# Introduction to several models from stochastic geometry

Pierre Calka

*Computational Geometry Week 2015*

**Eindhoven**, *25 June 2015*

# Plan

From game to theory: Buffon, integral geometry, random tessellations

From game to theory: 150 years of random convex hulls

Addendum: some more models

# Plan

From game to theory: Buffon, integral geometry, random tessellations

- Buffon's needle problem
- Example of a formula from integral geometry
- Poisson point process
- Poisson line tessellation
- Poisson-Voronoi tessellation

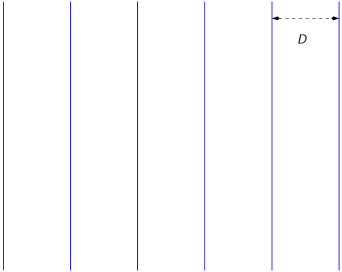
From game to theory: 150 years of random convex hulls

Addendum: some more models

# Roots of geometric probability

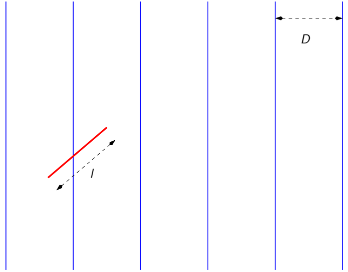
**Georges-Louis Leclerc, Comte de Buffon** (1733)

Probability $p$ that a needle of length $\ell$ dropped on a floor made of parallel strips of wood of same width $D > \ell$ will lie across a line?

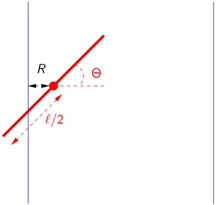


**Georges-Louis Leclerc, Comte de Buffon** (1733)

Probability $p$ that a needle of length $\ell$ dropped on a floor made of parallel strips of wood of same width $D > \ell$ will lie across a line?



# Roots of geometric probability

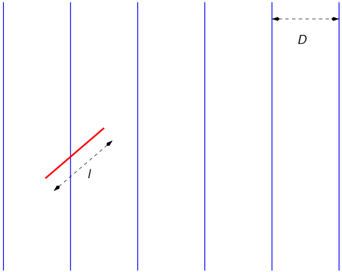



$R$ and $\Theta$ independent r.v., uniformly distributed on $]0, \frac{D}{2}[$ and $]-\frac{\pi}{2}, \frac{\pi}{2}[$.
There is intersection when $2R \leq \ell \cos(\Theta)$.

$$p = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{\frac{\ell}{2}\cos(\theta)} \frac{\mathrm{d}r\mathrm{d}\theta}{\frac{D}{2}\pi} = \frac{2\ell}{\pi D}$$

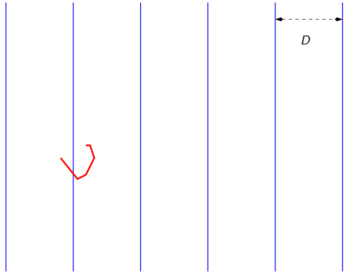
# Roots of geometric probability

$$p = p([0, \ell]) = \frac{2\ell}{\pi D}$$



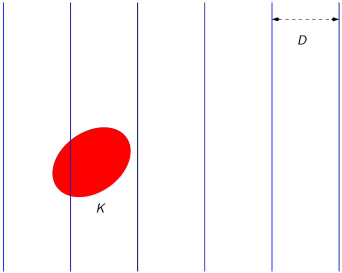
# Roots of geometric probability

Same question when dropping a polygonal line?



# Roots of geometric probability

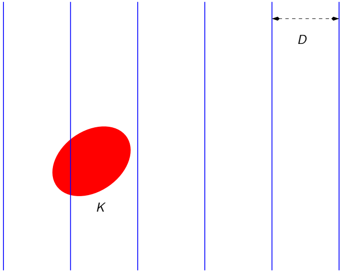
Same question when dropping a convex body $K$?



# Roots of geometric probability

$$p(\partial K) = \frac{\mathrm{per}(\partial \mathrm{K})}{\pi D}$$

where $\mathrm{per}(\partial \mathrm{K})$ : perimeter of $\partial K$



# Roots of geometric probability

*Notation*

• $p_k(\mathscr{C})$ probability to have exactly $k$ intersections of $\mathscr{C}$ with the lines
• $f(\mathscr{C}) = \sum_{k\geq 1} kp_k(\mathscr{C})$ mean number of intersections

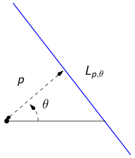
*Several juxtaposed needles*

• $f([0,\ell])$, $\ell > 0$, additive and increasing so $f([0,\ell]) = \alpha\ell$, $\alpha > 0$
• Similarly, $f(\mathscr{C}) = \alpha \mathrm{per}(\mathscr{C})$
• $f(\text{Circle of diameter } D) = 2 = \alpha\pi D$
• If $\mathscr{C}$ is the boundary of a convex body $K$ with $\mathrm{diam(K)} < \mathrm{D}$, $f(\mathscr{C}) = 2p(\mathscr{C})$

# Extensions in integral geometry

$K$ convex body of $\mathbb{R}^2$

$L_{p,\theta} = p(\cos(\theta), \sin(\theta)) + \mathbb{R}(-\sin(\theta), \cos(\theta))$, $p \in \mathbb{R}$, $\theta \in [0, \pi)$

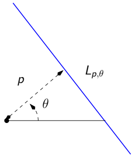


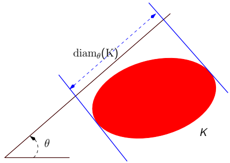

$$\mathrm{per}(\partial \mathrm{K}) = \int_{\theta=0}^{\pi} \int_{\mathrm{p}=-\infty}^{+\infty} \mathbf{1}(\mathrm{L}_{\mathrm{p},\theta} \cap \mathrm{K} \neq \emptyset) \mathrm{dpd}\theta$$

# Extensions in integral geometry

$K$ convex body of $\mathbb{R}^2$

$L_{p,\theta} = p(\cos(\theta), \sin(\theta)) + \mathbb{R}(-\sin(\theta), \cos(\theta))$, $p \in \mathbb{R}$, $\theta \in [0, \pi)$



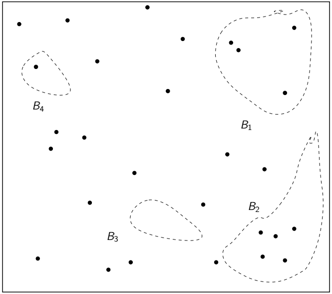




$$\mathrm{per}(\partial K) = \int_{\theta=0}^{\pi} \int_{p=-\infty}^{+\infty} \mathbf{1}(L_{p,\theta} \cap K \neq \emptyset) \, dp \, d\theta$$

**Cauchy-Crofton formula**

$$\mathrm{per}(\partial K) = \int_{\theta=0}^{\pi} \mathrm{diam}_\theta(K) \, d\theta$$

# Random points



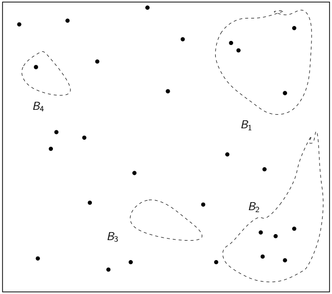

- $W$ convex body
- $\mu$ probability measure on $W$
- $(X_i, i \geq 1)$ independent $\mu$-distributed variables

$$\mathcal{E}_n = \{X_1, \cdots, X_n\} \quad (n \geq 1)$$

- $\#(\mathcal{E}_n \cap B_1)$ number of points in $B_1$
  - $\#(\mathcal{E}_n \cap B_1)$ binomial variable
    $\mathbb{P}(\#(\mathcal{E}_n \cap B_1) = k) = \binom{n}{k}\mu(B_1)^k(1-\mu(B_1))^{n-k}$, $0 \leq k \leq n$
  - $\#(\mathcal{E}_n \cap B_1), \cdots, \#(\mathcal{E}_n \cap B_n)$ not independent

$(B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R}^2),\ B_i \cap B_j = \emptyset,\ i \neq j)$

# Poisson point process




Poisson point process with intensity measure $\mu$ : locally finite subset $\mathbf{X}$ of $\mathbb{R}^d$ such that

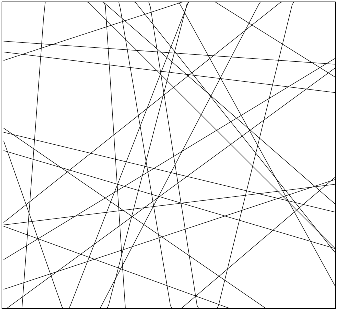
- $\#(\mathbf{X} \cap B_1)$ Poisson r.v. of mean $\mu(B_1)$

  $$\mathbb{P}(\#(\mathbf{X} \cap B_1) = k) = e^{-\mu(B_1)} \frac{\mu(B_1)^k}{k!}, \ k \in \mathbb{N}$$

- $\#(\mathbf{X} \cap B_1), \cdots, \#(\mathbf{X} \cap B_n)$ independent

  $(B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R}^d), \ B_i \cap B_j = \emptyset, \ i \neq j)$

# Poisson line tessellation



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^2$ of intensity measure $\mathrm{d}p\mathrm{d}\theta$
- For $(p,\theta) \in \mathbf{X}$, polar line

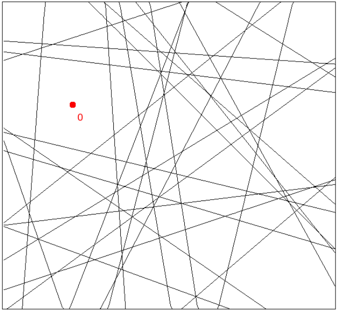
$$L_{p,\theta} = p(\cos(\theta), \sin(\theta)) + (\cos(\theta), \sin(\theta))^{\perp}$$

- Tessellation:
  set of connected components of
  $$\mathbb{R}^d \setminus \bigcup_{(p,\theta)\in\mathbf{X}} L_{p,\theta}$$

*Properties*: invariance under translations and rotations

*References*: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

# Poisson line tessellation



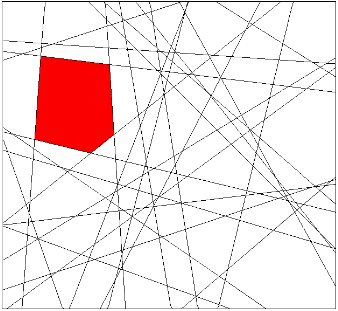

- $\mathbf{X}$ Poisson point process in $\mathbb{R}^2$ of intensity measure $\mathrm{d}p\mathrm{d}\theta$
- For $(p,\theta) \in \mathbf{X}$, polar line

$$L_{p,\theta} = p(\cos(\theta), \sin(\theta)) + (\cos(\theta), \sin(\theta))^{\perp}$$

- Tessellation:
  set of connected components of
  $\mathbb{R}^d \setminus \bigcup_{(p,\theta)\in\mathbf{X}} L_{p,\theta}$

*Properties*: invariance under translations and rotations
*References*: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

# Poisson line tessellation



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^2$ of intensity measure $\mathrm{d}p\mathrm{d}\theta$
- For $(p, \theta) \in \mathbf{X}$, polar line

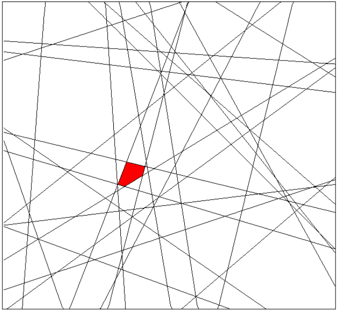
$$L_{p,\theta} = p(\cos(\theta), \sin(\theta)) + (\cos(\theta), \sin(\theta))^{\perp}$$

- Tessellation: set of connected components of $\mathbb{R}^d \setminus \bigcup_{(p,\theta)\in\mathbf{X}} L_{p,\theta}$

*Properties*: invariance under translations and rotations

*References*: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

# Poisson line tessellation



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^2$ of intensity measure $\mathrm{d}p\mathrm{d}\theta$
- For $(p, \theta) \in \mathbf{X}$, polar line

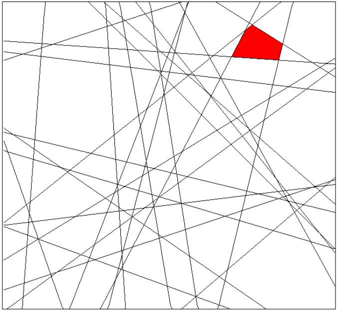
$$L_{p,\theta} = p(\cos(\theta), \sin(\theta)) + (\cos(\theta), \sin(\theta))^{\perp}$$

- Tessellation:
  set of connected components of
  $\mathbb{R}^d \setminus \bigcup_{(p,\theta)\in\mathbf{X}} L_{p,\theta}$

*Properties*: invariance under translations and rotations

*References*: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

# Poisson line tessellation



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^2$ of intensity measure $\mathrm{d}p\mathrm{d}\theta$
- For $(p, \theta) \in \mathbf{X}$, polar line

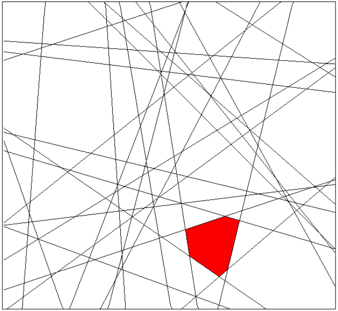
$$L_{p,\theta} = p(\cos(\theta), \sin(\theta)) + (\cos(\theta), \sin(\theta))^{\perp}$$

- Tessellation:
  set of connected components of
  $$\mathbb{R}^d \setminus \bigcup_{(p,\theta)\in\mathbf{X}} L_{p,\theta}$$

*Properties*: invariance under translations and rotations

*References*: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

# Poisson line tessellation



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^2$ of intensity measure $\mathrm{d}p\mathrm{d}\theta$
- For $(p,\theta) \in \mathbf{X}$, polar line

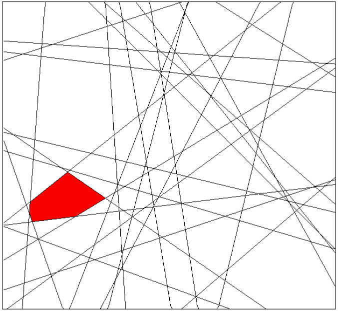
$$L_{p,\theta} = p(\cos(\theta), \sin(\theta)) + (\cos(\theta), \sin(\theta))^\perp$$

- Tessellation:
  set of connected components of
  $$\mathbb{R}^d \setminus \bigcup_{(p,\theta)\in\mathbf{X}} L_{p,\theta}$$

*Properties*: invariance under translations and rotations

*References*: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

# Poisson line tessellation



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^2$ of intensity measure $\mathrm{d}p\mathrm{d}\theta$
- For $(p, \theta) \in \mathbf{X}$, polar line

$$L_{p,\theta} = p(\cos(\theta), \sin(\theta)) + (\cos(\theta), \sin(\theta))^{\perp}$$

- Tessellation:
  set of connected components of
  $$\mathbb{R}^d \setminus \bigcup_{(p,\theta)\in\mathbf{X}} L_{p,\theta}$$

*Properties*: invariance under translations and rotations

*References*: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

# Questions of interest

- Asymptotic study of the population of cells (means, extremes): number of vertices, edge length in a window...

- Study of a particular cell

  zero-cell $C_0$ containing the origin

  typical cell $\mathcal{C}$ *chosen uniformly at random*

  Means, moments and distribution of functionals of the cell (area, perimeter...),
  asymptotic sphericality

**J. Møller** (1986), **I. N. Kovalenko** (1998), **D. Hug, M. Reitzner & R. Schneider** (2004)

# Mean number of vertices per cell

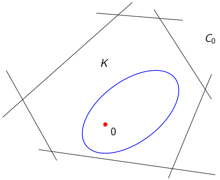
• Each vertex from the tessellation is contained in exactly 4 cells.

• Each vertex is the highest point from a unique cell with probability 1.

• There are as many vertices as there are cells.

*Conclusion.* The mean number of vertices of a typical cell is 4.

# Probability to belong to the zero-cell



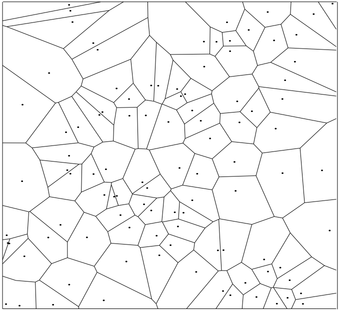

**Consequence of the Cauchy-Crofton formula**:

$K$ convex body containing 0, $C_0$ cell of the tessellation containing 0

$$\begin{aligned} \mathbb{P}(K \subset C_0) &= \exp\left(-\iint \mathbf{1}(L_{p,\theta} \cap K \neq \emptyset)\mathrm{d}p\mathrm{d}\theta\right) \\ &= \exp(-\mathrm{per}(\partial \mathrm{K})) \end{aligned}$$

*Remark.* In higher dimension, the perimeter is replaced by the mean width.

# Poisson-Voronoi tessellation



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^2$ of intensity measure $\mathrm{d}x$
- For every nucleus $x \in \mathbf{X}$, the cell associated is

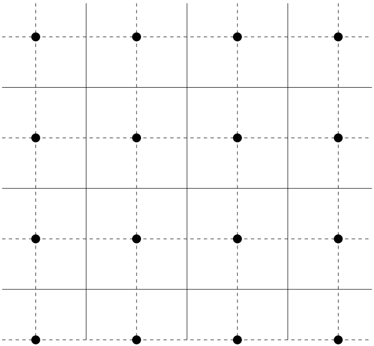
$$C(x|\mathbf{X}) := \{y \in \mathbb{R}^2 : \\ \|y - x\| \leq \|y - x'\| \ \forall x' \in \mathbf{X}\}$$

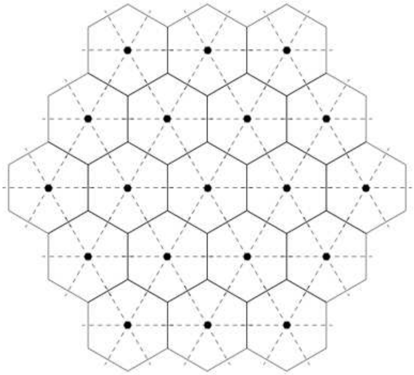
- Tessellation:
  set of cells $C(x|\mathbf{X})$

*Properties*: invariance under translations and rotations

*References*: **Descartes** (1644), **Gilbert** (1961), **Okabe et al.** (1992)

# Deterministic Voronoi grids





# Mean number of vertices per cell

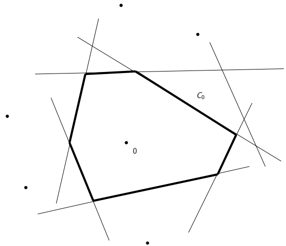
- Each vertex from the tessellation is contained in exactly 3 cells.

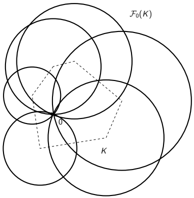
- Each vertex is the highest or lowest point from a unique cell with probability 1.

- There are twice as many vertices as there are cells.

*Conclusion.* The mean number of vertices of a typical cell is 6.

# Probability to belong to the zero-cell




$K$ convex body containing 0, $C_0$ Voronoi cell $C(0|\mathbf{X} \cup \{0\})$

$$\mathbb{P}(K \subset C_0) = \exp(-V_d(\mathcal{F}_0(K)))$$

where $V_d$ is the volume and $\mathcal{F}_0(K) = \cup_{x \in K} B(x, \|x\|)$ flower of $K$

# Plan

From game to theory: Buffon, integral geometry, random tessellations

From game to theory: 150 years of random convex hulls

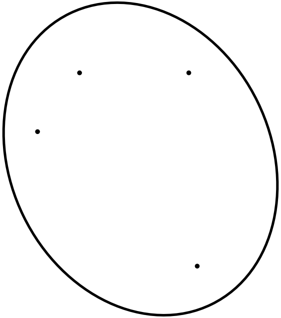
- Sylvester's problem
- Extension of Sylvester's problem
- Uniform model
- Gaussian model
- Asymptotic spherical shape
- Mean and variance estimates

Addendum: some more models

# Sylvester's problem

**J. J. Sylvester**, *The Educational Times*, Problem 1491 (1864)

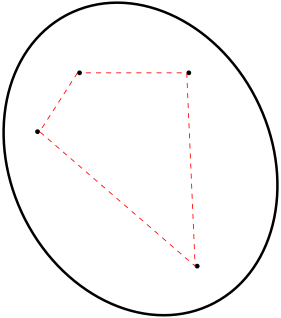
Probability $p(K)$ that 4 independent points uniformly distributed in a convex set $K \subset \mathbb{R}^2$ with finite area are the vertices of a convex quadrilateral?



# Sylvester's problem

**J. J. Sylvester**, *The Educational Times*, Problem 1491 (1864)

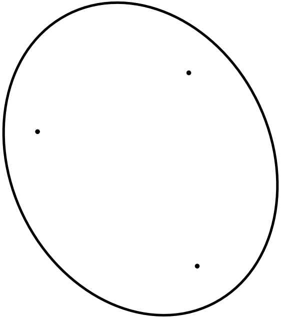
Probability $p(K)$ that 4 independent points uniformly distributed in a convex set $K \subset \mathbb{R}^2$ with finite area are the vertices of a convex quadrilateral?



# Sylvester's problem

**J. J. Sylvester**, *The Educational Times*, Problem 1491 (1864)

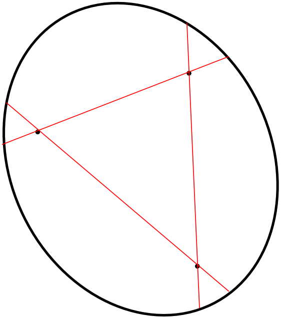
Probability $p(K)$ that 4 independent points uniformly distributed in a convex set $K \subset \mathbb{R}^2$ with finite area are the vertices of a convex quadrilateral?



# Sylvester's problem

**J. J. Sylvester**, *The Educational Times*, Problem 1491 (1864)

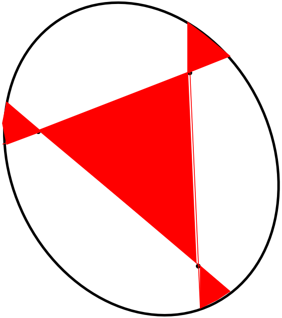
Probability $p(K)$ that 4 independent points uniformly distributed in a convex set $K \subset \mathbb{R}^2$ with finite area are the vertices of a convex quadrilateral?



# Sylvester's problem

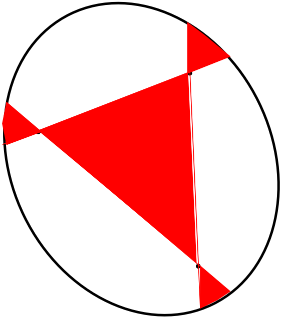
**J. J. Sylvester**, *The Educational Times*, Problem 1491 (1864)

Probability $p(K)$ that 4 independent points uniformly distributed in a convex set $K \subset \mathbb{R}^2$ with finite area are the vertices of a convex quadrilateral?



# Sylvester's problem

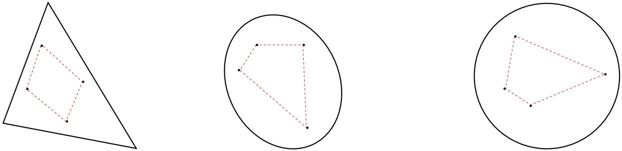
**B. Efron** (1965) : $p(K) = 1 - \frac{4\overline{A}(Triangle)}{A(C)}$



# Sylvester's problem

**W. Blaschke** (1923) :

$$\frac{2}{3} \leq p(K) \leq 1 - \frac{35}{12\pi^2} \approx 0.70448$$

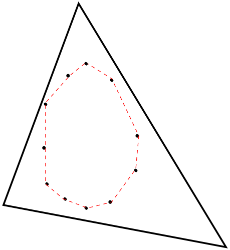


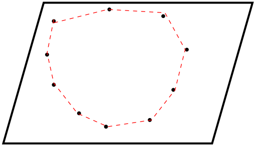
# Extension of Sylvester's problem

Probability that $n$ independent points uniformly distributed in a convex set of $\mathbb{R}^2$ with finite area are the vertices of a convex polygon?

**P. Valtr** (1996) :

$$p_n(\mathcal{T}) = \frac{2^n(3n-3)!}{[(n-1)!]^3(2n)!} \qquad p_n(\mathcal{P}) = \left[\frac{1}{n!}\binom{2n-2}{n-1}\right]^2$$



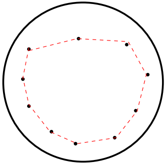


# Extension of Sylvester's problem

**I. Bárány** (1999) :

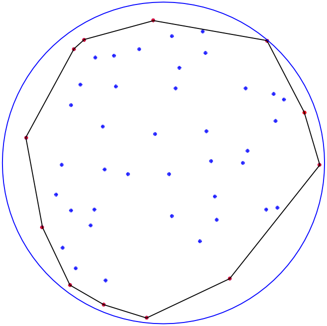
$$\log p_n(K) \underset{n\to\infty}{=} -2n\log n + n\log\left(\frac{1}{4}e^2\frac{PA(K)^3}{A(K)}\right) + o(n)$$

where $PA(K)$ is the *affine perimeter* of $K$



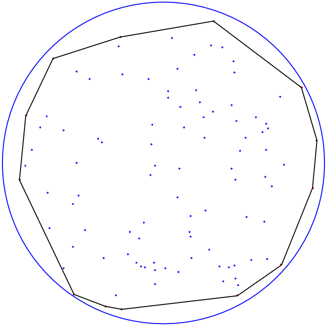
$$\log p_n(D) \underset{n\to\infty}{=} -2n\log n + n\log(2\pi^2 e^2) + o(n)$$

# Random convex hulls



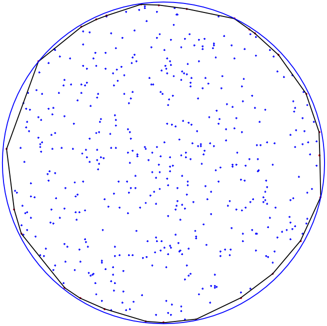
- $K$ convex body of $\mathbb{R}^d$
- $K_n$: convex hull of $n$ independent points, uniformly distributed in $K$

# Random convex hulls



- $K$ convex body of $\mathbb{R}^d$
- $K_n$: convex hull of $n$ independent points, uniformly distributed in $K$

# Random convex hulls



- $K$ convex body of $\mathbb{R}^d$
- $K_n$: convex hull of $n$ independent points, uniformly distributed in $K$

**Considered functionals**

$f_k(\cdot)$: number of $k$-dimensional faces, $0 \leq k \leq d$

$V_d(\cdot)$: volume

# Explicit calculations

**J. G. Wendel** (1962): when $K$ is symmetric,

$$\mathbb{P}\{0 \notin K_n\} = 2^{-(n-1)} \sum_{k=0}^{d-1} \binom{n-1}{k} \quad {\scriptstyle (n \geq d)}$$

**B. Efron** (1965) : $f_0(\cdot)$: # vertices, $V_d(\cdot)$: volume

$$\mathbb{E} f_0(K_n) = n \left( 1 - \frac{\mathbb{E} V_d(K_{n-1})}{V_d(K)} \right)$$

**C. Buchta** (2005) : identities between higher moments

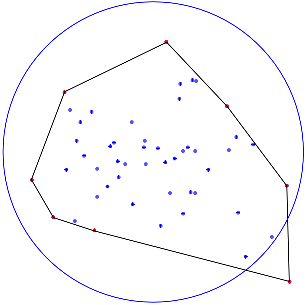
Conclusion: very few non asymptotic calculations are possible!

# Proof of Efron's relation

$X_1, \cdots, X_n$ independent and uniformly distributed in $K$:

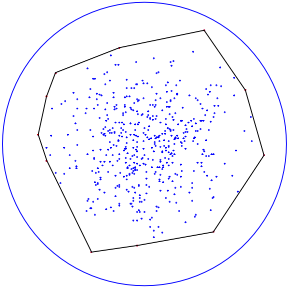
$$
\begin{aligned}
\mathbb{E} f_0(K_n) &= \mathbb{E} \sum_{k=1}^{n} \mathbf{1}_{\{X_k \notin \text{Conv}(X_i, i \neq k)\}} \\
&= n\mathbb{E}[\mathbb{E}[\mathbf{1}_{\{X_n \notin \text{Conv}(X_1, \cdots, X_{n-1})\}} | X_1, \cdots, X_{n-1}]] \\
&= n\mathbb{E}\left[1 - \frac{V_d(\text{Conv}(X_1, \cdots, X_{n-1}))}{V_d(K)}\right] \\
&= n\left(1 - \frac{\mathbb{E} V_d(K_{n-1})}{V_d(K)}\right)
\end{aligned}
$$

# Gaussian model



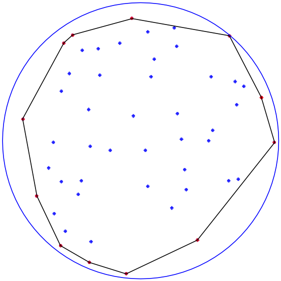
- $\Phi_d(x) := \frac{1}{(2\pi)^{d/2}} e^{-\|x\|^2/2}$, $x \in \mathbb{R}^d$, $d \geq 2$
- $K_n$ : convex hull of $n$ independent points with common density $\Phi_d$

# Gaussian model

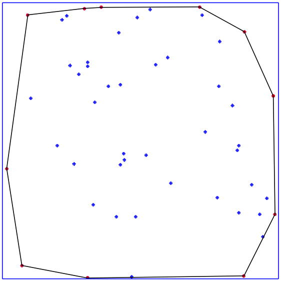


- $\Phi_d(x) := \frac{1}{(2\pi)^{d/2}} e^{-\|x\|^2/2}$, $x \in \mathbb{R}^d$, $d \geq 2$
- $K_n$ : convex hull of $n$ independent points with common density $\Phi_d$

# Simulations of the uniform model

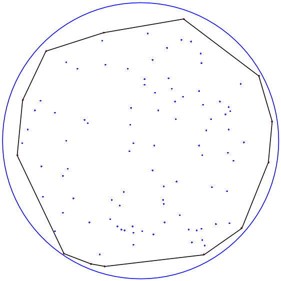


$K_{50}$, $K$ disk

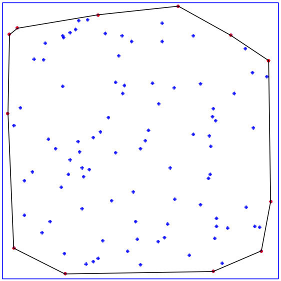


$K_{50}$, $K$ square

# Simulations of the uniform model

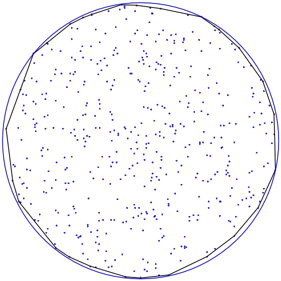


$K_{100}$, $K$ disk

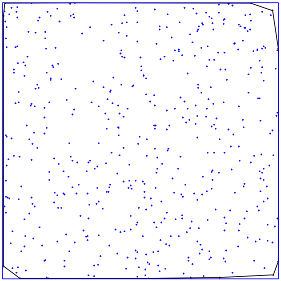


$K_{100}$, $K$ square

# Simulations of the uniform model

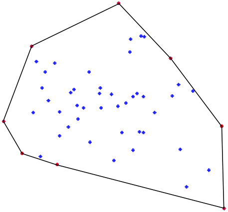


$K_{500}$, $K$ disk

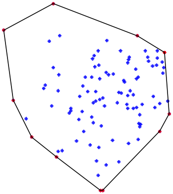


$K_{500}$, $K$ square

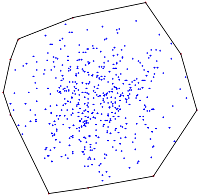
# Simulations of the Gaussian model



$K_{50}$

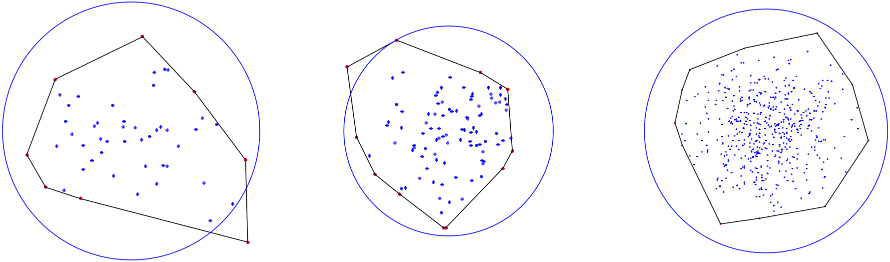


$K_{100}$



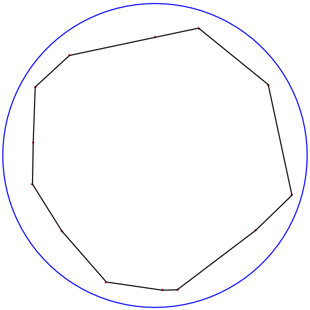
$K_{500}$

# Gaussian polytopes: spherical shape

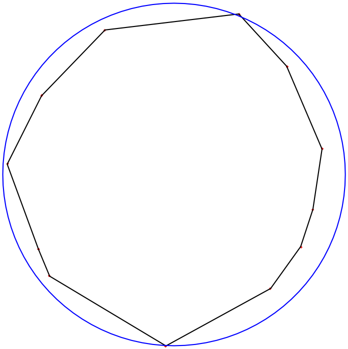


$K_{50}$ $K_{100}$ $K_{500}$

# Gaussian polytopes: spherical shape



$K_{5000}$

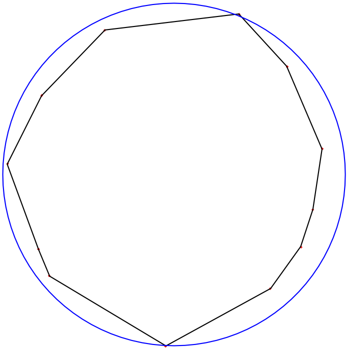


$K_{50000}$

# Asymptotic spherical shape

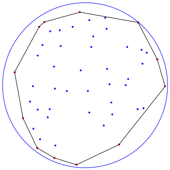
**Geffroy** (1961) :

$$d_H(K_n, B(0, \sqrt{2\log(n)})) \underset{n\to\infty}{\to} 0 \text{ a.s.}$$

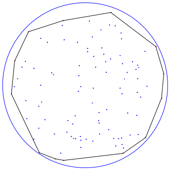


$K_{50000}$

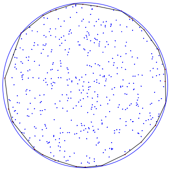
# Comparison between uniform and Gaussian



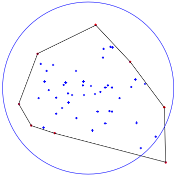
$K_{50}$ uniform/disk



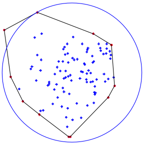
$K_{100}$ uniform/disk



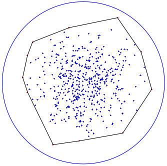
$K_{500}$ uniform/disk



$K_{50}$ Gaussian

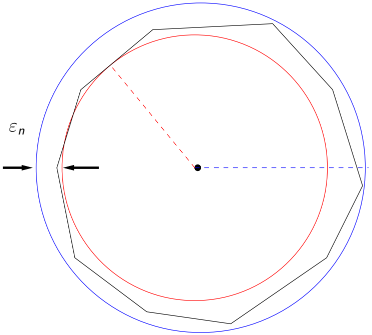


$K_{100}$ Gaussian



$K_{500}$ Gaussian

# Closeness to the spherical shape




*Uniform case in the ball:*

$$\varepsilon_n \underset{n\to\infty}{\approx} c_d \frac{\log(n)}{n^{\frac{2}{d+1}}}$$

*Gaussian case:*

$$\varepsilon_n \underset{n\to\infty}{\approx} c'_d \frac{\log(2\log(n))}{\sqrt{2\log(n)}}$$

# Asymptotic means

**A. Rényi & R. Sulanke** (1963), **H. Raynaud** (1970), **R. Schneider & J. Wieacker** (1978), **I. Bárány & C. Buchta** (1993)

| | $\mathbb{E}[f_k(K_n)]$ | $V_d(K) - \mathbb{E}[V_d(K_n)]$ or $\mathbb{E}[V_d(K_n)]$ |
|---|---|---|
| Uniform, smooth | $\sim c_{d,k}^{(1)}(K)\ n^{\frac{d-1}{d+1}}$ | $\sim c_{d,d}^{(4)}(K)\ n^{-\frac{2}{d+1}}$ |
| Gaussian | $\sim c_{d,k}^{(2)}\ \log^{\frac{d-1}{2}}(n)$ | $\sim c_{d,d}^{(5)}\ \log^{\frac{d}{2}}(n)$ |
| Uniform, polytope | $\sim c_{d,k}^{(3)}(K)\ \log^{d-1}(n)$ | $\sim c_{d,d}^{(6)}(K)\ n^{-1}\log^{d-1}(n)$ |

$c_{d,k}^{(i)}$, $0 \leq k \leq d$, explicit constants depending on $d$, $k$ and $K$

# Variance estimates

**M. Reitzner** (2005), **V. Vu** (2006), **I. Bárány & V. Vu** (2007), **I. Bárány & M. Reitzner** (2009)

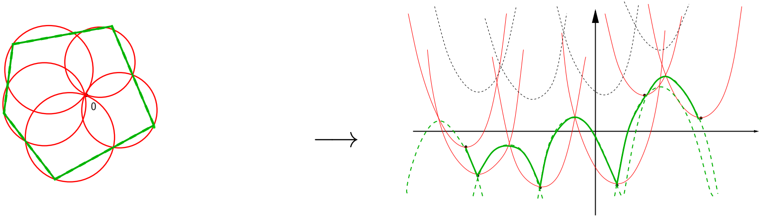
| | $\mathrm{Var}[f_k(K_n)]$ | $\mathrm{Var}[V_d(K_n)]$ |
|---|---|---|
| Uniform, smooth | $\Theta(n^{\frac{d-1}{d+1}})$ | $\Theta(n^{-\frac{d+3}{d+1}})$ |
| Gaussian | $\Theta(\log^{\frac{d-1}{2}}(n))$ | $\Theta(\log^{\frac{d-3}{2}}(n))$ |
| Uniform, polytope | $\Theta(\log^{d-1}(n))$ | $\Theta(n^{-2}\log^{d-1}(n))$ |

# Contributions

- Limiting variances for $f_k(K_\lambda)$ and $V_d(K_\lambda)$: existence and explicit calculation of the constants
- Asymptotic normality of the distributions of $f_k(K_\lambda)$ and $V_d(K_\lambda)$
- Limiting shape of $K_\lambda$ for the uniform model in the ball and the Gaussian model

*Joint works with* **T. Schreiber** (Toruń, Poland) and **J. E. Yukich** (Lehigh, USA)

# Asymptotic shape

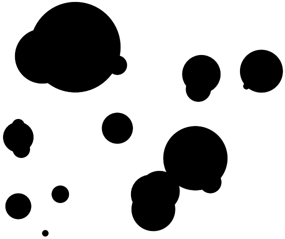



$\Pi^{\uparrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \geq \frac{\|v\|^2}{2}\}, \quad \Pi^{\downarrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \leq -\frac{\|v\|^2}{2}\}$

| | |
|---|---|
| Half-space | translate of $\Pi^{\downarrow}$ |
| Sphere containing $O$ | translate of $\partial\Pi^{\uparrow}$ |
| Convexity | Parabolic convexity |
| Extreme point | $(x + \Pi^{\uparrow})$ not completely covered |
| $k$-face of $K_\lambda$ | Parabolic $k$-face |
| $R_\lambda V_d$ | $V_d$ |

# Some more models

- Random geometric graphs: nearest-neighbor, Delaunay, Gabriel...
- Boolean model



Thank you for your attention!