Analysing and bounding error sources in numerical neural network simulations

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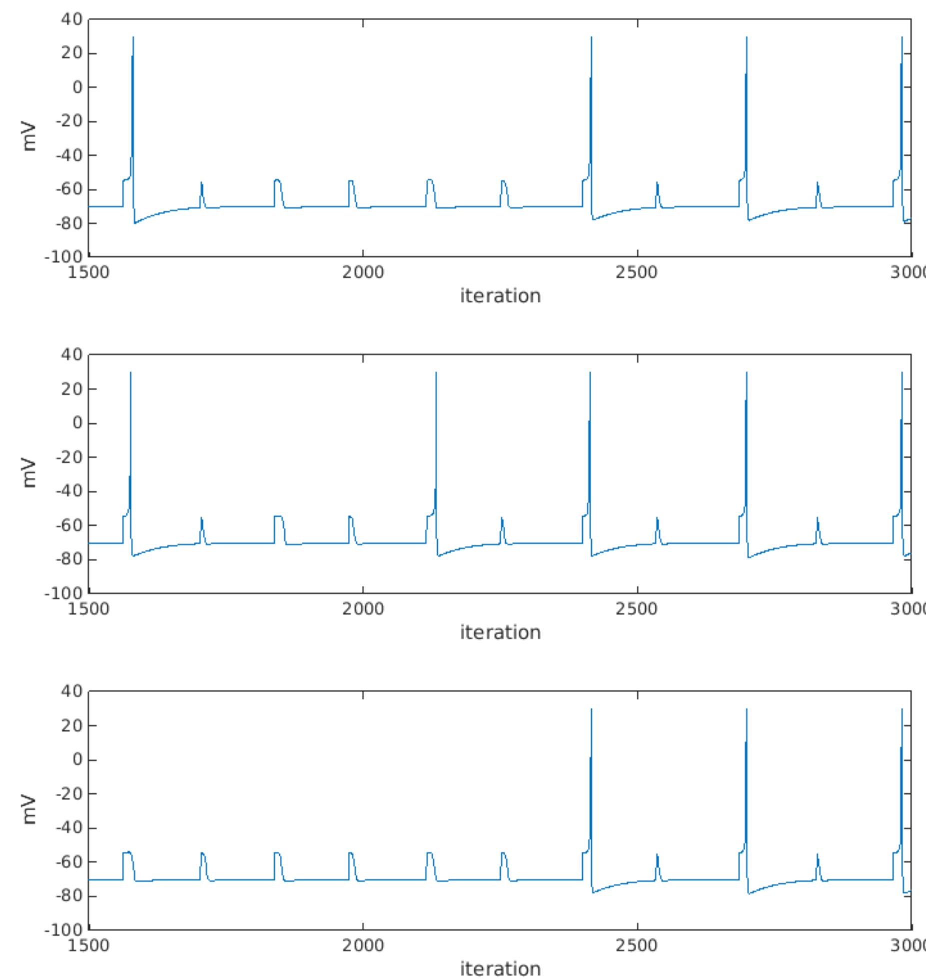
Numerical Error in Neural Networks

- **Reproducibility is desirable, but tricky**
- **Different results with different compilers and architecture**
- **Even different in the same environment**
- **Need a way to measure *how* reproducible a simulation is…**



Numerical Error in Neural Networks

Rounding error: $$x \, op \, y = \left(x \; \widehat{op} \; y \right)\left(1+\delta\right)$$

Nondeterminism: $$\left(\left(x \, \widehat{op} \; y \right) \; \widehat{op} \; z \right) \neq \left(x \, \widehat{op} \left(y \, \widehat{op} \, z \right)\right)$$

Truncation error: $$x_{t+h} = x_t + \left.\frac{dx}{dt}\right|_t h + O\left(h^2\right)$$

- **Lots of nondeterminism in parallel hardware**
- **Errors quickly amplified in unstable systems**

Interval Arithmetic (IA)

- **Replace variables and operators with IA equivalents:**

$$x \in \bar{x}, y \in \bar{y}, z \in \bar{z}$$
$$\bar{x} = [x_{lo}, x_{hi}]$$
$$\bar{y} = [y_{lo}, y_{hi}]$$
$$\bar{z} = \bar{x} - \bar{y} = [(x_{lo} - y_{hi}), (x_{hi} - y_{lo})]$$

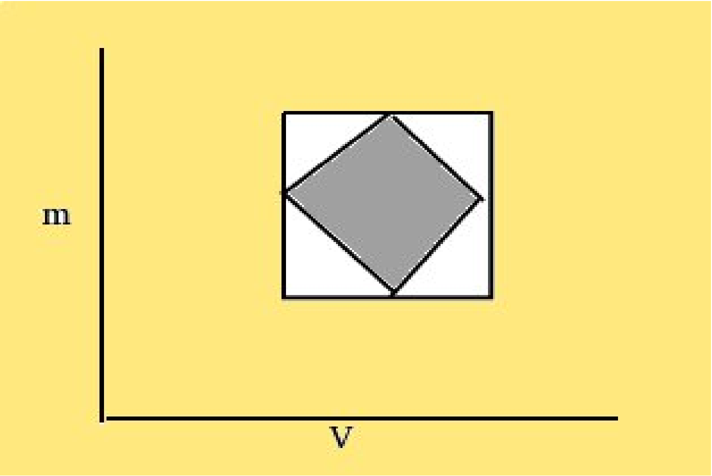
- **Cannot track correlations:**

$$p - p = 0$$
$$\bar{p} = [1, 2]$$
$$\bar{p} - \bar{p} = [(1-2), (2-1)] = [-1, 1]$$

- **Interval 'explosion' is an issue in long chained computations, such as numerical integration**

Interval Arithmetic (IA)

- **The ‘wrapping effect’**



- **IA tells us the axis-aligned white box is the reachable region**
- **If *m* and *V* are correlated, the real reachable region might in fact be the grey diamond**

Affine Arithmetic (AA)

- **Solution: encode correlations within representation**
- **Intervals are encoded as first-order polynomials with a centre: x_0, and n deviation terms: $x_i \varepsilon_i$**

$$\hat{x} = x_0 + x_1 \varepsilon_1 + \ldots + x_n \varepsilon_n$$

- **All ε_i are unknown, shared amongst all affine intervals and represent the uncertainty from an error source**

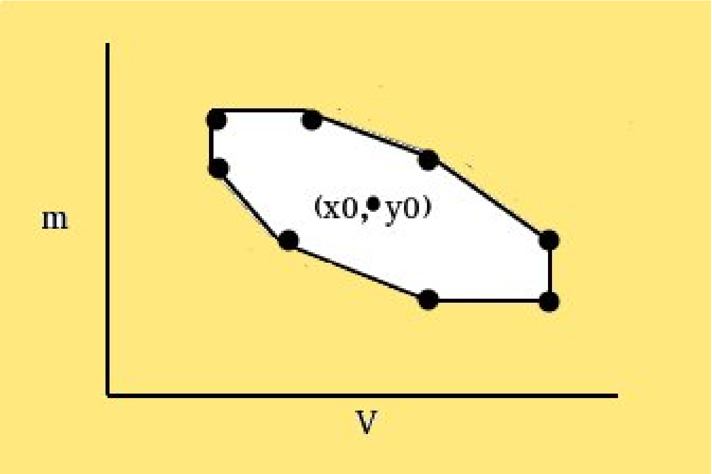
$$\varepsilon_i \in [-1, 1]$$

- **x and y are correlated iff:**

$$\exists k > 0 : |x_k| \neq 0 \wedge y_k \neq 0$$

Affine Arithmetic (AA)

- **A convex hull 'wraps' the reachable region better**
- **Deviation terms correspond to edges**



- **Numerical error is placed in a new deviation term $x_k \varepsilon_k$**

$$\hat{x} = x_0 + x_1 \varepsilon_1 + \ldots + x_n \varepsilon_n + x_k \varepsilon_k$$

- **Nonlinear operations are approximated linearly (Chebyshev), and linearisation error is also added to $x_k \varepsilon_k$**

Affine Arithmetic (AA)

affine_1 (z, x, α, γ, δ)

$$z = (\alpha x_0 + \gamma)$$
$$+ (\alpha x_1)\varepsilon_1 + \ldots + (\alpha x_n)\varepsilon_n$$
$$+ (\delta)\varepsilon_{n+1}$$

affine_2 (z, x, y, α, β, γ, δ)

$$z = (\alpha x_0 + \beta y_0 + \gamma)$$
$$+ (\alpha x_1 + \beta y_1)\varepsilon_1 + \ldots + (\alpha x_n + \beta y_n)\varepsilon_n$$
$$+ (\delta)\varepsilon_{n+1}$$

Affine Arithmetic (AA)

exp_cheb (z, x)

$\alpha = (e^{x.hi} - e^{x.lo}) / (x.hi - x.lo)$

$d_a = e^{x.lo} - \alpha x.lo$

$d_b = e^{x.hi} - \alpha x.hi$

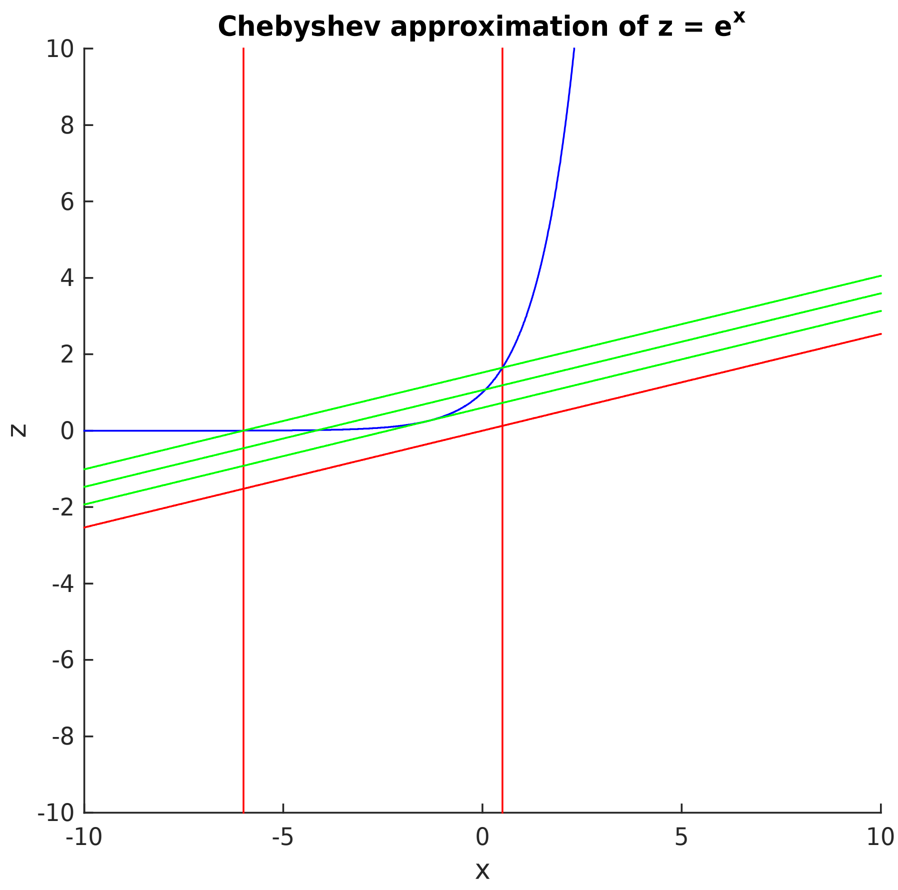
$d_{min} = e^{u} - \alpha u$

$d_{max} = \max(d_a, d_b)$

$\gamma = \text{mid}([d_{min}, d_{max}])$

$\delta = \text{rad}([d_{min}, d_{max}])$

affine_1(z, x, α, γ, δ)



The MPFA Library

- **Written in C**
- **Open Source (LGPLv3 license)**
- **Based on the GNU MPFR Library**
- **Link as a shared or static lib**
- **Uses the GNU Autotools, for simpler compilation, installation and portability**

  ./configure && make && sudo make install
- **Beta release very soon!**

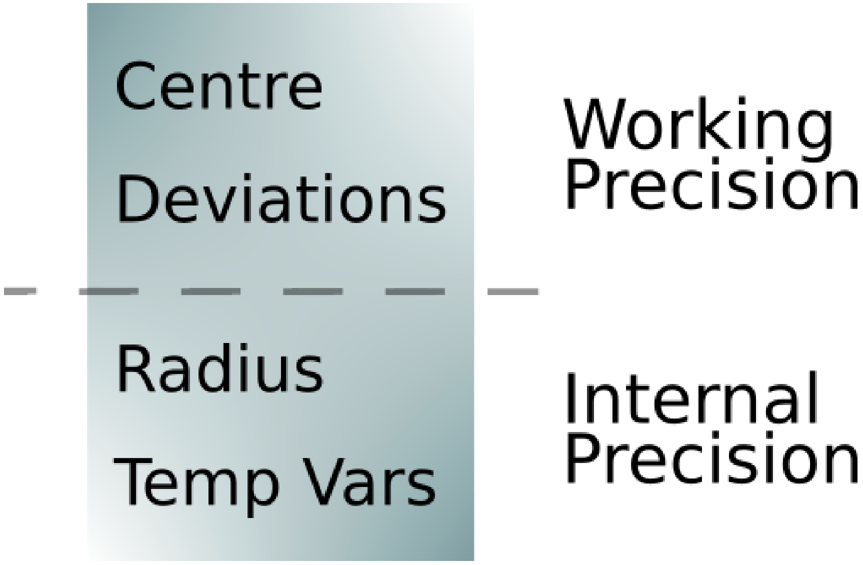
  Available: www.github.com/jamesturner246/mpfa

The MPFA Library

- **Benefits of MPFR**
  - Results guaranteed to be correctly rounded
  - Support for IEEE-754-2008 rounding modes
  - Precision is adjustable at runtime
  - No opaque compiler optimisation
  - Extensive testing
  - All the above, for **all** functions

The MPFA Library

- **Potentially lots of internal rounding error**
- **Centre and deviation terms are computed in working precision**
- **Intermediate variables and radius are computed in higher internal precision**



The MPFA Library

- **Many new deviation terms: 1 per operation**
- **Eventually need 'garbage collection'**

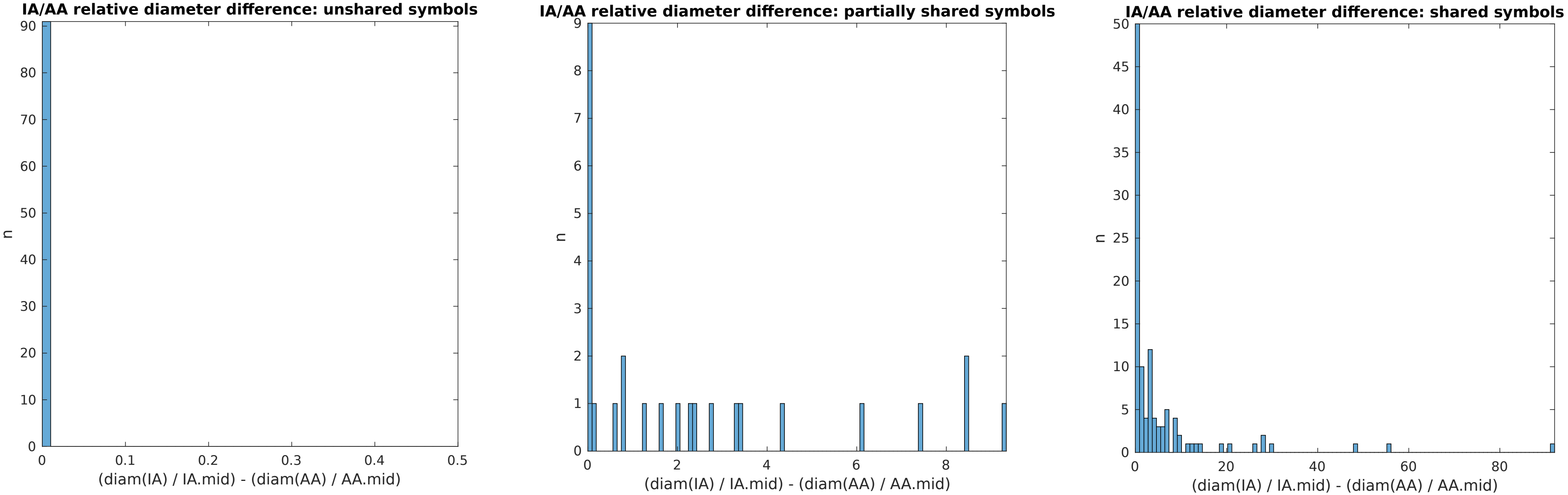
mpfa_condense_last_n (mpfa_t z, unsigned n)

- Condense the last *n* deviation terms
- Lossless (when used correctly)

mpfa_condense_small (mpfa_t z, double fraction)

- Condense terms smaller than some fraction of radius
- Lossy (some correlation information is lost)

The MPFA Library



mpfa_add and mpfi_add, 1,000,000 tests, 100 bins

Morris-Lecar Neuron Model

$$C\frac{dV}{dt}=I-g_L\left(V-V_L\right)-g_{Ca}M_{ss}\left(V\right)\left(V-V_{Ca}\right)-g_K N\left(V-V_K\right)$$

$$\frac{dN}{dt}=\frac{N_{ss}\left(V\right)-N}{\tau_N\left(V\right)}$$

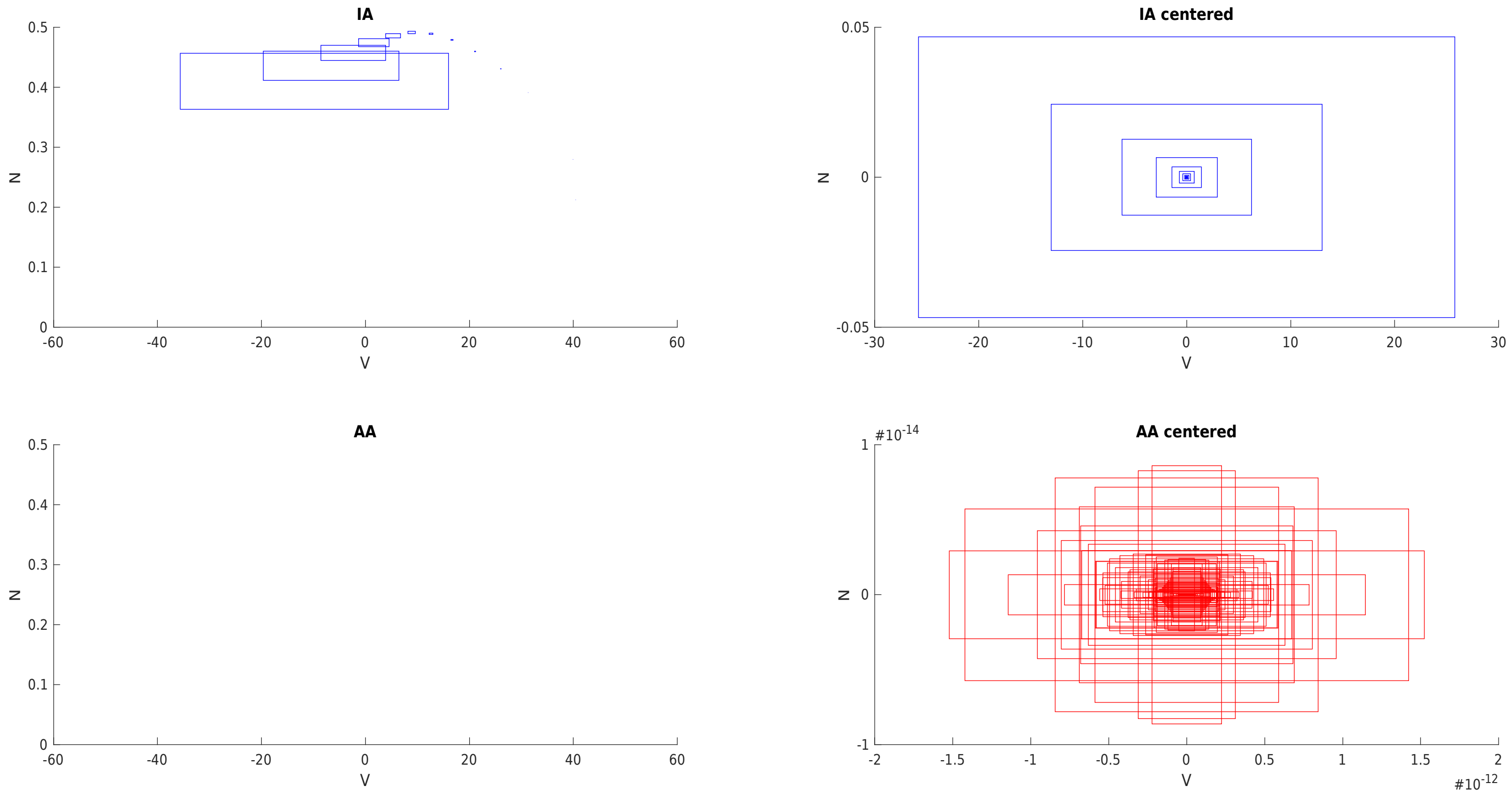
$$M_{ss}\left(V\right)=\frac{1}{2}\left(1+\tanh\left(\frac{V-V_1}{V_2}\right)\right)$$

$$N_{ss}\left(V\right)=\frac{1}{2}\left(1+\tanh\left(\frac{V-V_3}{V_4}\right)\right)$$

$$\tau_N\left(V\right)=1/\left(\phi\cosh\left(\frac{V-v_3}{2V_4}\right)\right)$$

Morris-Lecar Neuron Model

IA (top) and AA (bot). Iterations 1 to 83.



Thanks!

The MPFA Library

An arbitrary-precision affine arithmetic library based on GNU MPFR.

Open source permissive license (LGPLv3).

Download, contribute and give feedback at:

www.github.com/jamesturner246/mpfa