Logics

I. Decide whether the following formulas are valid or not (justify your answer).

1. \(((A \to B) \to A) \to A\)
2. \(B \lor (A \to B)\)

II. Consider a first-order language without functional symbol, and whose only relational symbols are \(Q\) and \(R\). Give a counter-model to the following formula.

\[(\forall x. (Q[x] \to R[x])) \to (\exists x. (Q[x] \land R[x]))\]

III. Let \(\mathcal{A}\) be the set of propositional variables. We consider the linear implicative fragment of the propositional calculus. The formulas of this fragment are inductively defined as follows:

i. If \(a \in \mathcal{A}\) then \(a\) is a formula.

ii. If \(\alpha\) and \(\beta\) are formulas then \((\alpha \to \beta)\) is a formula.

The notion of derivability is then defined by means of the following axiomatic system:

\begin{align*}
\text{Axiom schemes} & \quad \alpha \to \alpha \\
& \quad (\alpha \to \beta) \to (\beta \to \gamma) \to \alpha \to \gamma \\
& \quad (\alpha \to \beta \to \gamma) \to \beta \to \alpha \to \gamma
\end{align*}

\begin{align*}
\text{Inference rule} & \quad \alpha \to \beta \\
& \quad \frac{\alpha}{\beta}
\end{align*}

Let a valuation \(\xi : \mathcal{A} \to \mathbb{Z}\) be a function that assigns an integer to each propositional variable, and define the following interpretation:

i. \([a]_\xi = \xi(a)\) (for \(a \in \mathcal{A}\))

ii. \([\alpha \to \beta]_\xi = [\beta]_\xi - [\alpha]_\xi\)
III.A. Establish the following property:

For every implicative formula $\alpha$ and for every valuation $\xi$, if $\alpha$ is derivable in the linear implicative fragment of the propositional calculus, then $[\alpha]_\xi = 0$.

III.B. Establish that the following two formulas are not derivable in the linear implicative fragment of the propositional calculus:

$$\alpha \rightarrow \beta \rightarrow \alpha$$

$$\alpha \rightarrow (\alpha \rightarrow \alpha \rightarrow \beta) \rightarrow \beta$$