How Fast Can We Multiply Over $\mathrm{GF}(2)[x]$?

Paul Zimmermann
INRIA Lorraine/LORIA, Nancy, France

(thanks to Richard Brent, Pierrick Gaudry, Samuli Larvala, Emmanuel Thomé)

Followup to Mika's talk (preproceedings, p. 25)

Theorem. *The first digit of* $F_{5 \cdot 10^{87}}$ *is* 1.

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Theorem. *The first digit of* $F_{5\cdot 10^{87}}$ *is* 1.

Proof:

```
bash-3.00$ time ./fib 5e87
n=5000000000000000000000000000000000000000000\
  000000000000000000000000000000000000000000000
prec=302
length of Fib(n) in base 3 is
2190089397429712060570026179560455216382019945\
  882232014510953225378415664943464622853621
first digit is 1

user    0m0.004s
```

Credits: MPFR (`www.mpfr.org`), MPFI.

Plan of the talk

- **Theory**

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- **Algorithms**

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- **Numbers**

Motivation: Search for Primitive Trinomials

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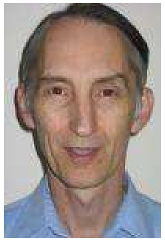

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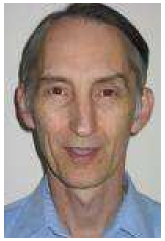

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June 26, 2000:

$$x^{859433} + x^{170340} + 1$$

Status so far

$$x^r + x^s + 1$$

r	s	when
756839	215747, 267428, 279695	June 2000
859433	170340, 288477	June 2000
3021377	361604, 1010202	July 2000 to April 2001: 13 GIPS-years
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As a comparison, RSA-155 (1999) took 8 GIPS years.

THEORY

GF(2)

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Multiplication table:

$\times$	0	1
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$\mathrm{GF}(2)[x]$

Polynomial ring:

$$a(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0,$$

where $a_i \in \{0, 1\}$.

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If $a_d \neq 0$, $d = \deg(a)$.

Irreducible Polynomial

Definition. $a(x) \in \mathrm{GF}(2)[x]$ is *irreducible* if

$$a(x) = b(x)c(x)$$

implies $b(x) = 1$ or $c(x) = 1$.

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Example 3. $x^4 + x^2 + x + 1$ is not either (irreducible over $\mathbb{Q}$):

$$x^4 + x^2 + x + 1 = (x^3 + x^2 + 1)(x + 1)$$

Why trinomials?

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Then try trinomials:

$$x^r + x^s + 1 \qquad \text{with } r > s > 0.$$

Primitive Trinomials

Definition. A polynomial $f(x) \in \mathrm{GF}(2)[x]$ is said *primitive* iff:

$\mathbf{(1)}$ $f(x)$ is *irreducible*;

$\mathbf{(2a)}$ x has order $2^r - 1$ modulo $f(x)$, where $r := \deg(f)$;

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$$x^3+x^2+x+1, x^3+x^2+1, x^3+1, x^{15} \equiv 1.$$

Example 2. $x^6 + x^3 + 1$ is irreducible but not primitive:

$$x^9 \equiv 1 \bmod (x^6 + x^3 + 1).$$

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Need to factor $2^r - 1 \ldots$

Easy if $2^r - 1$ is known to be prime:

$$f(x) \text{ irreducible} \Longrightarrow f(x) \text{ primitive}$$

Use Mersenne primes $2^r - 1$

Great Internet Mersenne Prime Search (GIMPS, `www.mersenne.org`).

George

Woltman

	r	date	$r \bmod 8$
M35	1398269	Nov 1996	5
M36	2976221	Aug 1997	5
M37	3021377	Jan 1998	1
M38	6972593	Jun 1999	1
M39	13466917	Nov 2001	5
M40?	20996011	Nov 2003	3
M41?	24036583	May 2004	7
M42?	25964951	Feb 2005	7
M43?	30402457	Dec 2005	1
M44?	32582657	Sep 2006	1

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Example. No irreducible trinomial of degree 8:

$$x^8 + x + 1 = (x^6 + x^5 + x^3 + x^2 + 1)(x^2 + x + 1)$$

$$x^8 + x^2 + 1 = (x^4 + x + 1)^2$$

$$x^8 + x^3 + 1 = (x^3 + x + 1)(x^5 + x^3 + x^2 + x + 1)$$

$$x^8 + x^4 + 1 = (x^2 + x + 1)^4$$

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$$x^8 + x^4 + 1 = (x^2 + x + 1)^4$$

In general, no irreducible trinomial of degree $r = 8k$.

Swan's Theorem (1962)

Previous work by von zur Gathen (2002), Dalen (1955), Dickson (1906), Stickelberger (1897), Pellet (1878), ...

Theorem. Suppose $r > s > 0$, $r - s$ odd. Then $x^r + x^s + 1$ has an even number of irreducible factors over $\mathrm{GF}(2)$ if and only if one of the following holds:

- r is even, $r \neq 2s$, $rs/2 \bmod 4 \in \{0, 1\}$;
- $2r \neq 0 \bmod s$, $r = \pm 3 \bmod 8$;
- $2r = 0 \bmod s$, $r = \pm 1 \bmod 8$.

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Corollary 1. If r is prime, $r = \pm 3 \bmod 8$, $s \notin \{2, r-2\}$, then $x^r + x^s + 1$ is reducible.

$\Longrightarrow$ need to check only $x^r + x^2 + 1$.

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Corollary 2. A trinomial of degree multiple of 8 cannot be irreducible.

$r = 8k$: pentanomials

Swan's theorem: no trinomial of degree $r = 8k$ can be irreducible.

How to perform efficient arithmetic in $\mathrm{GF}(2^r)$, say $\mathrm{GF}(2^{16})$?

Workaround: use a pentanomial

$$x^{16} + x^5 + x^3 + x + 1.$$

$r = 8k$: almost irreducible trinomials

(Richard Brent, PZ, 2003)

$$x^{19}+x^4+1 = (x^3+x+1)(x^{16}+x^{14}+x^{13}+x^{12}+x^9+x^7+x^6+x^5+x^2+x+1)$$

Perform all arithmetic modulo $x^{19} + x^4 + 1$.

Reduce mod $x^{16} + \cdots + 1$ only when a canonical form is needed.

ALGORITHMS

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Goal 2. (if possible) output a *certifi cate* which can be checked faster than the time to make it.

Certifi cates

Integer multiplication:

$$395718860534 \cdot 193139816415 \Rightarrow 76429068075489748865610$$

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Integer factorization:

$$179435405554681543034355 \Rightarrow 224241704651 \cdot 800187484459$$

One factor is a valid certificate.

Do not waste a factor of two!

One of Schönhage's **golden rules**.

$$x^r + x^s + 1 = a(x)b(x) \Longrightarrow 1 + x^{r-s} + x^r = x^r a(1/x)b(1/x)$$

$\Longrightarrow$ can restrict to $s \le r/2$.

Main Theorem

Theorem. The product of $\mathbf{ALL}$ irreducible factors of degree $\mathbf{dividing}$ k is $x^{2^k} + x$.

$$x^{2^1} + x = x(x+1)$$

$$x^{2^2} + x = x(x+1)(x^2+x+1)$$

$$x^{2^3} + x = x(x+1)(x^3+x+1)(x^3+x^2+1)$$

$$x^{2^4} + x = x(x+1)(x^2+x+1)(x^4+x+1)(x^4+x^3+1)(x^4+x^3+x^2+x+1)$$

The old algorithm

1. (sieving) for $k = 2$ to k_0, compute:

$$\gcd(x^{2^k} + x, x^r + x^s + 1)$$

If non trivial, output “divisible by degree k”

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For $r = 6972593$, we used $k_0 = 26$: 236244 trinomials (7%) survived Step 1.

Complexity: $O(r^2)$ for each full test.

The “new” algorithm

Perform a classical DDF (distinct degree factorization) with the “blocking strategy” (von zur Gathen and Shoup 1992, Kaltofen and Shoup 1998):

0. Partition $\{2, \ldots, \lfloor r/2 \rfloor\}$ into intervals $I_1, \ldots, I_m$.

1. for $j := 1$ to m do

    $a \leftarrow 1$; for k in I_j do

        $b \leftarrow x^{2^k} \bmod (x^r + x^s + 1)$ **[SQR]**

        $a \leftarrow a(b + x) \bmod (x^r + x^s + 1)$ **[MUL]**

    $g \leftarrow \gcd(a, x^r + x^s + 1)$ **[GCD]**

    if $g \neq 1$ then output “reducible with degree in I_j”

Output “irreducible”.

Complexity: $O(dM(r))$ if the smallest factor has degree d, assuming the GCD cost is not dominant.

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With R. Brent: a faster algorithm in $O(r^2 \log r \sqrt{M(r)/r})$, but no space in the margin. . .

NUMBERS

Binary Polynomials

$a(x) = a_{r-1}x^{r-1} + \cdots + a_1 x + a_0$ is stored in computer by the *binary polynomial*

$$a(2) = a_{r-1} \cdot 2^{r-1} + \cdots + a_1 \cdot 2 + a_0.$$

On a 8-bit computer, the trinomial $x^{19} + x^4 + 1$ is stored as:

$$\underbrace{\boxed{00001000}}_{x^3 \cdot x^{16}}\underbrace{\boxed{00000000}}_{0 \cdot x^8}\underbrace{\boxed{00010001}}_{(x^4+1) \cdot x^0}$$

Addition of Binary Polynomials

$x^{15}+x^{13}+x^{12}+x^{11}+x^{9}+x^{8}+x^{6}+x^{4}+x^{3}+x^{2}$	10111011	01011100
$x^{15}+x^{12}+x^{11}+x^{10}+x^{9}+x^{7}+x^{6}+x^{5}+x^{4}+x^{2}+x$	10011110	11110110
$x^{13}+x^{10}+x^{8}+x^{7}+x^{5}+x^{3}+x$	00100101	10101010

Multiplication by x^k

$a = x^{13} + x^{12} + x^{11} + x^9 + x^8 + x^6 + x^4 + x^3 + x^2$ | 00111011 | 01011100 |

$x^2 a = x^{15} + x^{14} + x^{13} + x^{11} + x^{10} + x^8 + x^6 + x^5 + x^4$ | 11101101 | 01110000 |

Multiplication

$$(x^6 + x^4 + x^3 + x^2)(x^5 + x^4 + x^3 + x + 1)$$

$$\begin{array}{r} \boxed{01011100} \\ \times \ \boxed{00111011} \\ \hline \end{array}$$

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\boxed{01011100} \\
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\hline
\boxed{01011100} \\
\boxed{01011100}
\end{array}
$$

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```
             01011100
          ×  00111011
          -----------
             01011100
            01011100
          01011100
```

Multiplication

$(x^6 + x^4 + x^3 + x^2)(x^5 + x^4 + x^3 + x + 1)$

$$
\begin{array}{r}
\boxed{01011100} \\
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\hline
\boxed{01011100} \\
\boxed{01011100} \\
\boxed{01011100} \\
\boxed{01011100}
\end{array}
$$

Multiplication

$(x^6 + x^4 + x^3 + x^2)(x^5 + x^4 + x^3 + x + 1)$

$$
\begin{array}{r}
\boxed{01011100} \\
\times\ \boxed{00111011} \\
\hline
\boxed{01011100} \\
\boxed{01011100} \\
\boxed{01011100} \\
\boxed{01011100} \\
+\ \boxed{01011100}
\end{array}
$$

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$$
\begin{array}{r}
\boxed{01011100} \\
\times \ \boxed{00111011} \\
\hline
\boxed{01011100} \\
\boxed{01011100} \\
\boxed{01011100} \\
\boxed{01011100} \\
+ \ \boxed{01011100} \\
\hline
\boxed{0000110001000100}
\end{array}
$$

$$x^{11} + x^{10} + x^6 + x^2$$

Squares are easy:

$$x^t + x^u + \cdots \qquad \Longrightarrow \qquad x^{2t} + x^{2u} + \cdots$$

GCDs reduce to multiplication: $O(M(r) \log r)$

$\Longrightarrow$ We have to improve multiplications!

Multiplication over $\mathrm{GF}(2)[x]$

- naive (quadratic) algorithm
- Karatsuba's algorithm
- Toom-Cook 3-way and higher order
- Fast Fourier Transform: segmentation, Cantor (BiPolAr), Schönhage

Schönhage's Algorithm

Schnelle Multiplikation von Polynomen über Körpern der Charakteristik 2, A. Schönhage, *Acta Inf. 7* (1977), 395–398.

Complexity $O(r \log r \log\log r)$.

High-level description:

one product $\bmod (x^{2N} + x^N + 1) \quad \Longrightarrow \quad 2K$ products $\bmod (x^{2L} + x^L + 1)$

Constraints: K power of 3, $L \geq N/K$, L multiple of K

Variant described here:

one product $\bmod (x^N + 1) \quad \Longrightarrow \quad K$ products $\bmod (x^{2L} + x^L + 1)$

Constraints: K power of 3, $L \geq N/K$, L multiple of $K/3$

Forward and backward transform: $O(K \log K)$ additions/shifts mod $x^{2L} + x^L + 1$.

Pointwise products: K products mod $x^{2L} + x^L + 1$.

The Algorithm

Input: a, b polynomials of degree $< N$

Parameters: K power of 3 dividing N, $M = N/K$, $L \geq M$ multiple of $K/3$.

1. Decompose a, b in base x^M:

$$a(x) = \sum_{i=0}^{K-1} a_i(x) x^{iM}$$

2. Forward transform with $\omega = x^{3L/K}$:

$$\hat{a}_i = \sum_{j=0}^{K-1} a_i(x) \omega^{ij} \bmod (x^{2L} + x^L + 1), 0 \leq i < K$$

3. Pointwise products:

$$\hat{c}_i = \hat{a}_i \hat{b}_i, \quad 0 \leq i < K$$

4. Backward transform:

$$c_\ell = \sum_{i=0}^{K-1} \hat{c}_i(x)\omega^{-\ell i} \bmod (x^{2L} + x^L + 1), 0 \leq \ell < K$$

5. Recomposition:

$$c(x) = \sum_{\ell=0}^{K-1} c_\ell x^{\ell M} \bmod (x^N + 1).$$

An example

Compute $a(x)b(x) \bmod (x^{15}+1)$:

$$a(x) = x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^{8}+x^{6}+x^{5}+x^{4}+x^{3}+x^{2}+1,$$

$$b(x) = x^{13}+x^{11}+x^{8}+x^{7}+x^{6}+x^{2}.$$

Take $K=3$, $L=5$:

$$a_2 = x^4+x^3+x^2+x+1, a_1 = x^3+x+1, a_0 = x^4+x^3+x^2+1$$

$$b_2 = x^3+x, b_1 = x^3+x^2+x, b_0 = x^2$$

Forward transform ($\omega = x^5$, mod $x^{10}+x^5+1$):

$$\hat{a}_2 = x^{20}a_2 + x^{10}a_1 + a_0 = x^9+x^7+x^4+x^2+x$$

$$\hat{a}_1 = x^{10}a_2 + x^{5}a_1 + a_0 = x^9+x^7+x$$

$$\hat{a}_0 = a_2 + a_1 + a_0 = x^3+1$$

An example

Forward transform ($\omega = x^5$, mod $x^{10} + x^5 + 1$):

$$\hat{a}_2 = x^9 + x^7 + x^4 + x^2 + x, \hat{a}_1 = x^9 + x^7 + x, \hat{a}_0 = x^3 + 1$$

$$\hat{b}_2 = x^7 + x^3 + x, \hat{b}_1 = x^7 + x^3 + x^2 + x, \hat{b}_0 = 0$$

Pointwise transforms:

$$\hat{c}_2 = x^6 + x^3, \hat{c}_1 = x^7 + x^6 + x^3, \hat{c}_0 = 0$$

Backward transform:

$$c_2 = x^6 + x^3, c_1 = x^7 + x^6 + x^3, c_0 = 0$$

Reconstruction:

$$c_2 x^{10} + c_1 x^5 + c_0 = x^{13} + x^{12} + x^{11} + x^8 + x^2 + x \bmod (x^{15} + 1)$$

Why does it work?

Let $R_L := \mathrm{GF}(2)[x]/(x^{2L} + x^L + 1)$.

$\omega = x^{3L/K} \implies \omega^{K/3} = x^L$ thus in R_L:

$$\omega^{2K/3} + \omega^{K/3} + 1 = 0 \tag{1}$$

From Eq. (1) it follows

$$\omega^K = 1 \quad \text{and} \quad \omega^{-1} = \omega^{K-1} \tag{2}$$

$$\begin{aligned} c_\ell := \sum_{i=0}^{K-1} \hat{c}_i(x)\omega^{-\ell i} &= \sum_{i=0}^{K-1} \omega^{-\ell i} \left(\sum_{j=0}^{K-1} \omega^{ij} a_i\right) \left(\sum_{k=0}^{K-1} \omega^{ik} b_k\right) \\ &= \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} a_j b_k \sum_{i=0}^{K-1} \omega^{i(j+k-\ell)}. \end{aligned}$$

Why does it work?

$$c_\ell = \sum_{j=0}^{K-1} \sum_{k=0}^{K-1} a_j b_k \sum_{i=0}^{K-1} \omega^{i(j+k-\ell)}$$

We have $-K < j + k - \ell < 2K$. If $t := j + k - \ell \neq 0 \bmod K$:

$$\sum_{i=0}^{K-1} \omega^{i(j+k-\ell)} = \frac{\omega^{Kt} + 1}{\omega^t + 1} = 0.$$

Otherwise $j + k - \ell \in \{0, K\}$, and $\omega^{i(j+k-\ell)} = 1$.

Thus $\sum_{i=0}^{K-1} \omega^{i(j+k-\ell)}$ is non-zero only when $j + k - \ell \in \{0, K\}$, in which case it equals $K = 1 \bmod 2$.

It follows:

$$c_\ell = \sum_{j+k=\ell} a_j b_k + \sum_{j+k=K+\ell} a_j b_k \pmod{x^{2L} + x^L + 1}.$$

Why does it work?

$$c_\ell = \sum_{j+k=\ell} a_j b_k + \sum_{j+k=K+\ell} a_j b_k \pmod{x^{2L} + x^L + 1}.$$

Recall $\deg(a_j), \deg(b_k) < M$: if $L \geq M$, then

$$c_\ell = \sum_{j+k=\ell} a_j b_k + \sum_{j+k=K+\ell} a_j b_k.$$

5. Recomposition:

$$c(x) = \sum_{\ell=0}^{K-1} c_\ell x^{\ell M} \bmod (x^N + 1).$$

$c(x)$ is simply the cyclic convolution of $a(x)$ and $b(x)$ mod $x^N + 1$.

Arithmetic Modulo $x^{2L} + x^L + 1$

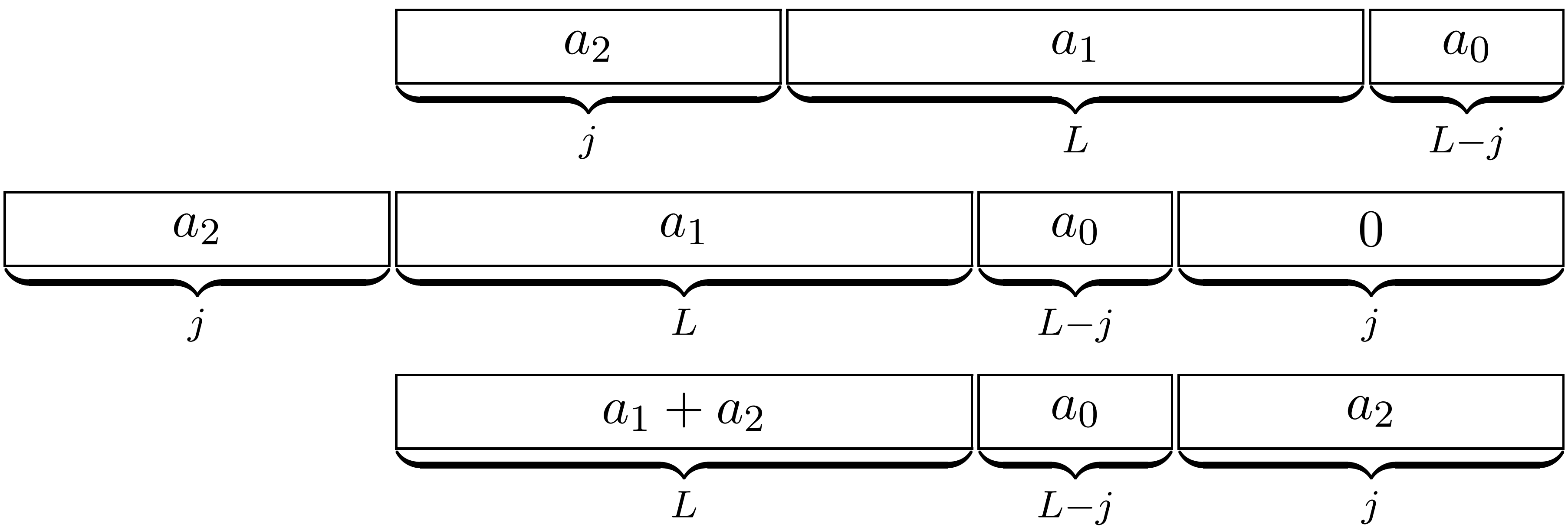
- addition: easy
- shift: multiplication by x^j, $0 \leq j < 3L$
- full multiplication

Shifts Modulo $x^{2L} + x^L + 1$

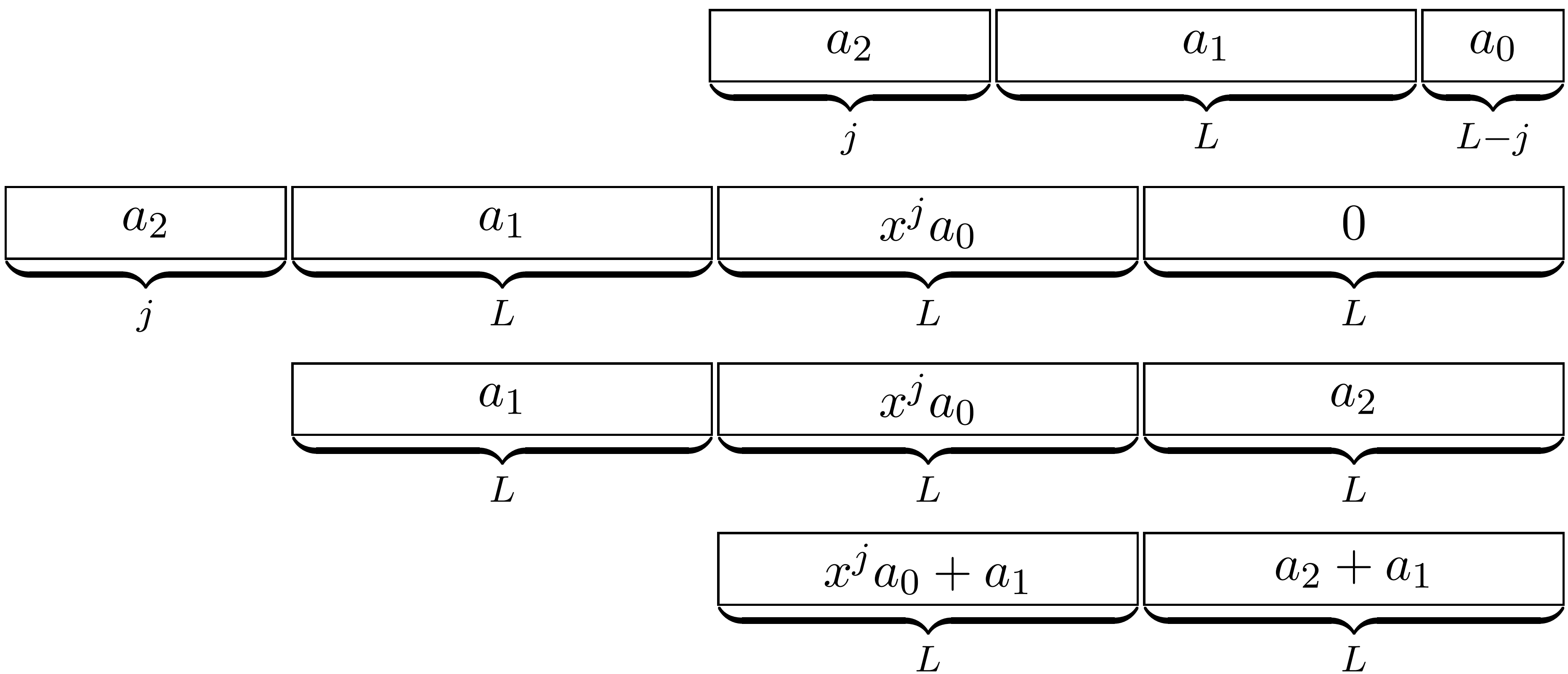
Input: a binary polynomial $a(x)$ of degree $< 2L$, $0 \le j < 3L$

Output: $x^j a(x) \bmod (x^{2L} + x^L + 1)$

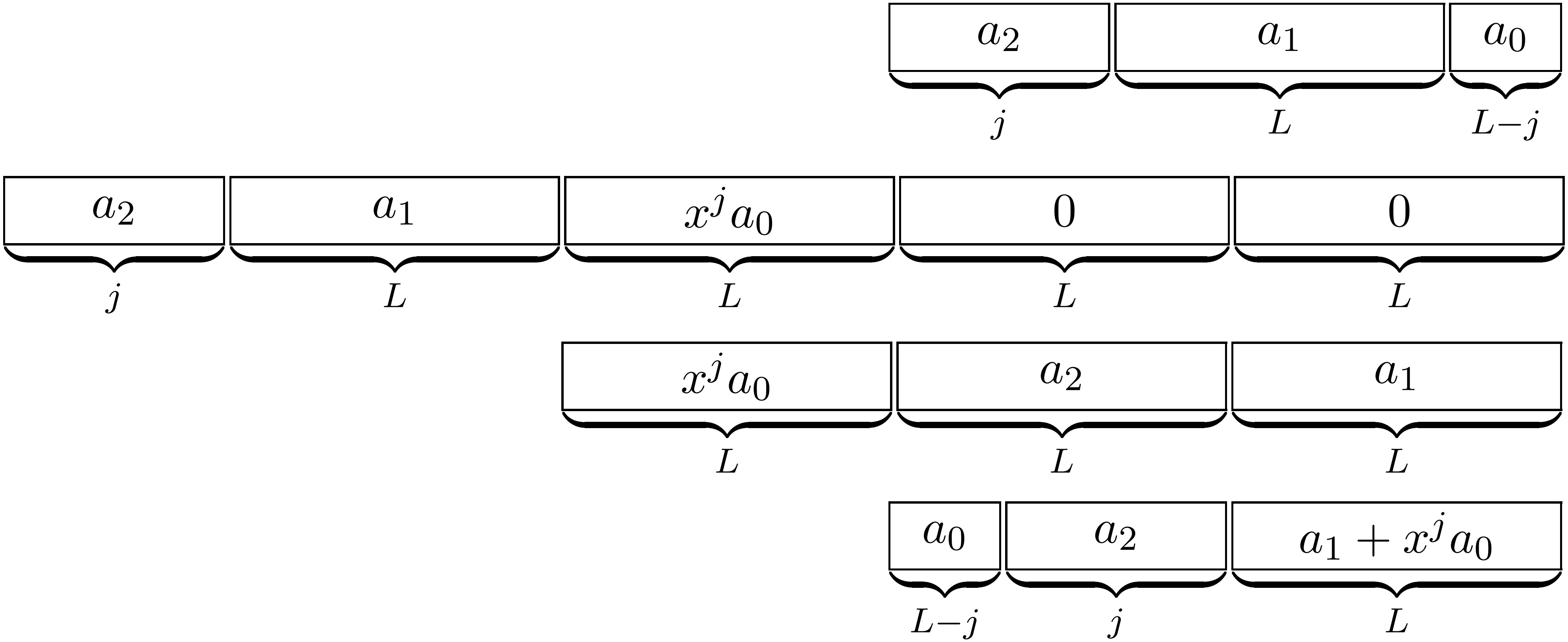
1. Shift of j, $0 \le j < L$:



Case 2: Shift of $L + j, 0 \le j < L$



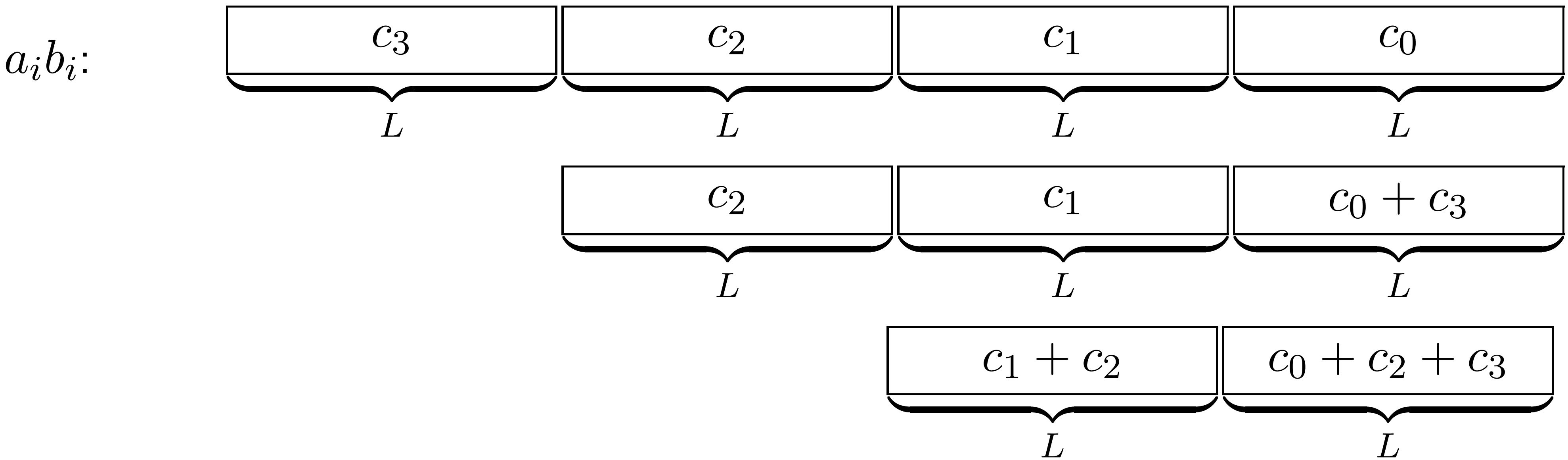
Case 3: Shift of $2L + j, 0 \leq j < L$



Multiplication mod $x^{2L} + x^L + 1$

3. Pointwise products

$$\hat{c}_i = \hat{a}_i \hat{b}_i (\bmod x^{2L} + x^L + 1)$$



Timings

Core 2 processor, 2.66Ghz, 4MB cache, 3GB memory.

r	Toom-Cook 3	Toom-Cook 4	FFTMul(K)	GCD
6972593	1.32s	1.01s	0.27s(6561)	12.1s
24036583	7.89s	6.30s	1.77s(6561)	55.3s
32582657	13.9s	8.11s	2.16s(6561)	78.4s

6972593 again

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- subquadratic GCD (still quite expensive)

24036583

We have started computations for $r = 24036583$ (M41?) on April 25.

Already done more than 10%.

No primitive trinomial so far.

But already found a (smallest) factor of degree almost one million!

Help welcome (preferably Opteron/Core 2)!

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Thank you for staying awake so far!