Logic

I. Give a valuation that satisfies the following propositional formula.

$$(A \vee B) \wedge (A \rightarrow B) \wedge (A \rightarrow \neg B)$$

II. Consider a first-order language without functional symbol, and whose only relational symbols are R and Q. Give a counter-model to the following formula.

$$(\forall x.R(x) \vee Q(x)) \rightarrow ((\forall x.R(x)) \vee (\forall x.Q(x)))$$

III. Give a sequent-calculus derivation of the following formula scheme.

$$(\neg \forall x.\alpha[x] \rightarrow \beta) \rightarrow ((\exists x.\alpha[x]) \wedge \neg \beta) \qquad \text{where } x \notin \mathrm{FV}(\beta)$$

IV. Consider the following possible natural-deduction derivation.

$$\dfrac{\dfrac{2 : \exists x.R(x) \qquad \dfrac{1 : R(x)}{\forall x.R(x)}\ \forall\text{-intro}}{\forall x.R(x)}\ (1)\ \exists\text{-elim}}{(\exists x.R(x)) \rightarrow (\forall x.R(x))}\ (2)\ \rightarrow\text{-intro}$$

Is this derivation correct? Justify your answer.

V. Given two first-order terms t and u, we write $t[x{:=}u]$ to denote the term obtained by substituting every occurrence of the first-order variable x in t by u. For instance, for $t = f(x, g(c, x), y)$, we have that

$$t[x{:=}u] = f(u, g(c, u), y)$$

Let $\mathcal{M} = \langle D, I \rangle$ be a first-order model. Establish the following property:

$$\llbracket t[x{:=}u] \rrbracket^{\mathcal{M}} \eta = \llbracket t \rrbracket^{\mathcal{M}} \eta[(\llbracket u \rrbracket^{\mathcal{M}} \eta)/x]$$

Hint: proceed by induction on the definition of a term.