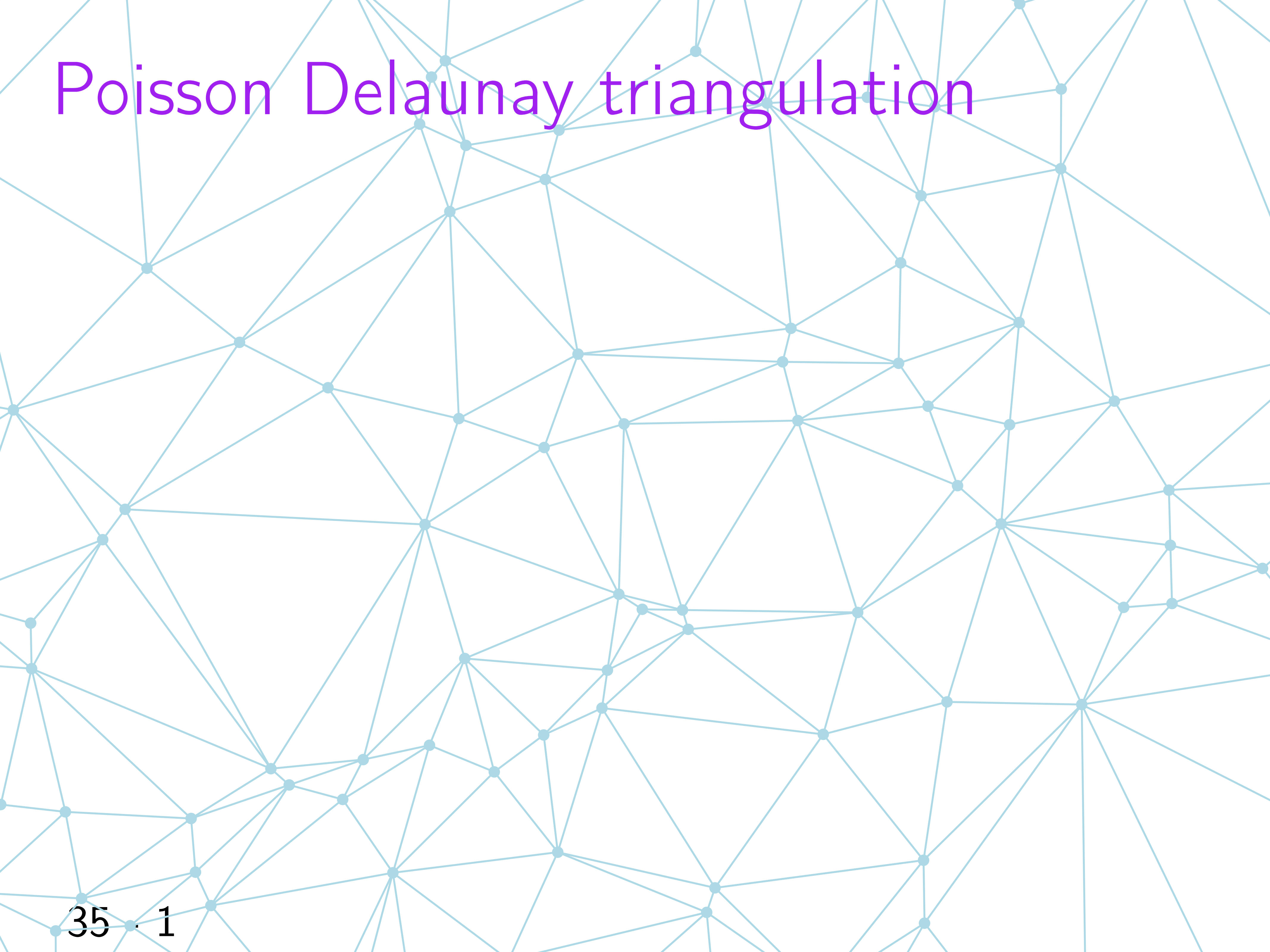
Probability and Delaunay triangulations

Poisson Delaunay triangulation



Poisson Delaunay triangulation

- Poisson distribution
- Slivnyak-Mecke formula
- Blaschke-Petkanschin variables substitution
- Stupid analysis of the expected degree
- Straight walk expected analysis
- Catalog of properties

Distribution in A independent from distribution in B.

when $A \cap B = \varnothing$

Unit uniform rate

$$\mathbb{P}\left[|X \cap A| = k\right] = \frac{\mathrm{vol}(A)^k}{k!} e^{-\mathrm{vol}(A)}$$

X a Poisson point process

Distribution in A independent from distribution in B.

when $A \cap B = \varnothing$

Very convenient

Unit uniform rate

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Distribution in A independent from distribution in B.

when $A \cap B = \varnothing$

Unit uniform rate

$$\mathbb{P}\left[|X \cap A| = k\right] = \frac{\mathrm{vol}(A)^k}{k!} e^{-\mathrm{vol}(A)}$$

$$\mathbb{P}\left[|X \cap A| = 0\right] = e^{-\mathrm{vol}(A)}$$

$$\mathbb{E}\left[|X \cap A|\right] = \sum_{0}^{\infty} k \frac{\mathrm{vol}(A)^k}{k!} e^{-\mathrm{vol}(A)} = \mathrm{vol}(A)$$

Slivnyak-Mecke formula

X a Poisson point process of density n

Sum ⟶ Integral

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$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[P(X,q)]}\right]$$

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Sum ⟶ Integral

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[P(X,q)]}\right] = n\int_{\mathbb{R}^2} \mathbb{P}\left[P(X\cup\{q\},q)\right]\,\mathrm{d}q$$

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e.g.,

$$\mathbb{E}\left[\sum_{q\in X} \mathbb{1}_{[NN_X(0)=q]}\right]$$

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$$= n\int_{\mathbb{R}^2} e^{-n\pi\|q\|^2}\,\mathrm{d}q$$

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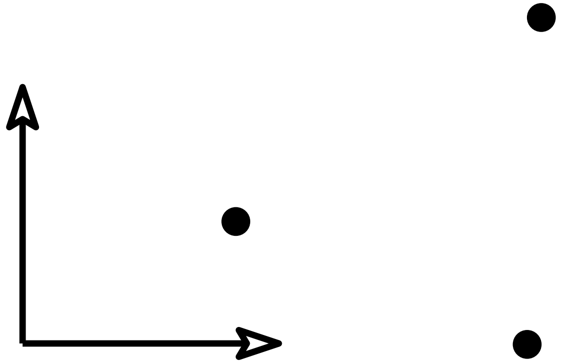
$$= n\int_{\mathbb{R}^2} e^{-n\pi\|q\|^2}\,\mathrm{d}q$$

$$= n\int_0^{2\pi}\int_0^{\infty} e^{-n\pi r^2} r\mathrm{d}\theta\,\mathrm{d}r = n\times 2\pi\times\frac{1}{2n\pi} = 1$$

Blaschke-Petkantschin variable substitution

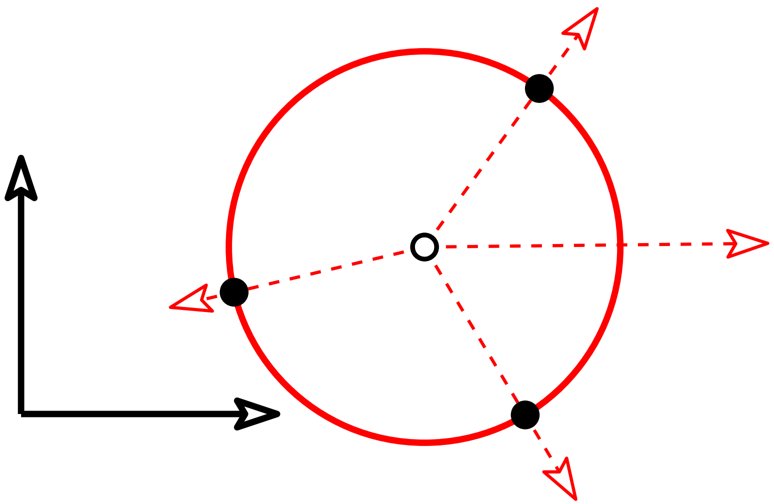
$$\int_{(\mathbb{R}^2)^3} f(p, q, t)\, dp\, dq\, dt$$



Blaschke-Petkantschin variable substitution

$$\int_{(\mathbb{R}^2)^3} f(p,q,t)\,dp\,dq\,dt$$

$$= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(p,q,t)|det(J)|\mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}x \mathrm{d}y \mathrm{d}r$$



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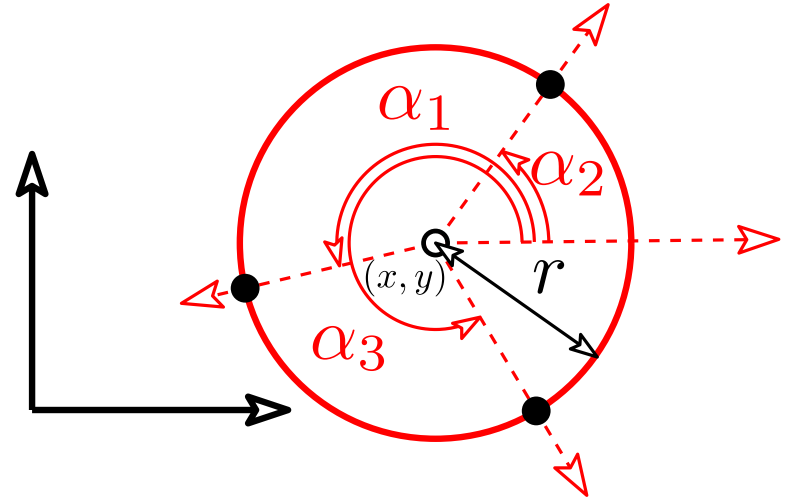
$$= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(p,q,t) 2r^3 area(\alpha_1\alpha_2\alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}x\mathrm{d}y\mathrm{d}r$$



Expected number of triangles in conflict with origin

X a Poisson point process of density n

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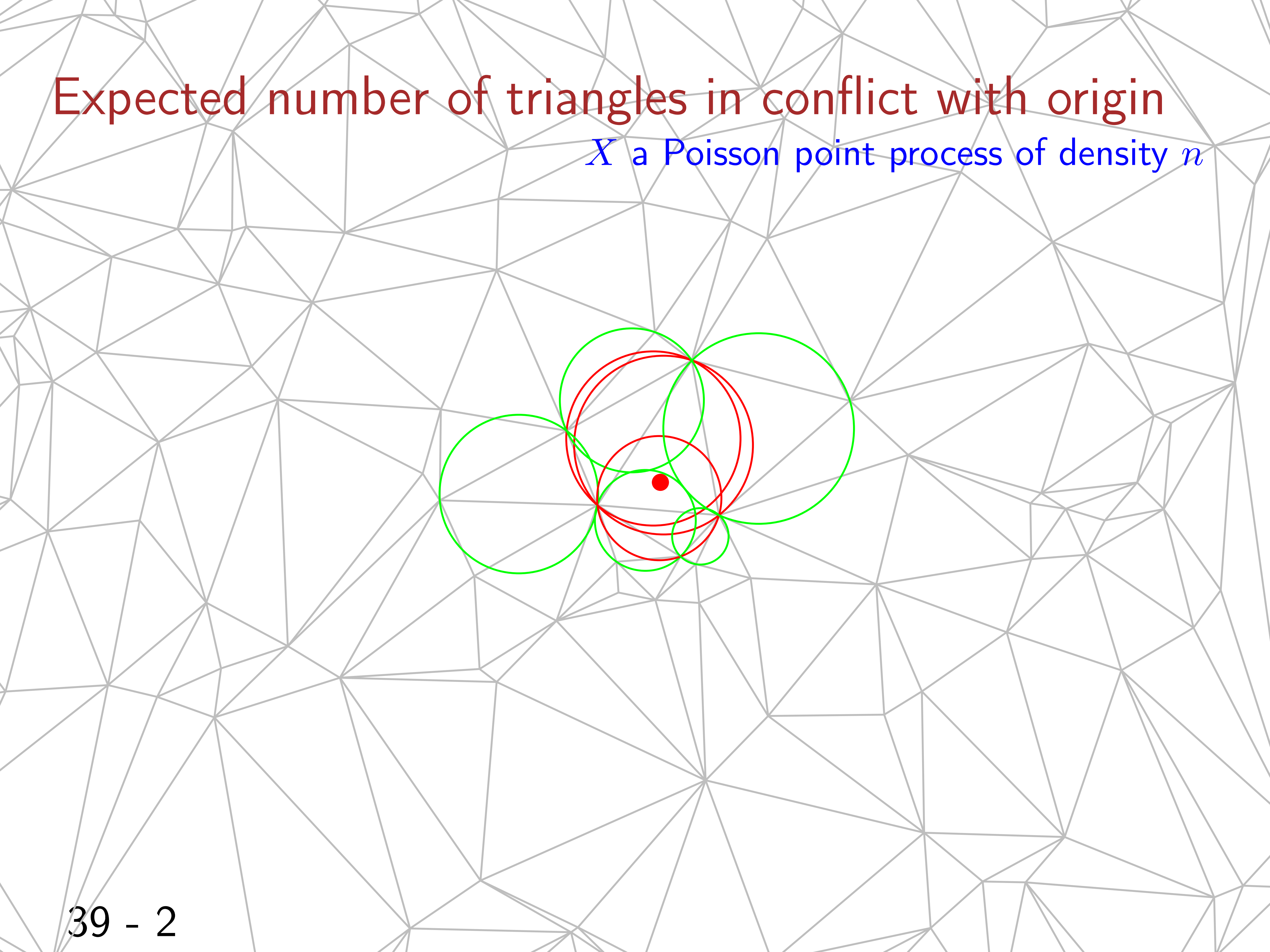

Expected number of triangles in conflict with origin

X a Poisson point process of density n

$$\mathbb{E}\left[\frac{1}{3}\sum_{p,q,t\in X^3}\mathbb{1}_{[pqt\in DT(X)]}\mathbb{1}_{[pqt\ \text{CCW}]}\mathbb{1}_{[O\in Disk(pqt)]}\right]$$

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Slivnyak-Mecke formula

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$$= \frac{n^3}{3}\int_0^\infty\int_0^r\int_0^{2\pi}\int_0^{2\pi}\int_{\alpha_1}^{\alpha_1+2\pi}\int_{\alpha_2}^{\alpha_1+2\pi} e^{-n\pi r^2}2r^3 area(\alpha_1\alpha_2\alpha_3)R\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1\mathrm{d}\theta\mathrm{d}R\mathrm{d}r$$

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$$=\frac{n^3}{3}\int_0^\infty e^{-n\pi r^2}r^3\left(\int_0^r R\mathrm{d}R\right)\left(\int_0^{2\pi}\mathrm{d}\theta\right)\mathrm{d}r\left(\int_0^{2\pi}\int_{\alpha_1}^{\alpha_1+2\pi}\int_{\alpha_2}^{\alpha 1+2\pi}2\,area(\alpha_1\alpha_2\alpha_3)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1\right)$$

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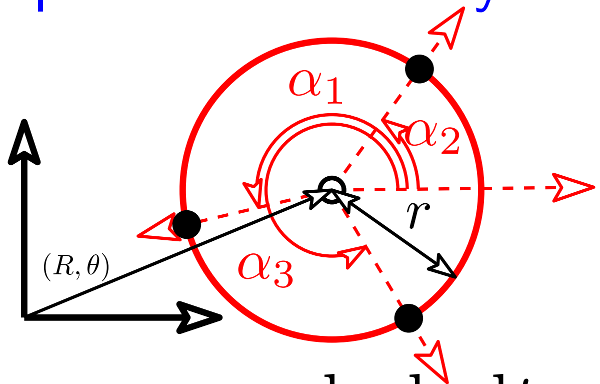

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```
Maple computation:
> assume(n>0):with(LinearAlgebra):
> int( exp(-n*Pi*r^ 2)*r^ 5,r=0..infinity);
```

$\frac{1}{n^3\pi^3}$

```
> 6*int(int(int(Determinant([[          1,            1,
                                [cos(alpha1),cos(alpha2),cos
                                [sin(alpha1),sin(alpha2),sin
     alpha3=alpha2..alpha1+2*Pi),alpha2=alpha1..alpha1+2
```

$12\pi^2$

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$$= \frac{n^3}{3}\int_0^\infty e^{-n\pi r^2}r^3 2\pi\frac{r^2}{2}\mathrm{d}r\ \ 12\pi^2 = \frac{n^3}{3}\pi\frac{1}{n^3\pi^3}12\pi^2 = 4$$

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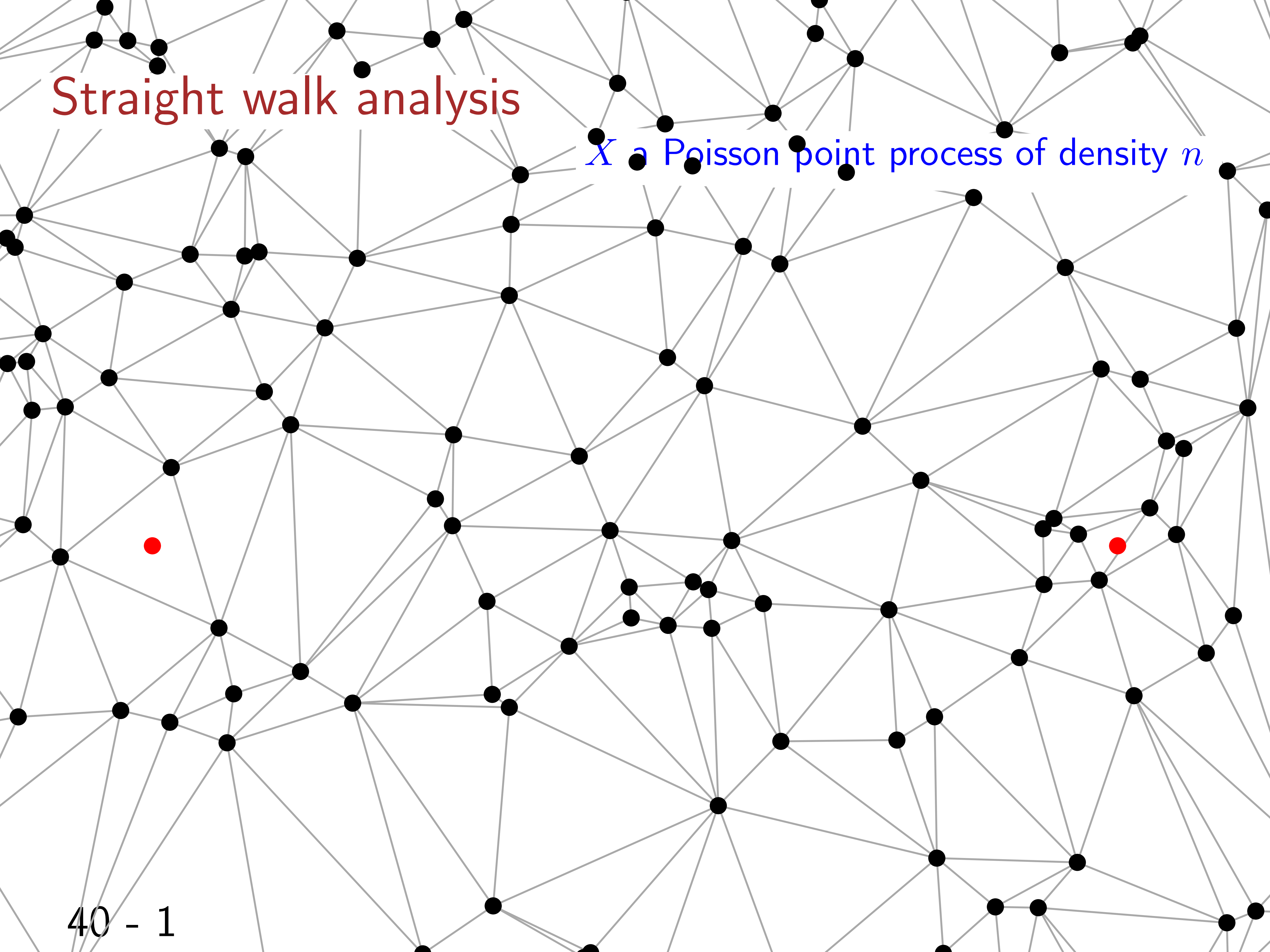
$$\Rightarrow\mathbb{E}\left[d^\circ_{DT(X\cup\{0\})}(0)\right]=6$$

(you already know ☺)

Straight walk analysis

X a Poisson point process of density n



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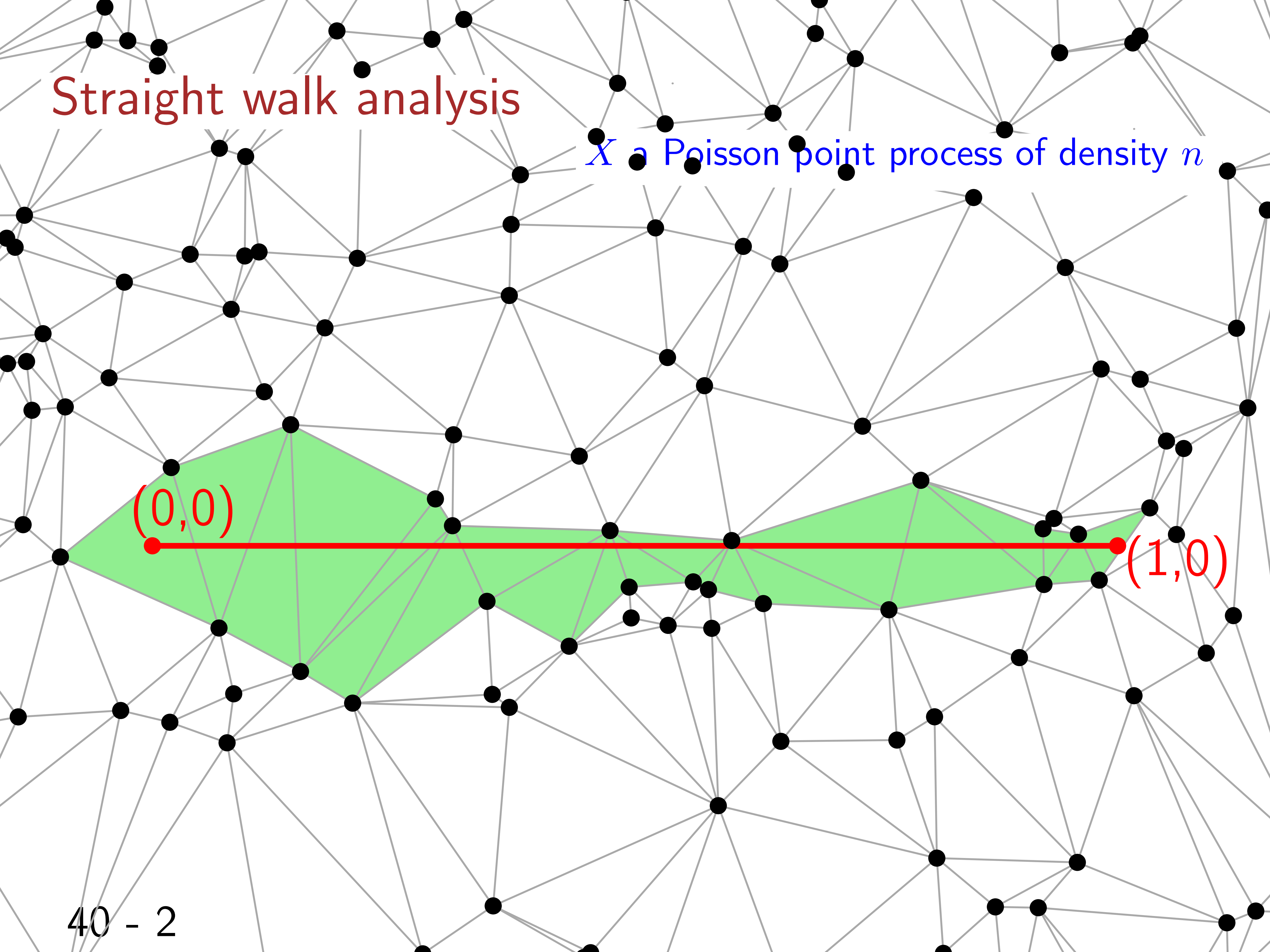

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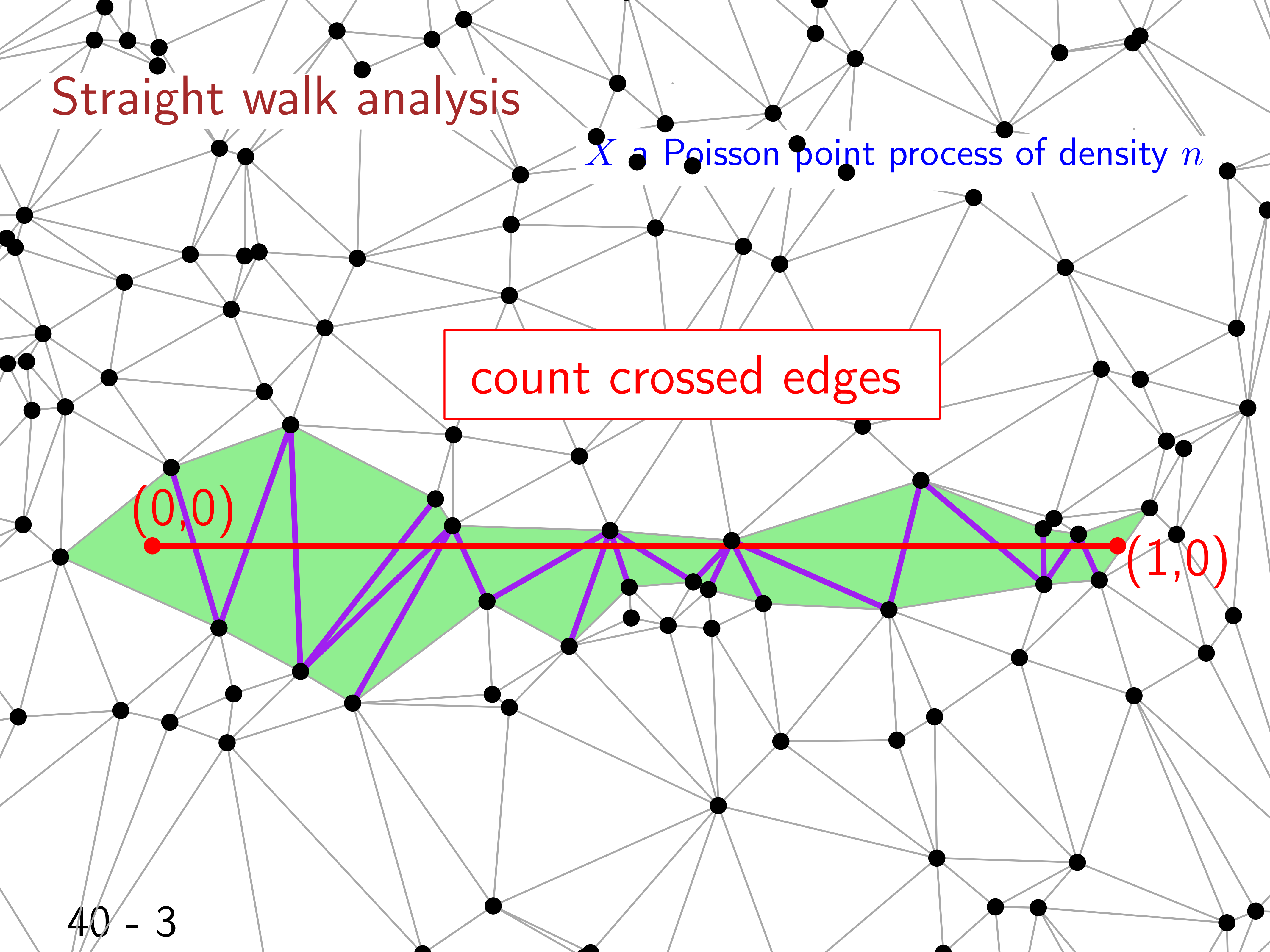

Straight walk analysis

$$\mathbb{E}\left[\overset{\color{blue}{X \text{ a Poisson point process of density } n}}{\frac{1}{2}\sum_{p,q,t\in X^3} \mathbb{1}_{[pqt\in DT(X)]}\mathbb{1}_{[p \text{ below},q \text{ above}]}\mathbb{1}_{[pq \text{ intersects segment}]}}\right]$$

Straight walk analysis

X a Poisson point process of density n

$$\mathbb{E}\left[\frac{1}{2}\sum_{p,q,t\in X^3}\mathbb{1}_{[pqt\in DT(X)]}\mathbb{1}_{[p\text{ below},q\text{ above}]}\mathbb{1}_{[pq\text{ intersects segment}]}\right]$$

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$$+\mathbb{E}\left[\frac{1}{2}\sum_{p,q,t\in X^3}\mathbb{1}_{[pqt\in DT(X)]}\mathbb{1}_{[p,t\text{ below},q\text{ above}]}\mathbb{1}_{[pq\text{ intersects segment}]}\right]$$

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$$= n^3 \int_{(\mathbb{R}^2)^3} \mathbb{P}\left[X \cap B(pqt) = \emptyset\right] \mathbb{1}_{[\text{"position"}]} \, dp \, dq \, dt$$

Slivnyak-Mecke formula

Straight walk analysis

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$$\simeq n^3 \int_0^\infty \int_0^1 \int_{-r}^{r} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} e^{-n\pi r^2} \mathbb{1}_{[\text{"position"}]}$$

$$\cdot\, r^3 2area(\alpha_1\alpha_2\alpha_3)\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}x\mathrm{d}y\mathrm{d}r$$

Blaschke-Petkantschin formula

Straight walk analysis

X a Poisson point process of density n

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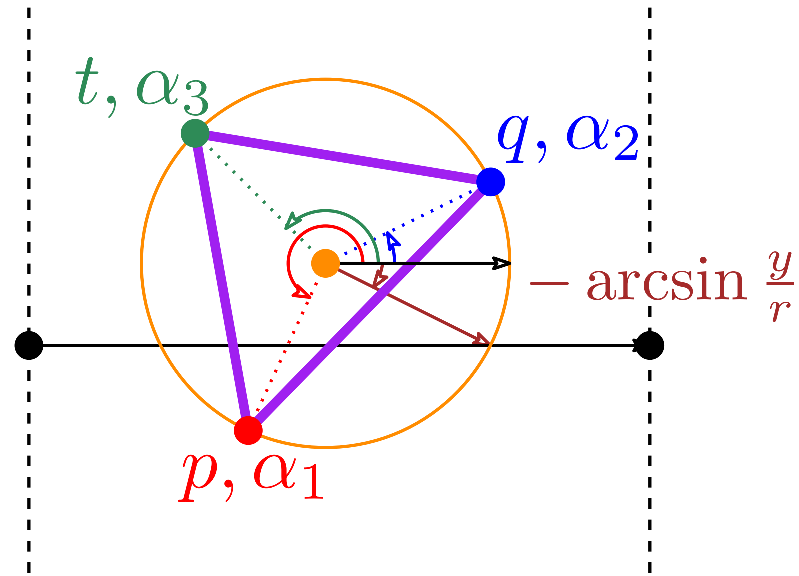
Straight walk analysis

$$\simeq n^3 \int_0^\infty \int_0^1 \int_{-r}^{r} \int_0^{2\pi}$$

$$\cdot\, r^3$$



$$\simeq n^3 \int_0^\infty \int_{-r}^{r} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} e^{-n\pi r^2} \mathbb{1}_{[\text{"position"}]}$$

$$\cdot\, r^3 2area(\alpha_1\alpha_2\alpha_3) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}y \mathrm{d}r$$

$$\simeq n^3 \int_0^\infty \int_{-1}^{1} \int_{\pi+\arcsin h}^{2\pi-\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} e^{-n\pi r^2}$$

$rh = y$

$$\cdot\, r^3 2area(\alpha_1\alpha_2\alpha_3) \mathrm{d}\alpha_3 \mathrm{d}\alpha_2 \mathrm{d}\alpha_1 r \mathrm{d}h \mathrm{d}r$$

Straight walk analysis

X a Poisson point process of density n

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$$\simeq n^3 \int_0^\infty e^{-n\pi r^2} r^3 \mathrm{d}r$$

$$\times \int_{-1}^1 \int_{\pi+\arcsin h}^{2\pi-\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} 2 area(\alpha_1 \alpha_2 \alpha_3) \mathrm{d}\alpha_3 \mathrm{d}\alpha_2 \mathrm{d}\alpha_1 r \mathrm{d}h$$

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ask Maple !

$$\times \int_{-1}^{1} \int_{\pi+\arcsin h}^{2\pi-\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} \int_{-\arcsin h}^{\pi+\arcsin h} 2area(\alpha_1\alpha_2\alpha_3) \mathrm{d}\alpha_3 \mathrm{d}\alpha_2 \mathrm{d}\alpha_1 r \mathrm{d}h$$

$$\simeq n^3 \int_0^\infty e^{-n\pi r^2} r^3 \, \frac{512}{9} r \, \mathrm{d}r$$

Straight walk analysis

X a Poisson point process of density n

$$\mathbb{E}\left[\frac{1}{2}\sum_{p,q,t\in X^3} \mathbb{1}_{[pqt\in DT(X)]}\mathbb{1}_{[p \text{ below}, q \text{ above}]}\mathbb{1}_{[pq \text{ intersects segment}]}\right]$$

$$\simeq n^3\int_0^\infty e^{-n\pi r^2} r^3 \mathrm{d}r$$

$$\times \int_{-1}^{1}\int_{\pi+\arcsin h}^{2\pi-\arcsin h}\int_{-\arcsin h}^{\pi+\arcsin h}\int_{-\arcsin h}^{\pi+\arcsin h} 2area(\alpha_1\alpha_2\alpha_3)\mathrm{d}\alpha_3\mathrm{d}\alpha_2\mathrm{d}\alpha_1 r\mathrm{d}h$$

$$\simeq n^3\int_0^\infty e^{-n\pi r^2} r^3 \ \frac{512}{9} r \ \mathrm{d}r$$

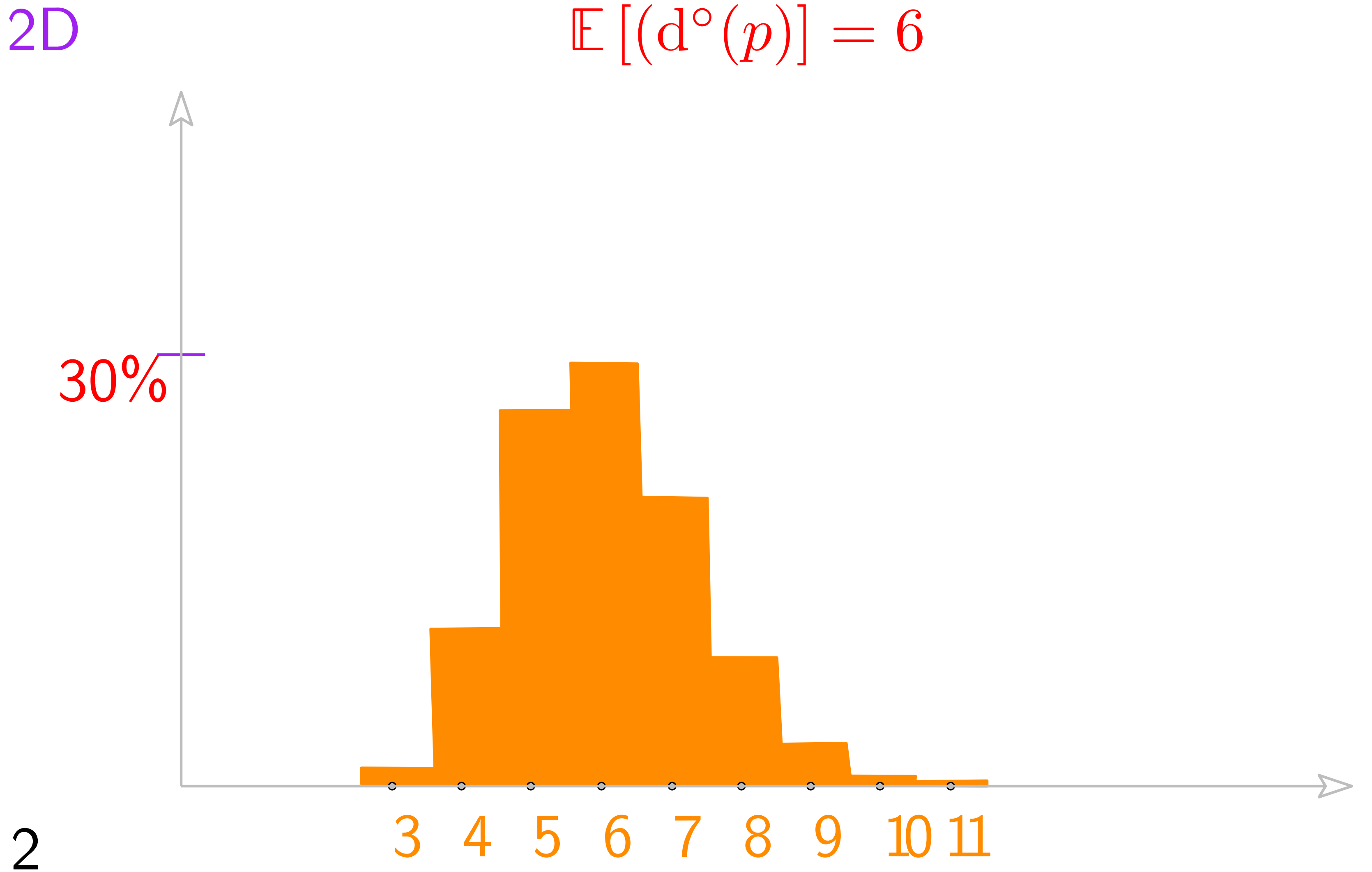
$$= \frac{512}{9} n^3 \frac{3}{8\pi^2 n^2\sqrt{n}} = \frac{64}{3\pi^2}\sqrt{n} \simeq 2.16\sqrt{n}$$

Sample of other probabilistic results

Expected degree

2D $\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = 6$

Expected degree



Expected degree

2D $\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = 6$

3D $\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = \frac{48\pi^2}{35} + 2 \simeq 15.535$

Expected degree

2D	$\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = 6$
3D	$\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = \frac{48\pi^2}{35} + 2 \simeq 15.535$
3D on a cylinder	$\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = \Theta(\log n)$

Expected degree

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3D $\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = \frac{48\pi^2}{35} + 2 \simeq 15.535$

3D on a cylinder $\mathbb{E}\left[(\mathrm{d}^\circ(p)\right] = \Theta(\log n)$

3D on a surface $O(1) \leq \mathbb{E}\left[(\mathrm{d}^\circ(p)\right] \leq O(\log n)$

generic

conjecture

Expected maximum degree

Poisson distribution intensity 1, window $[0, \sqrt{n}]^2$

$$\mathbb{E}\left[\max(\mathrm{d}^\circ(p)\right] = \Theta\left(\frac{\log n}{\log\log n}\right)$$

Expected maximum degree

Poisson distribution intensity 1, window $[0, \sqrt{n}]^2$

$$\mathbb{E}\left[\max(\mathrm{d}^\circ(p)\right] = \Theta\left(\frac{\log n}{\log\log n}\right)$$

no boundaries!

Expected maximum degree

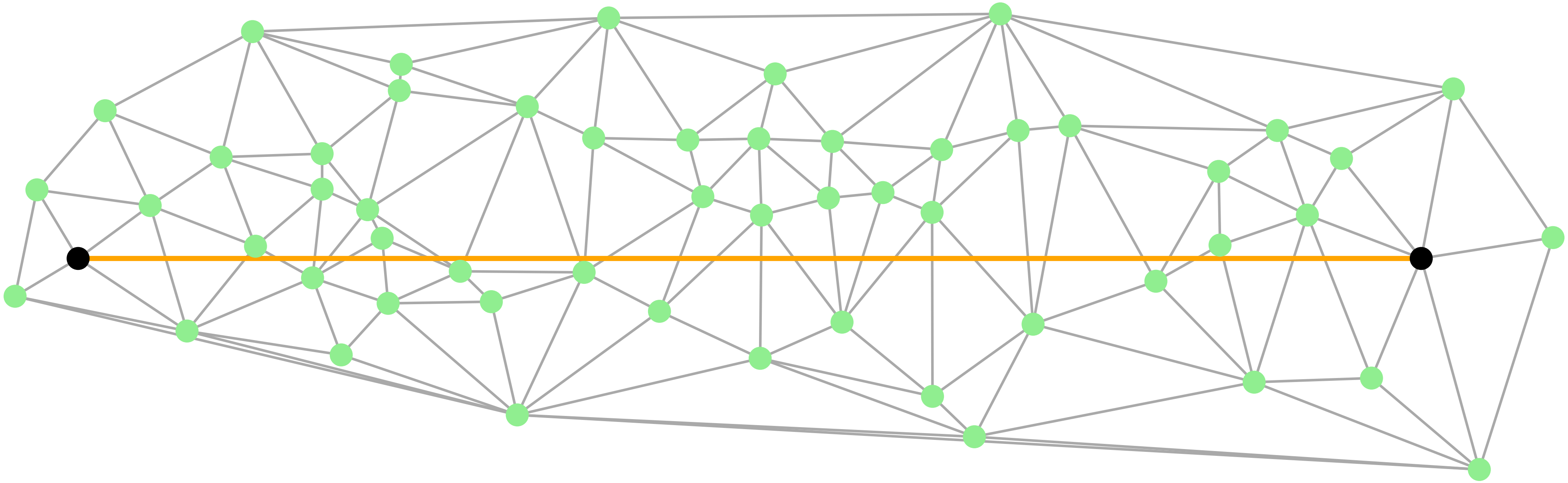
Poisson distribution intensity 1, window $[0, \sqrt{n}]^2$

$$\mathbb{E}\left[\max(\mathrm{d}^\circ(p)\right] = \Theta\left(\frac{\log n}{\log\log n}\right)$$

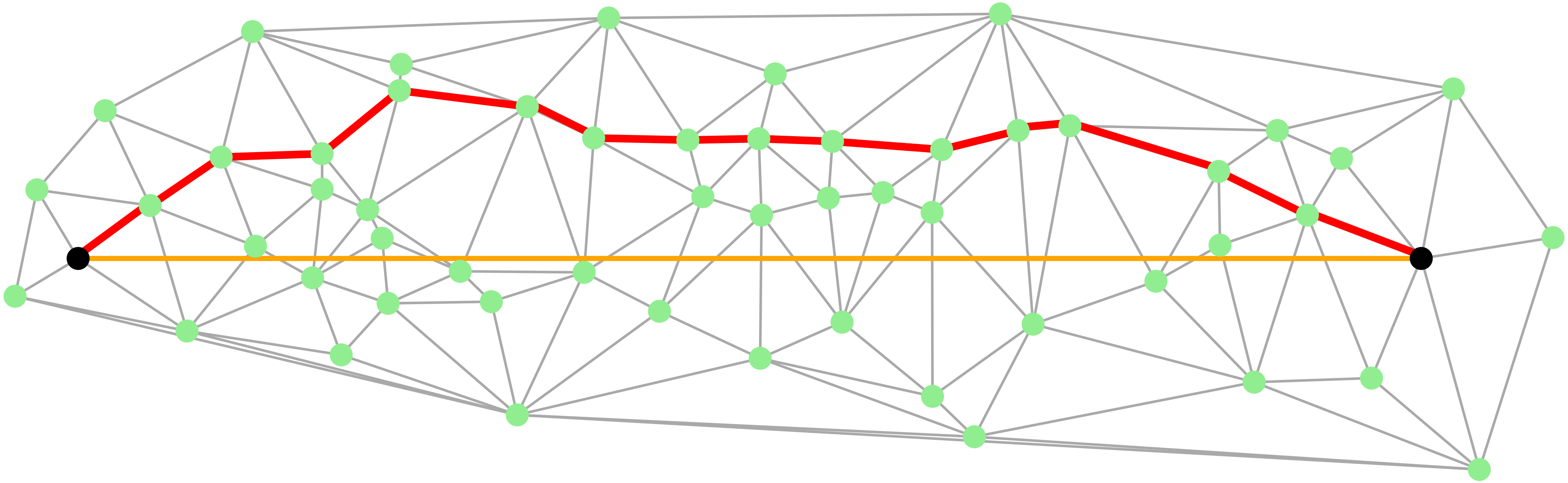
Poisson distribution intensity n, bounded domain

$$\mathbb{E}\left[\max(\mathrm{d}^\circ(p)\right] = O\left(\log^{2+\epsilon} n\right)$$

Walk between vertices

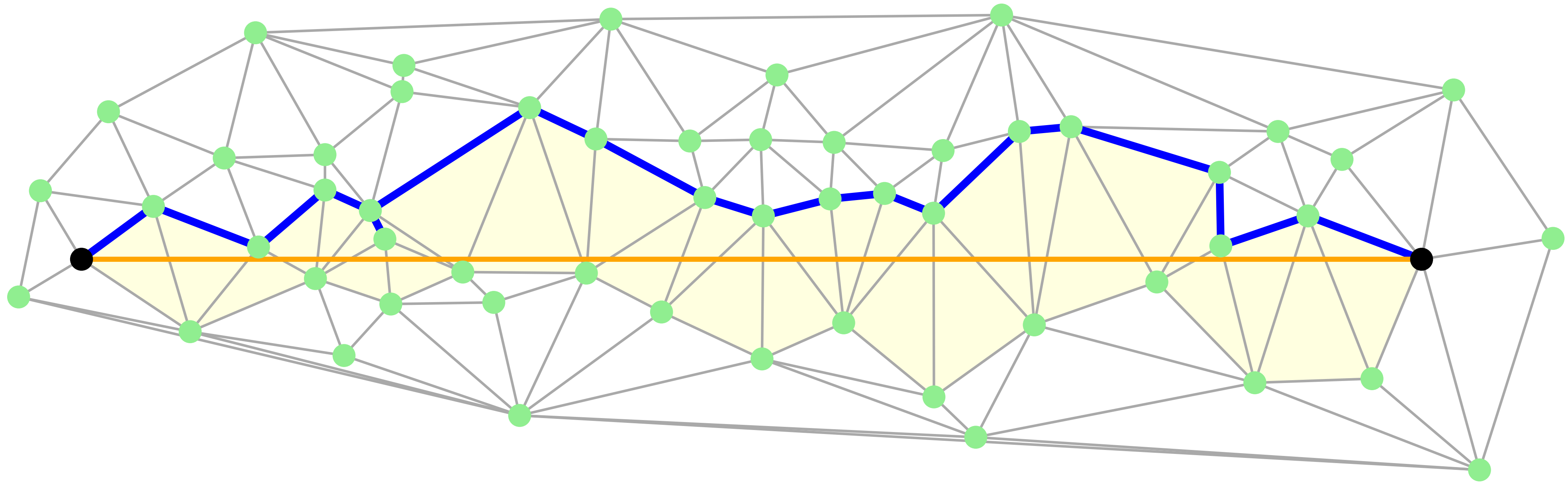


Walk between vertices



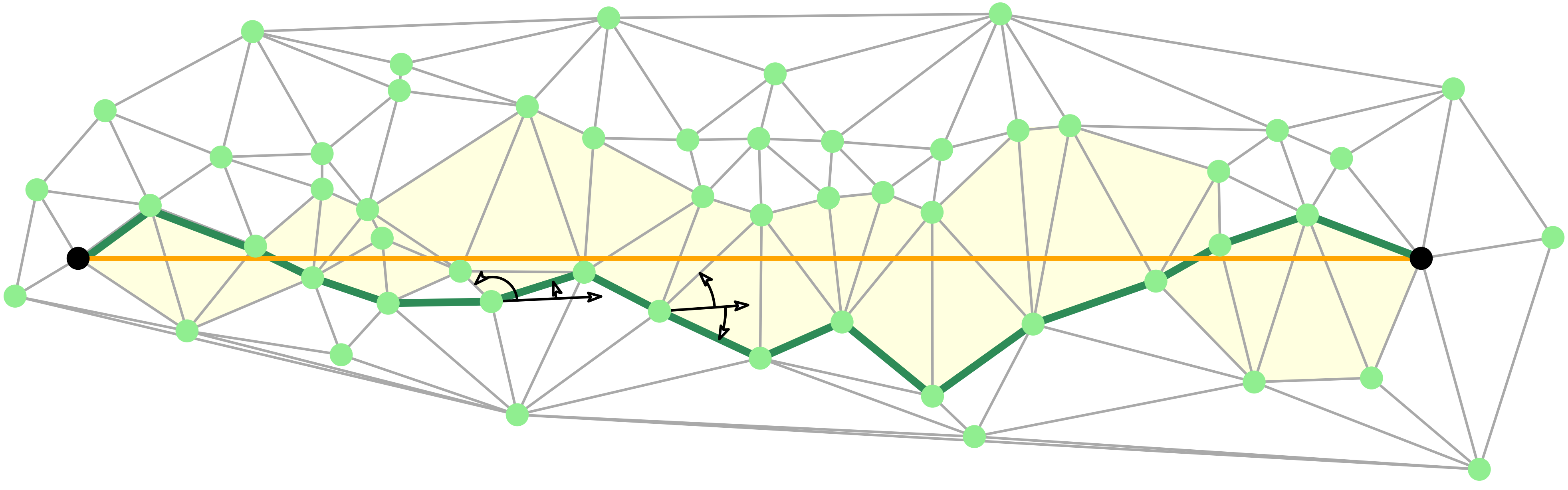
Shortest path

Walk between vertices



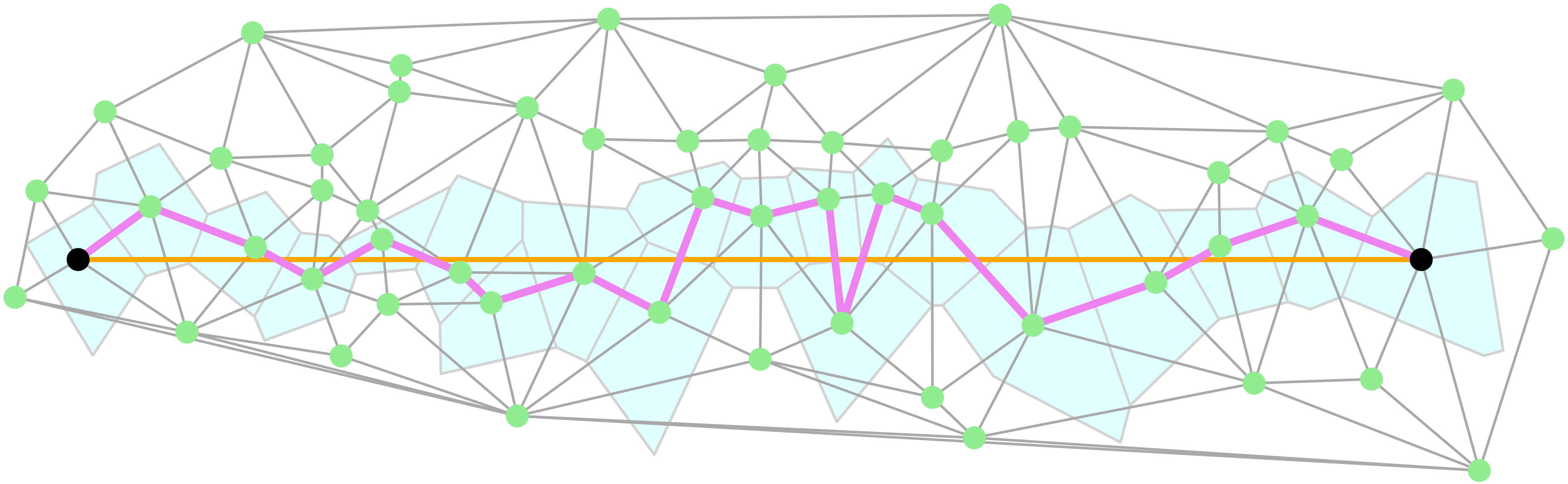
Upper path

Walk between vertices



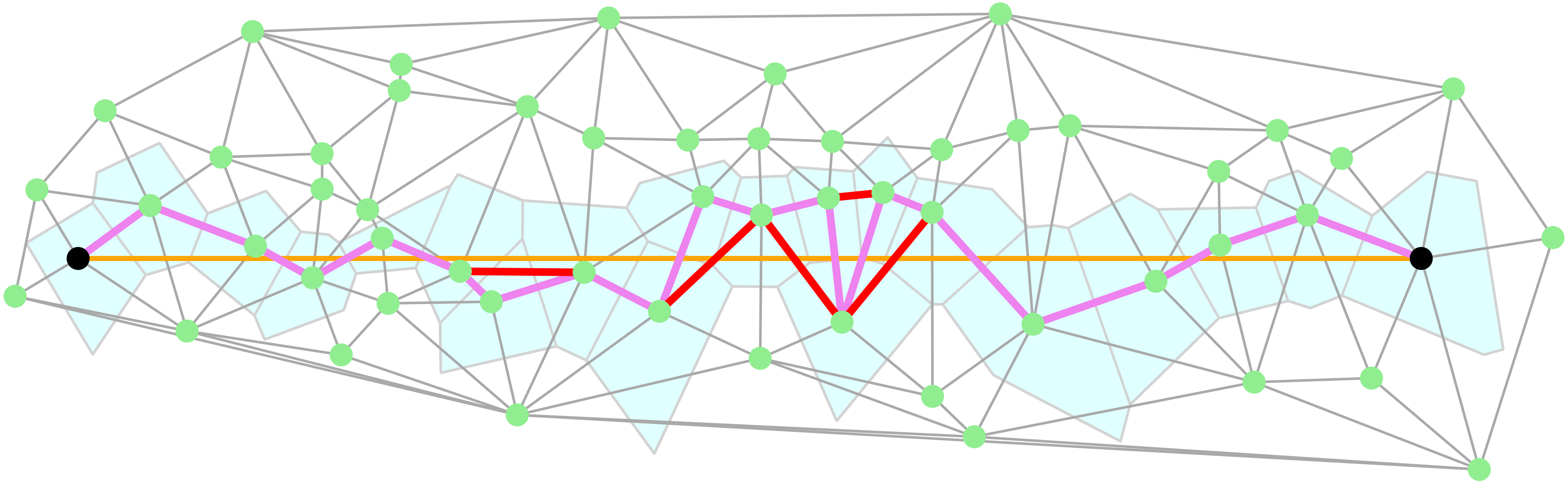
Compass walk

Walk between vertices



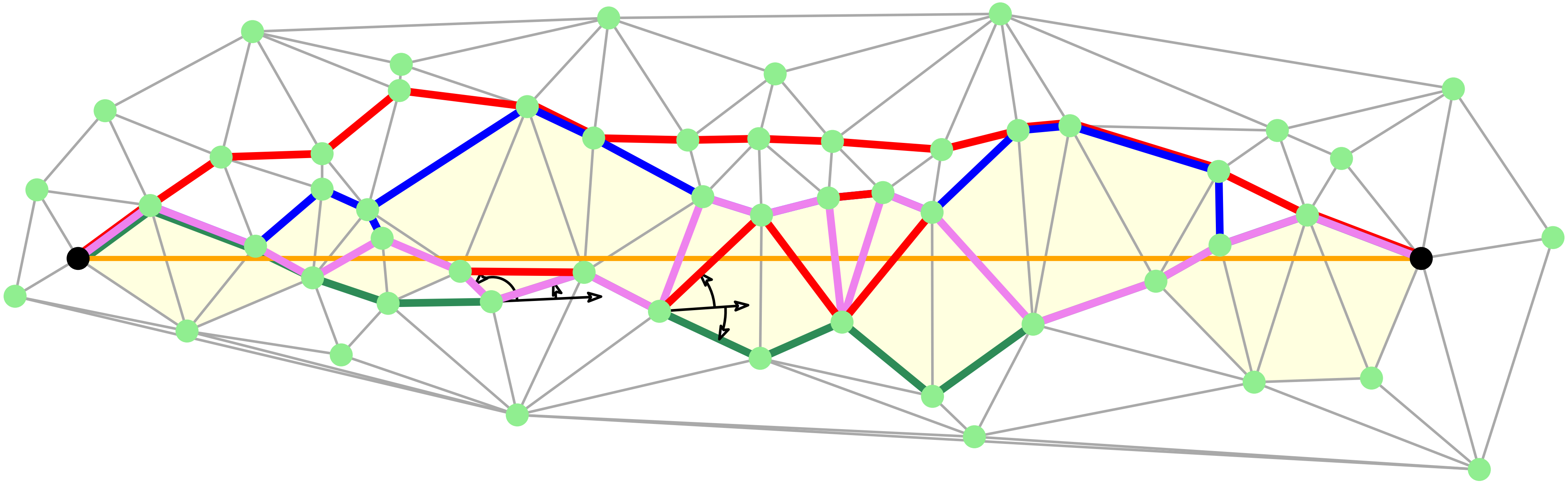
Voronoi path

Walk between vertices



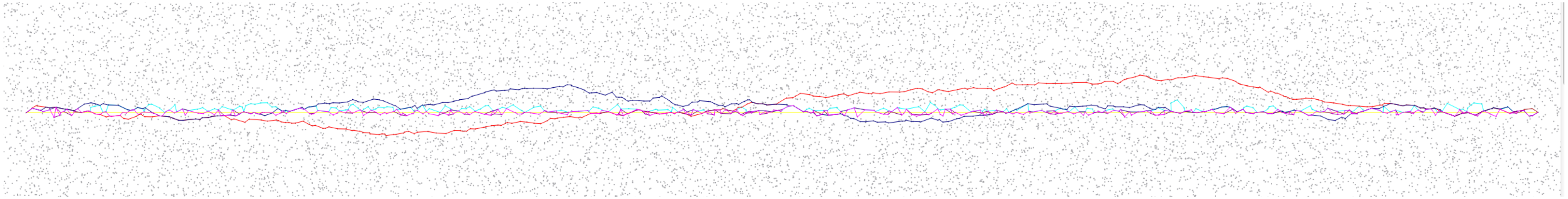
Voronoi path with shortcuts

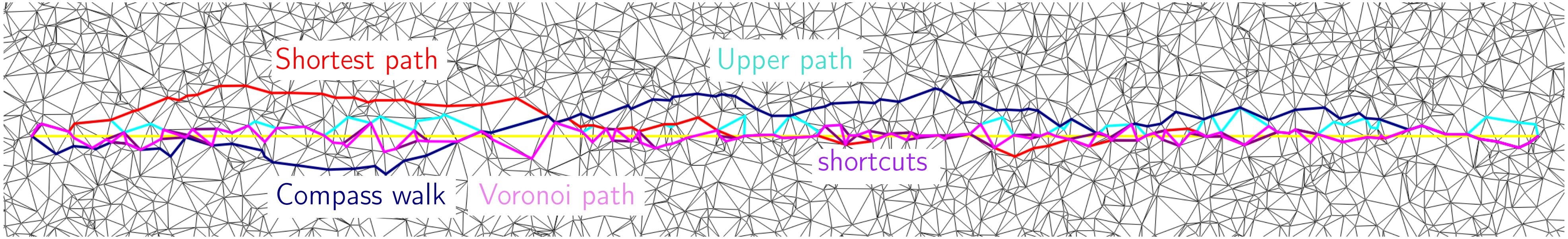
Walk between vertices



Shortest path
Upper path
Compass walk
Voronoi path with shortcuts

Walk between vertices





Expected length (experiments)

Euclidean length	1
Shortest path	1.04
Compass walk	1.07
Shortened V. path	1.16
Upper path	1.18
Voronoi path	1.27

Walk between vertices

Expected length (experiments)		theory
Euclidean length	1	
Shortest path	1.04	$\geq 1 + 10^{-11}$
Compass walk	1.07	
Shortened V. path	1.16	1.16 (numerical integration)
Upper path	1.18	$\frac{35}{3\pi^2} \simeq 1.18$
Voronoi path	1.27	$\frac{4}{\pi} \simeq 1.27$ [Baccelli et al., 2000]

The end