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A review of the ODE solvers of Axiom, Derive, Macsyma, Maple, Mathematica, MuPAD and Reduce

Frank Postel* Paul Zimmermann†

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Abstract

Using a wide set of more than 50 different kinds of ordinary differential equations and systems, we try to give an idea of the capabilities of Axiom, Derive, Macsyma, Maple, Mathematica, MuPAD and Reduce in solving differential equations.

Today, several computer algebra systems are available, including Axiom, Derive, Macsyma, Maple, Mathematica, MuPAD and Reduce. Frequently asked questions are of the following kind: which system is the best one to compute definite integrals ? to perform linear algebra computations ? To answer such questions, one should know the exact capabilities of each of the existing systems. This is not always possible, for two simple reasons: firstly it would be rather expensive to buy all these systems[1], and secondly it would take a lot of time to learn how to use all of them. For these two reasons, reviews of computer algebra system capabilities, like the one published by Michael Wester [10], are very useful to get a first idea quickly. However, general reviews like that of Michael Wester cannot study extensively a particular class of problems, like polynomial factorization, definite integration or linear algebra. We hope that this review will give to users of computer algebra systems a better idea of what each system can do. Moreover we hope that it will help developers to improve the current computer algebra systems.

About the problems. For this purpose, we decided to extend the work done by Michael Wester in the domain of differential equations (his review contained only 7 equations or systems). We consider here only ordinary differential equations (ODEs), for the simple reason that the current systems cannot deal with partial differential equations, except very simple examples. In addition, we consider only exact solutions, either in closed form, or in implicit form. Like Michael Wester, we have tested all the current general-purpose computer algebra systems. In order to be as fair as possible, we have chosen examples that cover all the range of exact methods known for ODEs. Our main reference about these methods is the excellent book of Daniel Zwillinger [12]. However we did not consider methods using symmetries of ODEs, as well as methods based on Lie groups; these methods are usually not implemented in the main ODE solver, but as separate libraries, for example the

*University of Paderborn, FB 17, D-33095 Paderborn, frankp@uni-paderborn.de

†INRIA Lorraine, BP 101, F-54600 Villers-lès-Nancy Cedex, Paul.Zimmermann@loria.fr

[1] Except for MuPAD which is free for non-commercial institutions.

program CRACK for REDUCE. About REDUCE, we have heard that some methods developed inside the Cathode European project are not yet implemented. We have distinguished three classes of problems: single equations without initial conditions are considered in section 1, among which we distinguish first-order, second-order, higher order and special equations; single equations with initial or boundary conditions are treated in section 2, and systems in section 3. We have solved all problems over the complex plane, i.e. we initialized MACSYMA with `domain: complex`. For all equations we use $y(x)$ as unknown function, i.e. x is the independent variable and y is the dependent variable. For systems we use t as independent variable, and $x, y, z, \ldots$ as dependent variables.

About the results. We used the last released version of each system, namely AXIOM 2.0a (1995), DERIVE version 3.11 (1994), MACSYMA 420 (1996), MAPLE V.4 (1996), MATHEMATICA 2.2.3 (1994), MUPAD 1.2.9 (1996) and REDUCE 3.6 (1995). For each system, we only consider the main function to solve differential equations. If an equation can be solved using Laplace transforms and the solver for algebraic equations, we do not consider this as a valid solution, starting from the point of view that the main ODE solver should try all possible choices.
We distinguish several possible results: The general solution for the equation or system is **complete and correct** (denoted by $\bullet$), is given **implicitly** whereas an explicit solution exists ($\circ$), could **not be found** ( ), could not be found whereby an **error occurs** during the computation (ε), is **wrong** ($\times$), could be much **simpler** ($\circ^1$), is not complete because some **solutions are missing**, or the solutions are only given for special values of the parameters ($\circ^2$), is not complete because **singular solutions are forgotten** ($\circ^3$), contains **integral signs**, i.e. is a Liouvillian solution ($\circ^4$), is given in **parametric form** ($\bullet^5$), **contains** some **wrong** or incorrect solutions ($\circ^6$), or could **not be computed** after one hour cpu time (τ). The last one needs a extra remark: we have found problems where the computations need more than one hour on a SPARCStation 10; we have then interrupted these computations, regarding the comparison to the other systems, whose computations needed often less than one minute for each problem.
It should be noted that the interpretation of AXIOM's results is not trivial: for linear equations, a list with a particular solution and a basis of the solution space of the homogeneous equation is returned; for nonlinear equations an implicit solution is always returned in the form of an expression $F(x, y(x))$ where it is meant that $F(x, y(x)) = C$ where C is a constant. DERIVE always returns an implicit equation, even if it is linear in the unknown function $y(x)$.
For each class of equations, a table summarizes the results using the above signs. In each table, we indicate the kind of equation and, between square brackets, the kind of method that can be used to solve it. We use the following abbreviations for the equation types: "linear polynomial" means a linear equation with polynomial coefficients, "linear constant" means a linear equation with constant coefficients. The complete demos and their output are available by anonymous ftp from `math-ftp.uni-paderborn.de` in `pub/MuPAD/paper/ode-demos`. We have used the following abbreviations for the system names: **Ax** for AXIOM, **De** for DERIVE, **Mc** for MACSYMA, **Mp** for MAPLE, **Mm** for MATHEMATICA, **Mu** for MUPAD and **Re** for REDUCE.
In the columns representing the results of MATHEMATICA we have given the results in parentheses when the built-in solver was not able to find a solution without the extensions included in the package `Calculus'DSolve'`.

1 Single equations without initial conditions

In this section we consider single equations without initial conditions: first order equations, second order equations, higher order equations and special equations.

1.1 Linear equations

Eq.	Kind of equation [method]	Ax	De	Mc	Mp	Mm	Mu	Re
(1)	linear polynomial [integr. factors]	•	∘	•	•	•	•	•
(2)	linear constant [integrating factors]	•	•	•	•	•	•	•
(3)	linear polynomial with parameters	•	•	•	ε	•	•	
(4)	linear polynomial [adjoint eq.]	•	•	•	•		•	
(5)	linear polynomial [polynomial sol.]	•	•	•	•		•	
(6)	dependent variable missing	•	×	•	•	(•)	•	
(7)	linear polynomial [Liouvillian sol.]	$\circ^4$	$\circ^4$	$\circ^4$	$\circ^4$	•	$\circ^4$	
(8)	linear polynomial [red. of order]	•	×	•	•			
(9)	linear [integrating factors]		•	•		(•)	ε	
(10)	radical solution			ε	ε		ε	
(11)	linear polynomial [undet. coeff.]	•	•	•	•		•	•
(12)	linear constant [var. of params]	•	•	•	•	•	$\circ^4$	$\circ^4$
(13)	linear constant, 7th order	•		•	•	•	•	•
(14)	linear polynomial [Euler]	$\circ^1$			•		•	•
(15)	linear polynomial [exact]	×			•		•	

Linear Bernoulli equations. Consider the equation from [12, p. 189]

$$(x^4 - x^3)y' + 2x^4 y = \frac{x^3}{3} + C, \tag{1}$$

which has the solution $[(2x-3)/12 + C/(2x^2) + C_2 e^{-2x}]/(x-1)^2$. It can also be seen as a Bernoulli equation.
The solution of the equation

$$-\frac{1}{2}y' + y = \sin x \tag{2}$$

is $C_1 e^{2x} + 2/5 \cos x + 4/5 \sin x$ [12, p. 195].

Linear change of variables. The solution of the equation

$$y''(ax+b)^2 + 4y'(ax+b)a + 2ya^2 = 0 \tag{3}$$

is $y(x) = \frac{C_1 x + C_2}{(ax+b)^2}$.

Adjoint equation. The equation

$$(x^2 - x)y'' + (2x^2 + 4x - 3)y' + 8xy = 1 \tag{4}$$

from [12, p. 188] can be solved using the adjoint equation method, that reduces it to a first order equation. Its solution is $[(2x-3)/12 + C_1/x^2 + C_2 e^{-2x}]/(x-1)^2$. Maple uses here Kovačić's algorithm.

Polynomial solutions. The solution of

$$(x^2 - x)y'' + (1 - 2x^2)y' + (4x - 2)y = 0 \tag{5}$$

is $y = C_1 e^{2x} + C_2 x^2$ [12, p. 188].

Dependent variable missing equations. The equation

$$y'' + 2xy' = 2x \tag{6}$$

has the solution $y = x + C_1 + C_2 \operatorname{erf}(x)$ [12, p. 193]. MAPLE solves this equation by recognizing a Riccati equation, and then finding polynomial solutions. MACSYMA recognizes that the dependent variable does not appear explicitly in the equation and uses a basic method to solve the equation. MUPAD outputs a constant solution, which is obviously wrong. DERIVE returns an incorrect solution, due to a wrong wronskian for confluent hypergeometric functions (this was noticed and fixed after the first version of our review).

Liouvillian solutions. Sometimes the solution of a differential equation can be expressed in terms of usual functions with some integral signs. Such a solution is called *Liouvillian*. The following equation

$$(x^3/2 - x^2)y'' + (2x^2 - 3x + 1)y' + (x - 1)y = 0, \tag{7}$$

taken from [3, p. 259], admits such a solution, that AXIOM, MAPLE, MACSYMA and MUPAD find, namely $e^{-1/x}/\sqrt{x^2 - 2x}(C_1 + C_2 \int e^{1/x} x^{-3/2}/\sqrt{x-2}\, dx)$. MATHEMATICA even returns an explicit solution using the imaginary error function **`Erfi`**.

Reduction of order. The equation

$$y'' - 2xy' + 2y = 3, \tag{8}$$

taken from [12, p. 331] has the solution $y(x) = \frac{3}{2} + Ax \int^x \frac{e^{t^2}}{t^2} dt + Bx$ (there is a typo in the solution given by [12], namely the term $A \int^x \frac{e^{t^2}}{t^2} dt$ must be $Ax \int^x \frac{e^{t^2}}{t^2} dt$). MUPAD does not take into account the inhomogeneous term. As for equation (6), DERIVE returns an incorrect solution due to a wrong internal formula.

Integrating factors. The following equation admits the integrating factor $\sqrt{x}$:

$$\sqrt{x}y'' + 2xy' + 3y = 0. \tag{9}$$

The solution is $C_1 x e^{-4x^{3/2}/3} + C_2 x e^{-4x^{3/2}/3} \int e^{4x^{3/2}/3}/x^2 dx$ [12, p. 307]. MAPLE cannot solve this equation, but returns a `DESol` object instead of `NULL`. MATHEMATICA uses generalized Laguerre polynomials to express the solution, whereas DERIVE uses the elliptic integral $E_n(z)$. AXIOM finds the particular solution $y = 0$ but returns an empty basis for the associated homogeneous equation !

Radical solution. The linear equation

$$a_2 y'' - a_0 y = 0 \tag{10}$$

with the polynomial coefficients [1]

$$\begin{aligned}
a_0 &= \frac{104}{25}x^{10} + \left(\frac{274}{25} - \frac{22}{15}\sqrt{-222}\right)x^8 + \left(\frac{7754}{75} - \frac{68}{15}\sqrt{-222}\right)x^6 \\
&\quad + \left(\frac{11248}{75} - \frac{194}{15}\sqrt{-222}\right)x^4 + \left(\frac{29452}{75} - \frac{296}{5}\sqrt{-222}\right)x^2 \\
&\quad - \frac{10952}{5} - \frac{148}{3}\sqrt{-222} \\
a_2 &= x^{12} + 2x^{10} + \frac{151}{3}x^8 + \frac{296}{3}x^6 + \frac{5920}{9}x^4 + \frac{10952}{9}x^2 + \frac{5476}{9},
\end{aligned}$$

admits $y = x(3x^2 + \sqrt{-222})(x^2+1)^{-\frac{3}{5}}\left(x^4 + \frac{74}{3}\right)^{\frac{1}{5}}$ as solution, the other one can be found by reduction of order. MAPLE fails with `Error, object too large` after having used several hours of computing time and more than 35MB of memory ! MACSYMA fails with `Division by 0`, then prints `Manyterm relation` and a series solution (which was not asked) after several hours too and 35MB too. MuPAD gives up with the error message `Error: illegal coefficients [gcd]`.

Undetermined coefficients. The solution of

$$y'' - \frac{2}{x^2}y = 7x^4 + 3x^3 \tag{11}$$

is $y = \frac{1}{4}x^6 + \frac{1}{6}x^5 + C_1x^2 + \frac{C_2}{x}$ [12, p. 354].

Variation of parameters. The equation

$$y'' + y = \csc x \tag{12}$$

has the solution $y = C_1 \sin x + C_2 \cos x + \sin x \log(\sin x) - x \cos x$ [12, p. 358]. REDUCE does not find a closed form for the term $\sin x \log(\sin x) - x \cos x$, which is expressed using two nested integrals. MuPAD cannot compute the integral $\displaystyle\int \left(\frac{x}{\sin(x)\sqrt{-2\cos(2x)+2}} - \frac{C}{\sin(x)\sqrt{-2\cos(2x)+2}} \right) dx$, but the given solution is correct.

Linear with constant coefficients. The solution of

$$y^{(7)} - 14y^{(6)} + 80y^{(5)} - 242y^{(4)} + 419y^{(3)} - 416y'' + 220y' - 48y = 0 \tag{13}$$

is $(C_0 + C_1x + C_2x^2)e^x + (C_3 + C_4x)e^{2x} + C_5e^{3x} + C_6e^{4x}$ [12, p. 205]. MATHEMATICA returns the correct solution after giving a warning that the solution probably could not be found.

Euler equations The following example, whose solution is $C_1x + C_2/x + C_3x^2 + C_4x^4$, was posted in `sci.math.symbolic` by Ming Liu in March 1995:

$$y'''' - \frac{4}{x^2}y'' + \frac{8}{x^3}y' - \frac{8}{x^4}y = 0 \tag{14}$$

The basis elements returned by AXIOM are rather complex, for example $(x^5 - 5x^3 + 5x^2 - 1)/x$.

Exact n-th order. The equation

$$(1 + x + x^2)y''' + (3 + 6x)y'' + 6y' = 6x \tag{15}$$

has the solution $(x^4/4 + C_1x^2 + C_2x + C_3)/(1 + x + x^2)$ [12, p. 244]. MuPAD returns a constant solution, which is obviously wrong. The particular solution returned by AXIOM is wrong; it does not take into account the inhomogeneous term $6x$.

1.2 Nonlinear equations

Eq.	Kind of equation [method]	Ax	De	Mc	Mp	Mm	Mu	Re
(16)	nonlinear [integrating factors]	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$(\bullet)$	$\bullet$	
(17)	$2y(y')^2 - 2xy' - y = 0$	ε	$\bullet^5$		$\circ^3$		$\circ^6$	
(18)	Bernoulli equation	$\circ$	$\times$	$\circ^2$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
(19)	general Bernoulli equation	$\times$	$\times$	$\bullet$	$\circ^2$	$\bullet$	$\bullet$	$\circ^2$
(20)	Clairaut's equation	ε	$\bullet$		$\bullet$	(τ)	$\circ$	
(21)	general Clairaut's equation	ε	$\bullet$		$\circ^1$		$\bullet$	
(22)	$y' = (3x^2 - y^2 - 7)/(e^y + 2xy + 1)$	$\bullet$	$\bullet$	$\bullet$	$\bullet$		$\bullet$	$\circ$
(23)	homogeneneous equation (deg. 4)		$\bullet$	$\bullet$	$\bullet$	$(\bullet)$	$\bullet$	$\varepsilon,\bullet$
(24)	nonlinear [factor]	ε	$\circ^2$	$\bullet$	$\bullet$	$(\bullet)$	$\bullet$	
(25)	$y' = x/(x^2y^2 + y^5)$	$\bullet$	$\bullet$	$\bullet$	$\bullet$		$\bullet$	
(26)	Lagrange equation	ε	$\bullet^5$		$\circ$	(τ)	$\circ$	
(27)	Lagrange equation	ε	$\bullet^5$		$\circ$		$\circ$	
(28)	Riccati equation		$\bullet$	$\circ^1$	$\bullet$	$(\bullet)$	$\bullet$	
(29)	Riccati equation	$\circ^3$	$\bullet$	$\circ^3$	$\circ^3$	$(\circ^3)$	$\bullet$	
(30)	separable equation	$\circ$	$\circ$	$\circ$	$\circ$	$\bullet$	$\circ$	$\circ$
(31)	equation of the form $x = f(y, y')$	ε	$\bullet^5$		$\circ$	$(\bullet)$	$\bullet$	
(32)	equation of the form $y = f(x, y')$	ε	$\bullet^5$		$\circ^1$		$\times$	
(33)	autonomous equation	ε	$\bullet$	$\circ$	$\circ$	$(\bullet)$	$\circ$	
(34)	autonomous equation	ε	$\bullet$	$\bullet$	$\bullet$	$(\bullet)$	$\bullet$	
(35)	nonlinear [differentiation]	ε	$\bullet$			(ε)	$\bullet$	
(36)	equidimensional-in-x equation	ε	$\circ$			$(\bullet)$	$\circ^1$	
(37)	equidimensional-in-y equation	ε	$\circ$			$(\bullet)$	$\times$	
(38)	exact	ε	$\circ$			$(\bullet)$	$\circ$	
(39)	nonlinear [factoring operator]	ε	$\circ$		$\bullet$	$(\bullet)$	$\bullet$	
(40)	scale invariant equation	ε	$\circ^1$			(ε)	$\times$	
(41)	autonomous, 3rd order	ε			$\circ^1$		$\circ^6$	
(42)	autonomous, 4th order	ε			$\bullet$		$\bullet$	

Integrating factors. The implicit solution of

$$y' = \frac{y}{y \log y + x} \tag{16}$$

is $2x/y - \log^2 y = C$ [12, p. 198].
The equation

$$2yy'^2 - 2xy' - y = 0 \tag{17}$$

has the singular solutions $y = 0$ and $y = \pm\frac{\sqrt{6}}{2}x$, and the general solution is given implicitly by the algebraic equation $4y^6 - 12x^2y^4 + (9x^4 - 24Cx)y^2 + (4Cx^3 - 8C^2) = 0$ [12, p. 207]. Maple gives two implicit solutions in terms of exp and arctanh, which correspond to the six implicit solutions, using the "general homogeneous" method. Axiom claims that this is not an appropriate ordinary differential equation. Derive gives the singular solutions in implicit form, and the general solution in parametric form. MuPAD returns the singular solutions but gives a wrong answer for the general solution.

Bernoulli equations. The solution of

$$y' + y = y^3 \sin x \tag{18}$$

is $y = \pm(C_1 e^{2x} + 2/5 \cos x + 4/5 \sin x)^{-\frac{1}{2}}$ [12, p. 195]. Maple returns the solution as an equation of degree 2 in $y(x)$. Macsyma returns an explicit solution which is one of the two branches. Mathematica gives the three solutions (containing the

trivial solution) in explicit form. DERIVE returns a wrong implicit solution, namely $1/y^2 = C_1 e^{-2x} + 2/5 \cos x - 4/5 \sin x$.
The general form of the Bernoulli equation

$$y' + P(x)y = Q(x)y^n \tag{19}$$

with $n \neq 1$ reduces to $u'/(1-n) + P(x)u = Q(x)$ with the change of function $u = y^{1-n}$. If $R(x) = (n-1)\int P(x)dx$, the solution is $u(x)^{1/(1-n)}$ where $u(x) = (1-n)e^{R(x)}\left[\int e^{-R(x)}Q(x)dx + C_1\right]$. MAPLE can solve the general form of the Bernoulli equation only for an integer value of n, REDUCE only for an integer value. MATHEMATICA, MACSYMA and MUPAD solve this equation for arbitrary n giving the result as unevaluated integrals to the power $1/(1-n)$, including the special cases $n = 0$ and $n = 1$, where the equation is respectively an inhomogeneous and a homogeneous linear equation; n can be any symbolic value, and also a complex value whereby then MATHEMATICA gives a warning during its calculations. AXIOM can only process the case where n is an integer, and returns a wrong result in this case (does not contain Q). As in the previous example, the DERIVE answer is wrong (it has the wrong sign for $R(x)$).

Clairaut's equations. The following equation taken from [12, p. 196] is of Clairaut type, i.e. can be written as $f(xy'-y) = g(y')$, with here $f(x) = x^2$ and $g(x) = x^2-1$:

$$(x^2-1){y'}^2 - 2xyy' + y^2 - 1 = 0. \tag{20}$$

There are two general solutions $y = Cx \pm \sqrt{C^2+1}$ and two singular solutions $y = \pm\sqrt{1-x^2}$. AXIOM claims that this is not an appropriate ordinary differential equation.
The symbolic Clairaut's equation

$$f(xy'-y) = g(y') \tag{21}$$

has the general implicit solution $f(xC - y) = g(C)$, where C is a constant. MAPLE returns two solutions, the above implicit one and a complex parametric solution which does not seem to be usable. MACSYMA ends up with `Division by 0` and returns `false`. AXIOM claims that the equation has order 0.

Equation of the form $y' = f(x,y)$. The equation

$$y' = \frac{3x^2 - y^2 - 7}{e^y + 2xy + 1} \tag{22}$$

from [12, p. 239] has an implicit solution, namely $xy^2 + y + e^y + 7x - x^3 = C_1$. REDUCE solves this equation if the denominator is put in front of y', and if it is in operator form.

Homogeneous equations. The first-order homogeneous equation

$$y' = \frac{2x^3y - y^4}{x^4 - 2xy^3} \tag{23}$$

admits an implicit solution too, given by $x^3 + y^3 = Cxy$ [12, p. 277]. REDUCE returns the error `y x invalid as integration variable` when using the functional notation `y(x)`, but returns the correct solution with the operator notation `y`. MUPAD uses the `RootOf` notation to describe the implicit solution.

Factoring the equation. The following equation can be factored into $(y' + y + x)(y' - x) = 0$:

$$y'(y' + y) = x(x + y) \tag{24}$$

The solutions are $x^2/2 + C_1$ and $1 - x + C_1 e^{-x}$ [12, p. 245]. AXIOM claims that this is not an appropriate ordinary differential equation. DERIVE returns one parametric solution equivalent to $1 - x + C_1 e^{-x}$ and the particular solution $1 - x$, but not $x^2/2 + C_1$.

Interchange variables. The solutions of

$$y' = \frac{x}{x^2 y^2 + y^5} \tag{25}$$

are given implicitly by $x^2 = C_1 e^{2y^3/3} - y^3 - 3/2$ [12, p. 309].

Lagrange equations. The following is a Lagrange equation, i.e. an equation of the form $y = xF(y') + G(y')$,

$$y = 2xy' - a{y'}^3, \tag{26}$$

whose implicit solution is $27ay^4 - (16x^3 + 144aC_1 x)y^2 + 64C_1 x^4 + 64a^2 C_1^3 + 128aC_1^2 x^2 = 0$ [12, p. 312]. AXIOM claims that this is not an appropriate ordinary differential equation. DERIVE returns a parametric solution and the particular solution $y = 0$. MUPAD uses nested `RootOf` notations to describe the implicit solution.
The following is another Lagrange equation,

$$y = 2xy' - {y'}^2, \tag{27}$$

whose implicit solution is $4y^3 - 3x^2y^2 + 12C_1 x^3 - 18C_1 xy + 9C_1^2 = 0$ [12, p. 313]. DERIVE returns a parametric solution and the particular solution $y = 0$.

Riccati equations. The equation

$$y' = e^x y^2 - y + e^{-x} \tag{28}$$

can be solved by the Riccati transformation [12, p. 334]. The solution is $-e^{-x}(\cos x - C \sin x)/(\sin x + C \cos x)$.
Another Riccati equation is [12, p. 334]:

$$y' = y^2 - xy + 1, \tag{29}$$

which has the solutions x and $x + \frac{e^{x^2/2}}{C - \int_0^x e^{t^2/2}\,dt}$. AXIOM, MAPLE, MATHEMATICA and MACSYMA return a solution that is equivalent to the second one, whereby MATHEMATICA uses for its representation the imaginary error function `Erfi`.

Separable equations. The equation

$$y' = \frac{9x^8 + 1}{y^2 + 1}. \tag{30}$$

taken from [12, p. 342] has the implicit solution $\frac{y^3}{3} + y = x^9 + x + C$. MATHEMATICA returns the three solutions in explicit form. REDUCE can solve the equation only in operator form.

Equations solvable for x or y. The equation

$$y = 2xy' + yy'^2. \tag{31}$$

can be rewritten to $x = -\frac{py}{2} + \frac{y}{2p}$, whereby $p = y'$. Its derivative in y yields an equation in p, whose solution can be resubstituted to a solution of (31), which is $2xC - y^2 + C^2 = 0$ [12, p. 350]. Axiom claims that this is not an appropriate ordinary differential equation. Derive returns the above solution in parametric form and the implicit singular solution $x^2y + y^3 = 0$.
The equation

$$x = yy' - xy'^2 \tag{32}$$

can be rewritten to $y = \frac{x+x(y')^2}{y'}$ [12, p. 351]. A parametric representation of its solution is $x = C\alpha e^{-\alpha^2/2}, y = C(\alpha^2+1)e^{-\alpha^2/2}$. Axiom claims that this is not an appropriate ordinary differential equation. MuPAD returns equations containing integral signs but it seems that they do not have a solution.

Autonomous equations. The solution of

$$y'' - y' = 2yy' \tag{33}$$

is $y = C_1 \tan(C_1x + C_2) - 1/2$ [12, p. 191]. Maple returns only an implicit solution of the form $x = 2\arctan((1 + 2y(x))/C_1)/C_1 - C_2$. Macsyma yields two implicit solutions. The first one is similar to the solution of Maple, where $C_1 = \sqrt{4k-1}$ and $4k-1 > 0$, which was asked to the user. If the user decides $4k-1 < 0$, Macsyma returns $x = \log\left(\frac{2y-C_1+1}{2y+C_1+1}\right)/C_1 - C_2$ with $C_1 = \sqrt{1-4k}$. Axiom produces an error, saying that this is not a linear ordinary differential equation ! Derive returns an implicit solution similar to the one returned by Macsyma for $4k - 1 < 0$, and an explicit one which is $y = (C_1(C_2+1)e^{C_2x} + C_2 - 1)/(2 - 2C_1e^{C_2x})$.
For the equation

$$\frac{y''}{y} - \frac{y'^2}{y^2} - 1 + \frac{1}{y^3} = 0, \tag{34}$$

taken from [12, p. 193], Maple and Macsyma give two implicit solutions $x = \int f(y)dy + C_2$ and $x = -\int f(y)dy + C_2$ where $f(y) = \sqrt{6}/2(3y^2 \log y + 1/y + 3C_1y^2)^{-1/2}$, and Derive returns only the first one. MuPAD gives only the three constant solutions. Axiom produces the error "not a linear ordinary differential equation".

Differentiation method. The following equation is taken from [12, p. 218]:

$$2yy'' - y'^2 = \frac{1}{3}(y' - xy'')^2 \tag{35}$$

The solutions are C_1/x, C_1x^3 and $ax^2 + bx + \frac{b^2}{3a}$. Axiom claims that this is not an appropriate ordinary differential equation.

Equidimensional in x equations. The solution of the equation

$$xy'' = 2yy' \tag{36}$$

is $y = C_1\tan(C_2 + C_1\log(x)) - \frac{1}{2}$ [12, p. 231]. Axiom produces the error "not a linear ordinary differential equation". MuPAD simplifies the problem to an equation (not a differential equation) which can not be solved by the system.

Equidimensional in y equations. The equation

$$(1-x)(yy''-{y'}^2)+x^2y^2=0 \tag{37}$$

has the solution $y=(x-1)^{(x-1)}e^{\frac{x^3}{6}+\frac{x^2}{2}+C_1x+C_2}$ [12, p. 233]. AXIOM produces the error "not a linear ordinary differential equation". MUPAD returns a constant as the general solution which is obviously wrong.

Exact second order. The solution of

$$xyy''+x{y'}^2+yy'=0 \tag{38}$$

is given by $y=\pm\sqrt{C_1\log x+C_2}$ [12, p. 241]. AXIOM produces the error "not a linear ordinary differential equation".

Factoring differential operators. The equation

$$(y'')^2-2y'y''+2yy'-y^2=0. \tag{39}$$

factors into $L(H(y))=0$ where $L(z)=(z')^2-z^2$ and $H(y)=y'-y$ and has the solutions $y_1=(C_1+C_2x)e^x$ and $y_2=C_1e^{-x}+C_2e^x$ [12, p. 247]. AXIOM claims that this is not an appropriate ordinary differential equation.

Scale invariant equations. The equation

$$x^2y''+3xy'+2y=\frac{1}{y^3x^4} \tag{40}$$

admits $y=\pm\frac{1}{x}\sqrt{\cosh(C_1)+\sinh(C_1)\sin(2\log(x)+C_2)}$ as solution [12, p. 339]. AXIOM claims that this is not an appropriate ordinary differential equation.

Autonomous, 3rd order. The solution of the equation

$$((y')^2+1)y'''-3y'(y'')^2=0 \tag{41}$$

in implicit form is $(y-C_1)^2-(x-C_2)^2-C_3^2$, the general parametric form of a circle [5, p. 602]. MUPAD gives up with the error `Recursive definition`. AXIOM produces the error "not a linear ordinary differential equation". MUPAD returns three solutions whereby the third one which is given implicitely is not correct.

Autonomous, 4th order. The equation

$$3y''y^{(4)}-5(y''')^2=0 \tag{42}$$

can be transformed to a Bernoulli equation [5, p. 604]. The solution is given implicitly by $(y+C_1x+C_2)^2=C_3x+C_4$. MUPAD gives up with the error `Recursive definition`. AXIOM produces the error "not a linear ordinary differential equation".

1.3 Special equations

Eq.	Kind of equation	Ax	De	Mc	Mp	Mm	Mu	Re
(43)	Delay equation	$\circ^4$		ε	$\times$	$\circ^4$	ε	$\circ^4$
(44)	with parameters	ε	$\bullet$	$\bullet$	$\bullet$		$\circ$	$\bullet$

Delay equations. The following is a delay equation (or mixed differential-difference equation) taken from [12, p. 210]:

$$y'(x) + ay(x-1) = 0 \tag{43}$$

Its solution can be expressed by $y(x) = \sum_{n=0}^{[x]+1} (-a)^n \frac{(x-n+1)^n}{n!}$. In DERIVE, there is no way to write such equations as $y(x)$ must be written `y`, $y'(x)$ must be written `y1`, ... MAPLE fails to recognize the delay term $y(x-1)$ and returns an invalid answer containing $(y(x))(x-1)$. AXIOM, MATHEMATICA and REDUCE give the solution $y(x) = C - a\int_0^x y(s-1)ds$. MACSYMA terminates with an error message:

```
(c3) ode( diff(y(x),x) + a*y(x-1) = 0, y(x), x);
Attempt to substitute y(x) for y in y(x - 1)
Illegal substitution for operator of expression
Returned to Macsyma Toplevel.
```

Functions with several parameters. Sometimes one may want to solve an ordinary differential equation for a function with several parameters. In the following equation from Stanly Steinberg [10],

$$\frac{d}{dx}y(x,a) = ay(x,a) \tag{44}$$

whose solution is $C_1 e^{ax}$, the variable a is simply a parameter, however the user may want to solve the above equation as it is, without having to rewrite it $\frac{d}{dx}y_a(x) = ay_a(x)$. AXIOM produces the error "equation has order 0". This equation makes no problem to DERIVE because of the particular syntax of the solver: `ODE(y1=a*y)`. MAPLE produces an error `Error, (in dsolve) Please try pdesolve`, but the function `pdesolve` correctly solves the equation.

2 Single equations with initial conditions

The `odesolve` package distributed with REDUCE 3.6 can not solve equations with boundary or initial conditions.

Eq.	Kind of equation [method]	Ax	De	Mc	Mp	Mm	Mu	Re
(45)	exact 4th order	•		•	•	•	•	
(46)	linear, polynomial coefficients		∘	•	•			
(47)	$x(y')^2 - y^2 + 1 = 0, y(0) = 1$	ε	∘	ε		(•)		
(48)	$y'' + y(y')^3 = 0, y(0) = 0, y'(0) = 2$	ε	$\circ^6$	×	•	$(\circ^6)$		

The solution of

$$y^{(4)} = \sin x, y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 0 \tag{45}$$

is $y(x) = \sin x - x + x^3/6$ [12, p. 186].
The problem

$$xy'' + y' + 2xy = 0,\ y(0) = 1, y'(0) = 0 \tag{46}$$

can be solved via Laplace transformation [2, p. 354]. The solution is $y(x) = J_0(\sqrt{2}x)$, where J_0 is the Bessel function of order 0. MATHEMATICA fails to solve the initial condition. In DERIVE, one must first solve the equation with `ODE`, then solve for the initial conditions using `APPLY_IC`.
The following first-order equation, taken from [3, p. 256], is rather easy to solve without any initial condition (MAPLE can do it), but it is much more difficult to find the value of the integration constant that satisfies the initial condition $y(0) = 1$:

$$x(y')^2 - y^2 + 1 = 0, \quad y(0) = 1. \tag{47}$$

The solutions are $y = \cosh(2\sqrt{x})$ and $y = 1$. MACSYMA fails with `Division by 0` to solve the initial condition. AXIOM claims that this is not an appropriate ordinary differential equation; if one separates the term y' by giving as equation $y' = \sqrt{\frac{y^2-1}{x}}$ instead, then it fails with `division by zero`. Here again, in DERIVE one must call explicitly `APPLY_IC`, which does not distribute over multiple solutions, but DERIVE finds the singular solution $y = 1$, which also satisfies the initial condition.
The following autonomous equation is taken from the on-line help of MACSYMA:

$$y'' + y(y')^3 = 0, \quad y(0) = 0, y'(0) = 2. \tag{48}$$

DERIVE returns an implicit solution given by the equation $y^3 + 3y = 6x$, but only one of its three solutions really satisfies the initial conditions. Same problem with MATHEMATICA: it returns the three explicit solutions corresponding to the roots of the above equation. MACSYMA returns an incorrect implicit solution, namely $2y^3 - 3y + 6x = 0$, which would give $y'' + y(y')^3 = -3(36x^2y + 18xy^2 + y - 18x)/(1 - 36x^2 + 324x^4)$. MUPAD fails to differentiate the general solution. AXIOM claims that this is not an appropriate ordinary differential equation. Only MAPLE returns the correct solution, i.e. $y = f - 1/f$ where $f = (3x + \sqrt{1 + 9x^2})^{1/3}$.

3 Systems of equations

The `odesolve` package distributed with REDUCE 3.6 can not solve systems of equations.

Eq.	Kind of system [method]	Ax	De	Mc	Mp	Mm	Mu	Re
(49)	nonlin., 1st order [int. comb.]	ε				$(\times)$	$\bullet$	
(50)	matrix Riccati system	ε		$\circ$			$\bullet$	
(51)	triangular			$\bullet$		$(\bullet)$	$\bullet$	
(52)	linear, constant coefficients	$\bullet$		$\bullet$	$\bullet$	$\bullet$	$\bullet$	
(53)	linear constant, 2nd order	ε		$\bullet$	ε	ε		
(54)	linear polynomial, 1st order	$\bullet$			$\bullet$			

Integrable combinations. Solutions for $x(t), y(t), z(t)$ of

$$x' = -3yz, y' = 3xz, z' = -xy \tag{49}$$

may be obtained by combining the relations $x^2 + 2y^2 + 3z^2 = C$ and $x^2 + y^2 = D$ [12, p. 284]. AXIOM fails with `not a linear ordinary differential equation`.

Matrix Riccati equations. The solution of

$$x' = a(t)(y^2 - x^2) + 2b(t)xy + 2cx, y' = b(t)(y^2 - x^2) - 2a(t)xy + 2cy \tag{50}$$

is $x(t) = e^{2ct}\left(\alpha(t)(x_0^2 + y_0^2) + x_0\right)/\Delta$ and $y(t) = e^{2ct}\left(\beta(t)(x_0^2 + y_0^2) + x_0\right)/\Delta$, where $x_0 = x(0), y_0 = y(0)$, $\Delta = (\beta^2(t) + \alpha^2(t))(x_0^2 + y_0^2) - 2\beta(t)y_0 + 2\alpha(t)x_0 + 1$ and $\alpha(t) = \int_0^t e^{2cs}a(s)ds, \beta(t) = \int_0^t e^{2cs}b(s)ds$ [12, p. 336]. AXIOM produces the error message `not an appropriate ordinary differential equation`.

Triangular system. The following system taken from [9] is triangular: the first equation can be solved for $x(t)$, then the corresponding solution can be substituted in the second equation, which can then be solved completely. Of course, as always, the intention is that the CAS solves the system automatically, e.g. recognizes that the system is triangular, and carries out all necessary steps itself without user intervention.

$$x' = x(1 + \frac{\cos t}{2 + \sin t}), \quad y' = x - y. \tag{51}$$

The solution is $x = C_1 e^t(2+\sin t)$, $y = C_1 e^t(1-\cos t/5+2\sin t/5)+C_2 e^{-t}$. MAPLE returns an answer in terms of `DESol`, but the order of the system is not even reduced.

Vector ordinary. The system

$$x' = 9x + 2y, y' = x + 8y \tag{52}$$

has the solution $x = C_1 e^{7t} + 2C_2 e^{10t}$, $y = -C_1 e^{7t} + C_2 e^{10t}$ [12, p. 361].

Higher order. For the system

$$x' - x + 2y = 0, x'' - 2y' = 2t - \cos(2t) \tag{53}$$

the variable y can be eliminated [5, p. 613]. The solution is $x = 2C_1 + 4C_2 e^{\frac{t}{2}} - t^2 - 4t + \frac{\sin(2t)+4\cos(2t)}{34}$, $y = C_1 + C_2 e^{\frac{t}{2}} - \frac{t^2}{2} - t + 2 + \frac{9\sin(2t)+2\cos(2t)}{68}$.
AXIOM fails with `not a first order linear system`. MATHEMATICA crashed by a segmentation fault. MAPLE fails with the error message: `unable to convert to an explicit first-order system`.

Inhomogeneous system. The solution of the system

$$\begin{aligned} x' &= -\frac{1}{t(t^2+1)}x + \frac{1}{t^2(t^2+1)}y + \frac{1}{t}, \\ y' &= -\frac{t^2}{t^2+1}x + \frac{2t^2+1}{t(t^2+1)}y + 1, \end{aligned} \tag{54}$$

taken from [4, p. 144], is given by $x = \log t + C_1 - \frac{C_2}{t}$, $y = t\log t + C_1 t + C_2 t^2$.

4 Concluding remarks

As explained in the introduction, we had two aims in writing such a review: firstly to help users to get an idea of the capabilities of each system in solving ODEs, secondly to point out to developers of computer algebra systems the advantages and/or drawbacks of their ODE solver, to help them to improve it. This second goal has been already fulfilled, if we consider the numerous exchanges that we had with the developers of all systems. Some problems or bugs that we mentioned in this review will be fixed in future versions or are already fixed. Some problems that were not solved are integrated in newest versions (for example, UNIX Macsyma 420 now solves the triangular system).

What system should I use? The best answer to this question is surely "try all of them" if one has all systems available, as it appears that all but one or two equations can be solved by at least one system. However several peculiarities should be observed: MAPLE is rather powerful for linear equations and general methods, but some kinds of ODEs like equidimensional equations are not recognized; MMA can solve only few equations without the `Calculus‘DSolve‘` package, this package is rather weak for linear equations with polynomial coefficients, but does recognize special kinds of equations that MAPLE does not; MACSYMA works well for linear equations but is weak like MAPLE for special kinds of equations; MUPAD is rather weak for nonlinear equations; AXIOM works very well for linear equations and equations of the form $y' = f(x, y)$, but cannot solve other equations; DERIVE can recognize and solve a very large class of equations but makes some stupid mistakes; finally REDUCE is rather weak for all kinds of equations (we were told this will be improved in a future version).

Possible improvements. We have noticed during our review several improvements that could be done in all systems. Most of these improvements are related to the fact that the given solutions are mainly formatted for an interactive output, and not for further processing by a program:

- **type of solutions**: no information is given about the type of the solutions (explicit, implicit, series expansion, ...);
- **basis of solutions**: except in AXIOM, it is not easy to get a basis for the vector space of solutions of a linear differential equation.
- **decision procedures**: when the solver returns no solution, it is not possible[2] to know whether the equation has no solution in a given class (e.g. rational or elementary functions).
- **partial solutions**: no warning is given when the output does not cover all solutions, as for example in MAPLE with the following equation which admits also $x^2/2$ as solution:

```
> dsolve((2*y(x)*diff(y(x),x$2)-x*D(y)(x))*(D(y)(x)-56*y(x)*x^2),y(x));

                                             3
                           y(x) = exp(56/3 x ) _C1
```

- **domain of computation**: except for MACSYMA, it is not possible to choose between real valued and complex solutions; in MACSYMA, this possibility seems to be well implemented (only few problems occur).
- **solving strategy**: except for MAPLE, MACSYMA and MUPAD, it is not possible to know which methods were used to solve a given system. Such information would allow the user to know which algorithms are really implemented in the system.

Therefore, improving information about *solving strategies* and *solutions* is, in our opinion, an important task for further improvements of the ODE solvers!

Other programs and packages. The `odesolve` package of REDUCE does not yet cover nonlinear equations of order 2 or more, but the package CRACK [11] written by Thomas Wolf (`T.Wolf@qmw.ac.uk`) can solve systems 52 and 54, while the program PSODE written by Y-K. Man (`ykman@bc.ied.edu.hk`) can solve problems 16 and 28. We also mention the very nice program named CONVODE written by Alain Moussiaux from the University of Namur in Belgium [7, 8]. This program written in REDUCE is callable by email: for example to solve the equation (1), just send to `convode@riemann.physmath.fundp.ac.be` a message as follows

```
depend y,x;
CONVODE({(x^4-x^3)*df(y,x) + 2*x^4*y = x^3/3 + C},{y},{x},{},{});
```

and you will get the answer by email, with details (in French) about the computation details. CONVODE implements all algorithms from Zwillinger's book [12] and can even solve partial differential equations. For more information about CONVODE, see `http://www.physique.fundp.ac.be/physdpt/administration/convode.html`.

[2]In MAPLE, such a decision procedure exists in fact for linear equations with polynomial coefficients, as explained in [6], but it is not documented, neither in the help page nor in the `userinfo` mechanism.

Future work. Our goal is to maintain an up-to-date version of our review for new versions of the different systems, and for possibly new systems. This up-to-date version will be available from `http://www.loria.fr/~zimmerma/ComputerAlgebra/`, together with the input files for each system. New test equations are most welcome. In particular we are looking for new examples that would test the implementation of algorithms that search for well-defined classes of solutions (rational, elementary, Liouvillian) or general-purpose algorithms such as factorization of differential operators or methods based on Lie groups.

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