Logics

I. Decide whether the following formulas are valid or not (justify your answer).

1. $((A \to B) \to A) \to A$
2. $B \vee (A \to B)$

II. Consider a first-order language without functional symbol, and whose only relational symbols are Q and R. Give a counter-model to the following formula.

$$(\forall x.(Q[x] \to R[x])) \to (\exists x.(Q[x] \wedge R[x]))$$

III. Let $\mathcal{A}$ be the set of propositional variables. We consider the linear implicative fragment of the propositional calculus. The formulas of this fragment are inductively defined as follows:

i. If $a \in \mathcal{A}$ then a is a formula.

ii. If α and β are formulas then $(\alpha \to \beta)$ is a formula.

The notion of derivability is then defined by means of the following axiomatic system:

Axiom schemes

$$\alpha \to \alpha$$

$$(\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma$$

$$(\alpha \to \beta \to \gamma) \to \beta \to \alpha \to \gamma$$

Inference rule

$$\frac{\alpha \qquad \alpha \to \beta}{\beta}$$

Let a valuation $\xi : \mathcal{A} \to \mathbb{Z}$ be a function that assigns an integer to each propositional variable, and define the following interpretation:

i. $[\![a]\!]_\xi = \xi(a) \quad$ (for $a \in \mathcal{A}$)

ii. $[\![\alpha \to \beta]\!]_\xi = [\![\beta]\!]_\xi - [\![\alpha]\!]_\xi$

III.A. Establish the following property:

> *For every implicative formula α and for every valuation ξ, if α is derivable in the linear implicative fragment of the propositional calculus, then* $[\![\alpha]\!]_\xi = 0$.

III.B. Establish that the following two formulas are not derivable in the the linear implicative fragment of the propositional calculus:

$$\alpha \to \beta \to \alpha$$

$$\alpha \to (\alpha \to \alpha \to \beta) \to \beta$$