Logics

I. Decide whether the following formulas are valid or not (justify your answer).

1. \( (A \land \neg B) \lor (A \rightarrow B) \)
2. \( (A \lor B) \rightarrow (A \rightarrow B) \)

II. Consider a first-order language without functional symbol, and whose only relational symbols are \( R \) and \( Q \). Give a counter-model to the following formula.

\[
((\exists x. R[x]) \land (\exists x. Q[x])) \rightarrow (\exists x. (R[x] \land Q[x]))
\]

III. Consider the linear implicative fragment of the propositional calculus. The formulas of this fragment are inductively defined as follows:

i. Every propositional variable \( a \) is a formula.

ii. If \( \alpha \) and \( \beta \) are formulas then \((\alpha \rightarrow \beta)\) is a formula.

The notion of derivability is then defined by means of the following sequent-calculus:

\[
\frac{a \vdash a}{\Gamma \vdash a} \quad \text{(where } a \text{ is a propositional variable)}
\]

\[
\frac{\Gamma \vdash \alpha \quad \Delta, \beta \vdash \gamma}{\Gamma, \Delta, (\alpha \rightarrow \beta) \vdash \gamma} \quad \frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash (\alpha \rightarrow \beta)}
\]

where \( \Gamma \) and \( \Delta \) denote multisets of formulas, and “\( \Gamma, \Delta \)” denotes the union of these multisets.

Establish the following property:

If \( \Gamma \vdash \alpha \) and \( \Delta, \alpha \vdash \beta \) then \( \Gamma, \Delta \vdash \beta \).

Hint: proceed by induction on the derivation of \( \Gamma \vdash \alpha \), using an auxiliary induction on the derivation of \( \Delta, \alpha \vdash \beta \) when needed.