Optimized Binary64 and Binary128 Arithmetic with GNU MPFR (common work with Vincent Lefèvre)

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Arith 24 conference, London, July 24, 2017

Introducing the GNU MPFR library

- a software implementation of *binary* IEEE-754 (decimal implementation provided by decNumber from Mike Cowlishaw)
- variable/arbitrary precision (up to the limits of your computer)
- *each variable* has its own precision: `mpfr_init (a, 35)`
- global user-defined exponent range (might be huge): `mpfr_set_emin (-123456789)`
- mixed-precision operations: $a \leftarrow b - c$ where a has 35 bits, b has 42 bits, c has 17 bits
- correctly rounded mathematical functions (exp, log, sin, cos, ...) as in Section 9 of IEEE 754-2008

History

- 2000: first public version;
- 2008: MPFR is used by GCC 4.3.0 for *constant folding*:

  ```
  double x = sin (3.14);
  ```
- 2009: MPFR becomes GNU MPFR;
- 2016: 4th developer meeting in Toulouse.
- `mpfr.org/pub.html` mentions 2 books, 27 PhD theses, 59 papers citing MPFR

```
SageMath version 7.6, Release Date: 2017-03-25
Type ‘‘notebook()’’ for the browser-based notebook interface.
Type ‘‘help()’’ for help.

sage: x=1/7; a=10^-8; b=2^24
sage: RealIntervalField(24)(x+a*sin(b*x))
[0.142857119 .. 0.142857150]
```

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Calcul mathématique avec
SAGE

Now in english!

This work

- concentrates on small precision (1 to 2 machine-words)
- all operands have same precision
- basic operations: add, sub, mul, div, sqrt
- get the fastest possible software implementation
- while keeping the same user interface

Correct Rounding

Definition: we compute the floating-point value closest to the exact result, with the given precision and rounding modes (following IEEE-754).

RNDN: to nearest (ties to even);

RNDZ: toward zero,   RNDA: away from zero;

RNDD: toward $-\infty$,   RNDU: toward $+\infty$.

Only one possible conforming result: the correct rounding.

Notations

MPFR uses GMP's `mpn` layer for the internal representation of significands.

limb: a GMP word (in general 32 or 64 bits)

We will assume here a *limb* has 64 bits.

In a 64-bit limb, we call "bit 1" the most significant bit, and "bit 64" the least significant one.

Representation of MPFR numbers (*mpfr_t*)

- precision $p \geq 1$ (in bits);
- sign (-1 or $+1$);
- exponent (between $E_{\min}$ and $E_{\max}$), also used to represent special numbers (NaN, $\pm\infty$, ± 0);
- significand (array of $\lceil p/64 \rceil$ *limbs*), defined only for *regular* numbers (neither NaN, nor $\pm\infty$ and ± 0, which are *singular* values).

The most significant bits are shown on the left.

Regular numbers are *normalized*: the most significant bit of the most significant *limb* should be 1.

Example, $x = 17$ with a precision of 10 bits and *limbs* of 6 bits is represented as follows:

$$\underbrace{10}_{\text{precision}} \quad \underbrace{+1}_{\textit{sign}} \quad \underbrace{5}_{\textit{exponent}} \quad \underbrace{\boxed{100010}}_{\text{limb 1}}\underbrace{\boxed{0000\textcolor{red}{00}}}_{\text{limb 0}}$$

Round bit and *sticky bit*

$$v = \underbrace{xxx\ldots yyy}_{m \text{ of } p \text{ bits}} \quad \underbrace{r}_{\textit{round bit}} \quad \underbrace{sss\ldots}_{\textit{sticky bit}}$$

The *round bit* r is the value of bit $p+1$ (where bit p is the least significant bit of the significand).

The *sticky bit* s is zero iff `sss...` is zero.

The *round bit* and *sticky bit* enable us to determine correct rounding for all rounding modes:

r	s	toward zero	to nearest	away from zero
0	0	m	m	m
0	1	m	m	$m+1$
1	0	m	$m+(m \bmod 2)$	$m+1$
1	1	m	$m+1$	$m+1$

The function mpfr_add

The function mpfr_add(a, b, c) works as follows $(a \leftarrow b + c)$:

- first check for singular values $(NaN, \pm Inf, \pm 0)$;
- if b and c have different signs, call the subtraction code;
- if a, b, c have same precision, call mpfr_add1sp;
- otherwise call the generic mpfr_add1 code described in: Vincent Lefèvre, *The Generic Multiple-Precision Floating-Point Addition With Exact Rounding (as in the MPFR Library)*, 6th Conference on Real Numbers and Computers 2004 - RNC 6, Nov 2004, Dagstuhl, Germany, pp.135-145, 2004.

The (new) function mpfr_add1sp

- if $p < 64$, call mpfr_add1sp1;
- if $p = 64$, call mpfr_add1sp1n;
- if $64 < p < 128$, call mpfr_add1sp2;
- otherwise execute the generic code for operands of same precision.

Note: $p = 128$ uses the generic code, prefer $p = 127$ if possible.

The function mpfr_add1sp1

Case 1, $e_b = e_c$:

$$b = \boxed{110100}$$

$$c = \boxed{111000}$$

```
a0 = (bp[0] >> 1) + (cp[0] >> 1);
bx ++;
rb = a0 & (MPFR_LIMB_ONE << (sh - 1));
ap[0] = a0 ^ rb;
sb = 0;
```

Since b and c are normalized, the most significant bits from $bp[0]$ and $cp[0]$ are 1.

Thus the addition of $bp[0]$ and $cp[0]$ *always* produces a carry, and the exponent of a is $e_b + 1$ (here `bx + 1`).

$$b = \boxed{110100}$$

$$c = \boxed{111000}$$

```
a0 = (bp[0] >> 1) + (cp[0] >> 1);
bx ++;
rb = a0 & (MPFR_LIMB_ONE << (sh - 1));
ap[0] = a0 ^ rb;
sb = 0;
```

The sum might have $p + 1$ significant bits, but since $p < 64$ ($p < 6$ in the example), it always fits into 64 bits.

`sh` is the number $64 - p$ of unused bits, here $6 - p = 2$.

The *round bit* is bit $p + 1$ from the sum, the *sticky bit* is always zero.

We might have an *overflow*, but no *underflow*.

The function mpfr_sub

The function mpfr_sub(a, b, c) works as follows ($a \leftarrow b - c$):

- first check for singular values ($NaN, \pm Inf, \pm 0$);
- if b and c have different signs, call the addition code;
- if b and c have same precision, call mpfr_sub1sp;
- otherwise call the generic code mpfr_sub1.

The function mpfr_sub1sp

- if $p < 64$, call mpfr_sub1sp1;
- if $p = 64$, call mpfr_sub1sp1n;
- if $64 < p < 128$, call mpfr_sub1sp2;
- otherwise execute the generic code for the subtraction of operands of same precision.

Note: as for addition, prefer $p = 127$ to $p = 128$ if possible.

The function mpfr_sub1sp1

- if exponents differ, swap *b* and *c* if necessary, so that $e_b \geq e_c$;
- case 1: $e_b = e_c$;
- case 2: $e_b > e_c$.

Case 1, $e_b = e_c$:

$$b = \boxed{110100}$$

$$c = \boxed{111000}$$

compute $bp[0] - cp[0]$ and store the result in $ap[0]$, which then equals $bp[0] - cp[0] \bmod 2^{64}$;

if $ap[0] = 0$, the result is 0;

if $ap[0] > bp[0]$, a borrow occurred, we thus have $|c| > |b|$: change $ap[0]$ into $-ap[0]$ and change the sign of a;

otherwise no borrow occurred, thus $|c| < |b|$;

compute the number k of leading zeros of $ap[0]$, shift $ap[0]$ by k bits to the left and decrease the exponent by k;

in this case the *round bit* and *sticky bit* are always 0.

We might have an *underflow*, but no *overflow*: $|a| \leq \max(|b|, |c|)$.

The function mpfr_mul(a,b,c)

$$a \leftarrow \circ(b \cdot c)$$

- if $p_a = p_b = p_c < 64$, call mpfr_mul_1;
- if $p_a = p_b = p_c = 64$, call mpfr_mul_1n;
- if $64 < p_a = p_b = p_c < 128$, call mpfr_mul_2;
- otherwise use the generic code.

The function mpfr_mul_1

$$a \leftarrow \circ(b \cdot c)$$

a, b, c: at most one *limb* (minus 1 bit):

$$h \cdot 2^{64} + \ell \leftarrow bp[0] \cdot cp[0] \qquad \text{(umul\_ppmm)}$$

Since $2^{63} \leq bp[0], cp[0] < 2^{64}$, we have $2^{62} \leq h$.

If $h < 2^{63}$, shift h, ℓ of one bit to the left, and decrease the exponent.

The *round bit* is bit $p + 1$ of h ($p < 64$).

The *sticky bit* is formed by the remaining bits from h (none if $p = 63$) and those of ℓ.

Both *underflow* and *overflow* might happen.

Beware: MPFR considers *underflow after* rounding (with an infinite exponent range).

Underflow before vs after rounding

Assume $bc = 0.\underbrace{111...111}_{p \text{ bits}}101 \cdot 2^{E_{\min}-1}$.

With underflow before rounding, there is an underflow since the exponent of bc is $E_{\min} - 1$.

With underflow after rounding, and rounding to nearest, $\circ(bc) = 0.100...000 \cdot 2^{E_{\min}}$, and there is no underflow since the exponent of $\circ(bc)$ is $E_{\min}$.

The function mpfr_div(a,b,c)

$$a \leftarrow \circ(b/c)$$

- if $p_a = p_b = p_c < 64$, call mpfr_div_1;
- if $p_a = p_b = p_c = 64$, call mpfr_div_1n;
- if $64 < p_a = p_b = p_c < 128$, call mpfr_div_2;
- otherwise use the generic code.

The function mpfr_div_1

$$a \leftarrow \circ(b/c)$$

We have $p_a = p_b = p_c < 64$:

1. $bp[0] \geq cp[0]$: one extra quotient bit;
2. $bp[0] < cp[0]$: no extra quotient bit.

Algorithm DivApprox1

```
Input: integers $u, v$ with $0 \le u < v$ and $\beta/2 \le v < \beta$.
Output: integer $q$ approximating $u\beta/v$.
1: compute an approximate inverse $i$ of $v$, verifying
       $$i \le \lfloor (\beta^2 - 1)/v \rfloor - \beta \le i + 1$$
2: $q = \lfloor iu/\beta \rfloor + u$
```

Note: here we have $\beta = 2^{64}$.

The computation of the approximate inverse is done by a variant of the GMP macro `invert_limb` (Möller and Granlund, *Improved division by invariant integers*, IEEE TC, 2011).

Theorem

The approximate quotient computed by Algorithm DivApprox1 satisfies

$$q \leq \lfloor \frac{u\beta}{v} \rfloor \leq q + 2.$$

Consequence: we can determine the correct rounding of u/v, except if the last `sh-1` bits from q are 000..000, 111..111 or 111..110.

In this (rare) case, to improve the worst case latency, we start from the approximation q.

The function mpfr_sqrt(r,u)

$$r \leftarrow \circ(\sqrt{u})$$

- if $p_r = p_u < 64$, call mpfr_sqrt1;
- if $p_r = p_u = 64$, call mpfr_sqrt1n;
- if $64 < p_r = p_u < 128$, call mpfr_sqrt2;
- otherwise use the generic code.

Algorithm RecSqrtApprox1

Input: integer d with $2^{62} \le d < 2^{64}$.
Output: integer v_3 approximating $s = \lfloor 2^{96}/\sqrt{d} \rfloor$.

```
1: $d_{10} = \lfloor 2^{-54} d \rfloor + 1$
2: $v_0 = \lfloor \sqrt{2^{30}/d_{10}} \rfloor$                    (*table lookup*)
3: $d_{37} = \lfloor 2^{-27} d \rfloor + 1$
4: $e_0 = 2^{57} - v_0^2 d_{37}$
5: $v_1 = 2^{11} v_0 + \lfloor 2^{-47} v_0 e_0 \rfloor$
6: $e_1 = 2^{79} - v_1^2 d_{37}$
7: $v_2 = 2^{10} v_1 + \lfloor 2^{-70} v_1 e_1 \rfloor$
8: $e_2 = 2^{126} - v_2^2 d$
9: $v_3 = 2^{33} v_2 + \lfloor 2^{-94} v_2 e_2 \rfloor$
```

Remark: if a *table lookup* is faster than a multiplication, we might tabulate v_0^2 at step 4.

Theorem

The value v_3 returned by RecSqrtApprox1 differs by at most 8 from the inverse square root:

$$v_3 \leq s := \lfloor 2^{96}/\sqrt{d} \rfloor \leq v_3 + 8.$$

Algorithm SqrtApprox1

Input: integer n with $2^{62} \leq n < 2^{64}$.
Output: integer r_0 approximating $\sqrt{2^{64}n}$.

```
1: compute an integer $x$ approximating $2^{63}/\sqrt{n}$ with
       $$x \leq 2^{63}/\sqrt{n}$$
2: $y = \lfloor \sqrt{n} \rfloor$        (reusing the approximation $x$)
3: $z = n - y^2$
4: $t = \lfloor 2^{-32} xz \rfloor$
5: $r_0 = y \cdot 2^{32} + t$
```

Theorem

If the approximation x at step 1 is the value v_2 of Algorithm RecSqrtApprox1, then Algorithm SqrtApprox1 returns r_0 such that

$$r_0 \leq \lfloor \sqrt{2^{64}n} \rfloor \leq r_0 + 7.$$

The function mpfr_sqrt1

Input: $2^{63} \leq u < 2^{64}$ representing a number of $p < 64$ bits (most significant bit set to 1).

- if the associated exponent is odd, shift u by one bit to the right;
- now $2^{62} \leq u < 2^{64}$. Call `__gmpfr_sqrt_limb_approx`, which implements SqrtApprox1, and computes r_0 such that

$$r_0 \leq \lfloor \sqrt{2^{64} u} \rfloor \leq r_0 + 7;$$

- if the `sh-1` least significant bits of r_0 are not 000..000, 111..111 (-1), 111..110 (-2), ..., 111..011 (-5), 111..010 (-6), 111..001 (-7), then we can determine the correct rounding;
- otherwise we compute $r = r_0 + i$ with $0 \leq i \leq 7$ such that

$$r = \lfloor \sqrt{2^{64} u} \rfloor$$

which is equivalent to:

$$0 \leq 2^{64} u - r^2 \leq 2r.$$

MPFR 3.1.5 compared to MPFR 4.0-dev

`araignee.loria.fr`, Intel(R) Core(TM) i5-6500 CPU @ 3.20GHz, with GMP 6.1.2 and GCC 6.3.0.
GMP and MPFR are configured with `-disable-shared`.

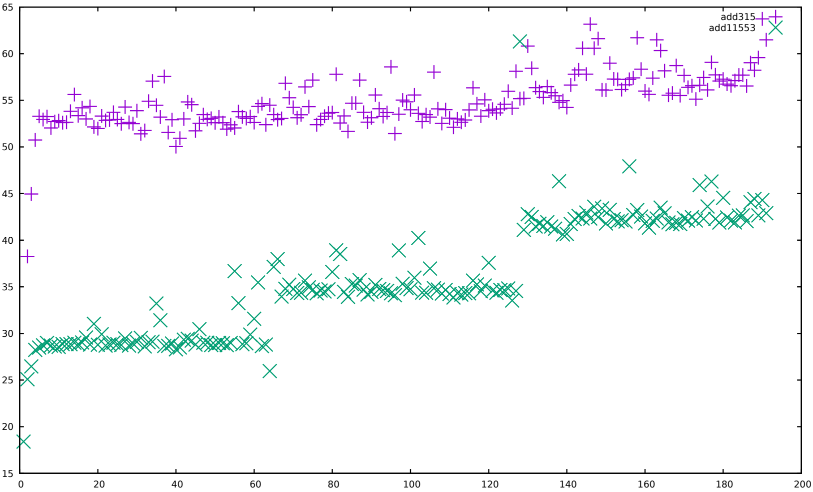
MPFR 3.1.5

bits	53	113
mpfr_add	52	53
mpfr_sub	49	52
mpfr_mul	49	63
mpfr_sqr	74	79
mpfr_div	134	146
mpfr_sqrt	171	268

MPFR 4.0-dev

bits	53	113
mpfr_add	**25**	**29**
mpfr_sub	**28**	**33**
mpfr_mul	**23**	**33**
mpfr_sqr	**21**	**29**
mpfr_div	**56 (64)**	**77 (102)**
mpfr_sqrt	**55 (56)**	**84 (133)**

Timings are in cycles.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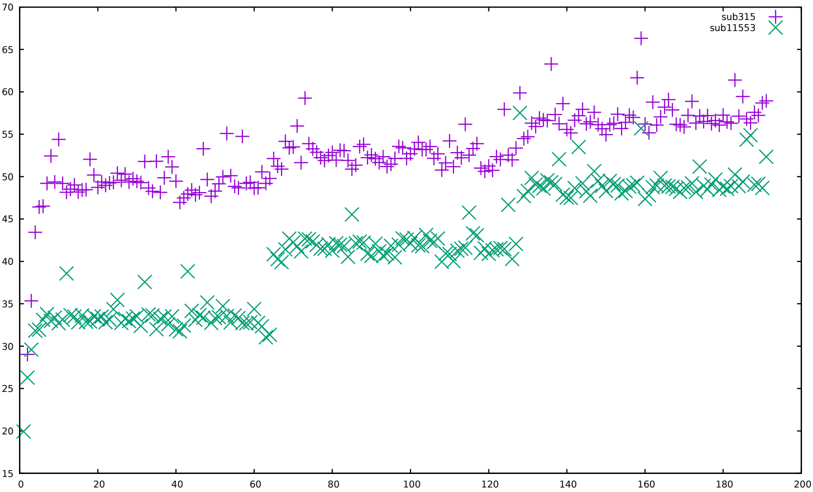
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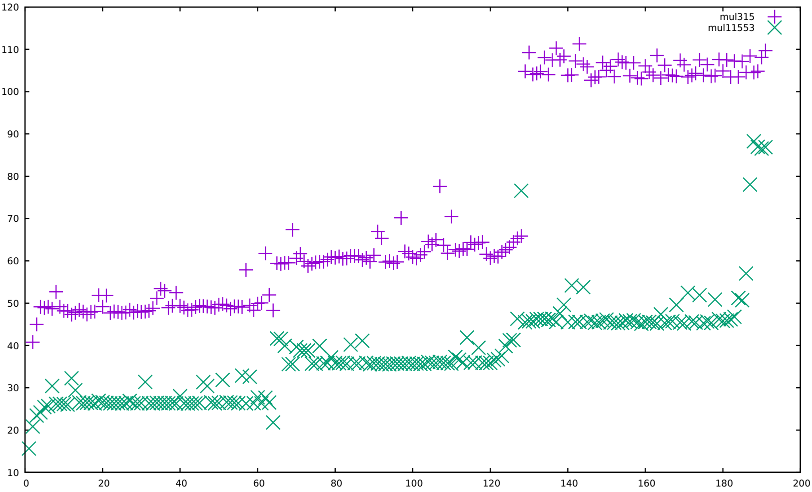
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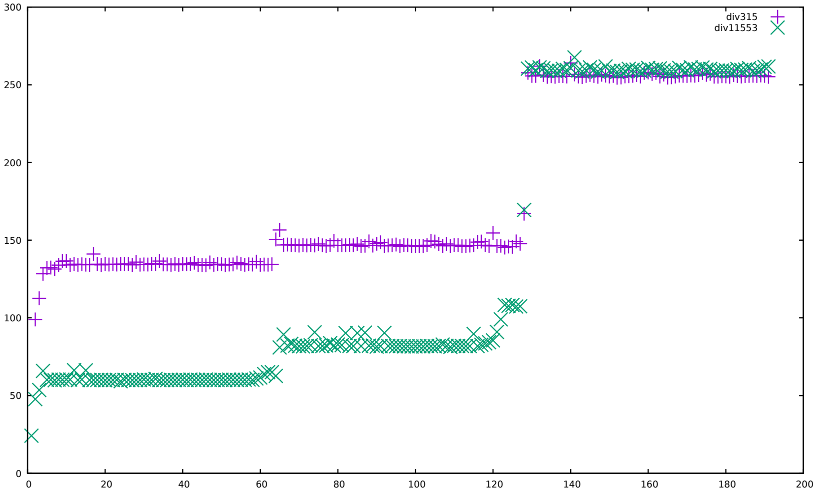
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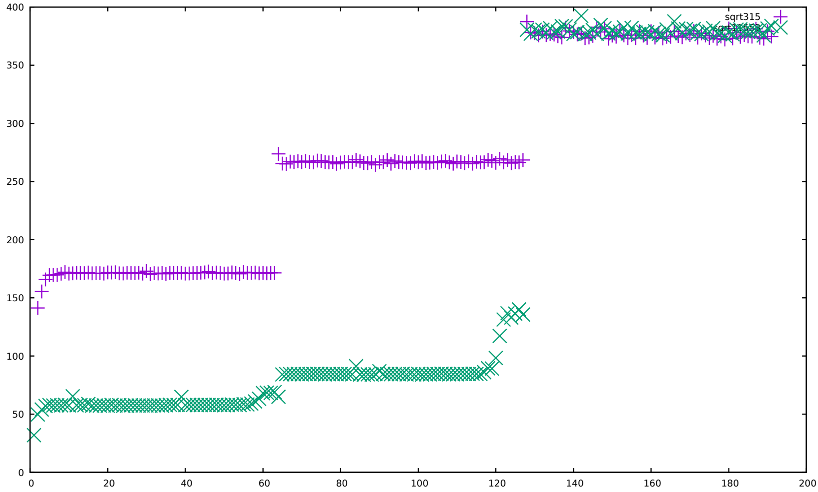
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Conclusion

Speedup by a factor 2 or more until 127 bits for $\div$, $\sqrt{}$, until 191 bits for $+, -, \times$.

Will be available in MPFR 4, already available in the development version!

New algorithms for division and square root, with small and tight error bounds.

Also in paper (and MPFR 4): new RNDF rounding mode (faithful)

Detailed and public code and proofs, ready for a formal proof. Any volunteers to find a bug?