Logics

I. Decide whether the following formulas are valid or not (justify your answer).

1. $(A \wedge \neg B) \vee (A \rightarrow B)$
2. $(A \vee B) \rightarrow (A \rightarrow B)$

II. Consider a first-order language without functional symbol, and whose only relational symbols are R and Q. Give a counter-model to the following formula.

$$((\exists x.R[x]) \wedge (\exists x.Q[x])) \rightarrow (\exists x.(R[x] \wedge Q[x]))$$

III. Consider the linear implicative fragment of the propositional calculus. The formulas of this fragment are inductively defined as follows:

i. Every propositional variable a is a formula.

ii. If α and β are formulas then $(\alpha \rightarrow \beta)$ is a formula.

The notion of derivability is then defined by means of the following sequent-calculus:

$$a \vdash a \quad \text{(where } a \text{ is a propositional variable)}$$

$$\frac{\Gamma \vdash \alpha \qquad \Delta, \beta \vdash \gamma}{\Gamma, \Delta, (\alpha \rightarrow \beta) \vdash \gamma} \qquad\qquad \frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash (\alpha \rightarrow \beta)}$$

where Γ and Δ denote multisets of formulas, and "Γ, Δ" denotes the union of these multisets.

Establish the following property:

If $\Gamma \vdash \alpha$ and $\Delta, \alpha \vdash \beta$ then $\Gamma, \Delta \vdash \beta$.

Hint: proceed by induction on the derivation of $\Gamma \vdash \alpha$, using an auxiliary induction on the derivation of $\Delta, \alpha \vdash \beta$ when needed.