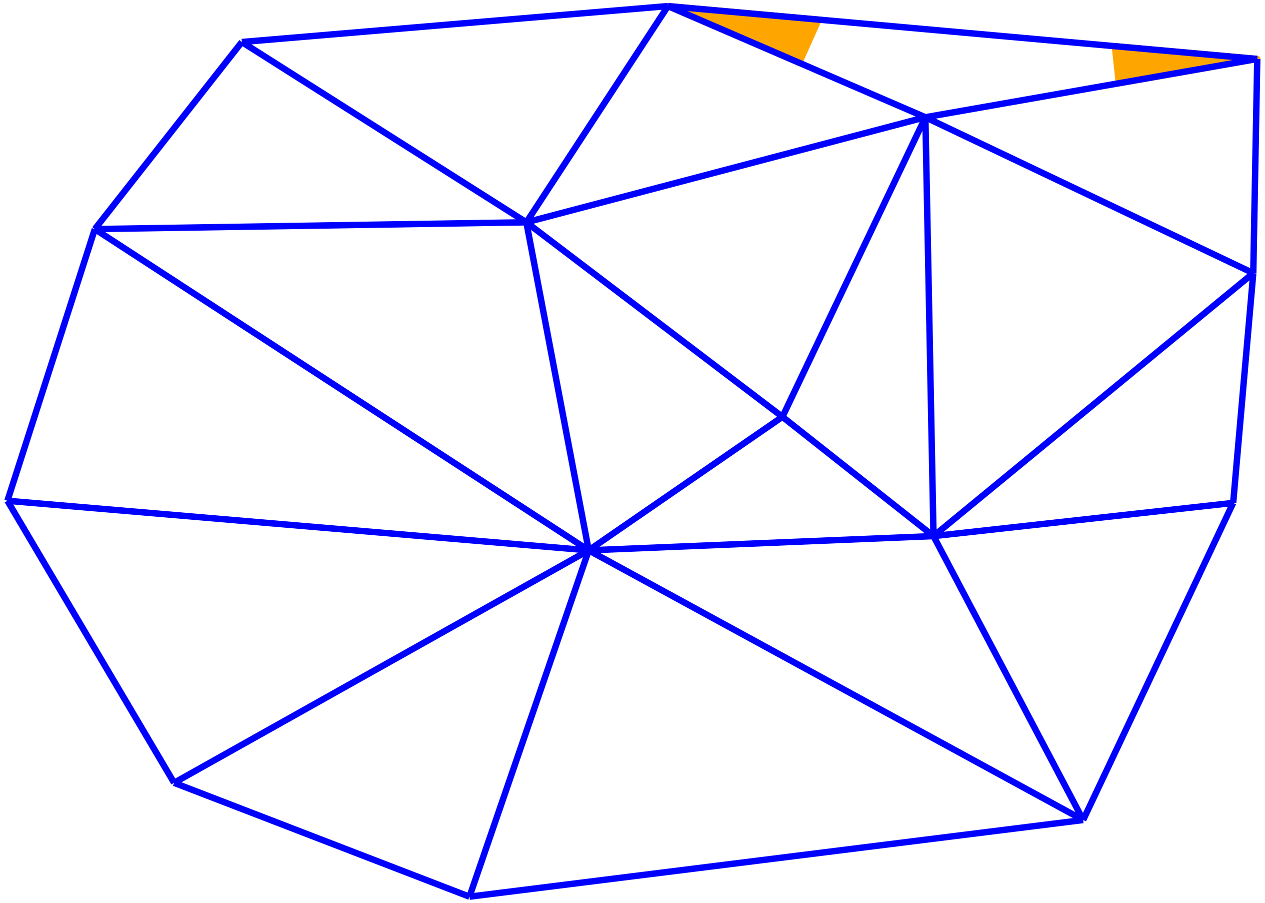


Delaunay Triangulation: max-min angle

Map: Triangulations $\longrightarrow \mathbb{R}^{6n-3k-4}$

smallest angle α_1

second smallest angle α_2



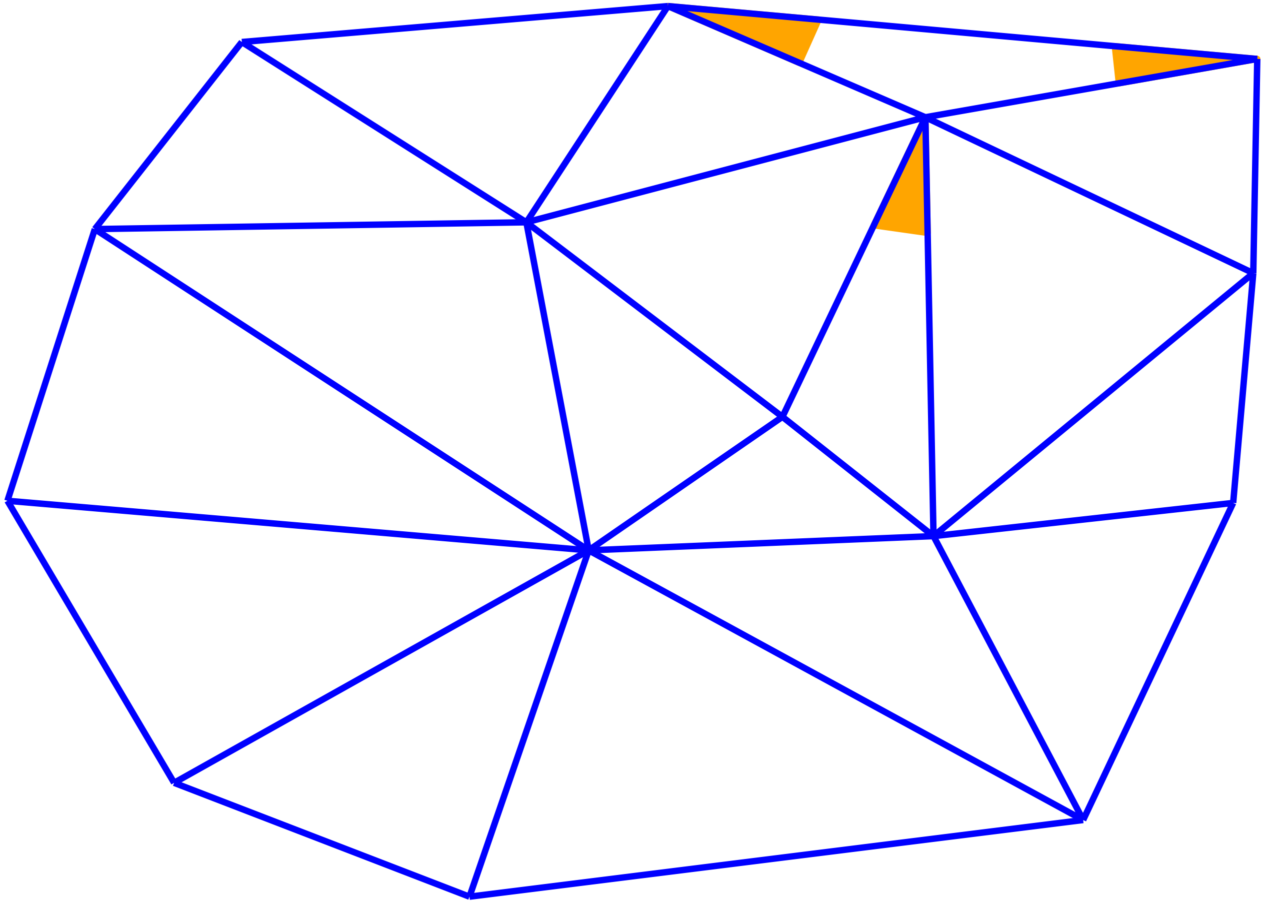
Delaunay Triangulation: max-min angle

Map: Triangulations $\longrightarrow \mathbb{R}^{6n-3k-4}$

smallest angle α_1

second smallest angle α_2

third smallest angle α_3



Delaunay Triangulation: max-min angle

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$(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{6n-3k-4})$

smallest angle α_1

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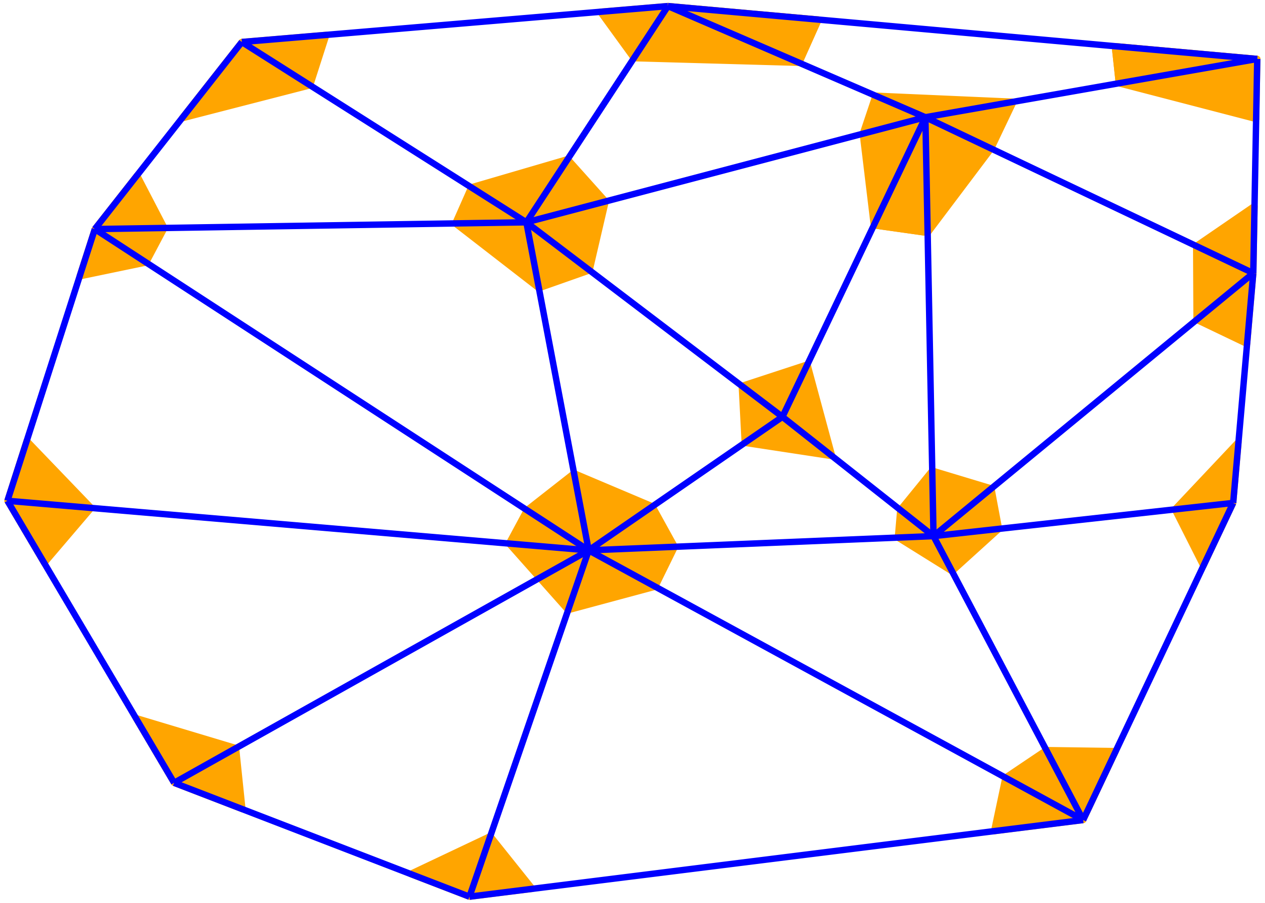

Delaunay Triangulation: max-min angle

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$(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{6n-3k-4})$

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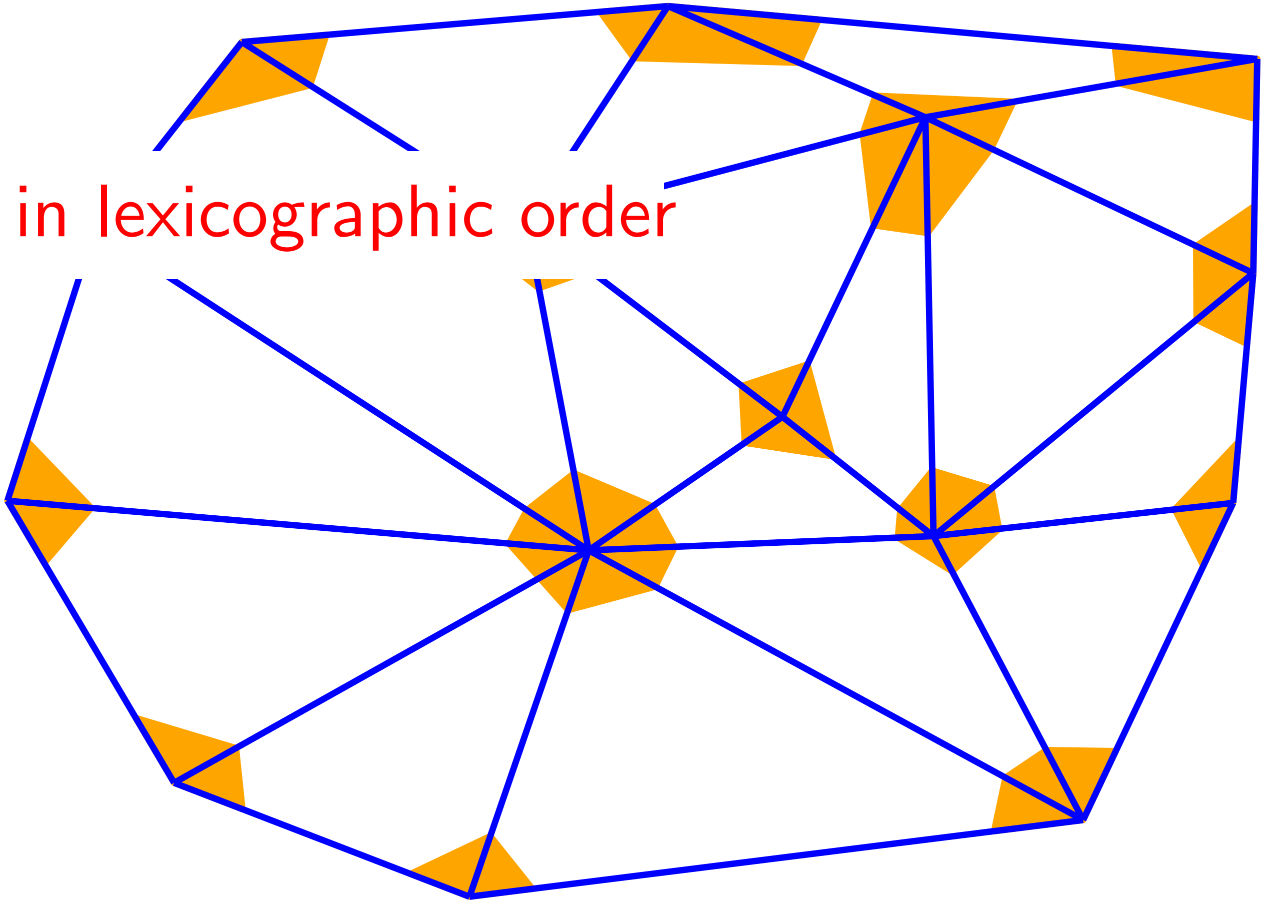

sort triangulations in lexicographic order

Delaunay Triangulation: max-min angle

Theorem:

Delaunay maximizes minimum angles (in lexicographic order)

Delaunay Triangulation: max-min angle

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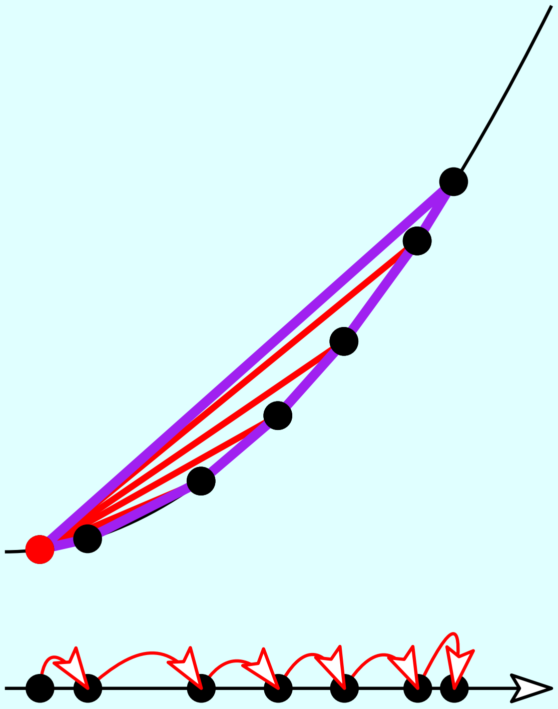
$\Longrightarrow T =$ Delaunay

Delaunay Triangulation: lower bound

Convex hull

Lower bound

A stupid algorithm for sorting numbers



$O(n)$ project on parabola

$f(n)$ compute convex hull

$O(n)$ find lowest point

$O(n)$ enumerate x coordinates in ccw CH order

Lower bound on sorting

$\Longrightarrow f(n) + O(n) \geq \Omega(n \log n)$

Delaunay Triangulation: lower bound

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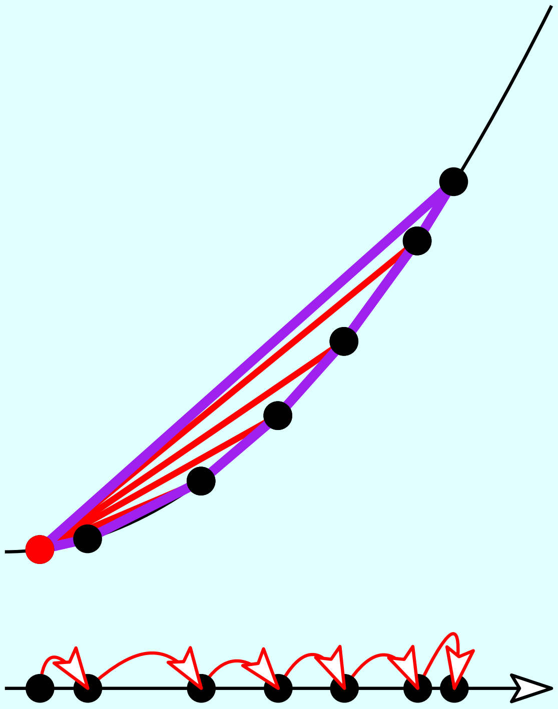

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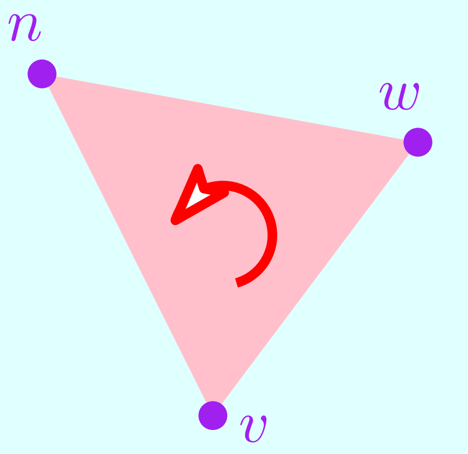
Delaunay Triangulation: incircle predicate

Convex hull

Orientation predicate

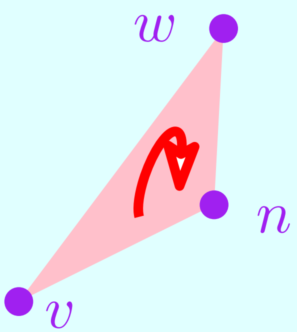
vwn + ?

$$\begin{vmatrix} x_w - x_v & x_n - x_v \\ y_w - y_v & y_n - y_v \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} > 0$$



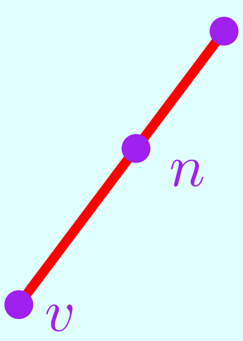
vwn - ?

$$\begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} < 0$$

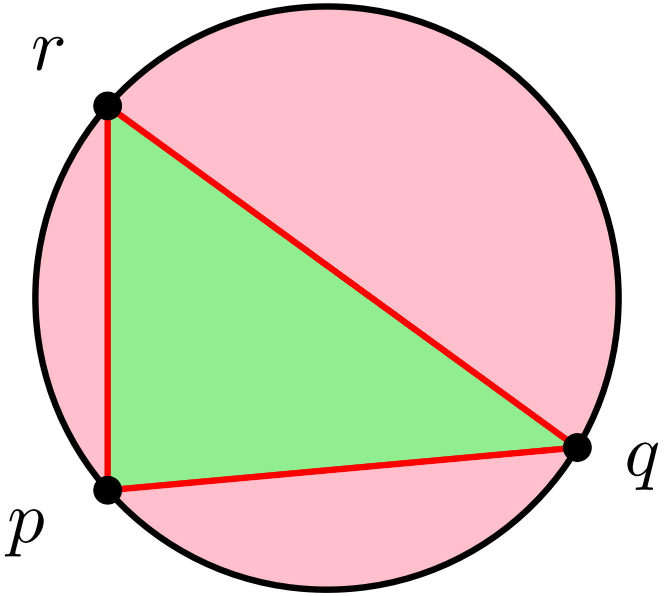


vwn 0 ?

$$\begin{vmatrix} 1 & 1 & 1 \\ x_v & x_w & x_n \\ y_v & y_w & y_n \end{vmatrix} = 0$$



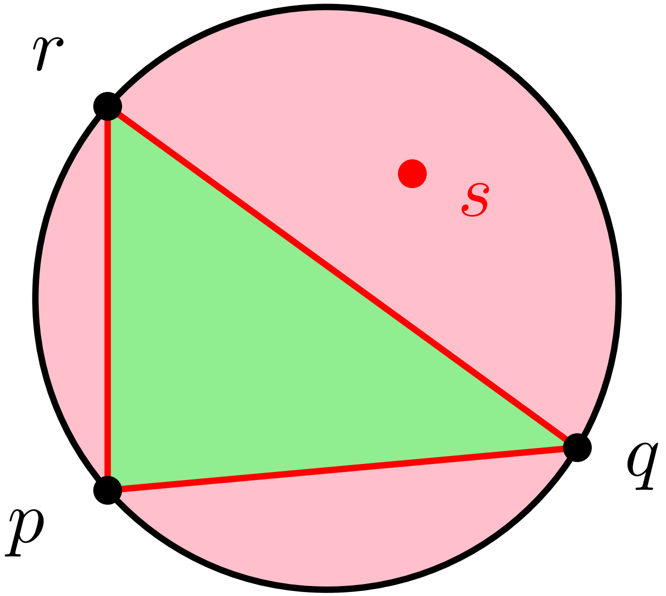
Delaunay Triangulation: incircle predicate



pqr ccw triangle

query s

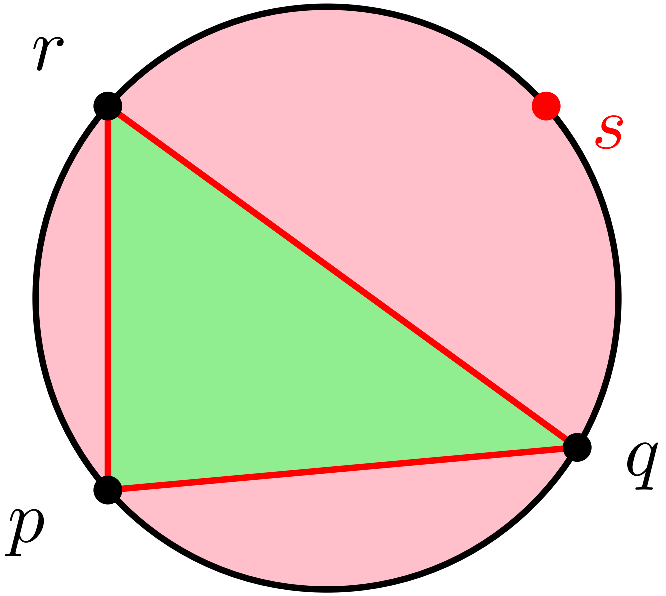
Delaunay Triangulation: incircle predicate



pqr ccw triangle

query s inside circumcircle

Delaunay Triangulation: incircle predicate



pqr ccw triangle

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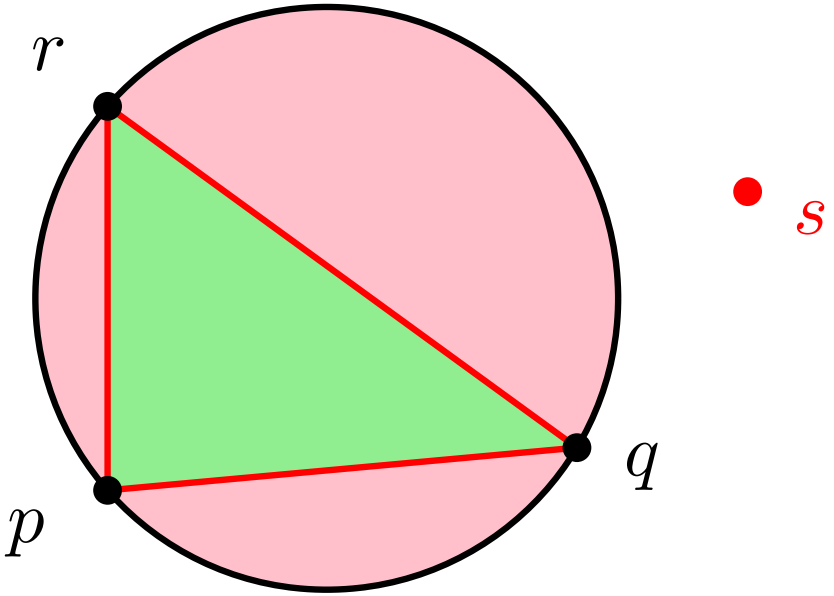

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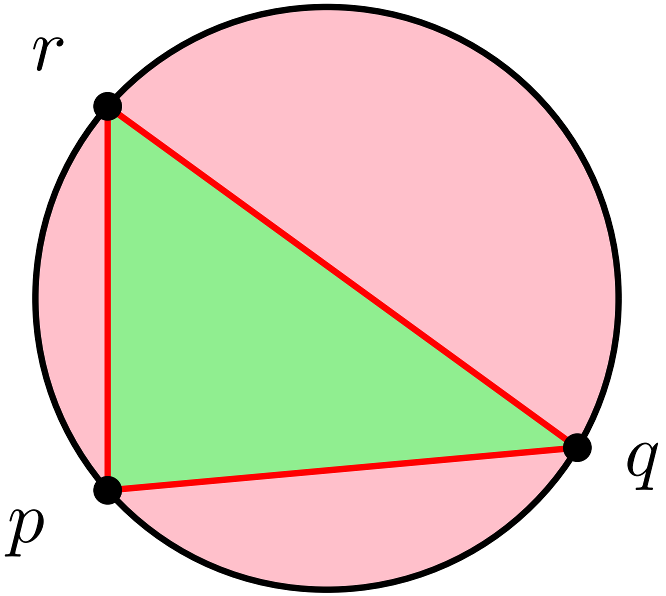

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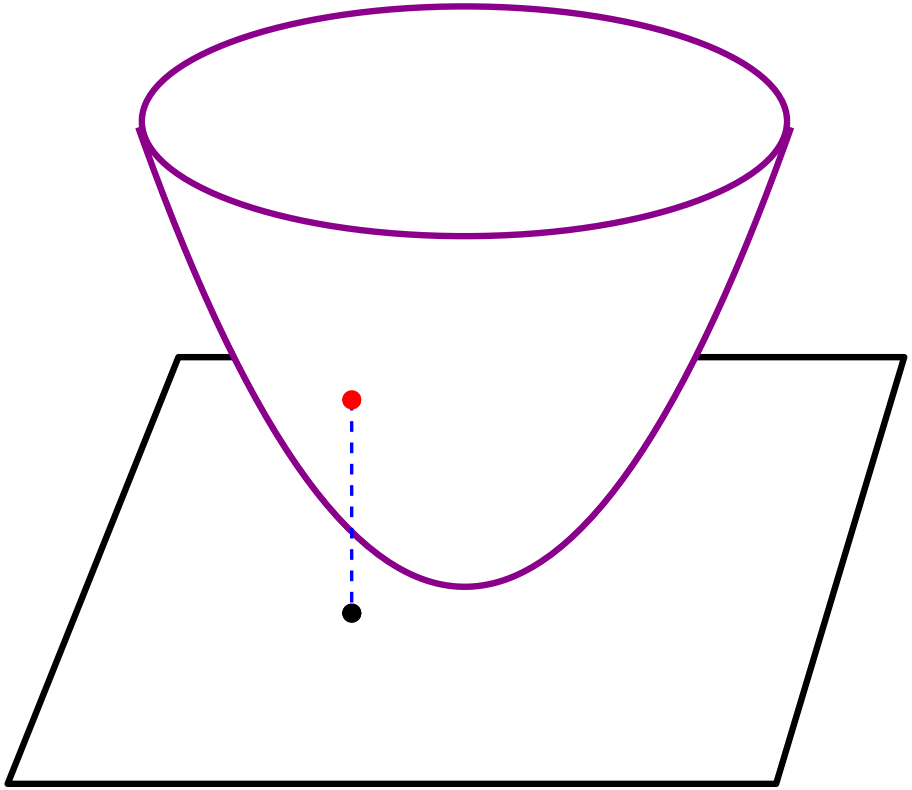
Delaunay Triangulation: incircle predicate

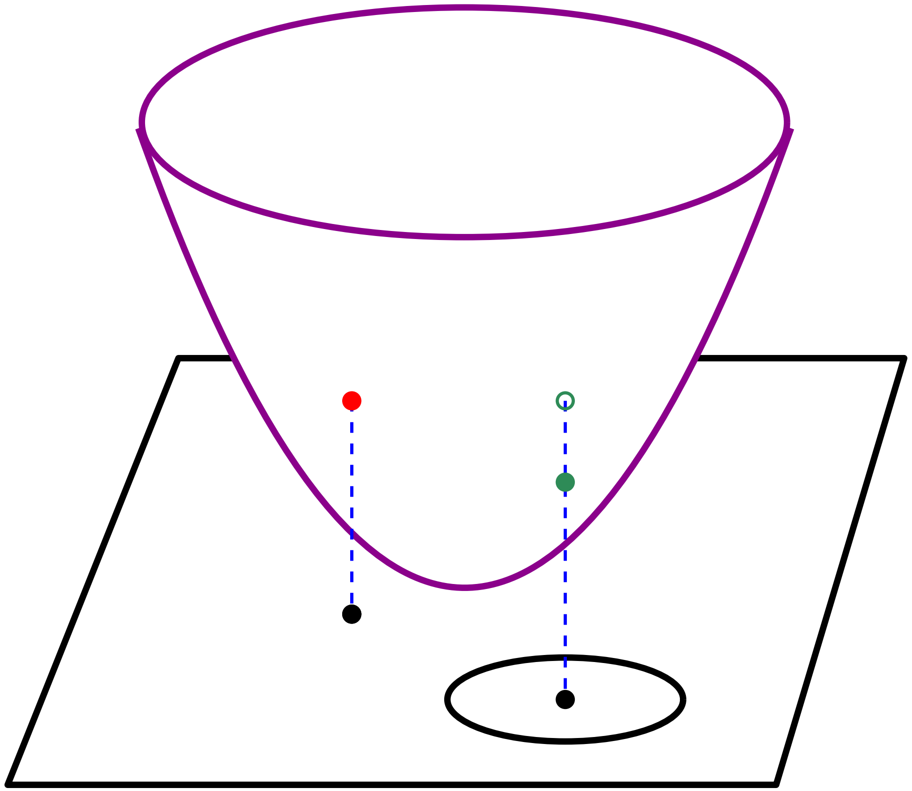
Space of circles



$$p = (x, y) \rightsquigarrow p^\star = (x, y, x^2 + y^2)$$

Delaunay Triangulation: incircle predicate

Space of circles



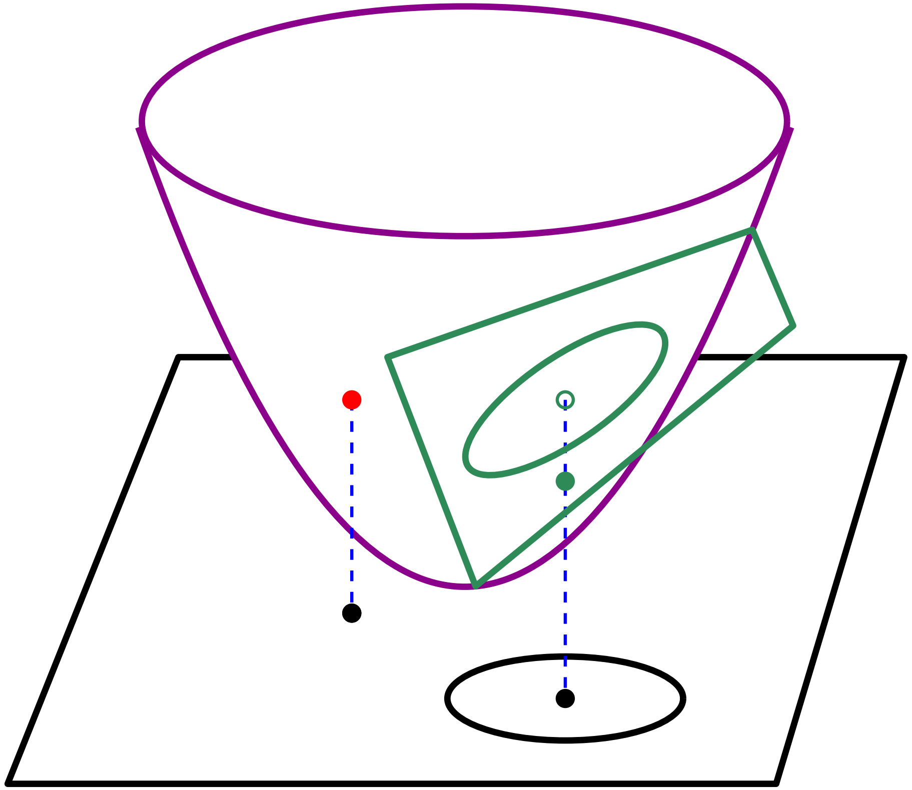
$$p = (x, y) \rightsquigarrow p^\star = (x, y, x^2 + y^2)$$

$$C : x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

$$\rightsquigarrow C^\star = (a, b, a^2 + b^2 - r^2)$$

Delaunay Triangulation: incircle predicate

Space of circles



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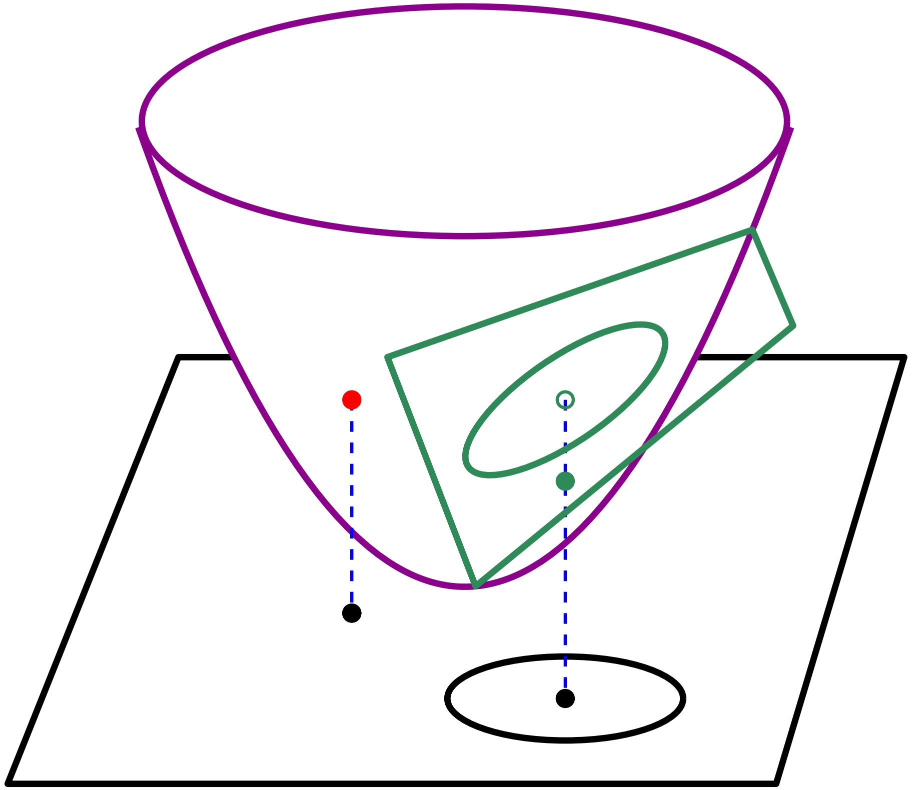
Delaunay Triangulation: incircle predicate

Space of circles

$p \in C \iff p^\star \in C^\dagger$



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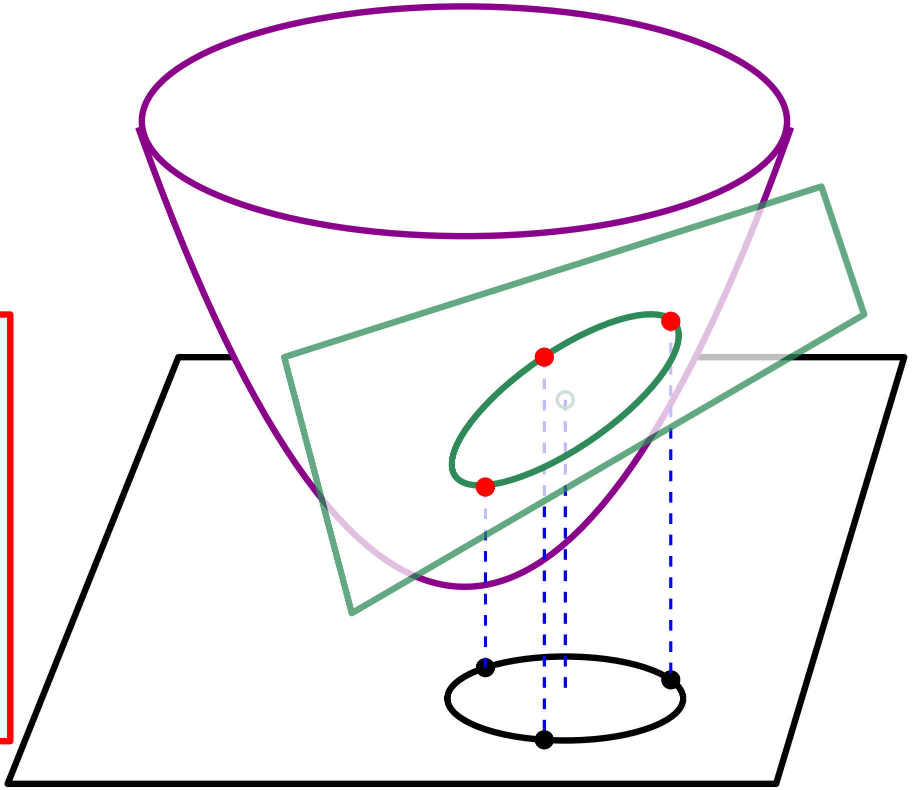

circle through pqr

$\leadsto$ plane through $p^\star q^\star r^\star$

$p = (x, y) \leadsto p^\star = (x, y, x^2 + y^2)$

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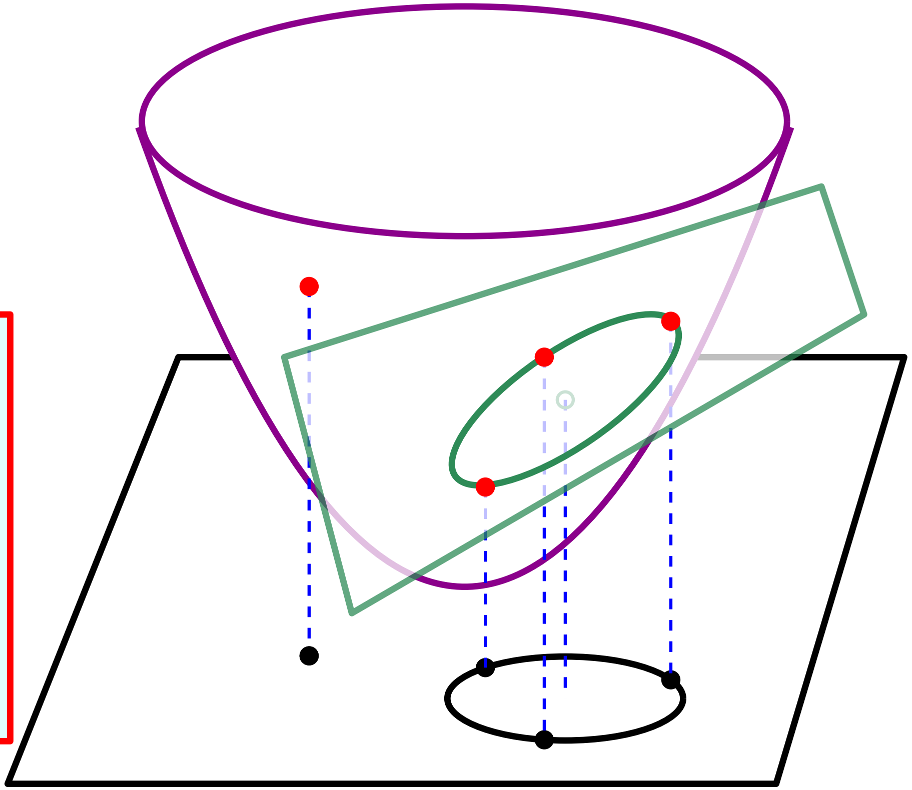
Delaunay Triangulation: incircle predicate

Space of circles

$p \in C \iff p^\star \in C^\dagger$

s inside/outside of circle through pqr

$\rightsquigarrow$ plane through $p^\star q^\star r^\star$ above/below $s^\star$



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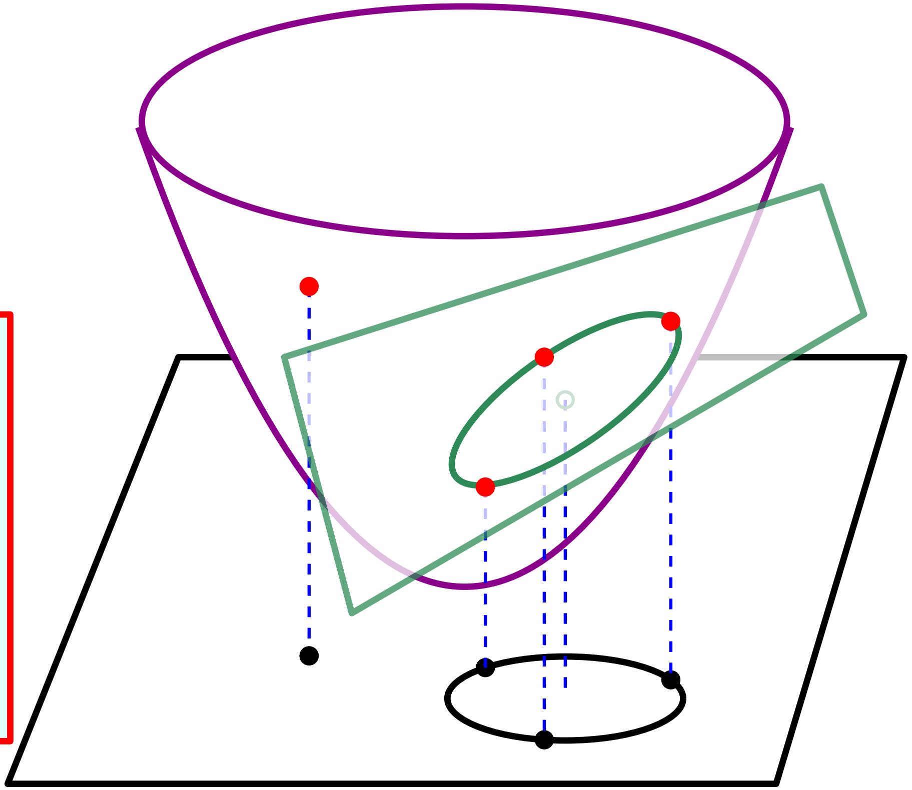
Space of circles



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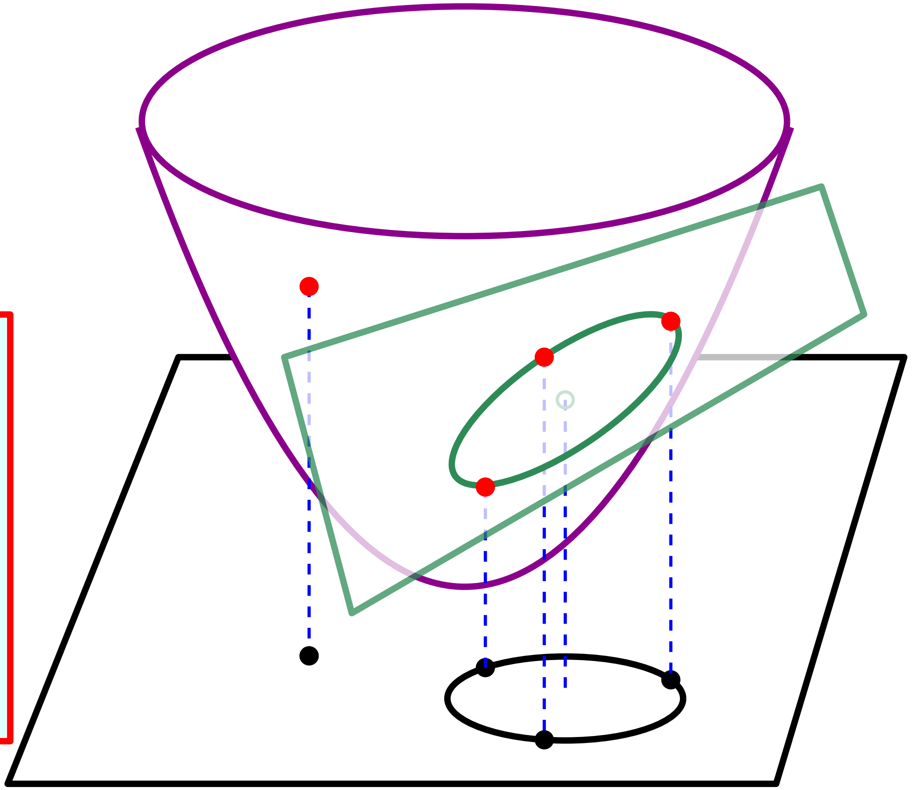

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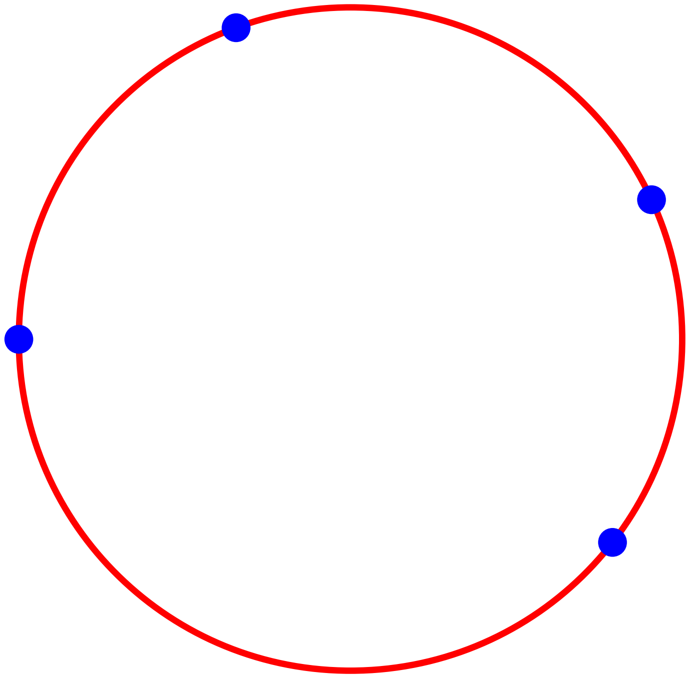
$$\operatorname{sign}\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_p & x_q & x_r & x_s \\ y_p & y_q & y_r & y_s \\ x_p^2+y_p^2 & x_q^2+y_q^2 & x_r^2+y_r^2 & x_s^2+y_s^2 \end{vmatrix}$$

Delaunay Triangulation: incircle predicate

Degeneracies



Delaunay Triangulation: incircle predicate

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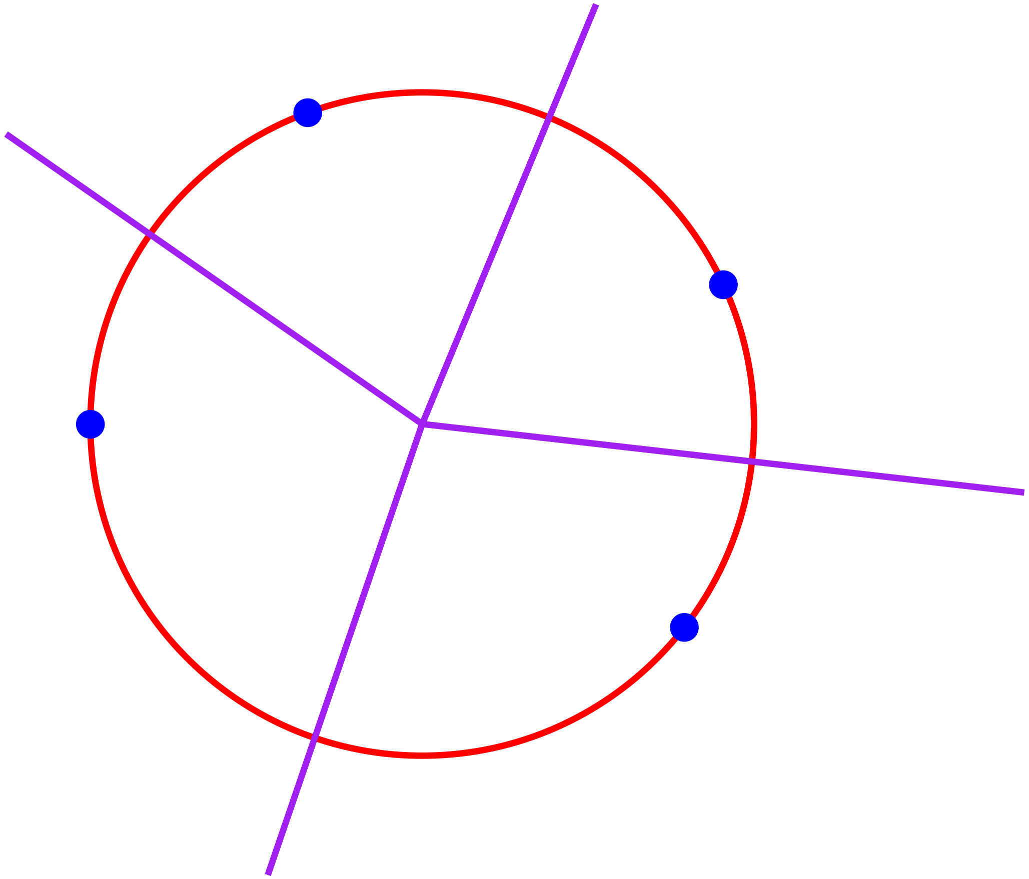
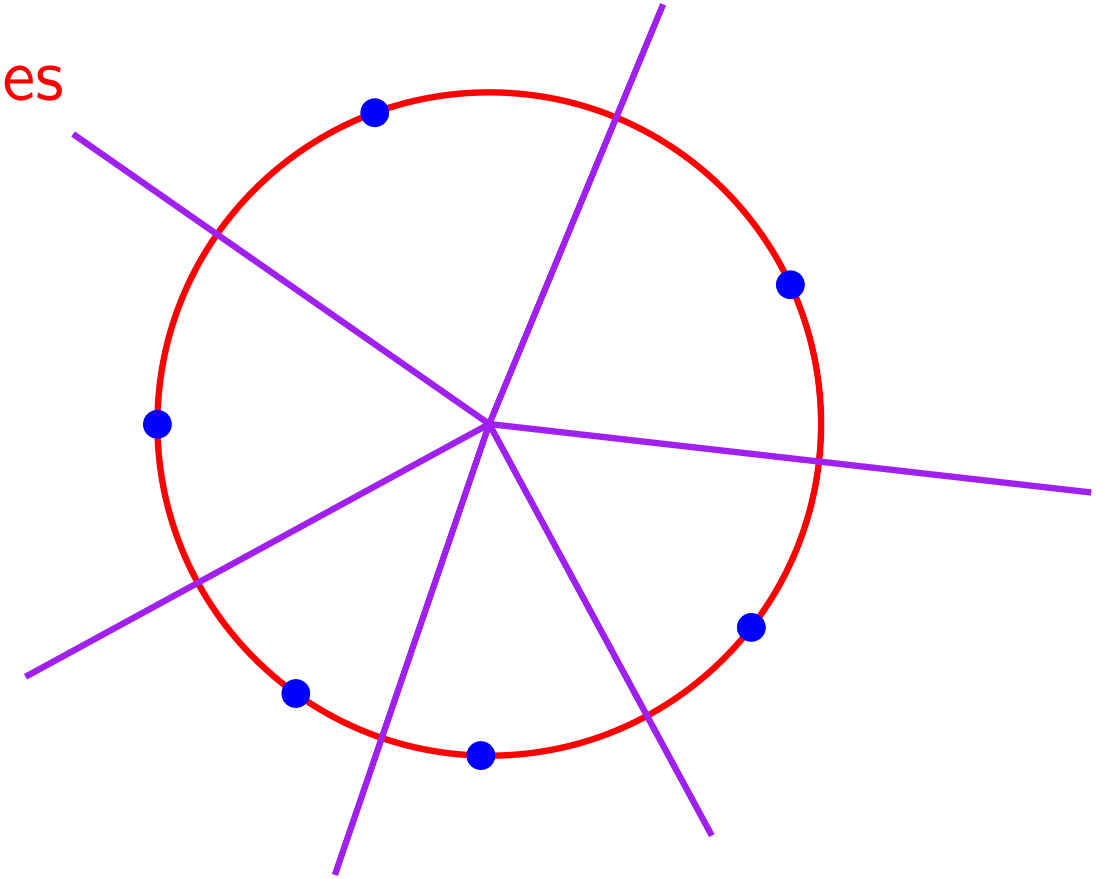


Degree 4 vertex in Voronoi diagram

Delaunay Triangulation: incircle predicate

Degeneracies



Degree ~~4~~ d vertex in Voronoi diagram

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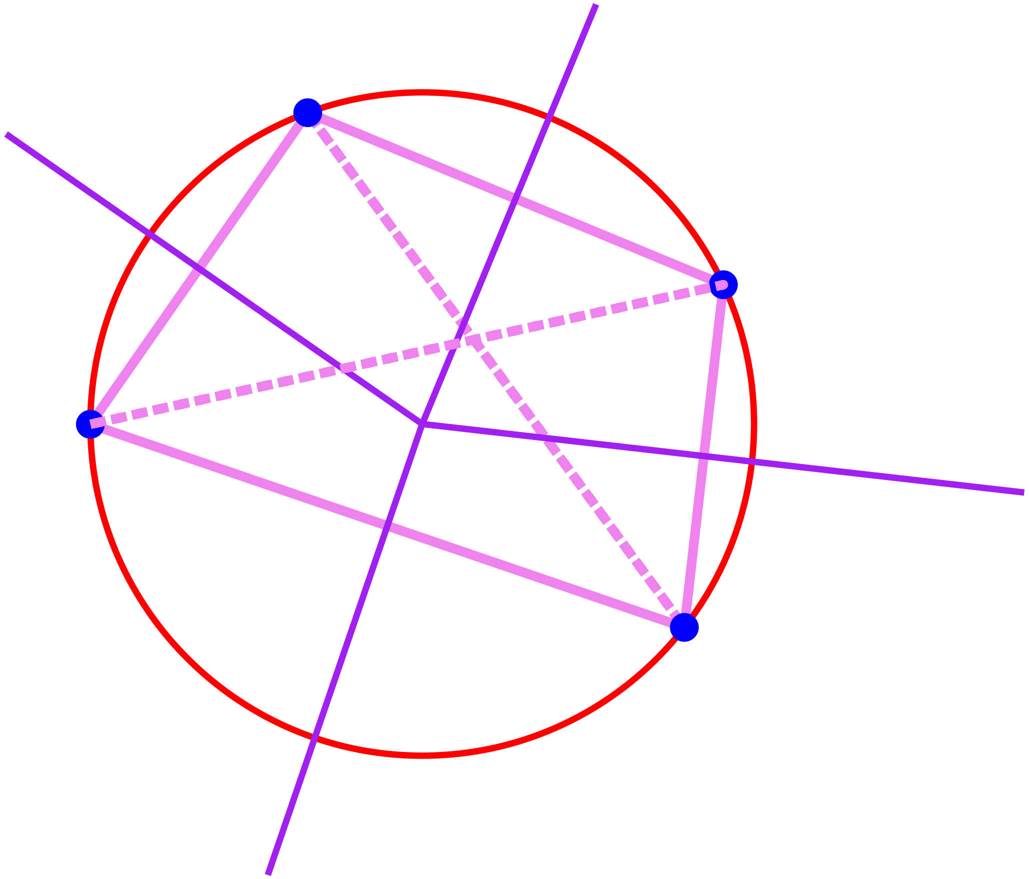
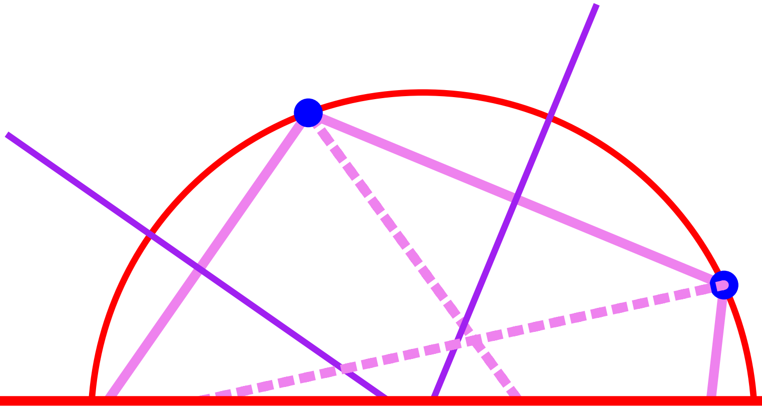


Degree 4 vertex in Voronoi diagram

Delaunay quad ? random diagonal ?

Delaunay Triangulation: incircle predicate

Degeneracies



Teaser robustness lecture

assume no degeneracies for a while

Degree 4 vertex in Voronoi diagram

Delaunay quad ? random diagonal ?

Delaunay Triangulation:

Teaser CGAL lecture

Data structure for (Delaunay) triangulation ?

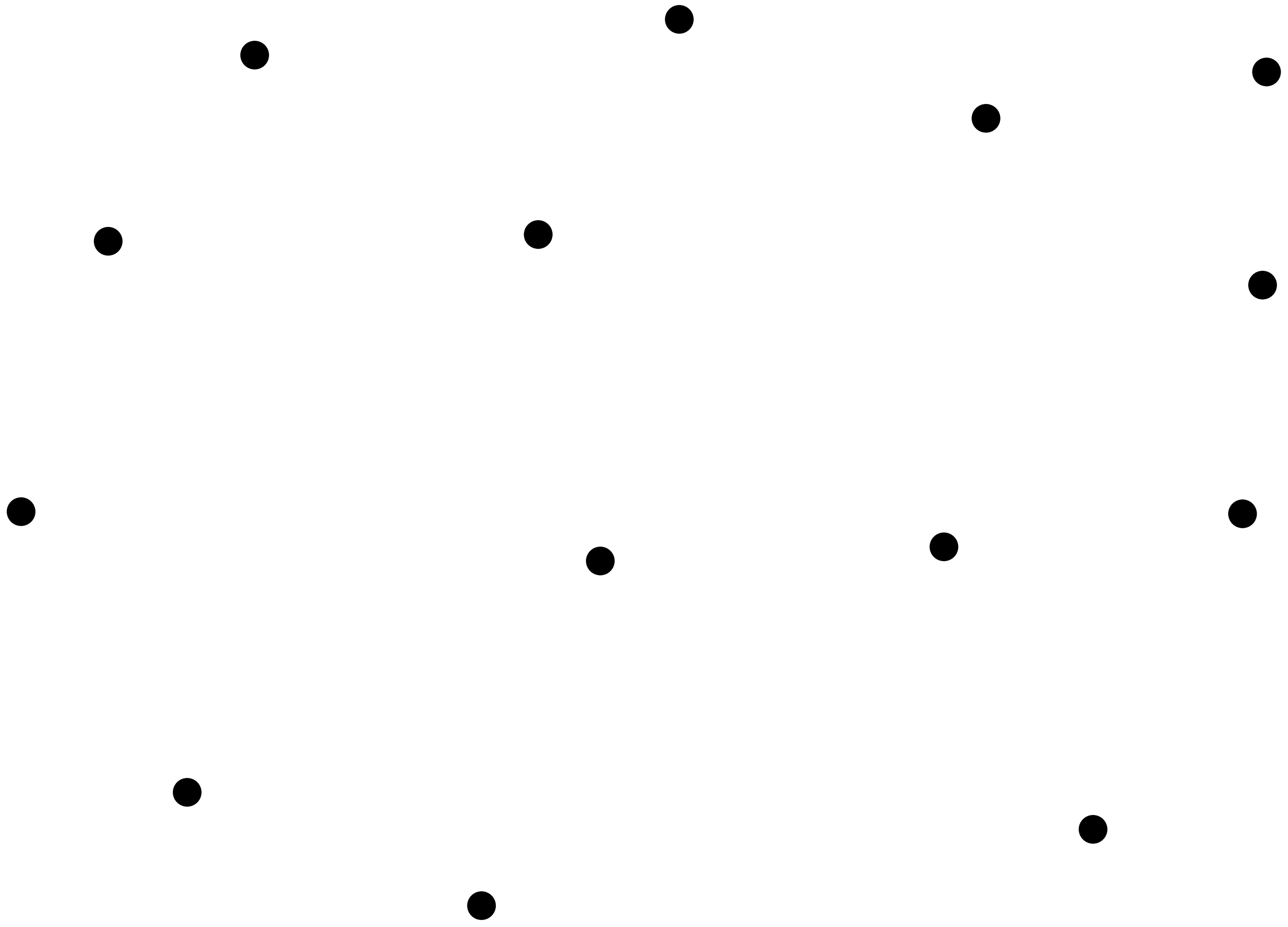
Representing incidences

Representing hull boundary

Representing user's data put colors in triangles

...

Delaunay Triangulation: Diagonal flipping



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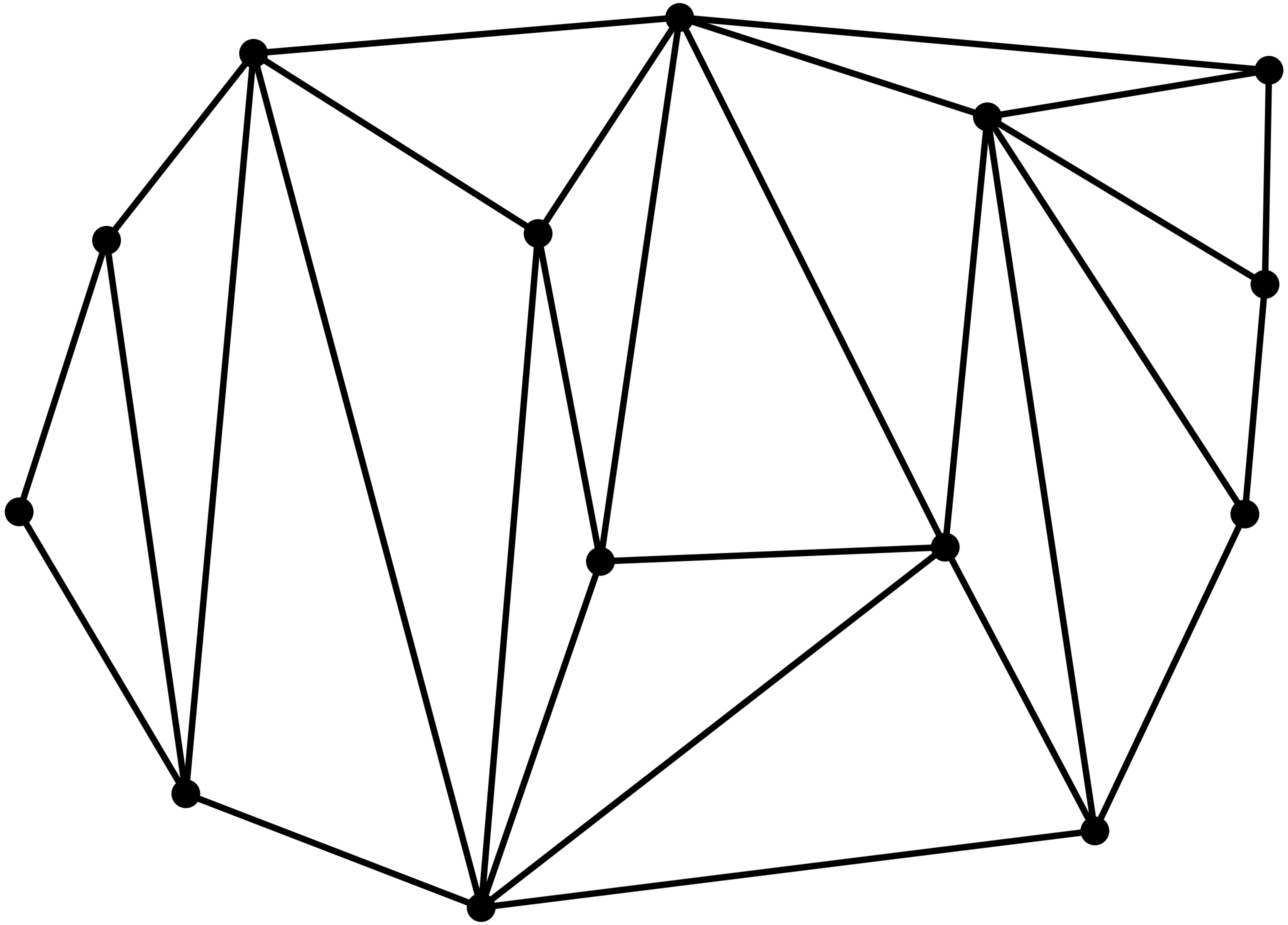
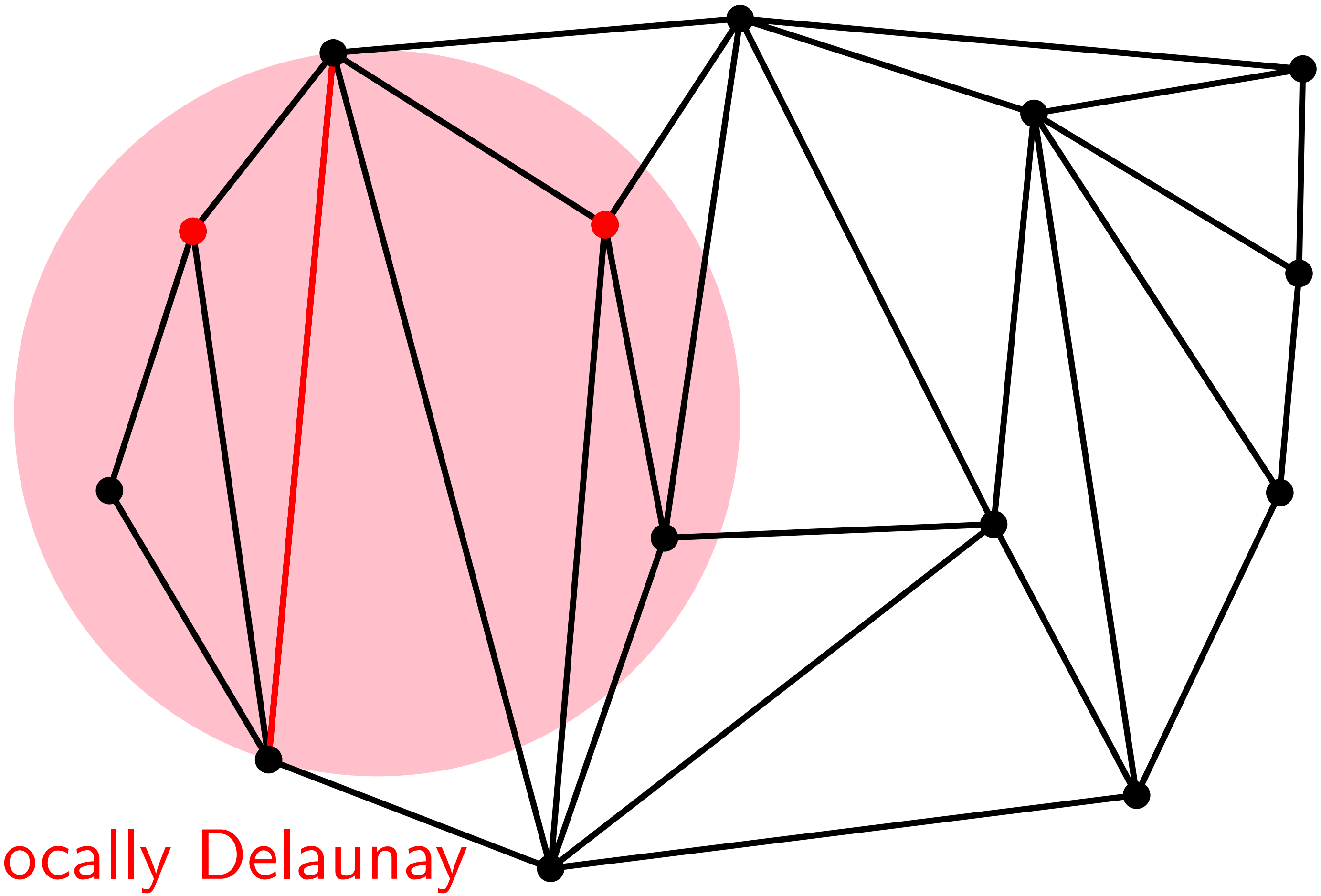
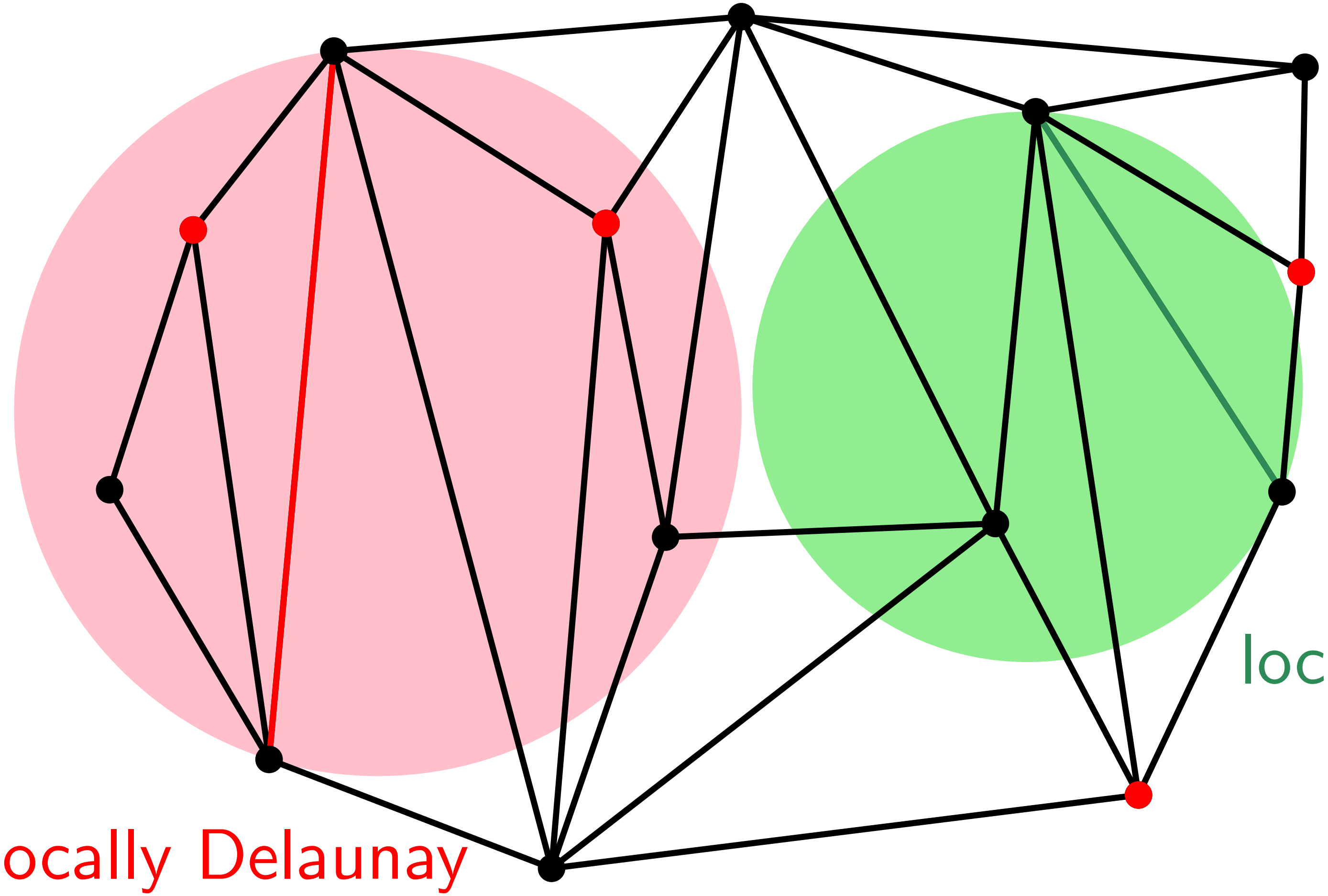


Delaunay Triangulation: Diagonal flipping



non locally Delaunay

Delaunay Triangulation: Diagonal flipping



locally Delaunay

non locally Delaunay

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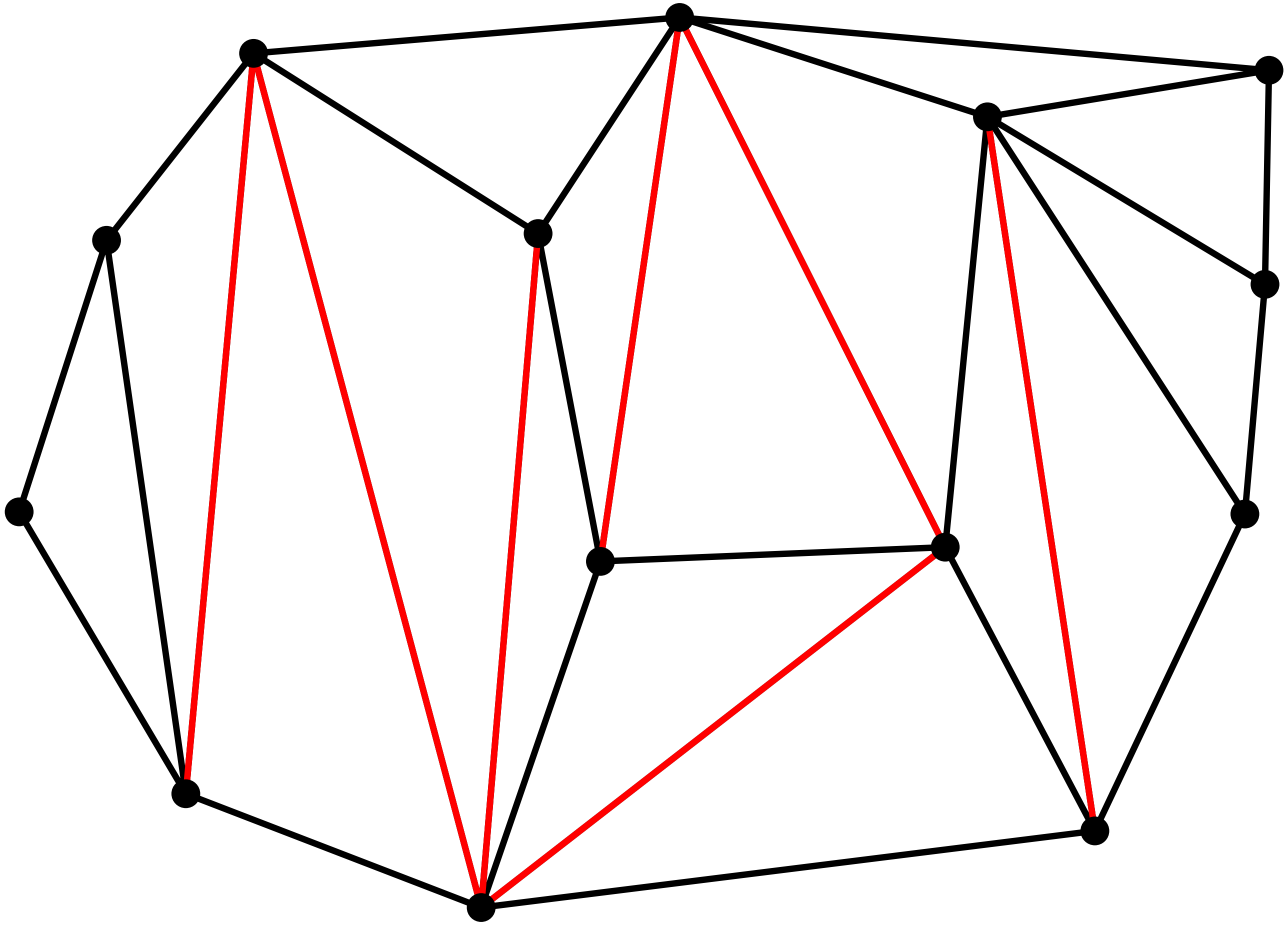

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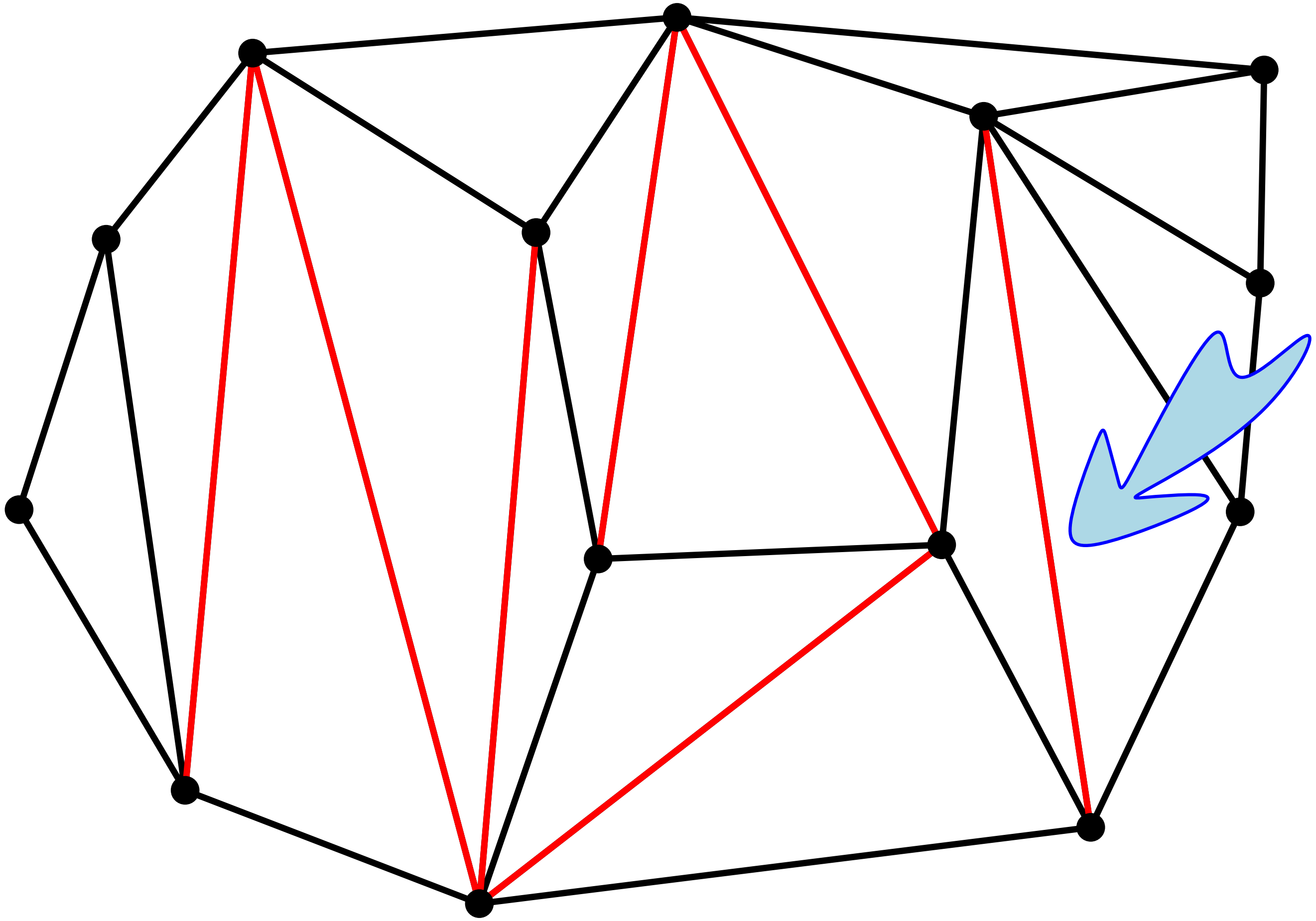

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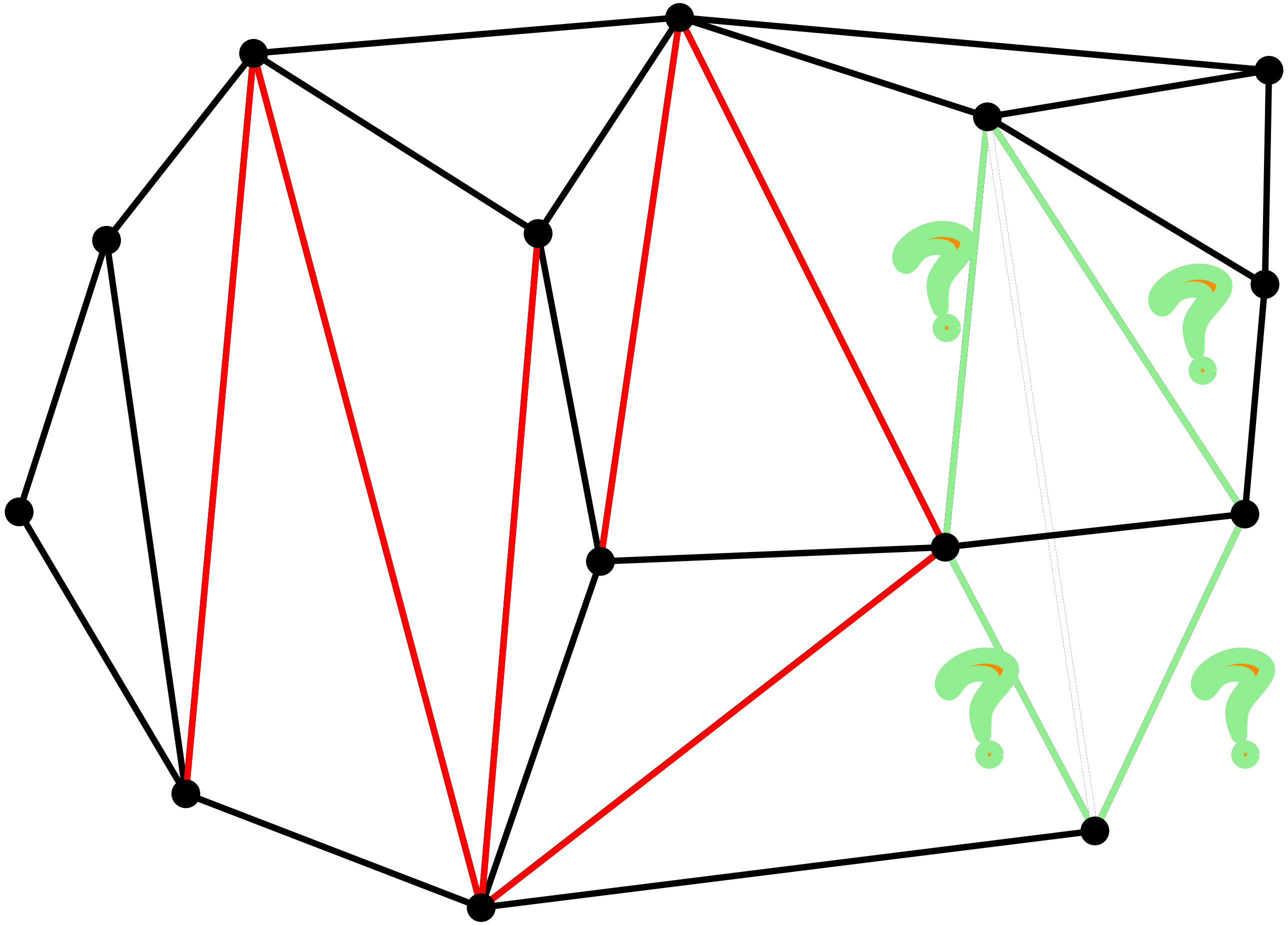
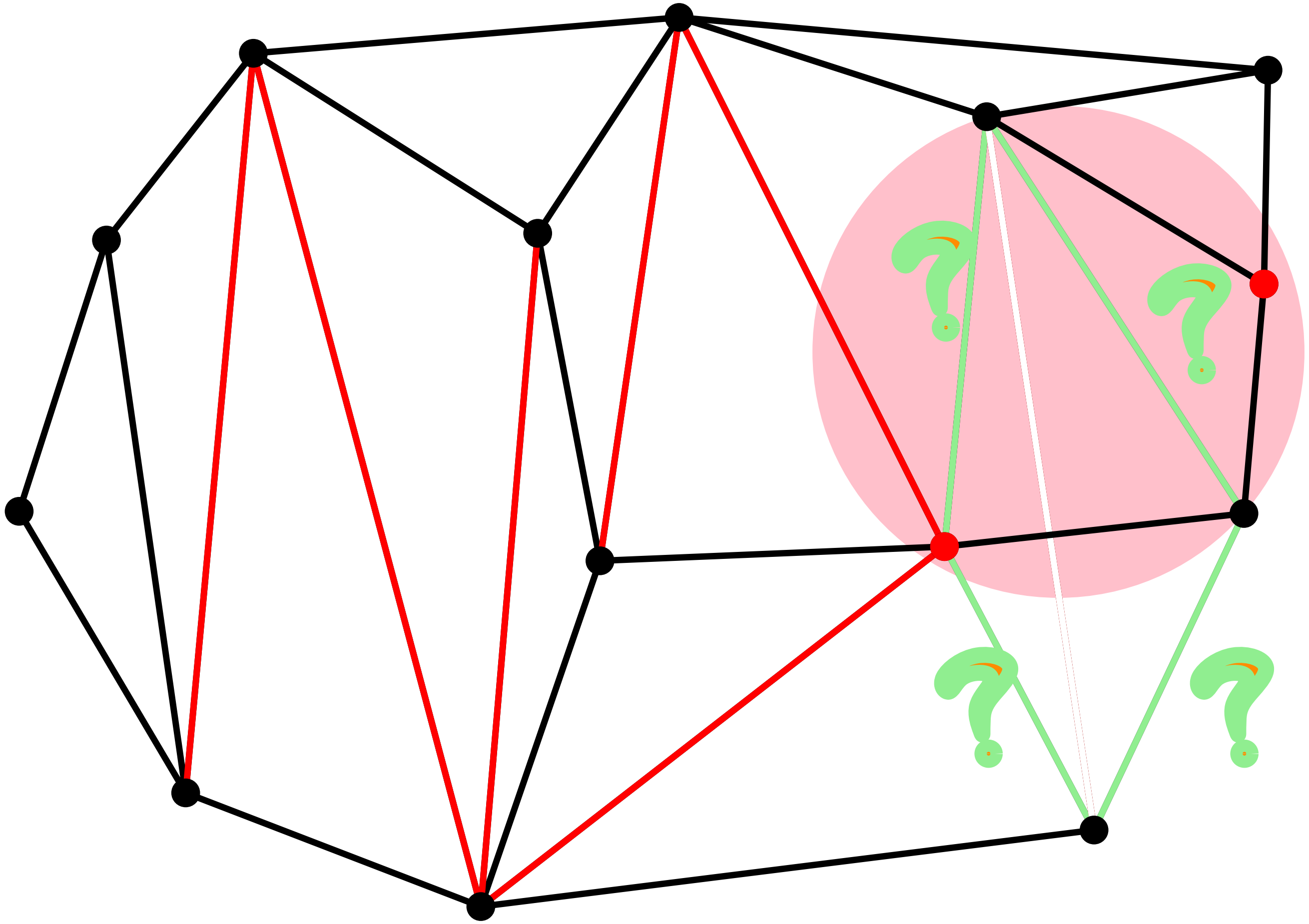
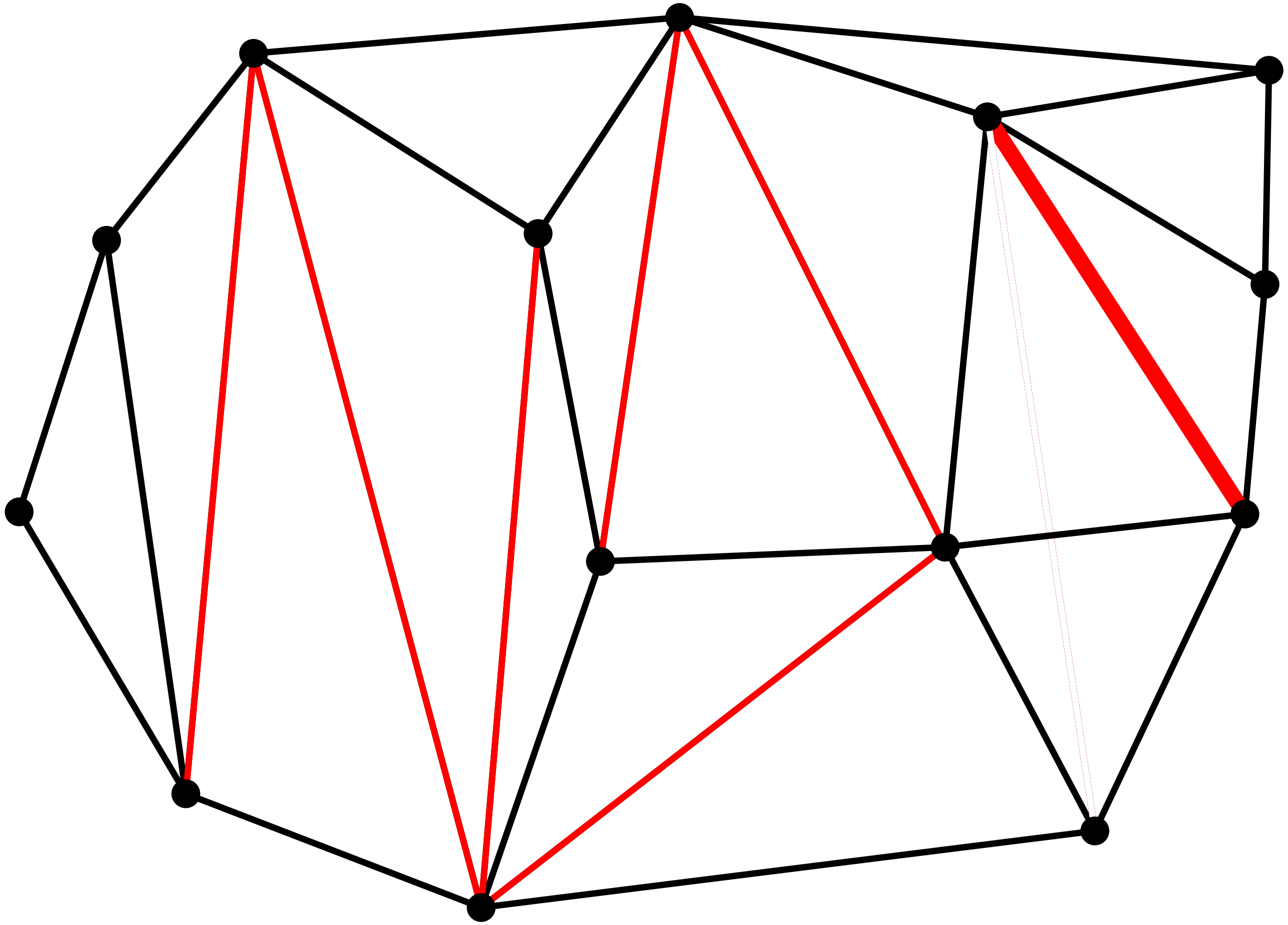


Delaunay Triangulation: Diagonal flipping

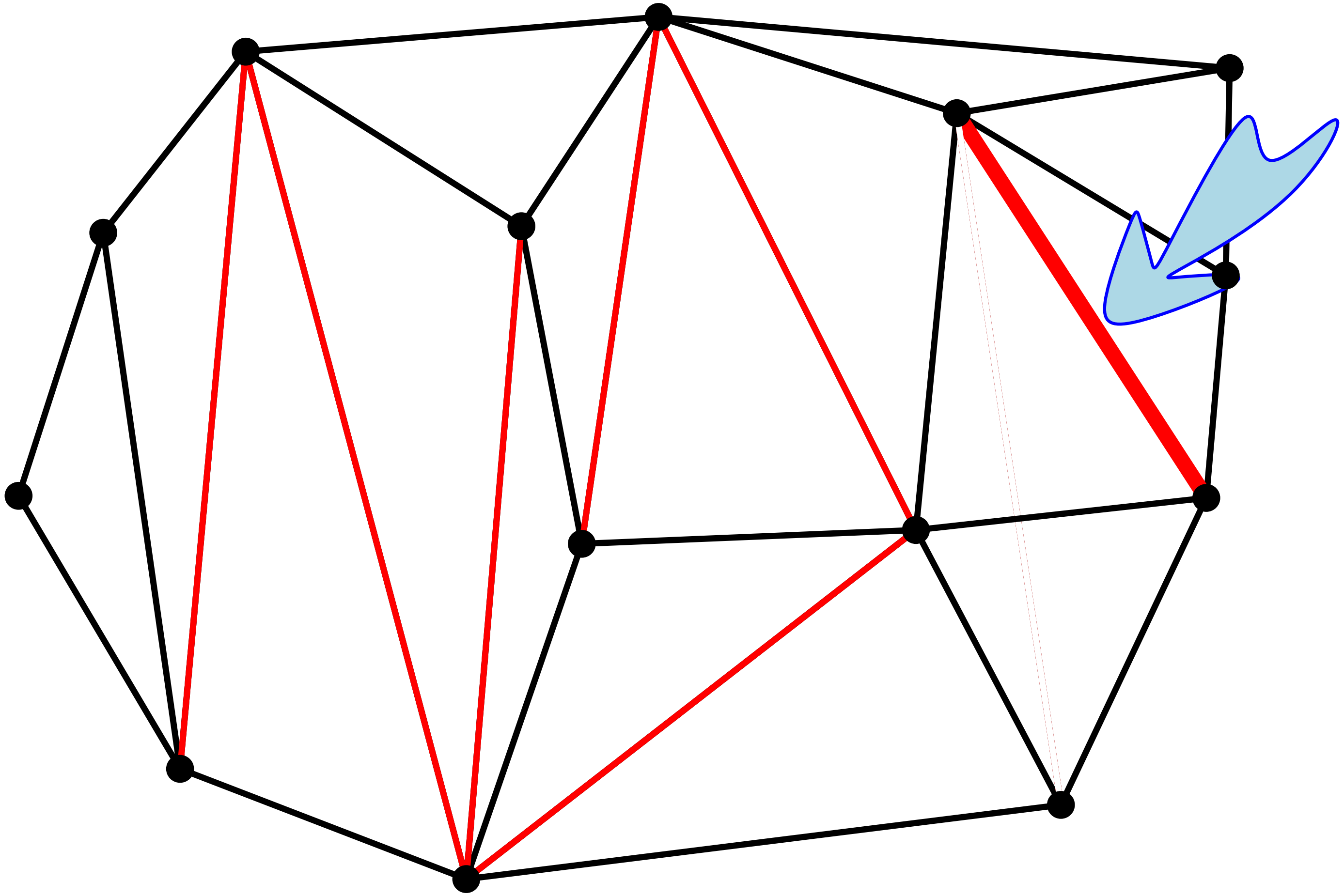


check edges of quadrilateral

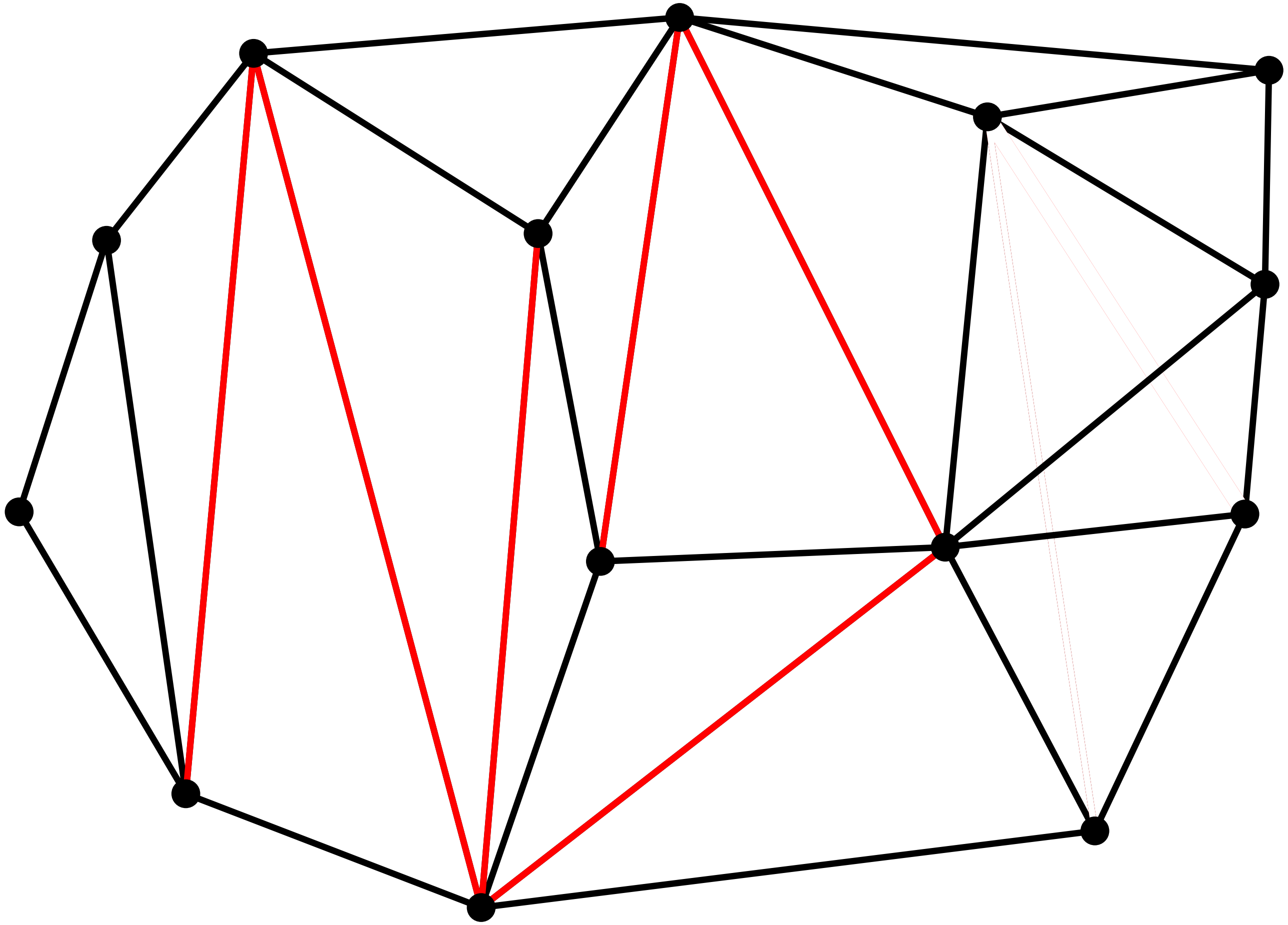
Delaunay Triangulation: Diagonal flipping



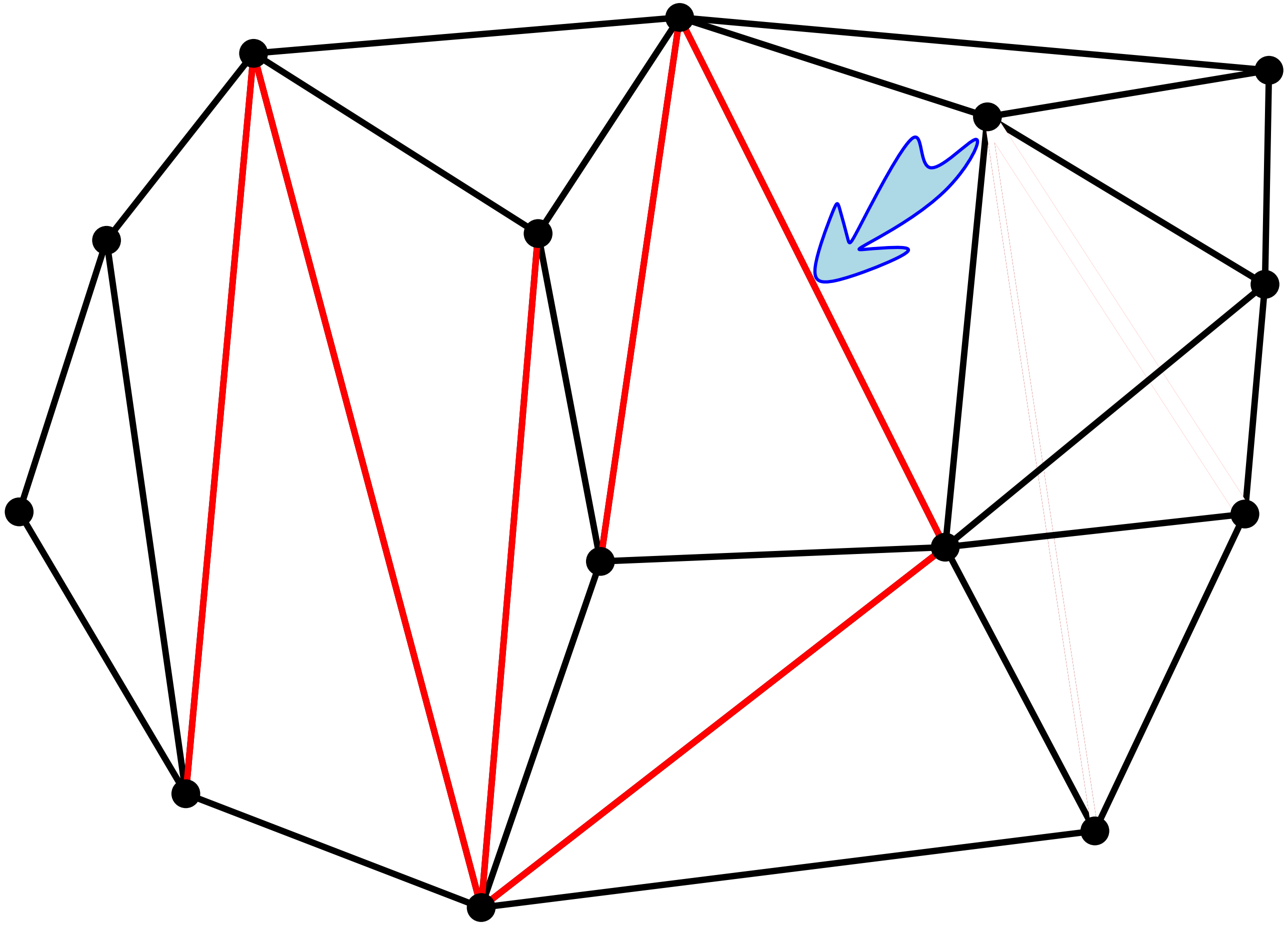
Delaunay Triangulation: Diagonal flipping



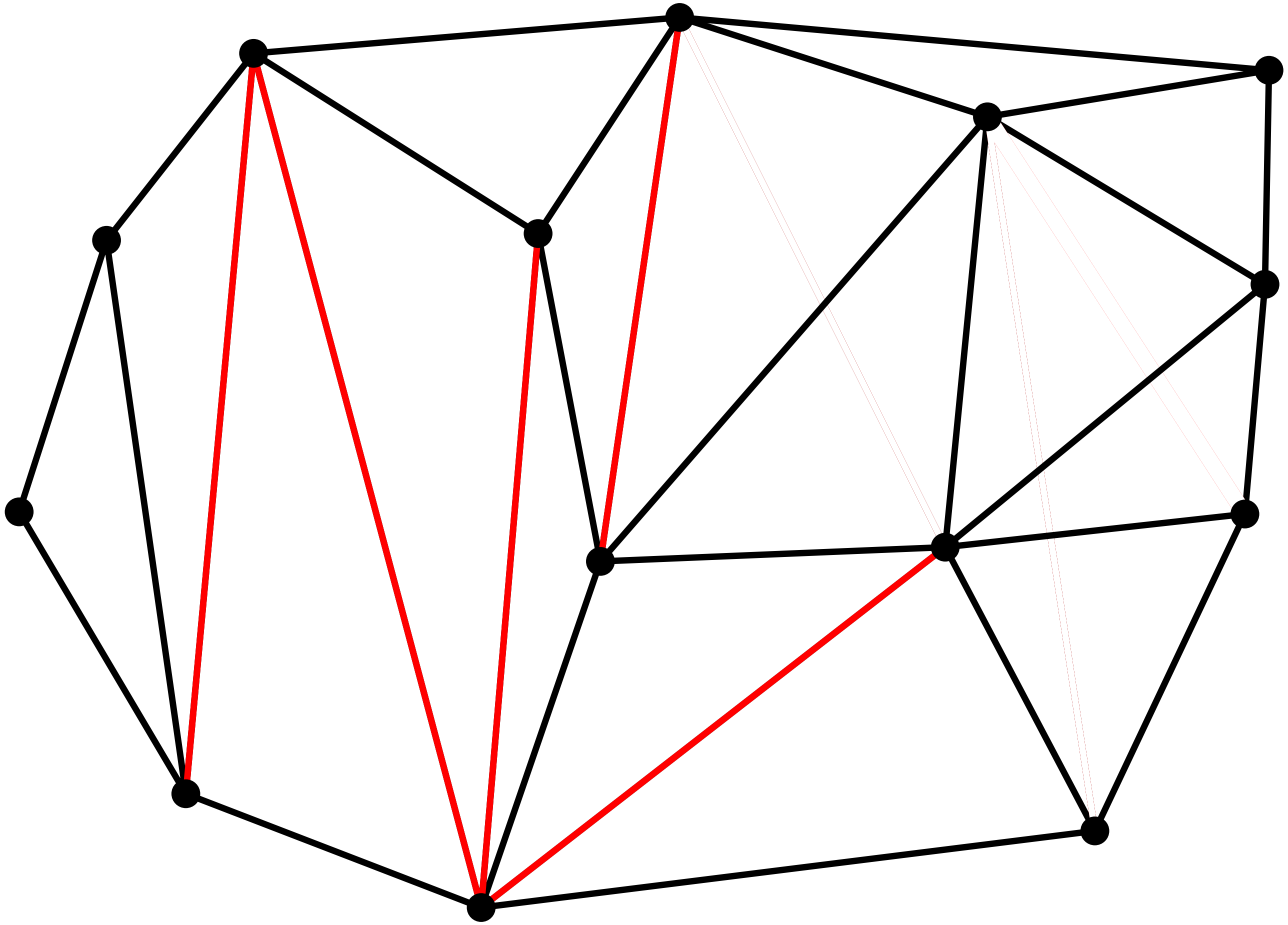
Delaunay Triangulation: Diagonal flipping



Delaunay Triangulation: Diagonal flipping



Delaunay Triangulation: Diagonal flipping



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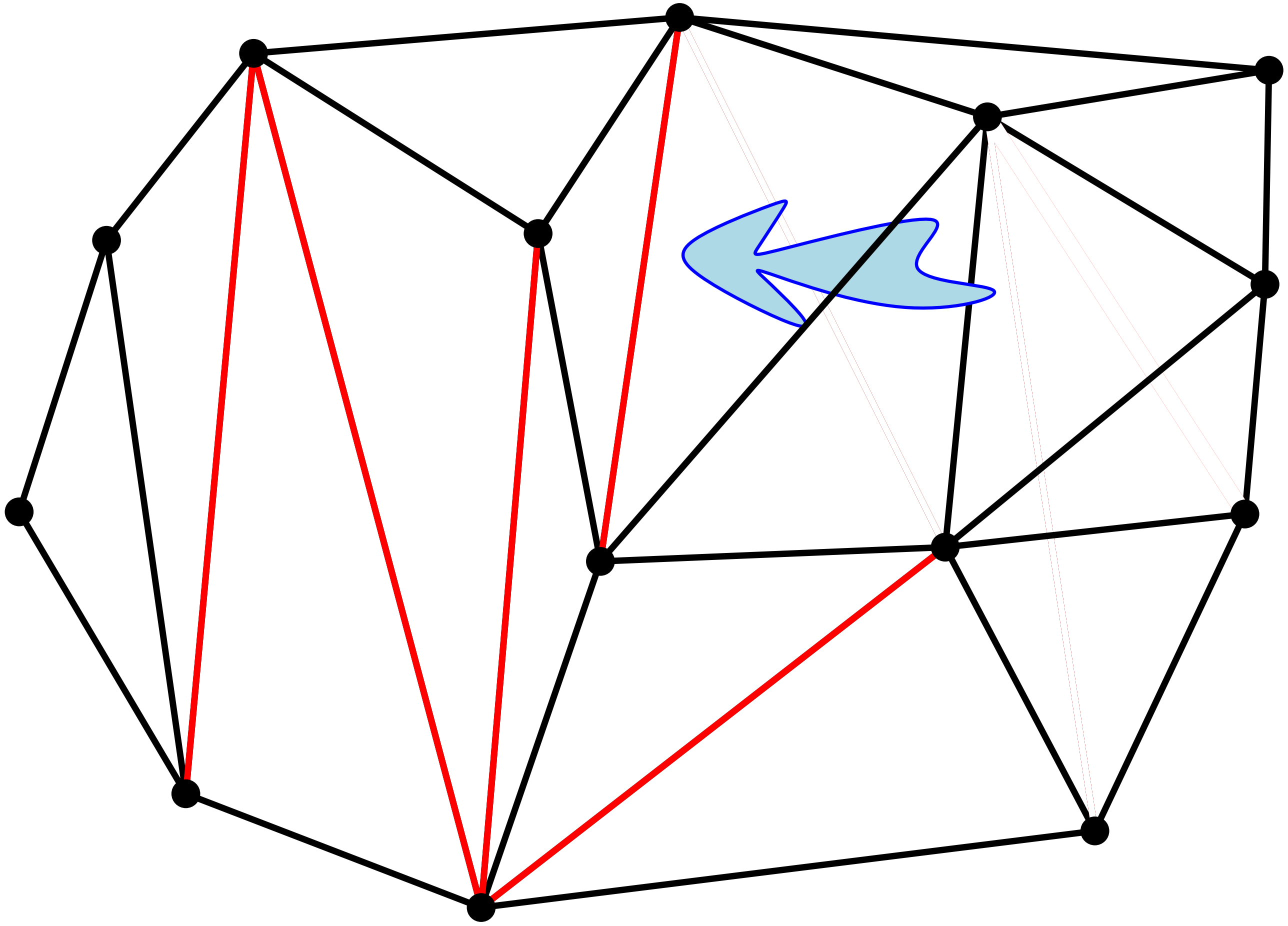
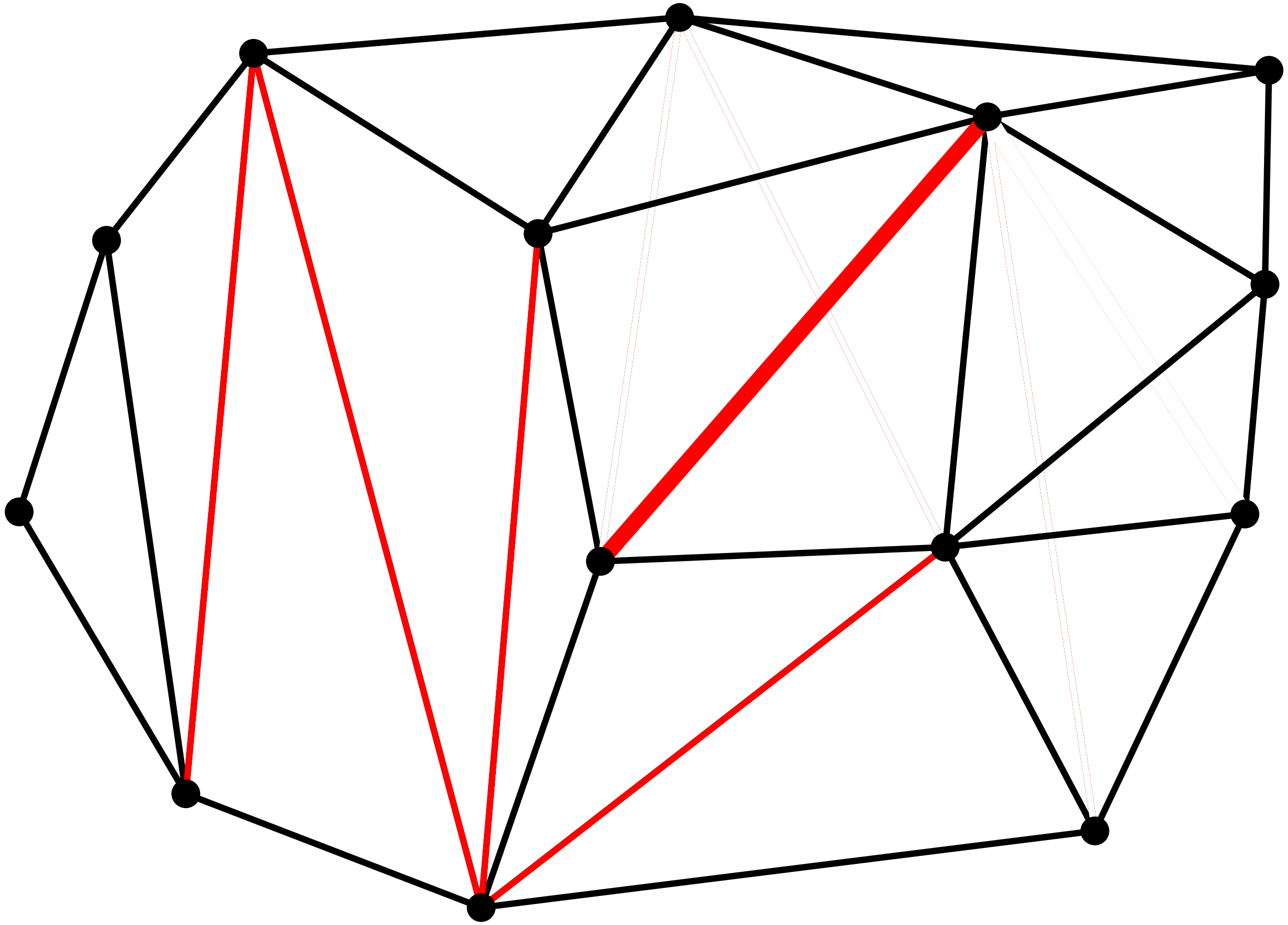
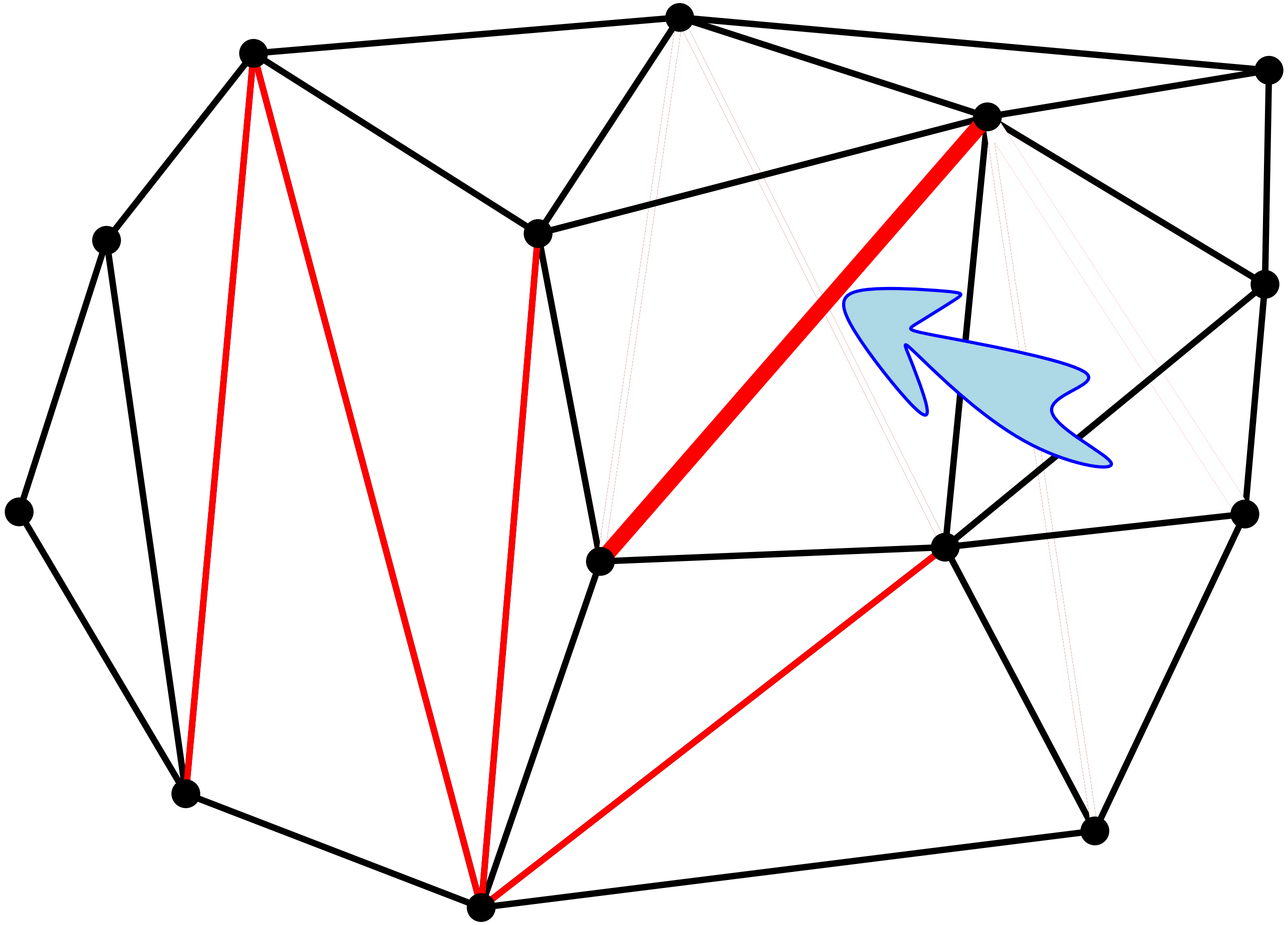


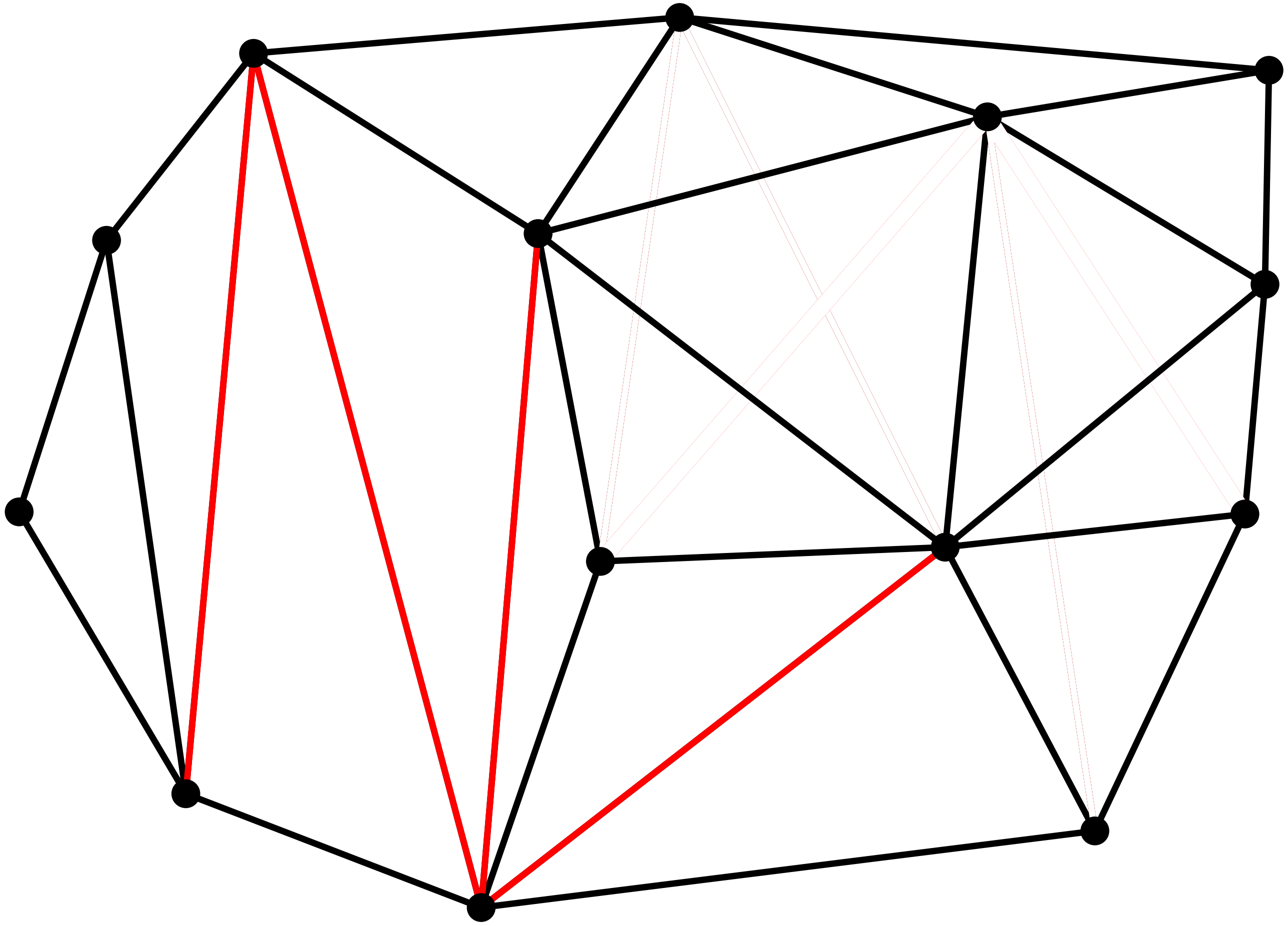
Delaunay Triangulation: Diagonal flipping



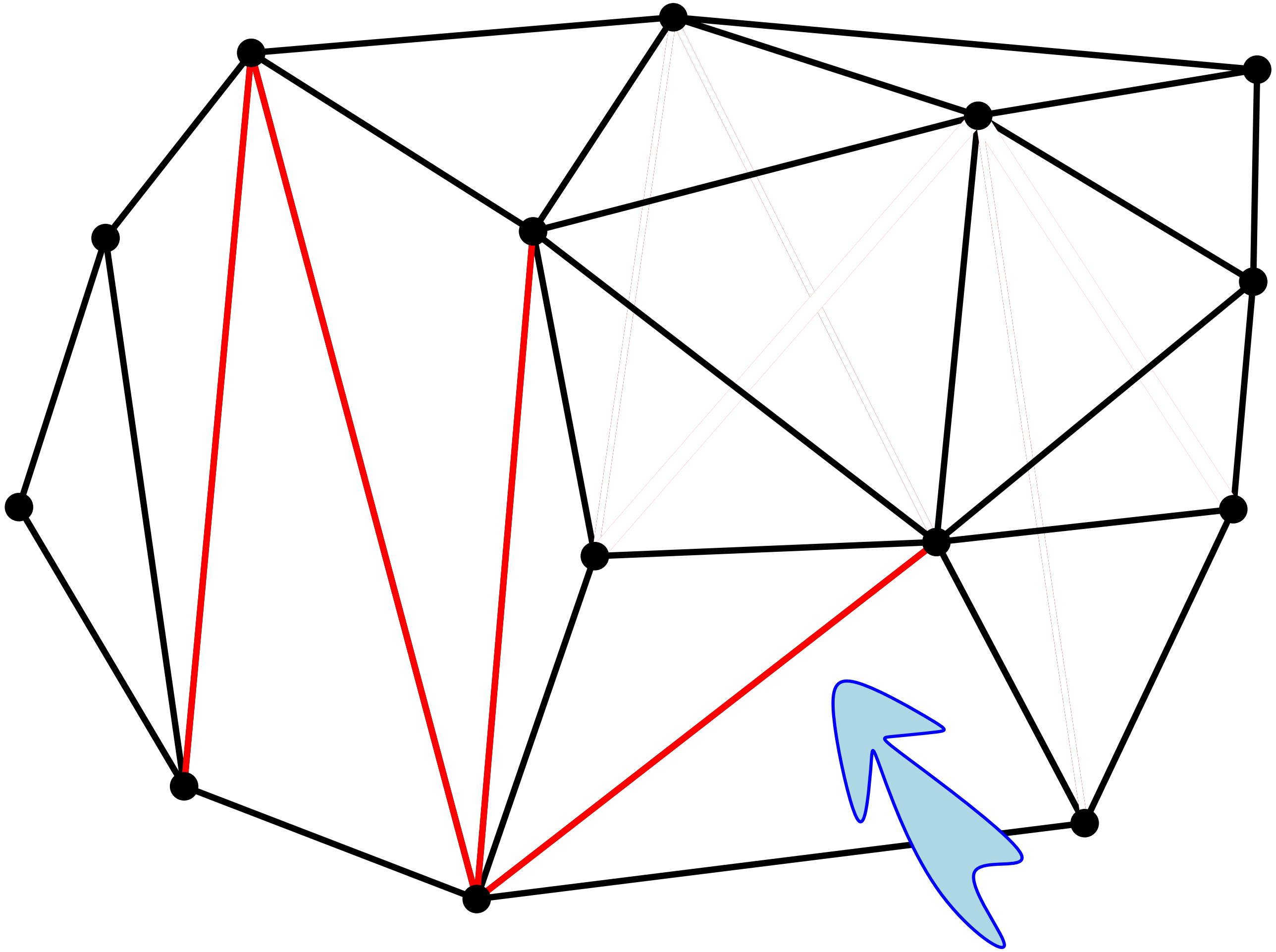
Delaunay Triangulation: Diagonal flipping



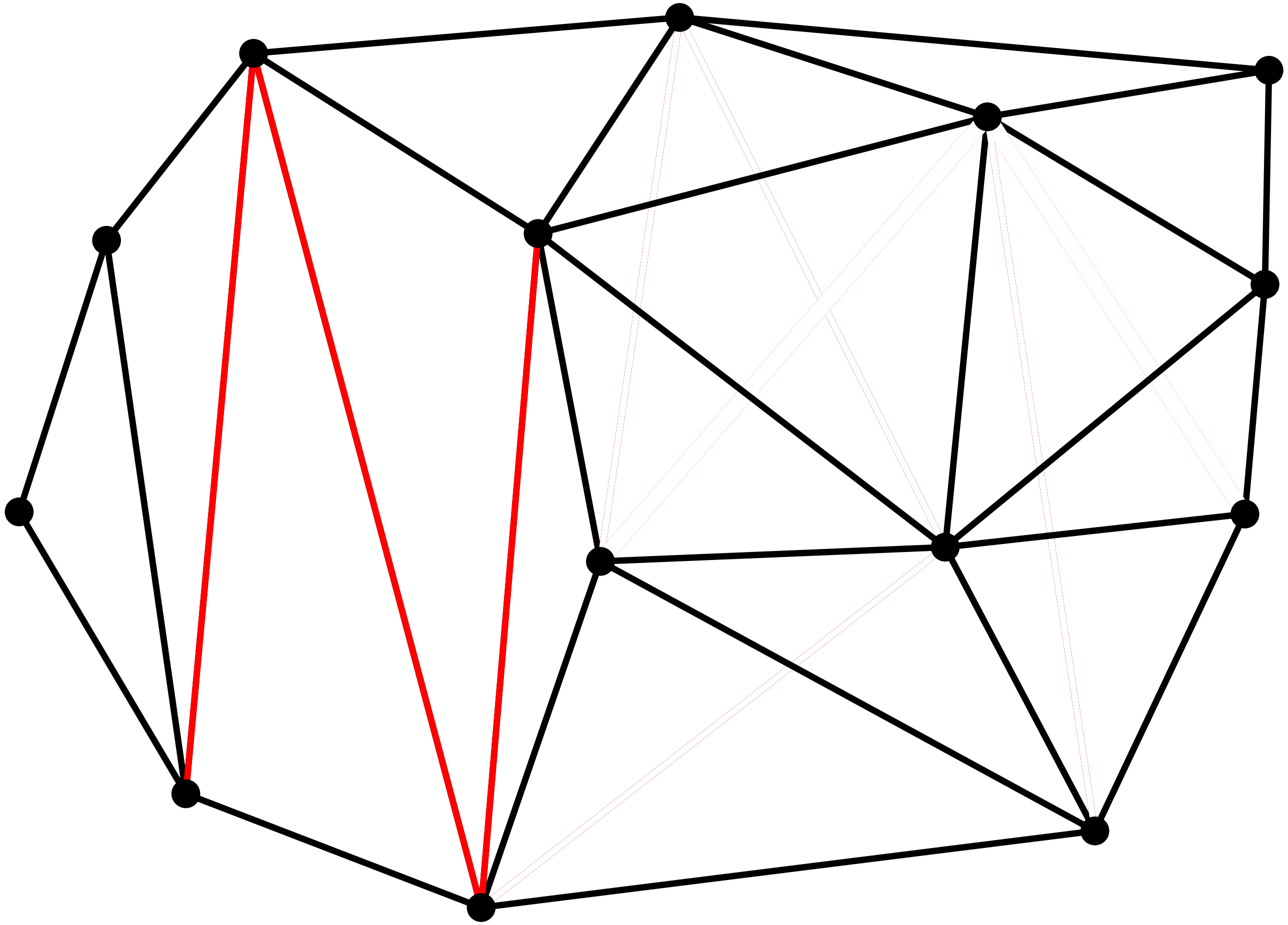
Delaunay Triangulation: Diagonal flipping



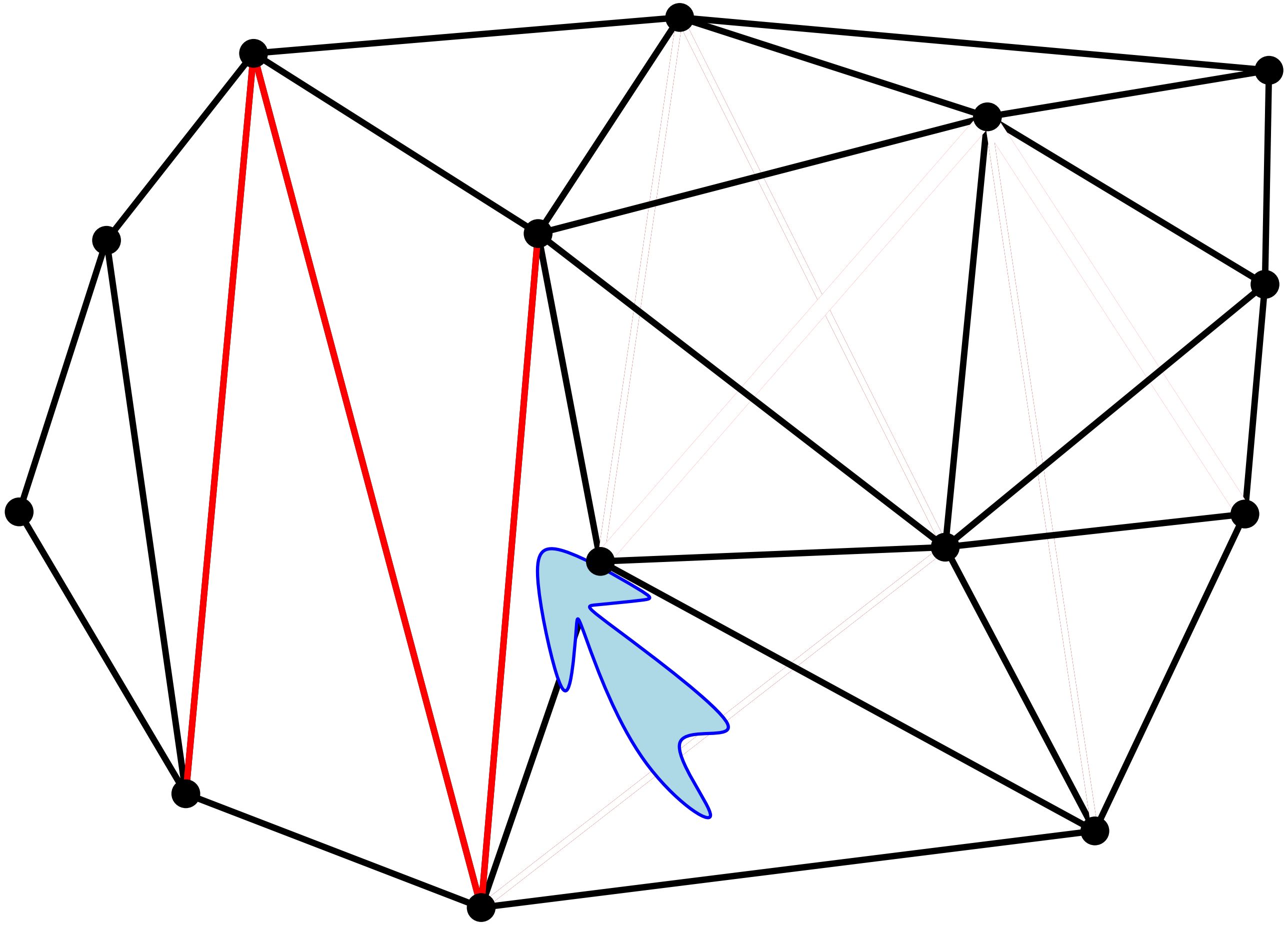
Delaunay Triangulation: Diagonal flipping



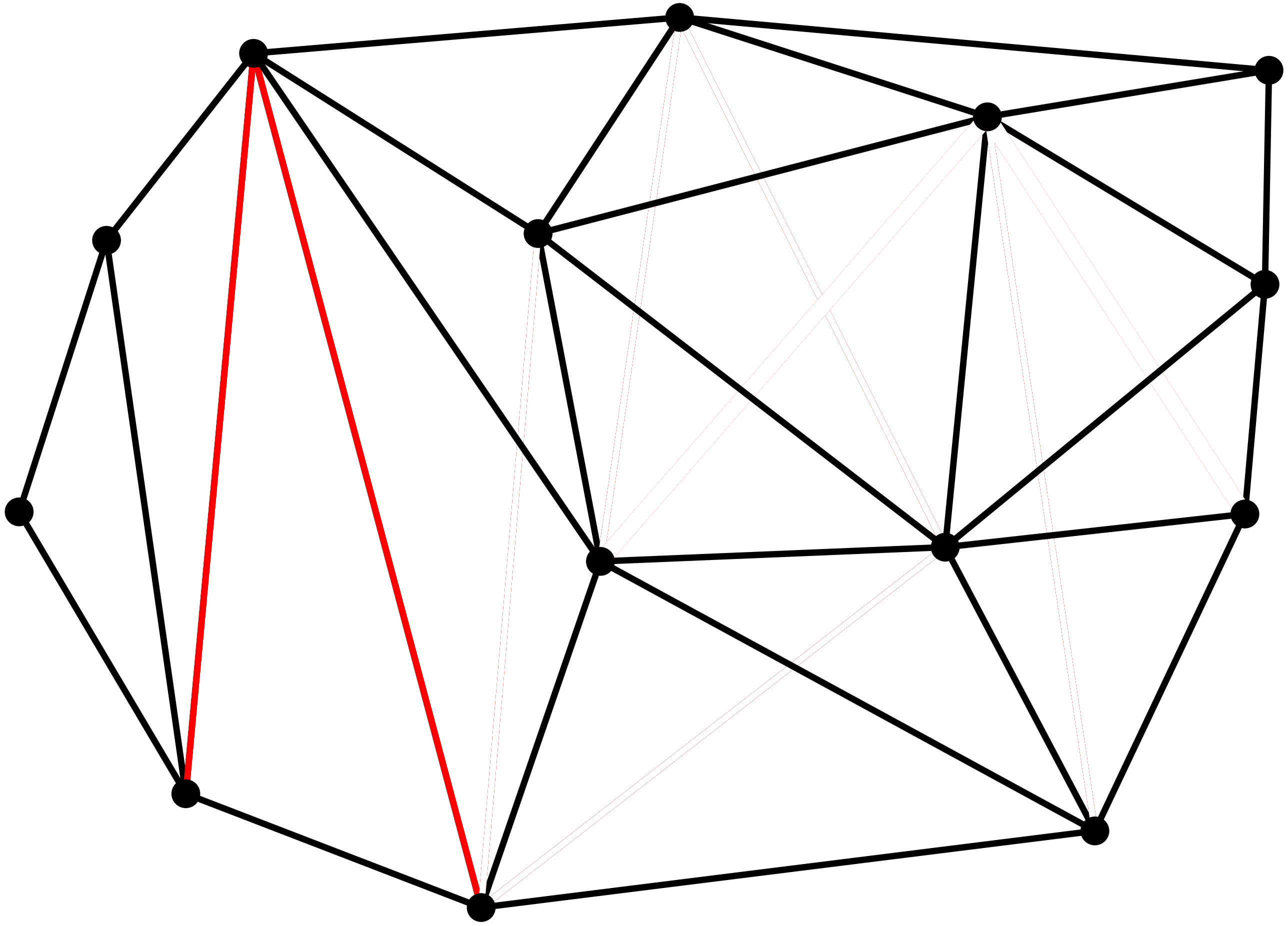
Delaunay Triangulation: Diagonal flipping



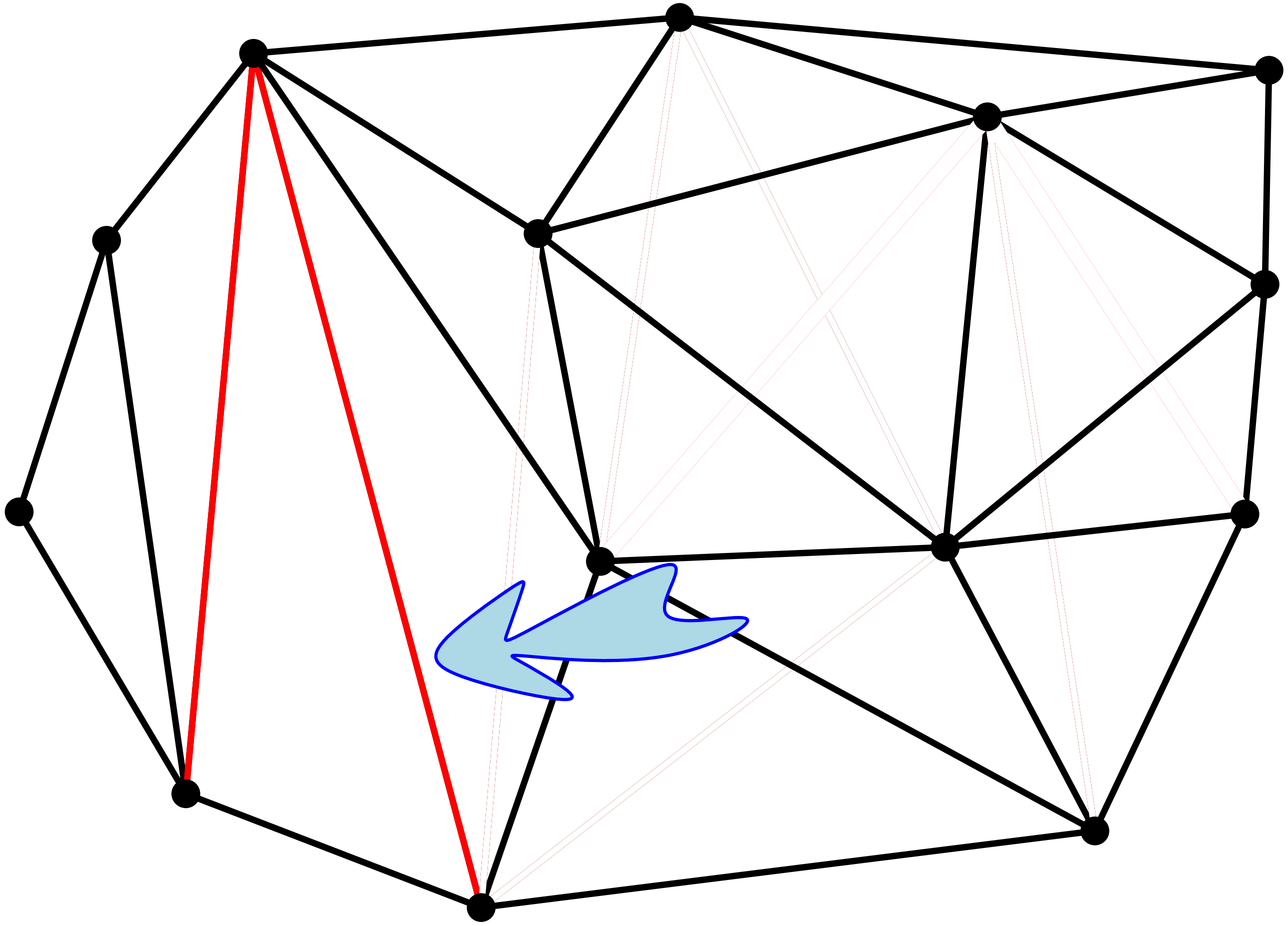
Delaunay Triangulation: Diagonal flipping



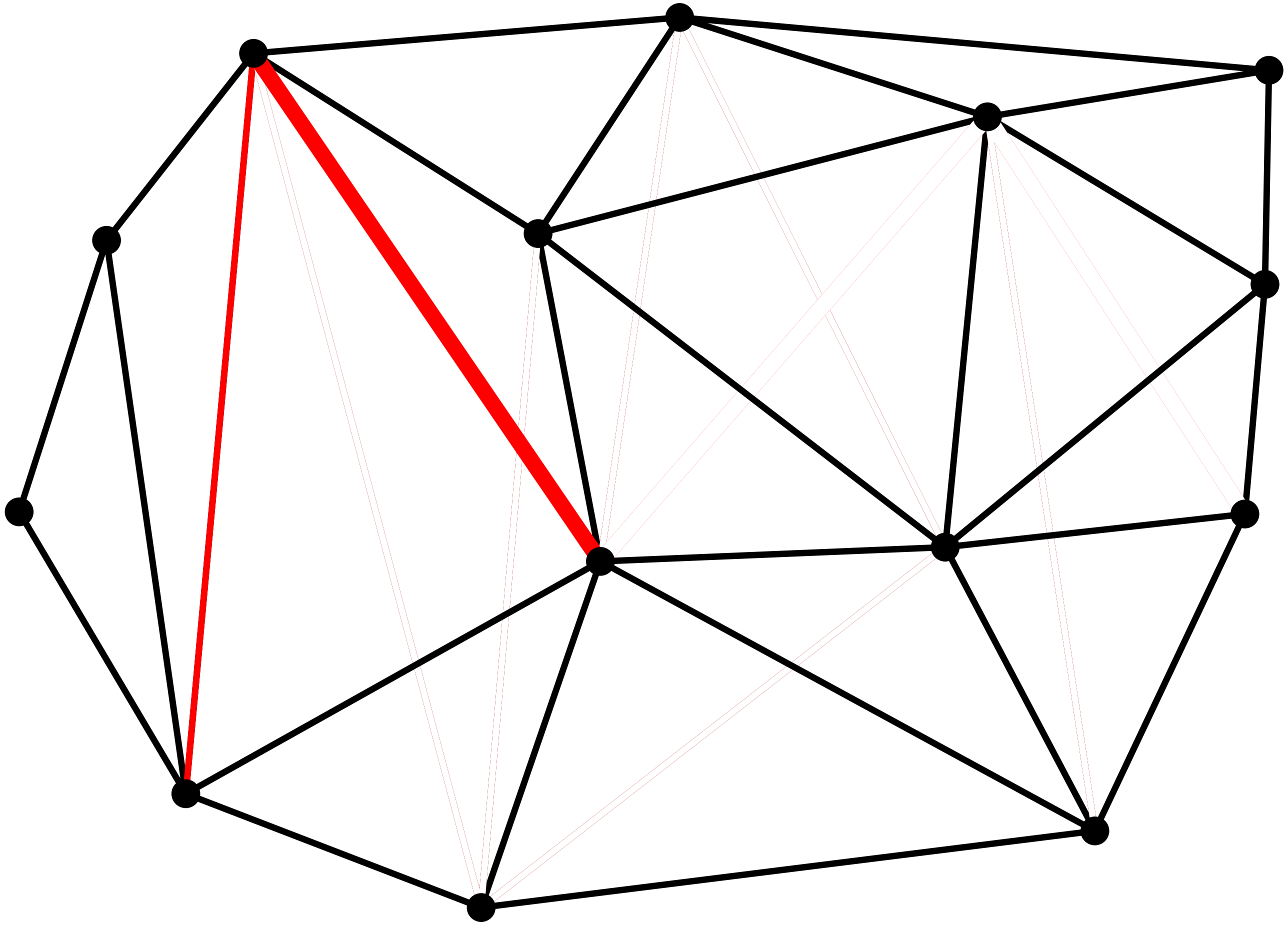
Delaunay Triangulation: Diagonal flipping



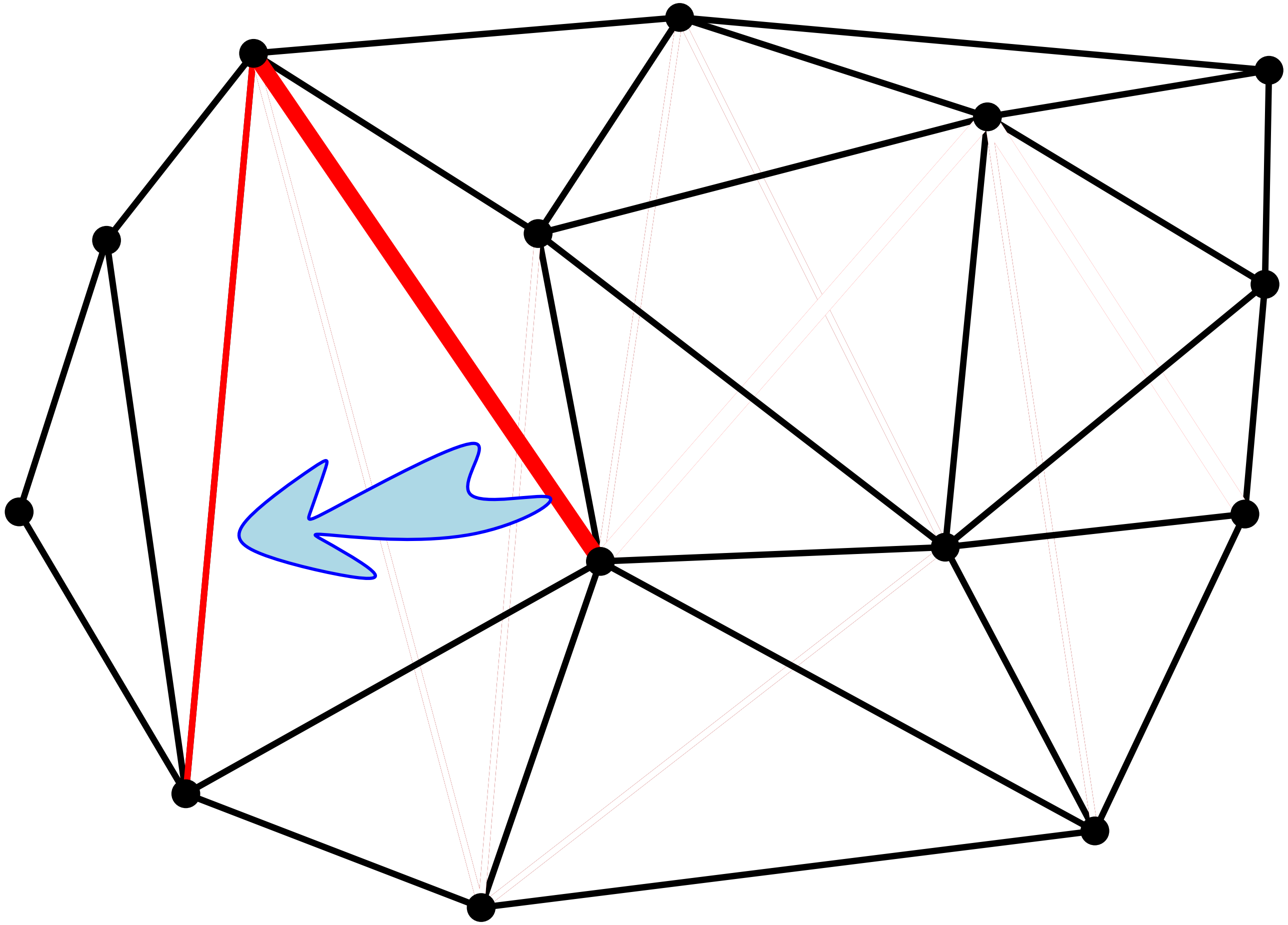
Delaunay Triangulation: Diagonal flipping



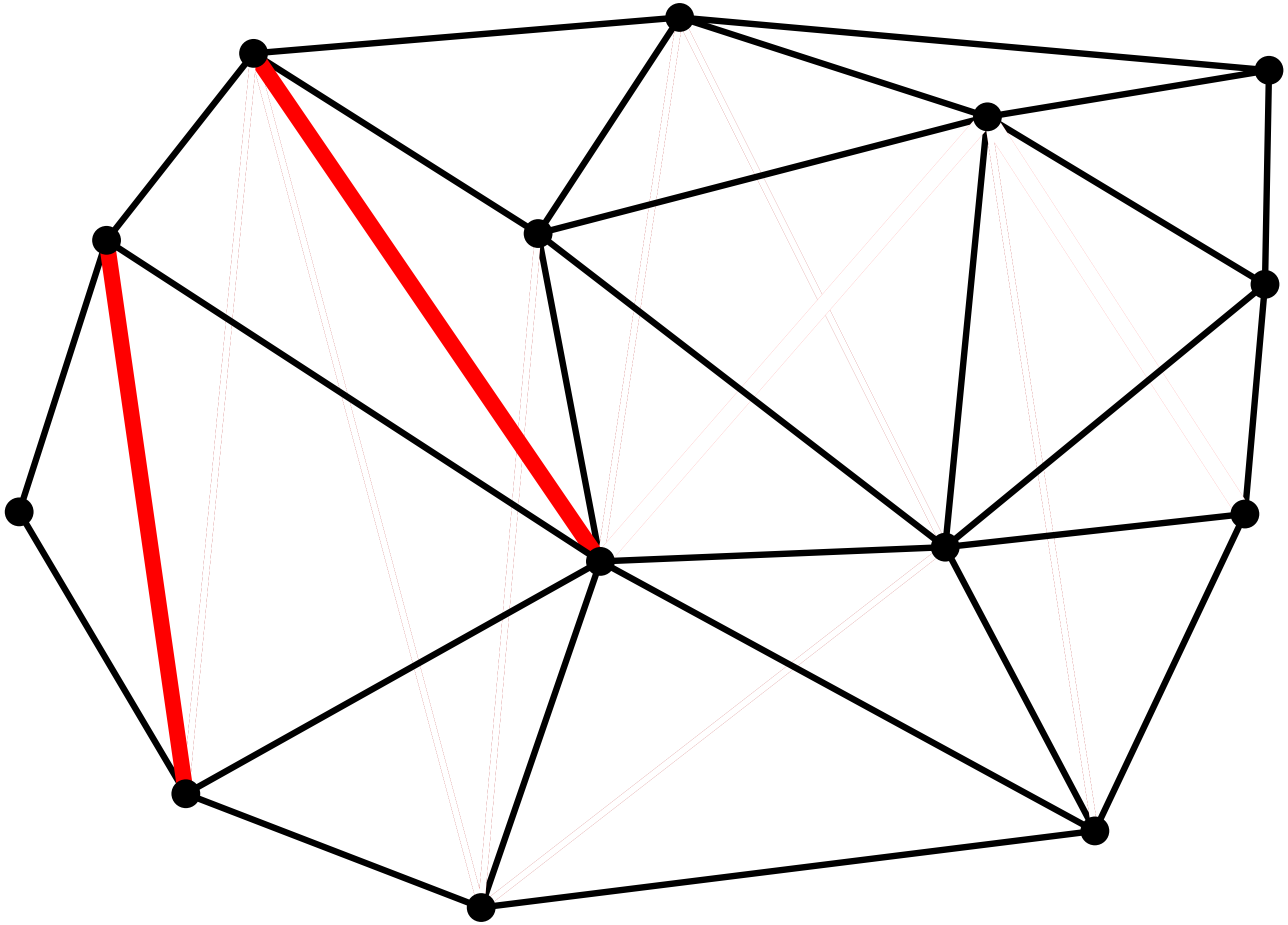
Delaunay Triangulation: Diagonal flipping



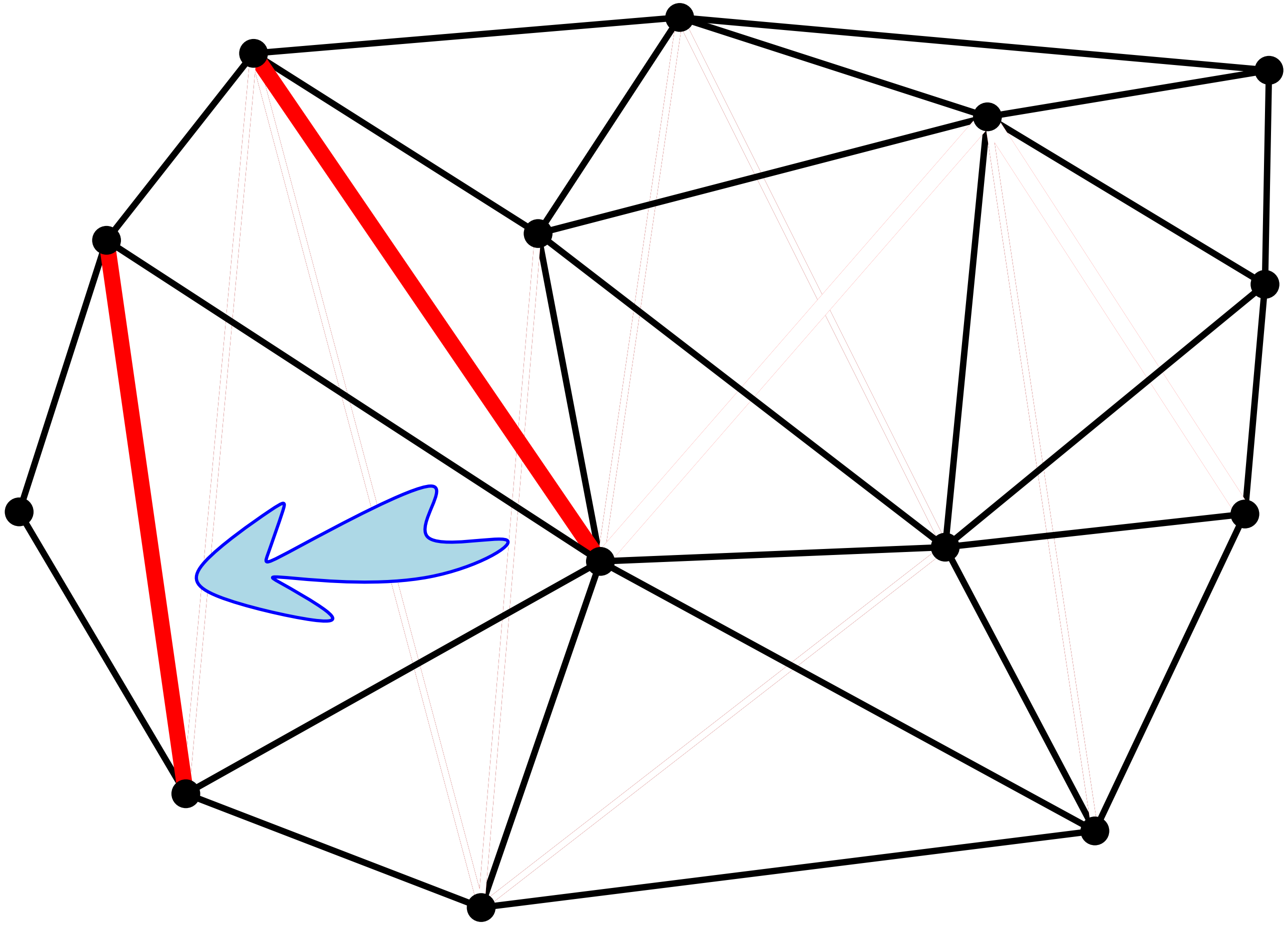
Delaunay Triangulation: Diagonal flipping



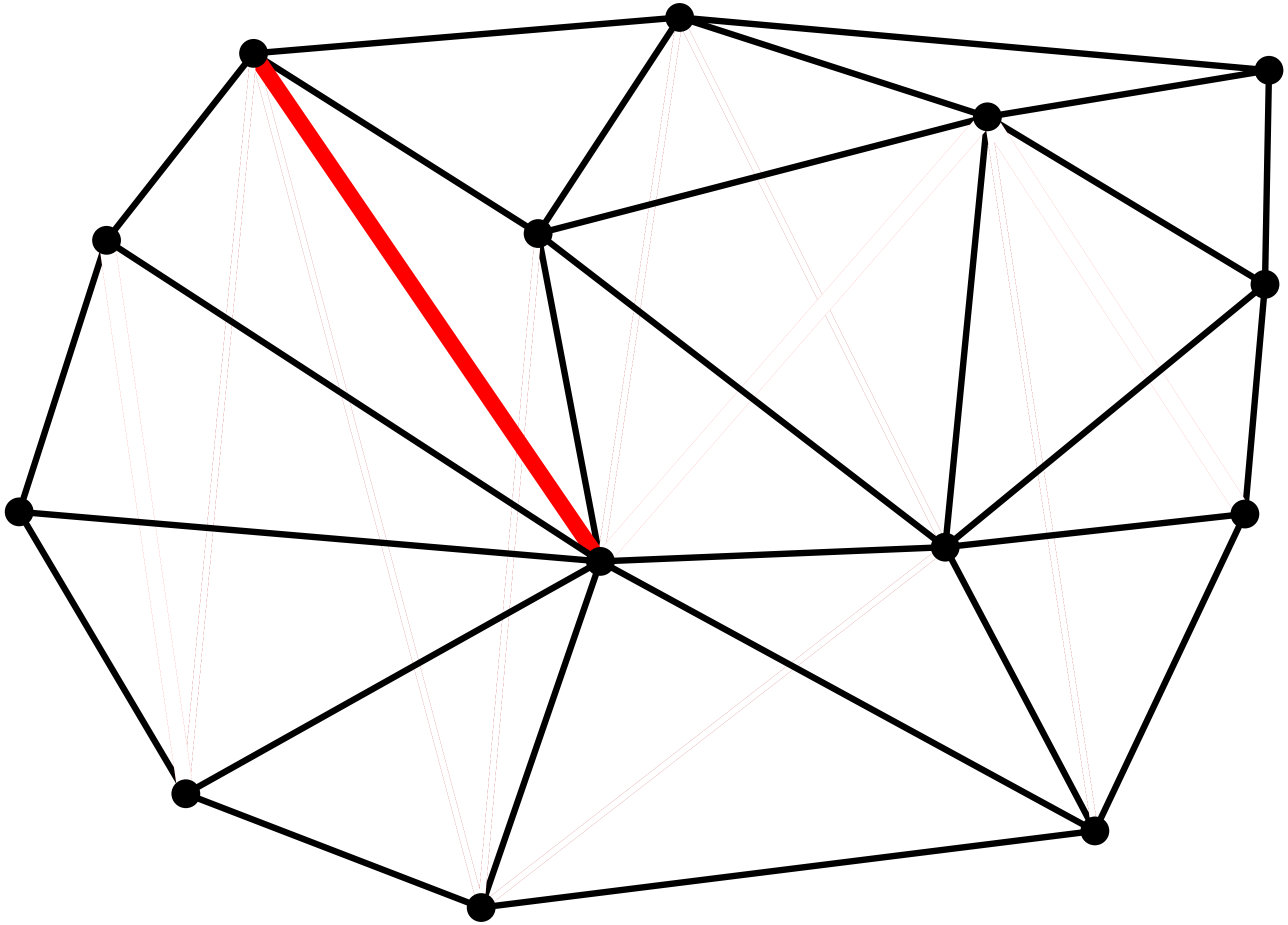
Delaunay Triangulation: Diagonal flipping



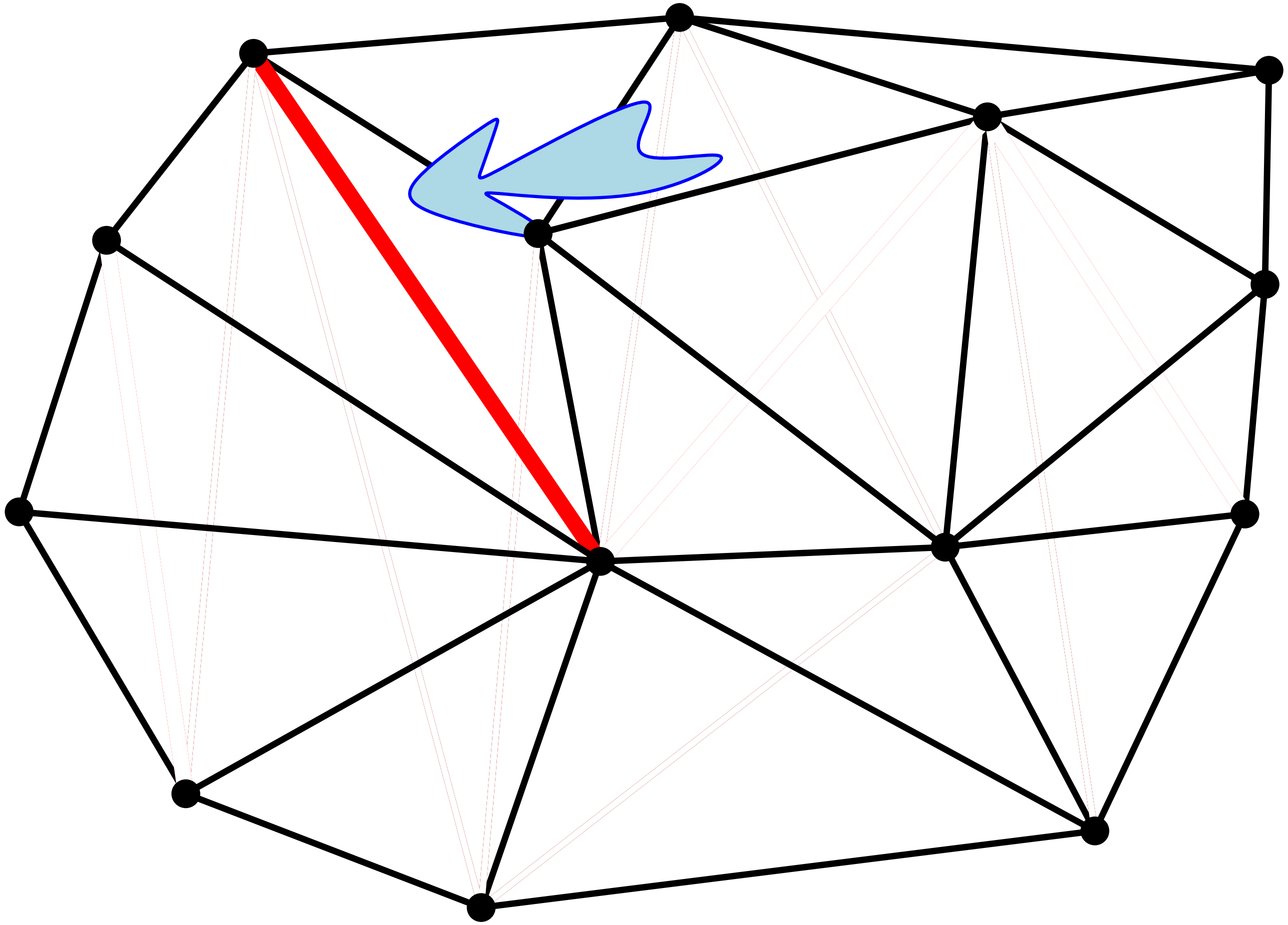
Delaunay Triangulation: Diagonal flipping



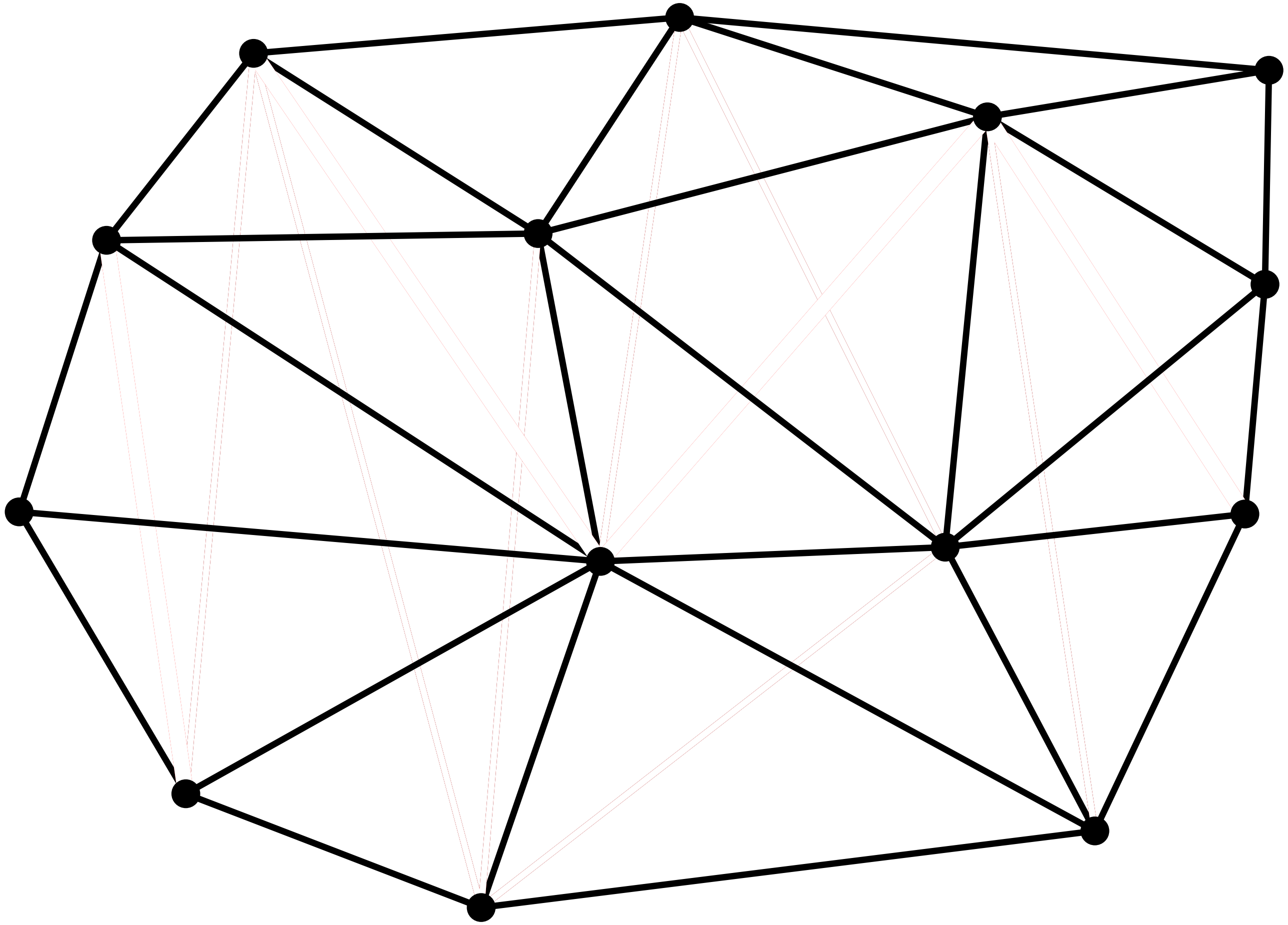
Delaunay Triangulation: Diagonal flipping



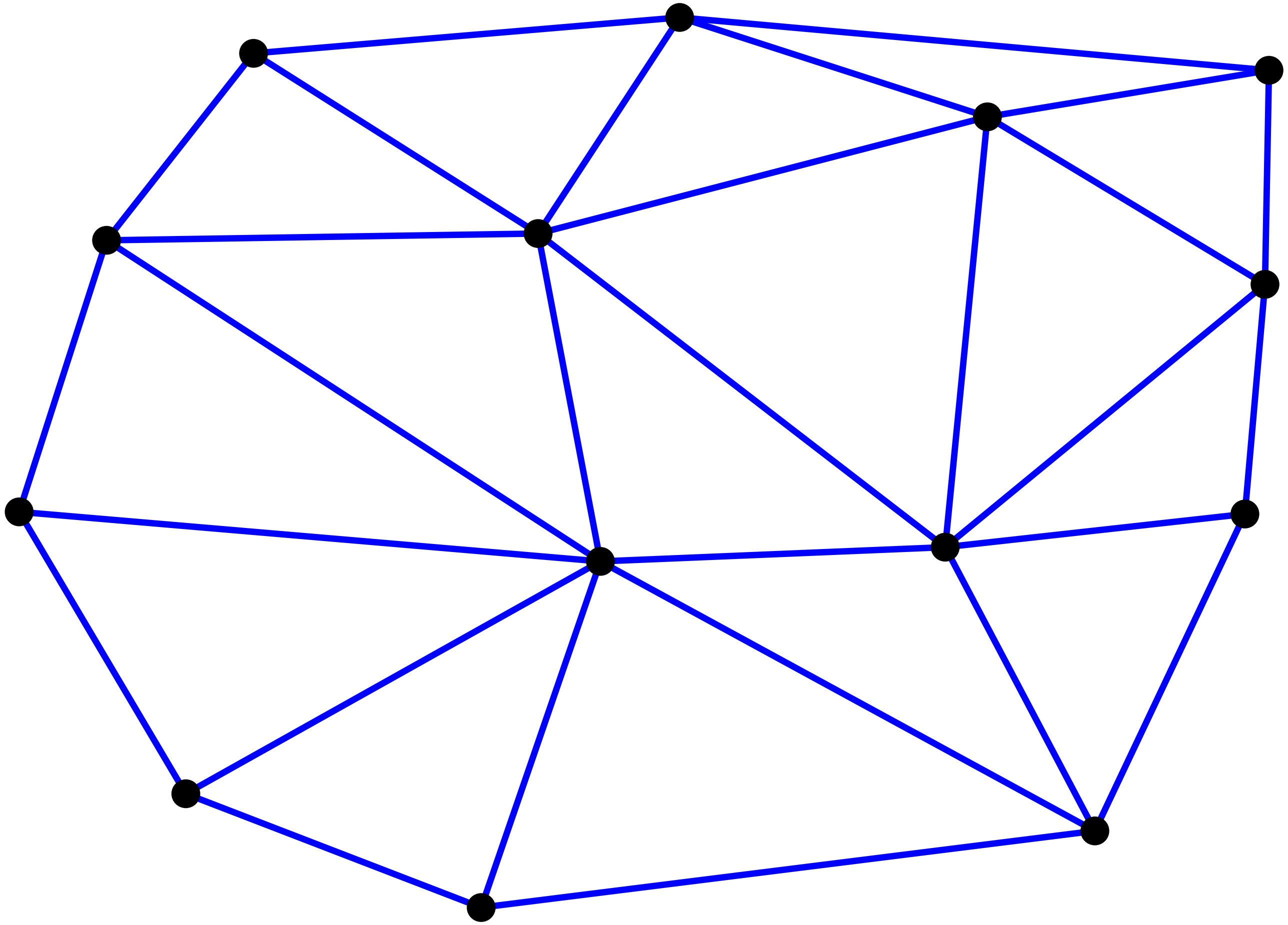
Delaunay Triangulation: Diagonal flipping



Delaunay Triangulation: Diagonal flipping



Delaunay Triangulation: Diagonal flipping



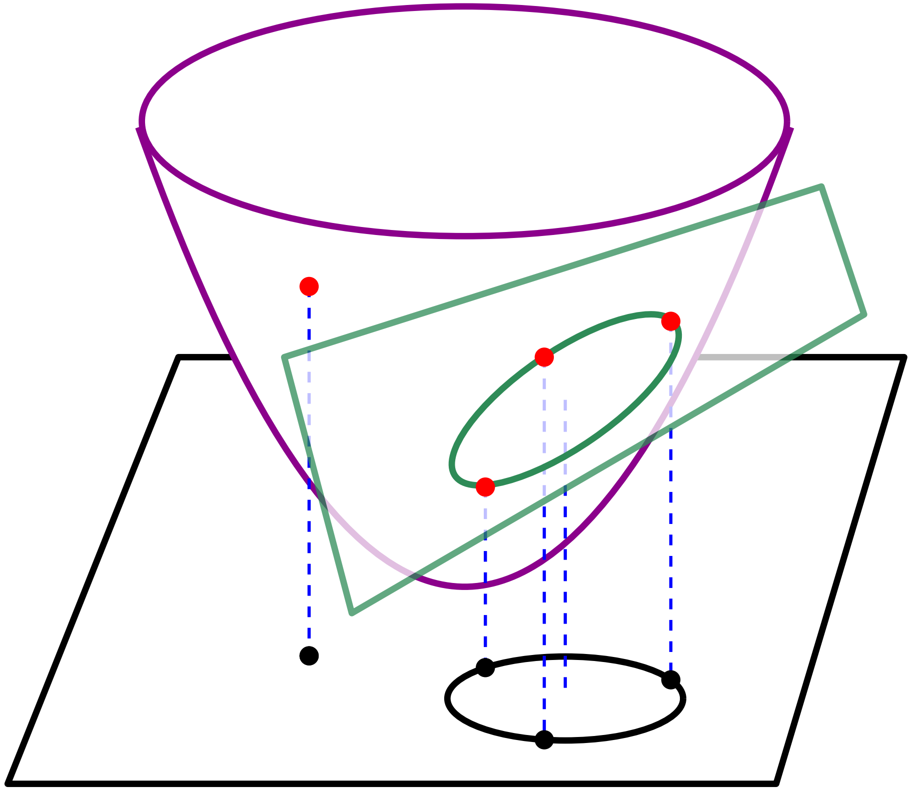
Delaunay is obtained

Delaunay Triangulation: Diagonal flipping

Complexity ?

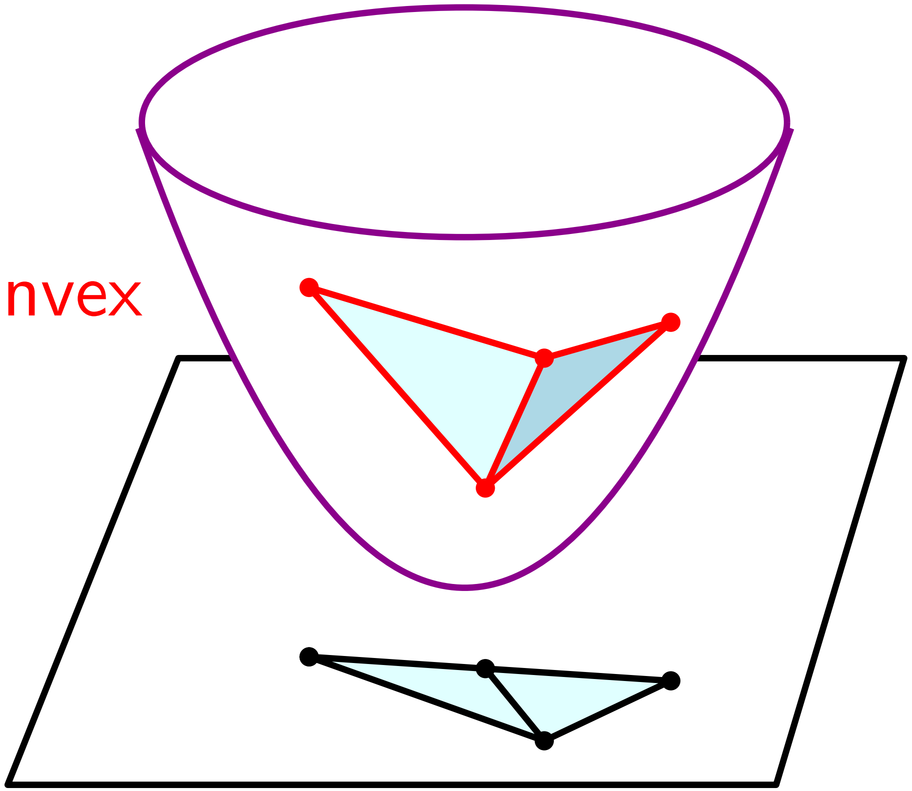
Delaunay Triangulation: Diagonal flipping

Complexity ?



Delaunay Triangulation: Diagonal flipping

Complexity ?

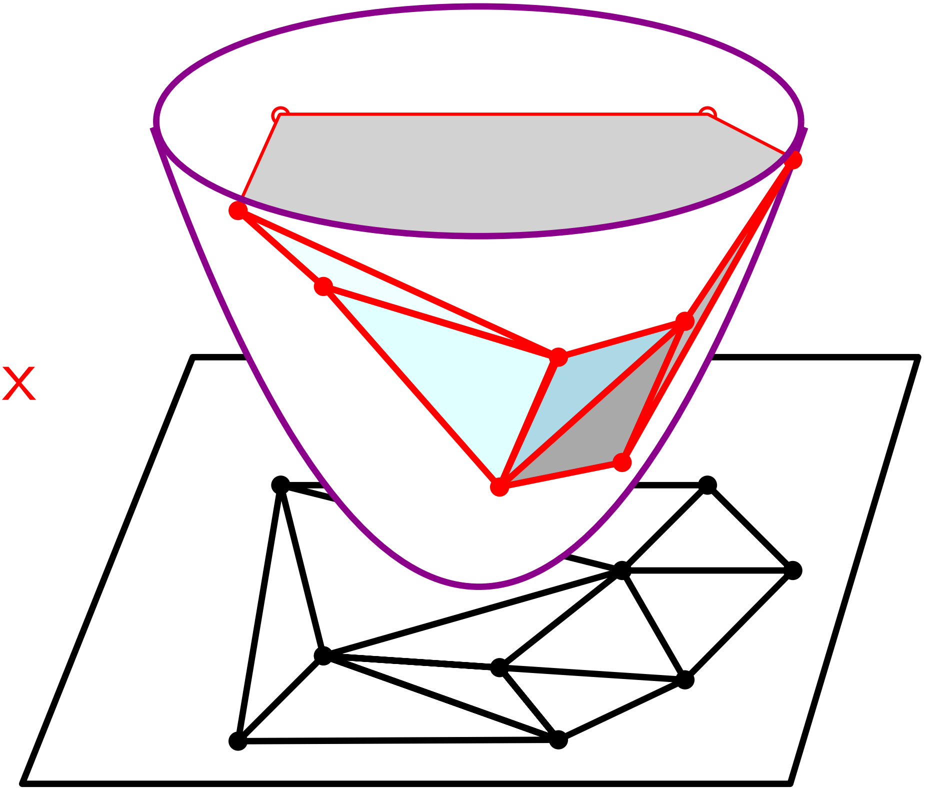


Locally Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Convex

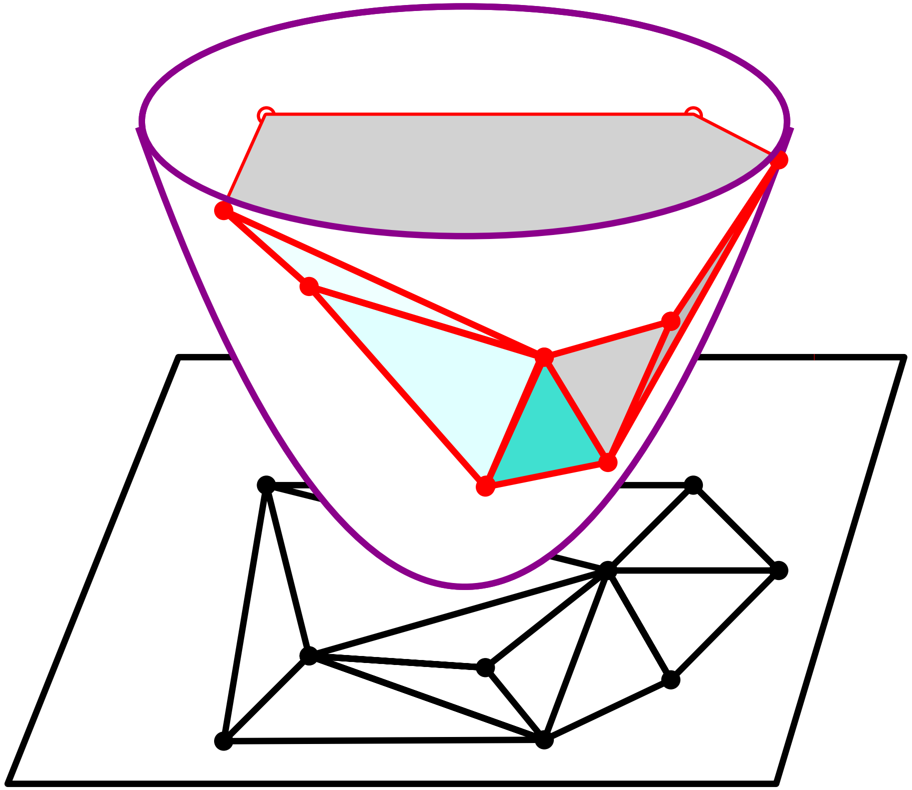


Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

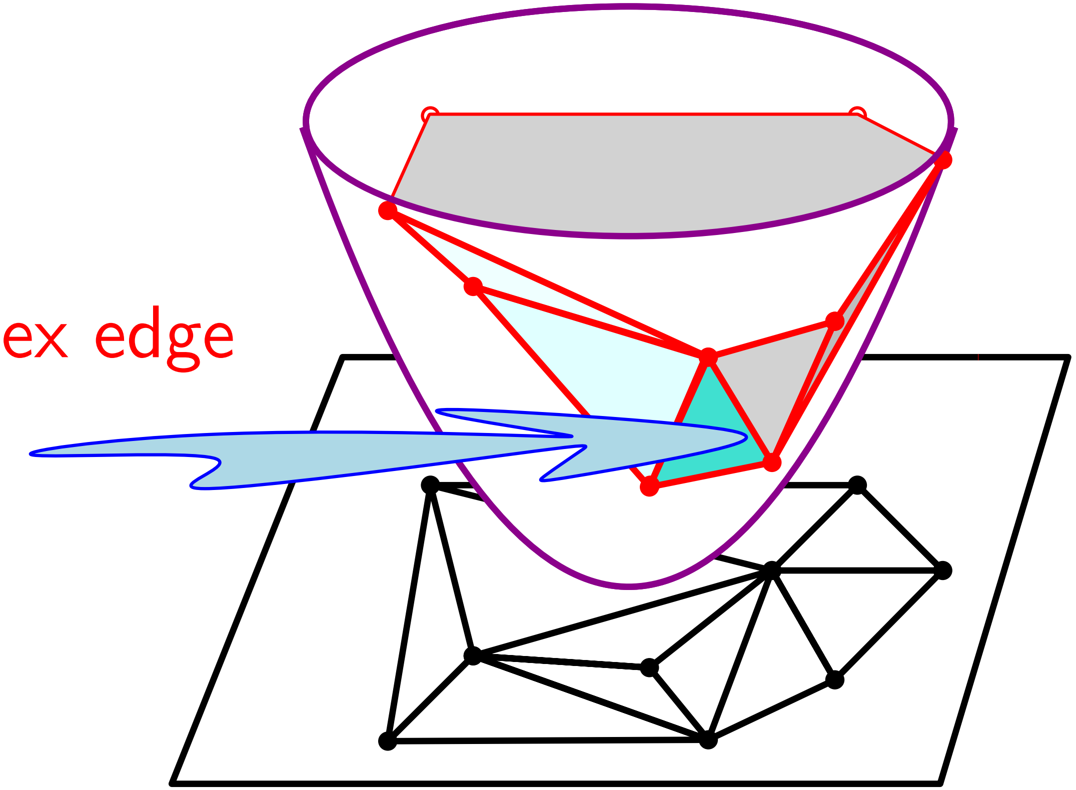


Non Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex edge



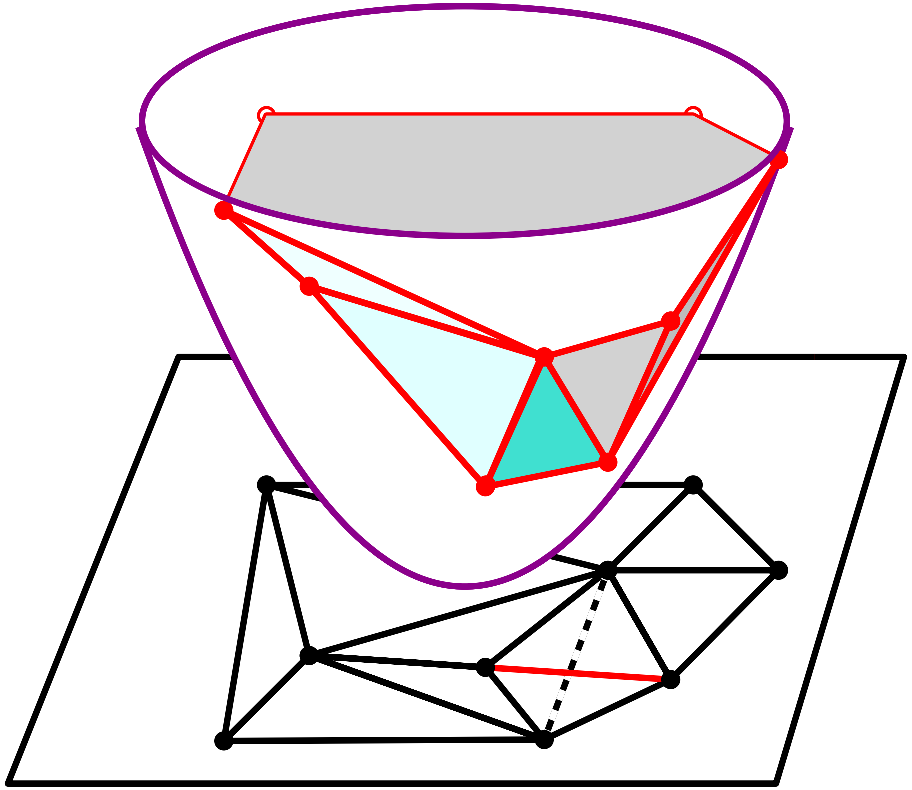
Non Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip



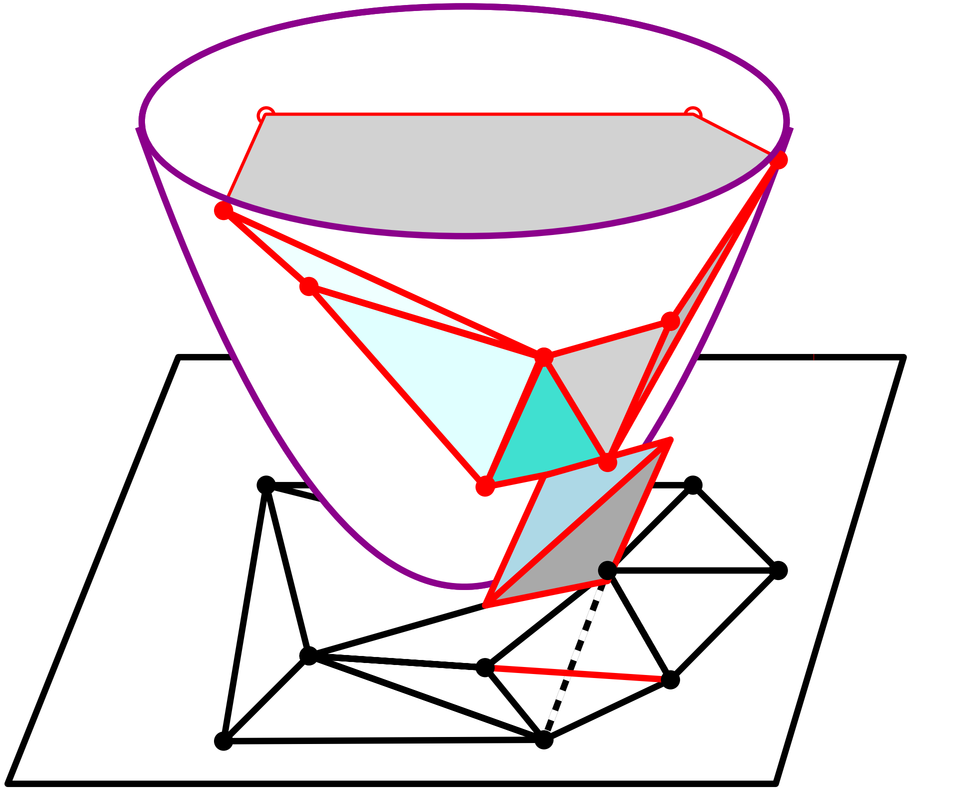
Non Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip



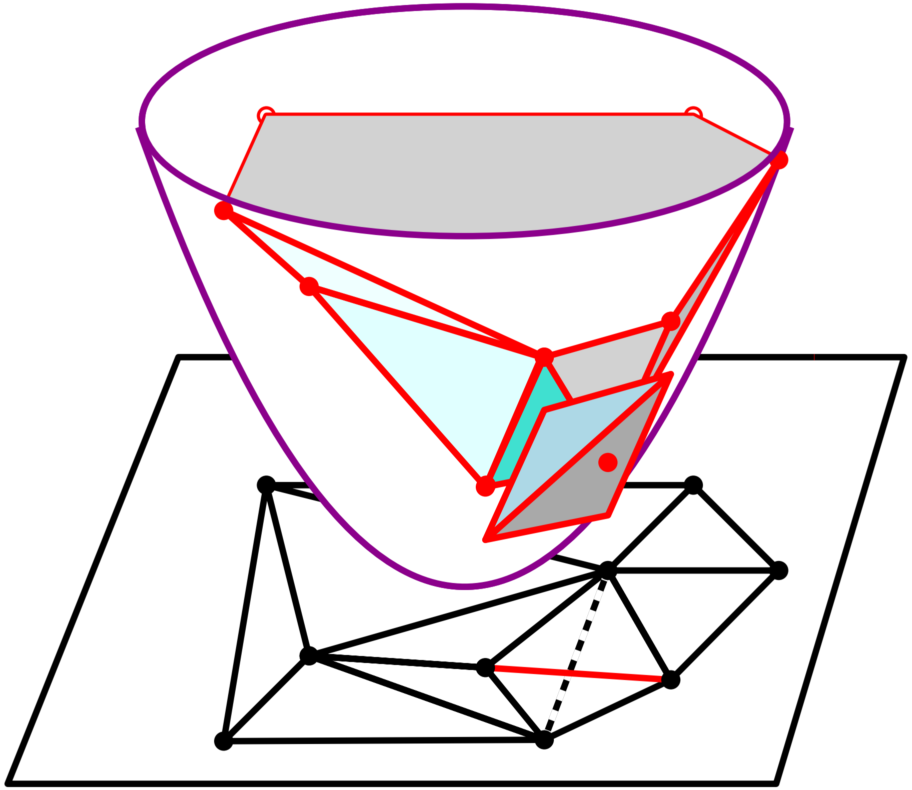
Non Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip



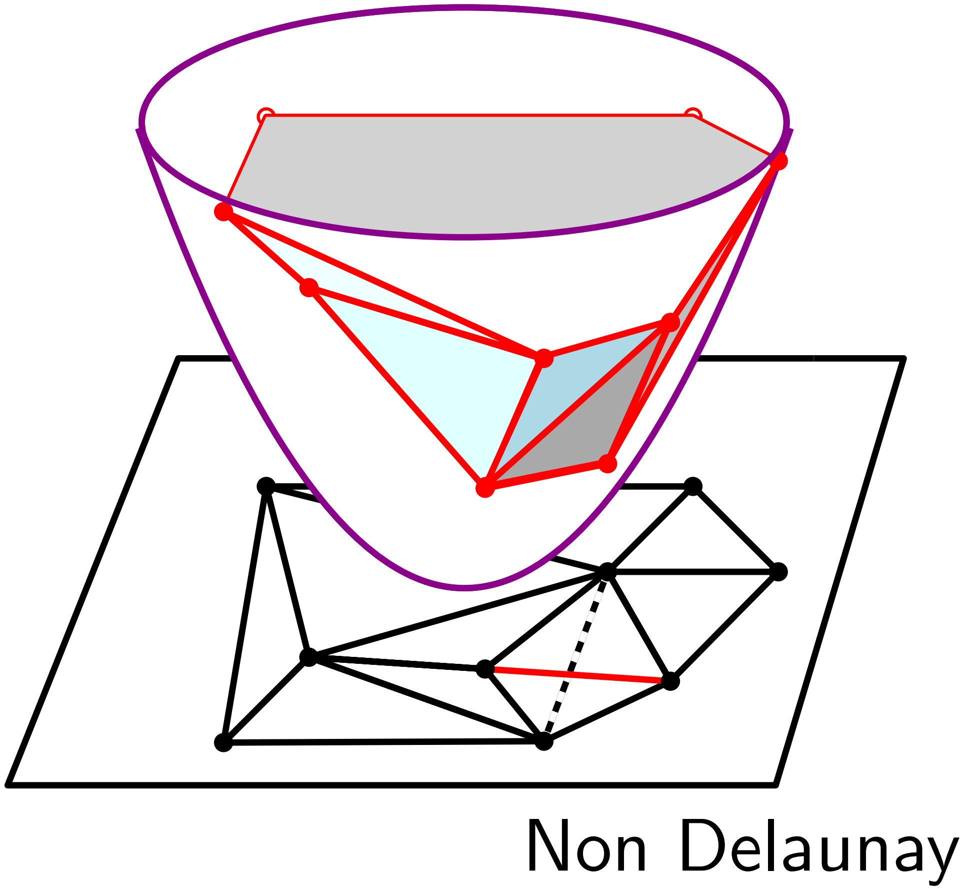
Non Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip

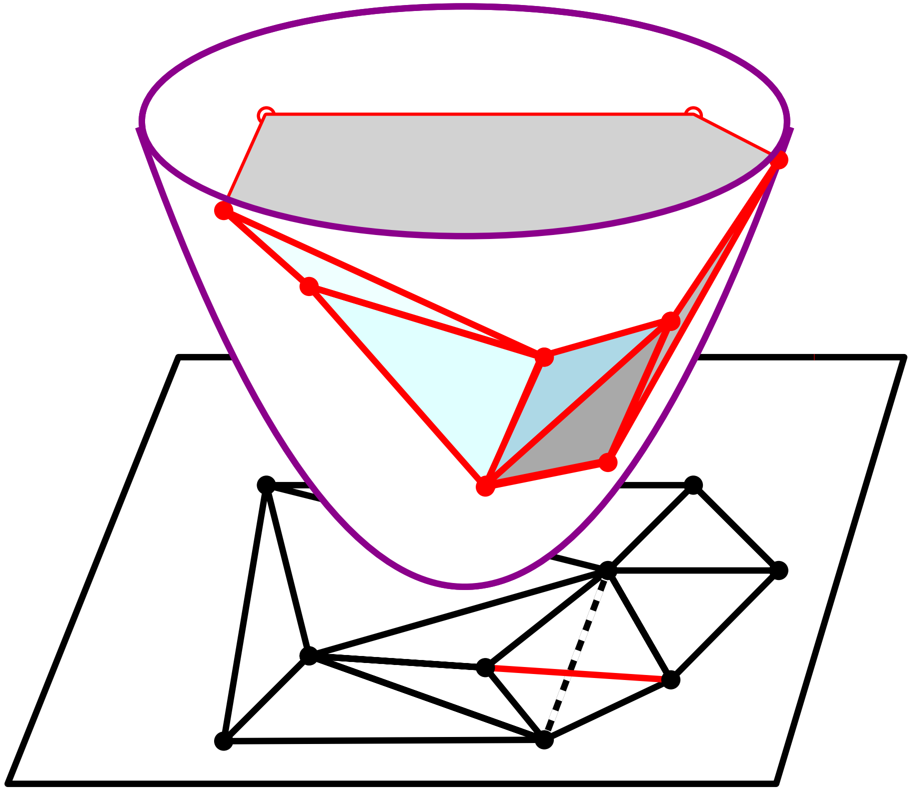


Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip



An hidden edge cannot be visible again

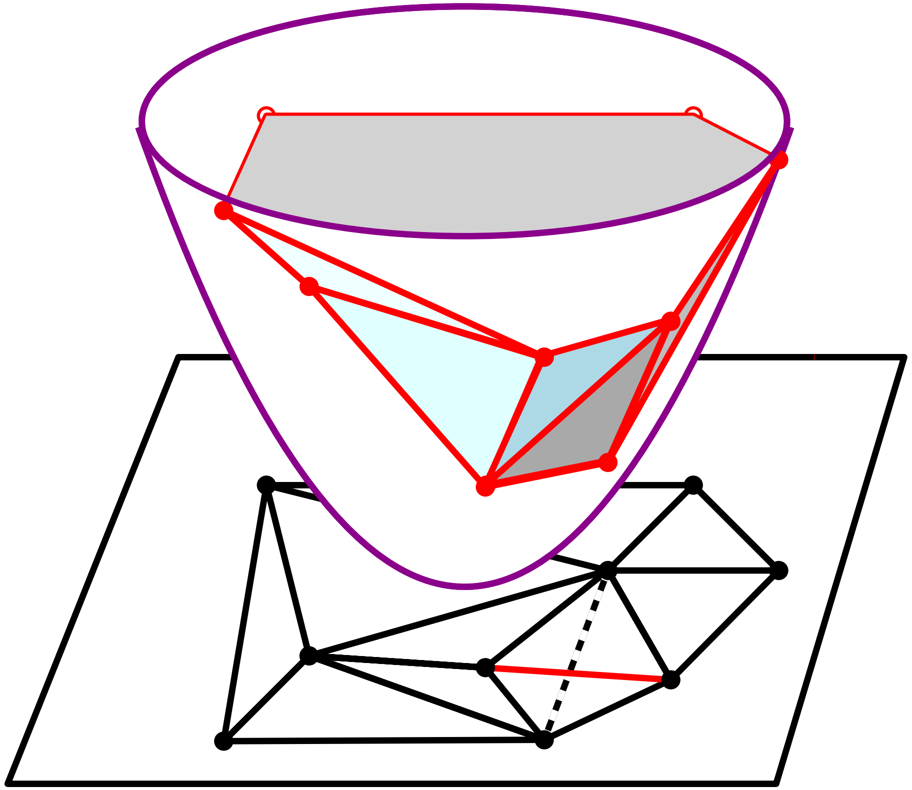
Non Delaunay

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip



An hidden edge cannot be visible again

Non Delaunay

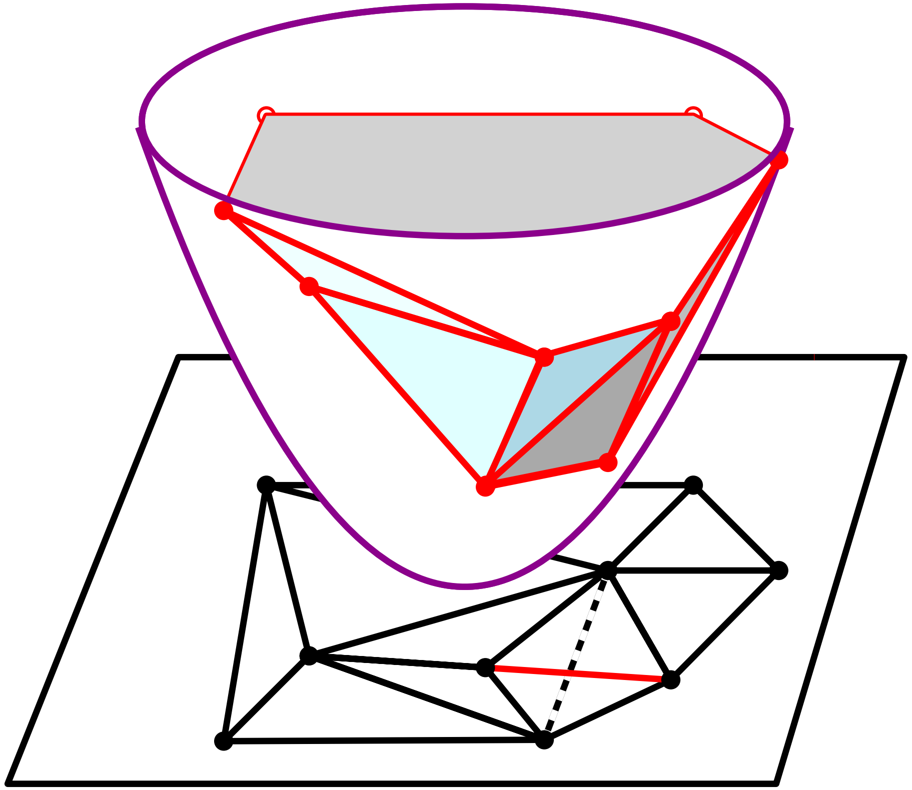
At most $\frac{n(n-1)}{2}$ edges

Delaunay Triangulation: Diagonal flipping

Complexity ?

Non convex

Flip



Non Delaunay

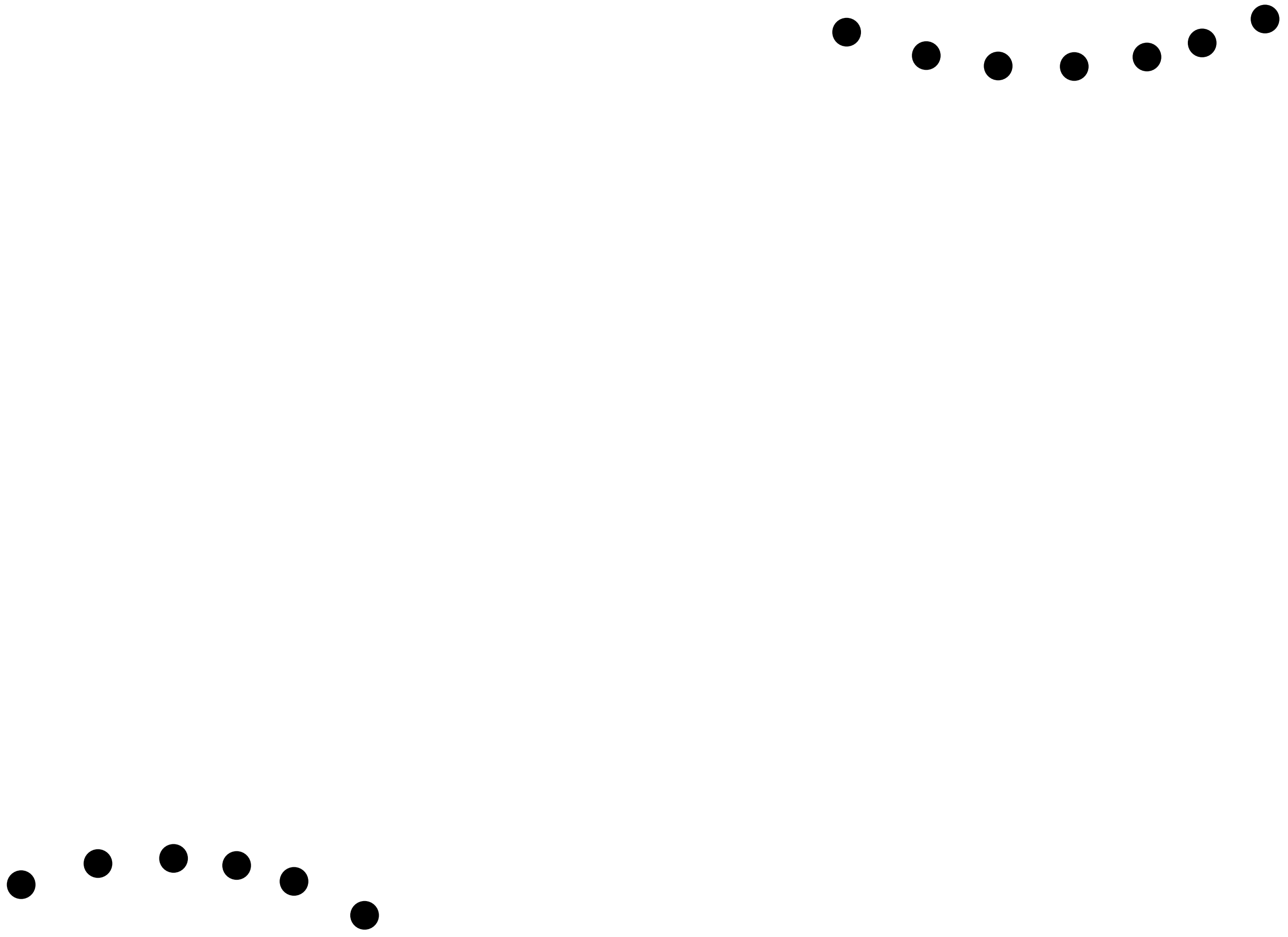
An hidden edge cannot be visible again

At most $\frac{n(n-1)}{2}$ edges

Complexity of diagonal flipping is $O(n^2)$

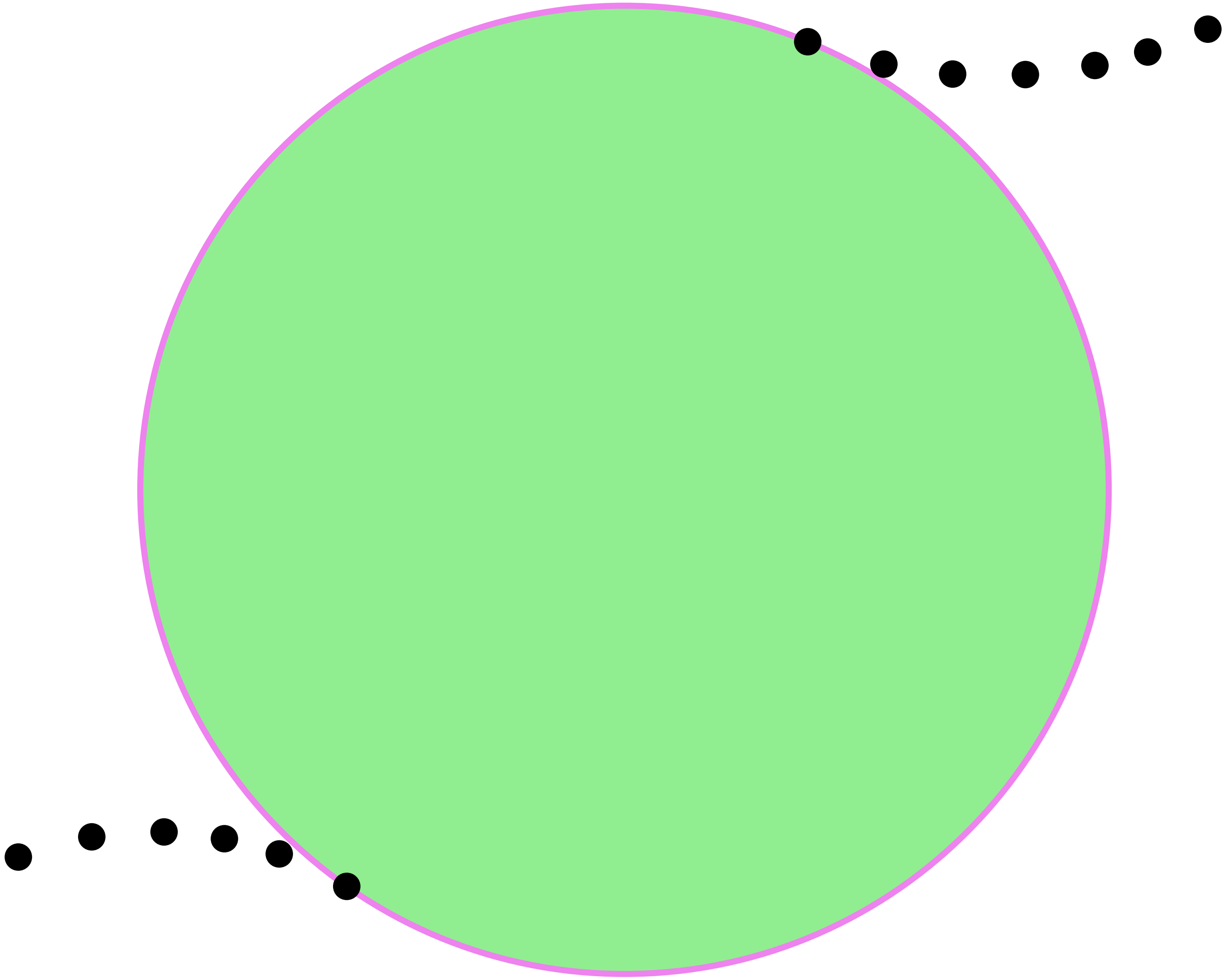
Delaunay Triangulation: Diagonal flipping

Complexity ?



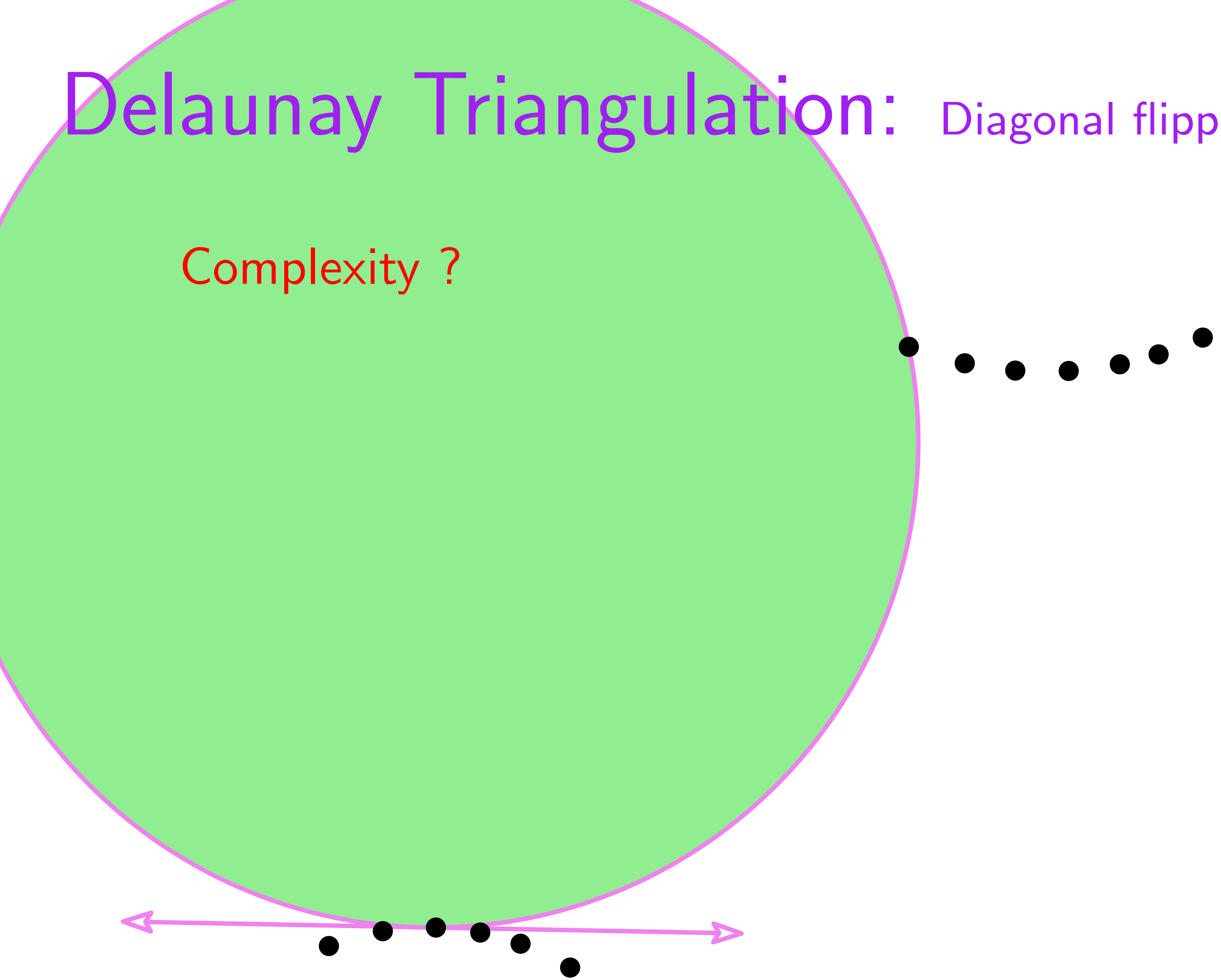
Delaunay Triangulation: Diagonal flipping

Complexity ?



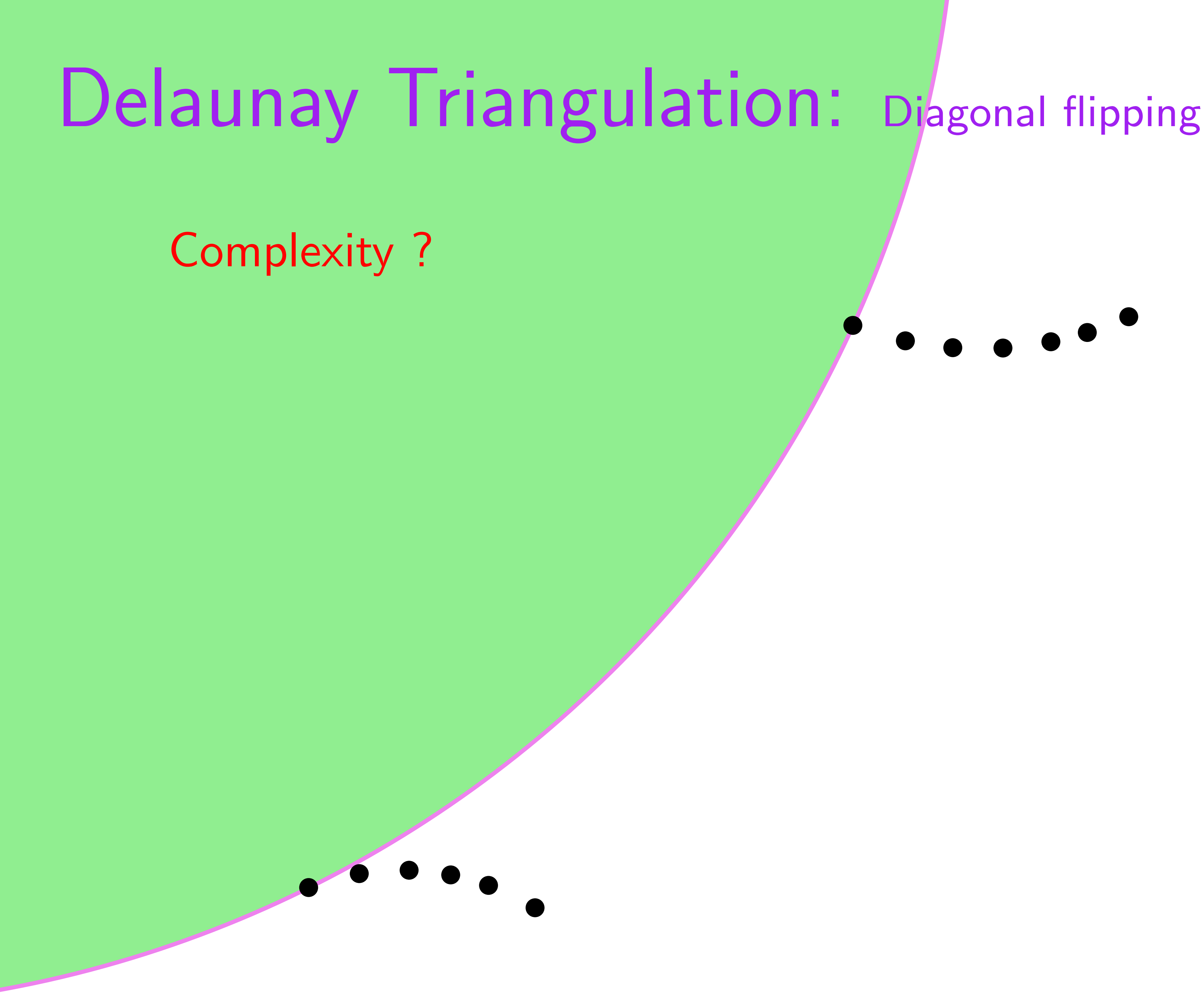
Delaunay Triangulation: Diagonal flipping

Complexity ?



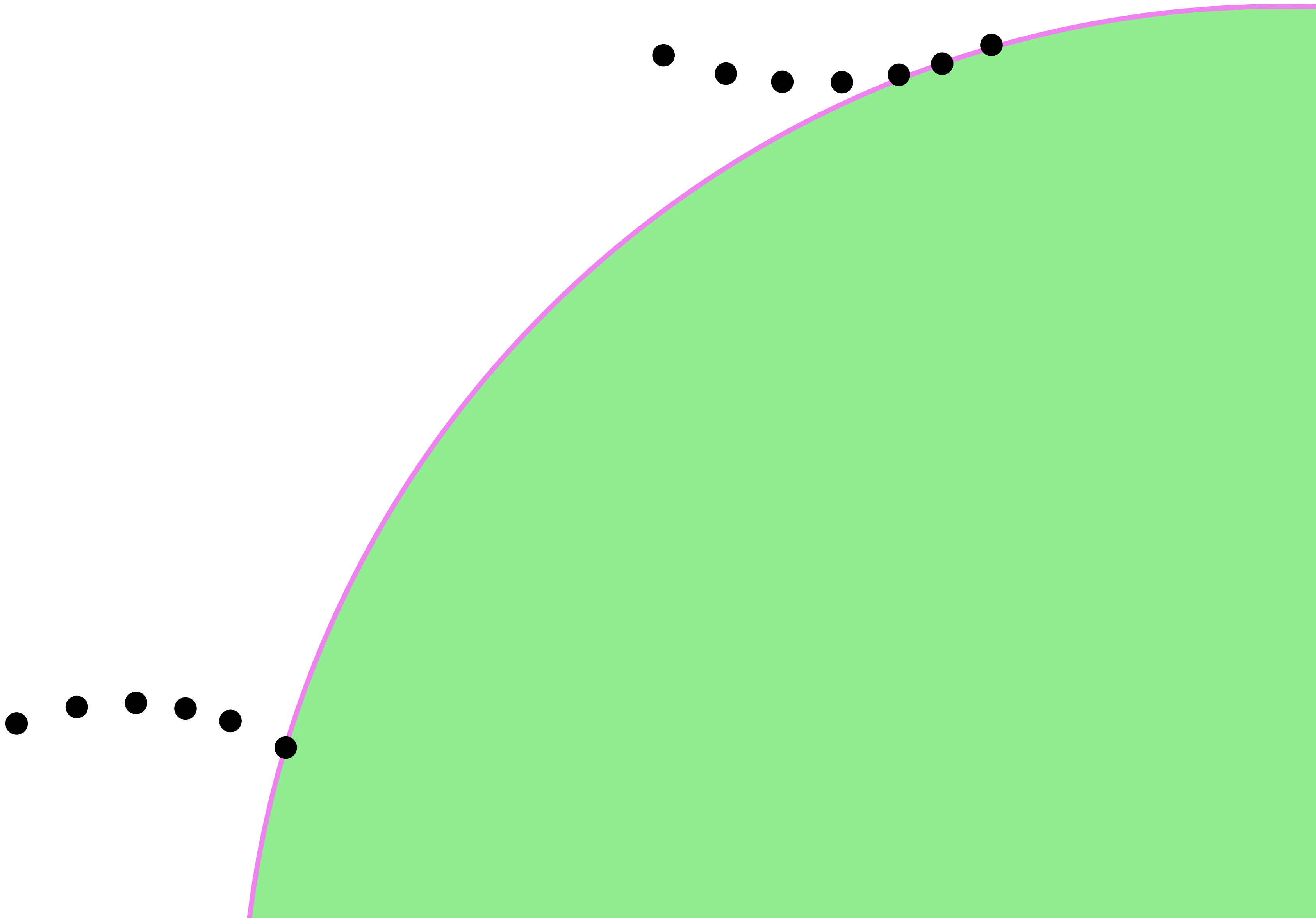
Delaunay Triangulation: Diagonal flipping

Complexity ?



Delaunay Triangulation: Diagonal flipping

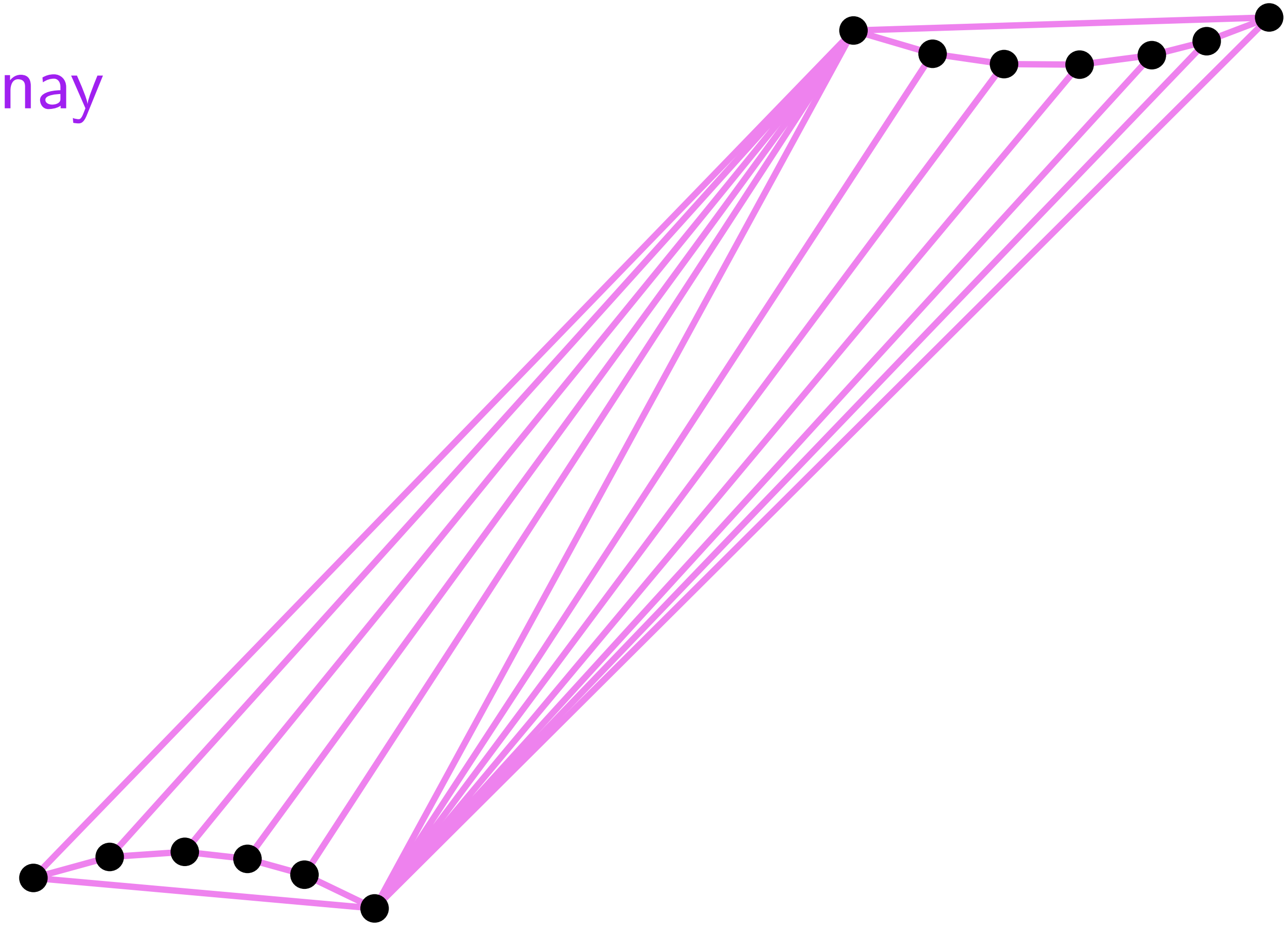
Complexity ?



Delaunay Triangulation: Diagonal flipping

Complexity ?

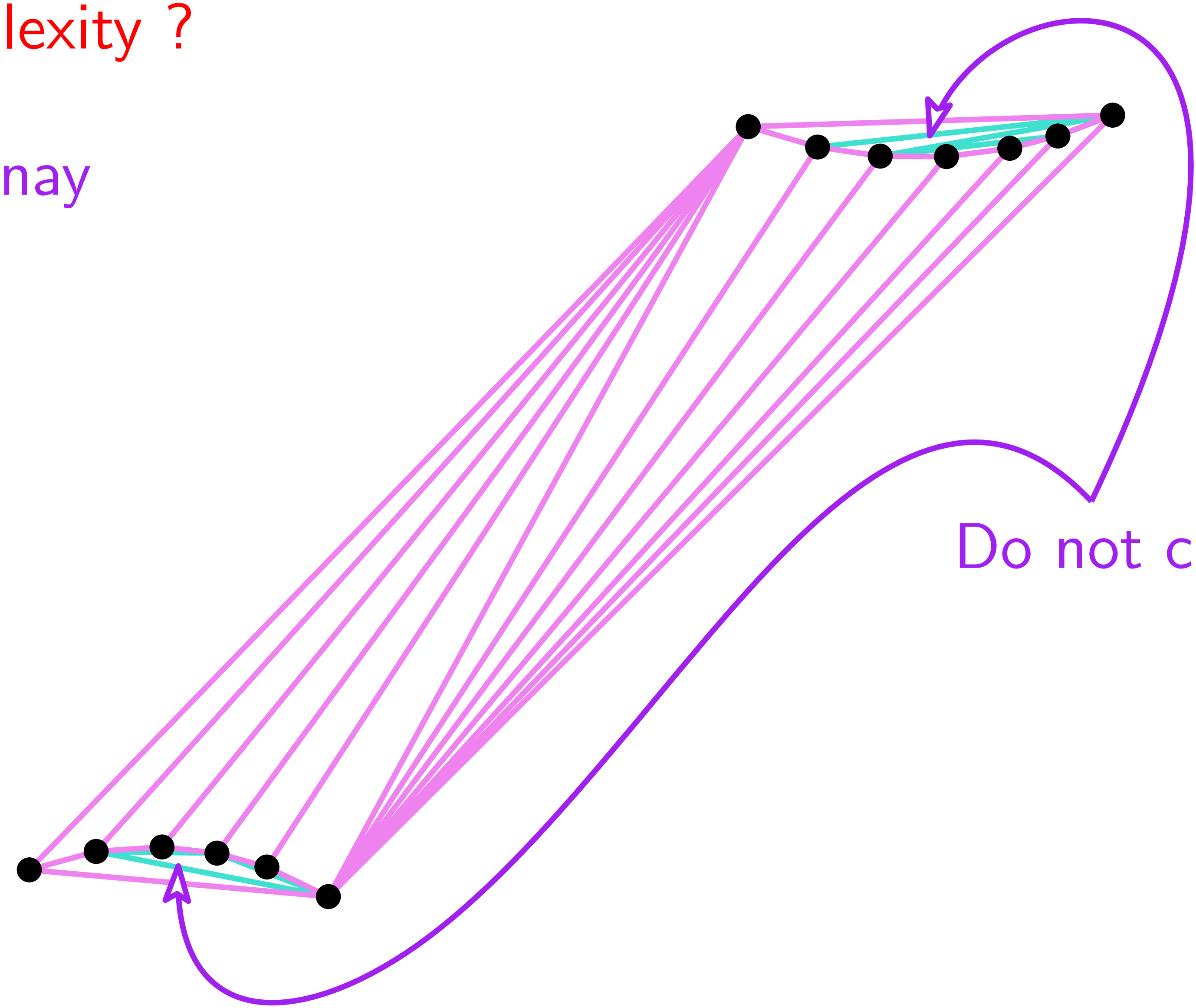
Delaunay



Delaunay Triangulation: Diagonal flipping

Complexity ?

Delaunay

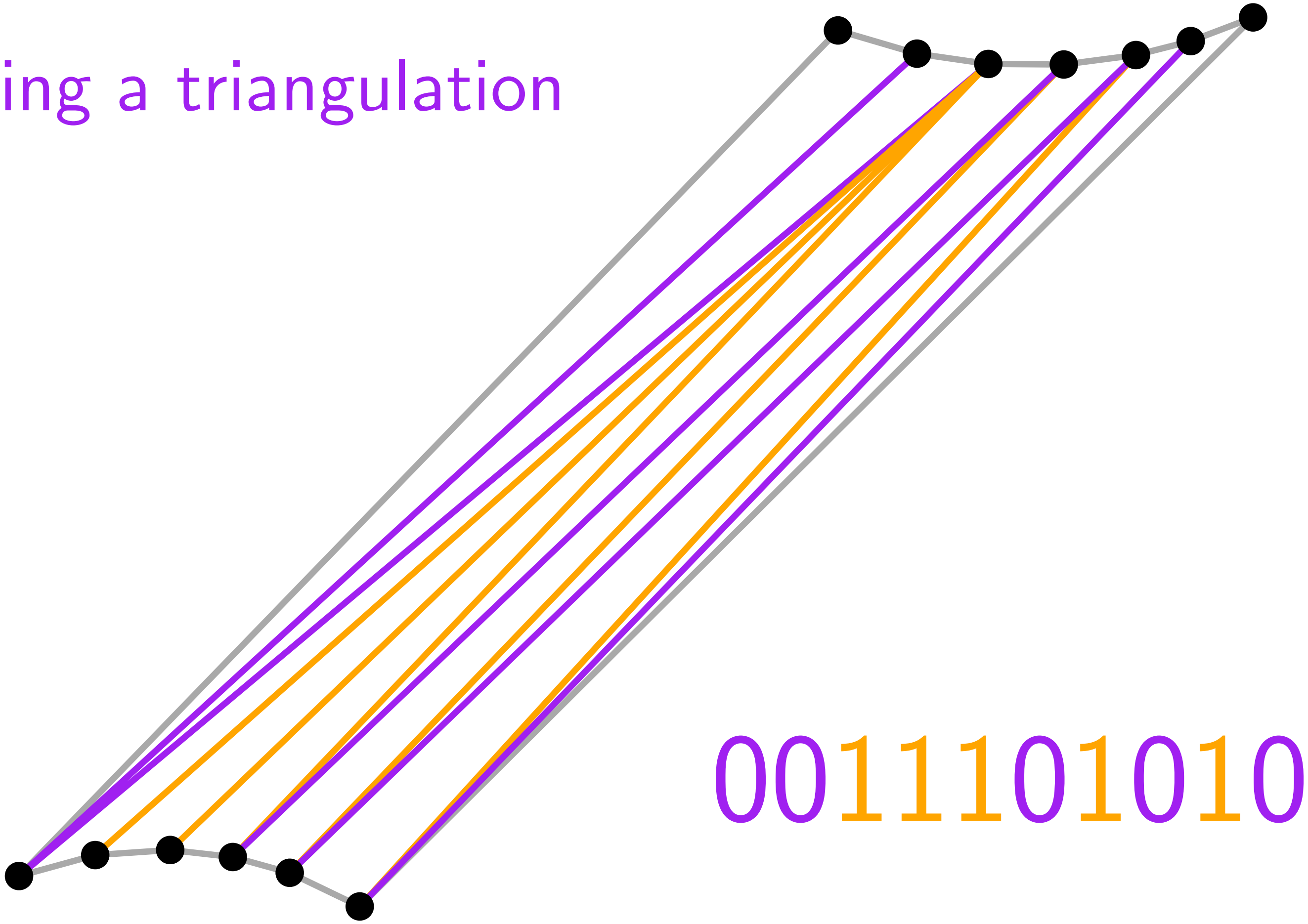


Do not care

Delaunay Triangulation: Diagonal flipping

Complexity ?

Encoding a triangulation

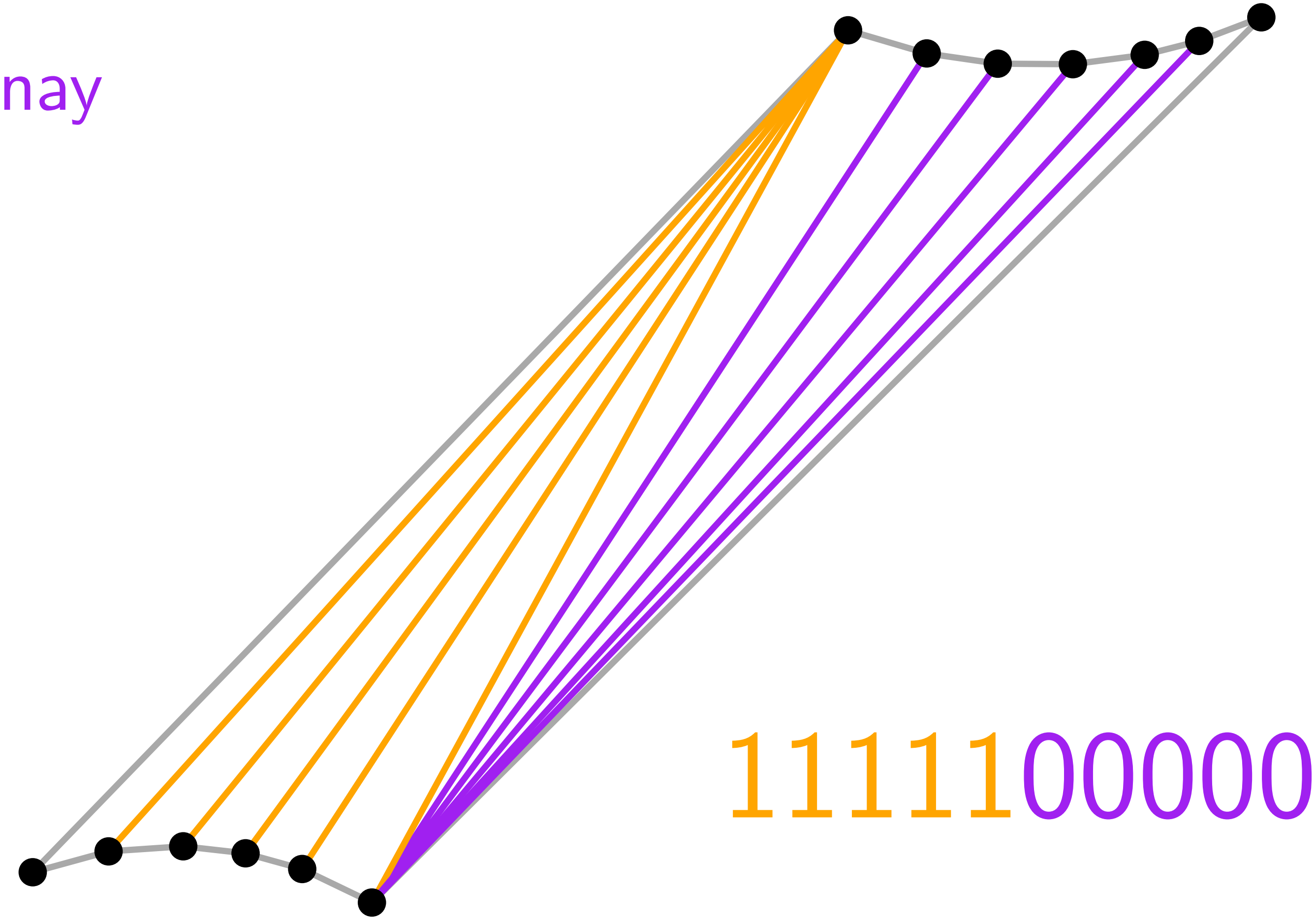


0011101010

Delaunay Triangulation: Diagonal flipping

Complexity ?

Delaunay



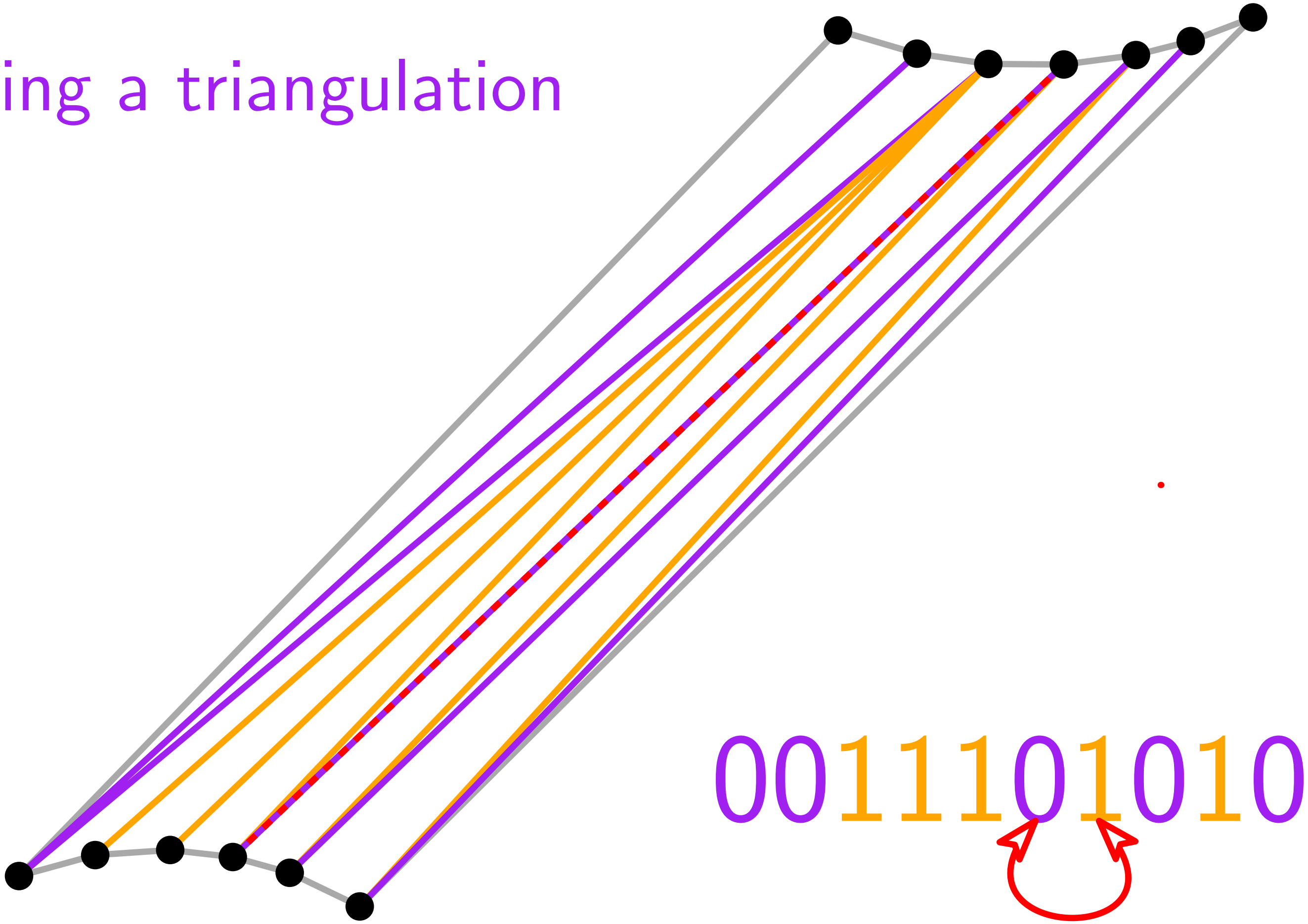
1111100000

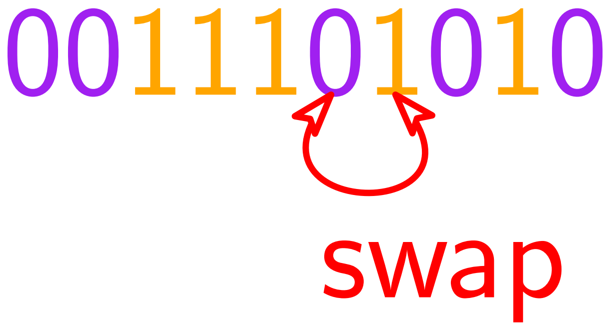
Delaunay Triangulation: Diagonal flipping

Complexity ?

Encoding a triangulation

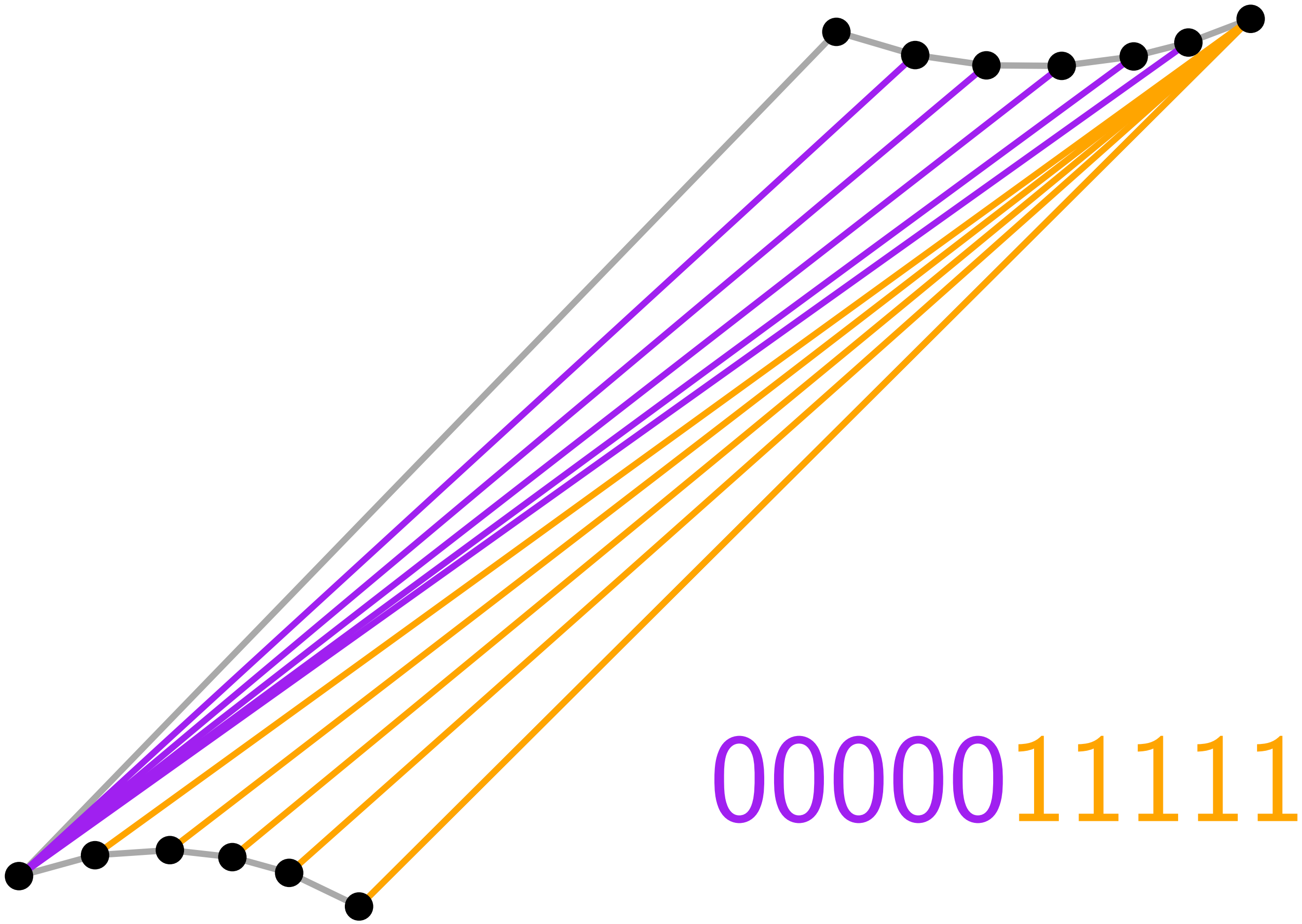
Flip





Delaunay Triangulation: Diagonal flipping

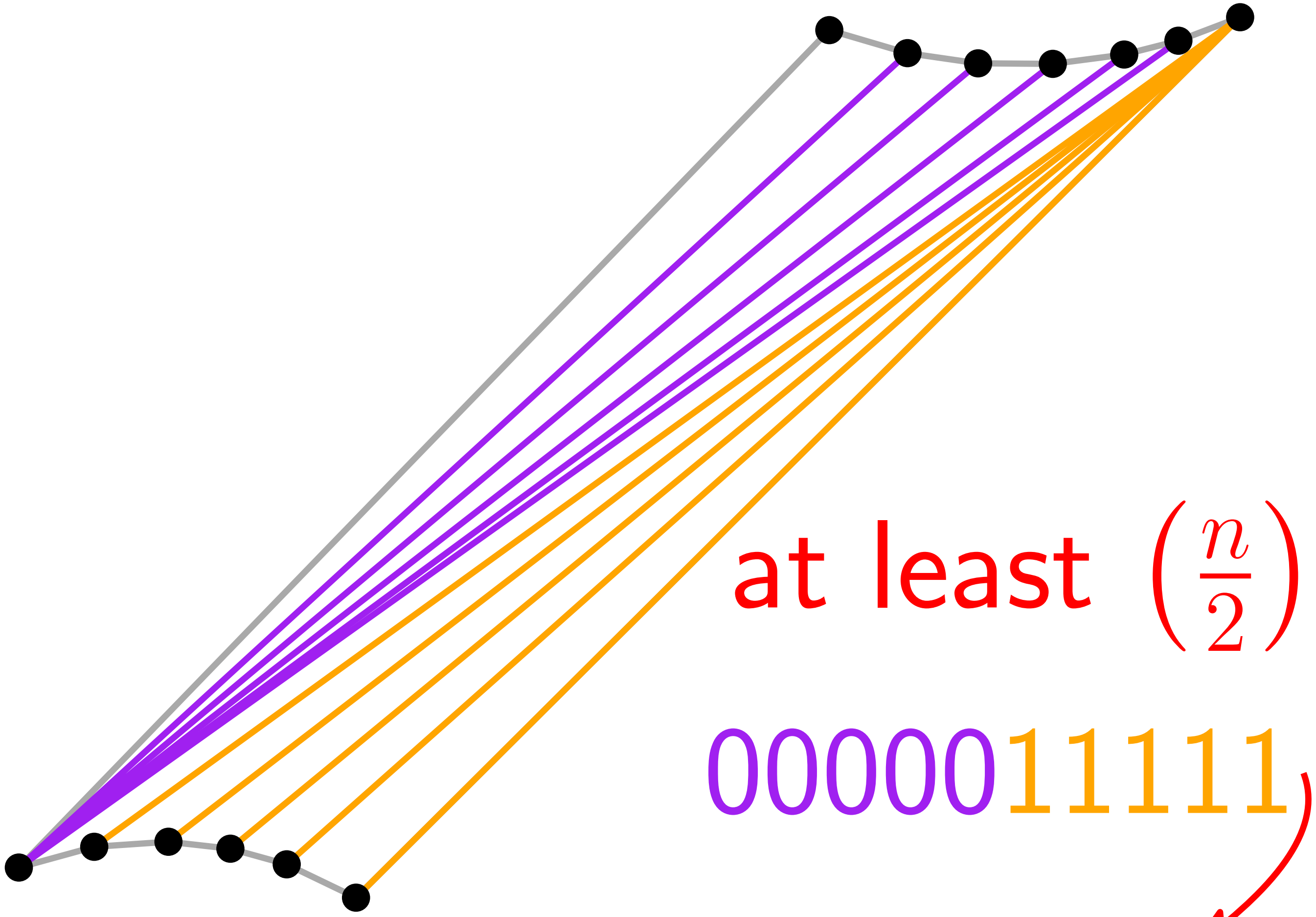
Complexity ?



0000011111

Delaunay Triangulation: Diagonal flipping

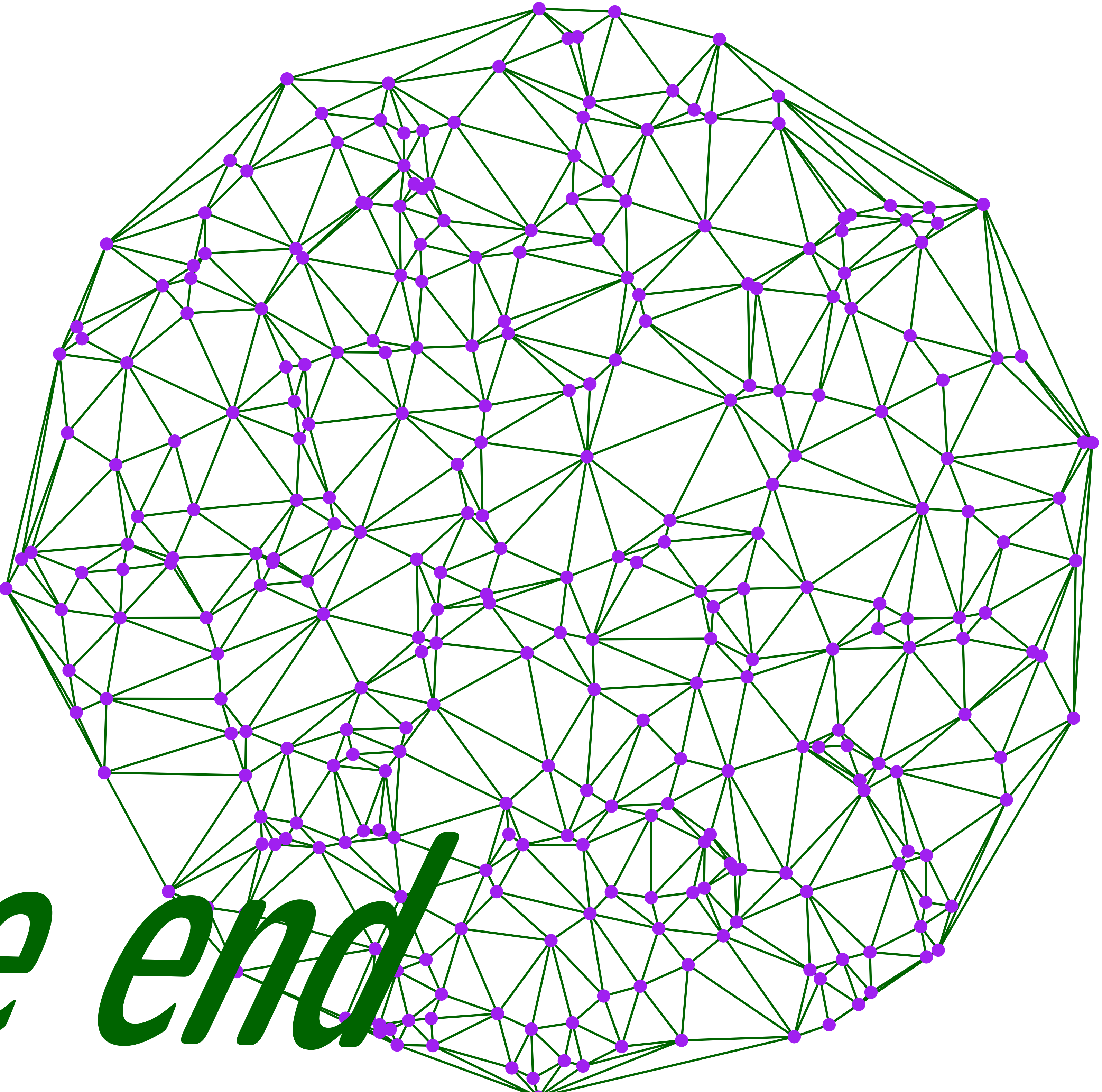
Complexity ?



at least $\left(\frac{n}{2}\right)^2$ flips

0000011111

Delaunay 1111100000



The end