Logics

I. Decide whether the following formula is valid or not (justify your answer).

$$(((a \rightarrow b) \rightarrow a) \land (a \rightarrow c)) \rightarrow c$$

II. Prove that the following formula is derivable.

$$\neg(\alpha \lor \beta) \rightarrow (\neg\alpha \land \neg\beta)$$

III. Consider a first-order language without functional symbol, and whose only relational symbol is $R$. Then, consider the formula $\phi$, which is defined as follows:

$$\phi = \phi_0 \rightarrow (\phi_1 \rightarrow \phi_2)$$

where:

$$\phi_0 = \forall x. (\exists y. R(x, y))$$
$$\phi_1 = \forall x. (\forall y. (\forall z. R(x, y) \rightarrow (R(y, z) \rightarrow R(x, z)))$$
$$\phi_2 = \exists x. R(x, x)$$

Define a model $\mathcal{M} = \langle D, I \rangle$ as follows:

$$D = \mathbb{N}$$
$$I(R) = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$$

Show that $\mathcal{M}$ is a counter-model of $\phi$.

IV. Let $\mathcal{F}$ be a ranked alphabet of function symbols, and let $\mathcal{X}$ be an alphabet of variables. The set of terms $\mathcal{T}$, built over $\mathcal{F}$ and $\mathcal{X}$, is inductively defined as follows:

1. If $x \in \mathcal{X}$, then $x \in \mathcal{T}$;
2. If $c \in \mathcal{F}$ is of arity 0, then $c \in \mathcal{T}$;
3. If $f \in \mathcal{F}$ is of arity $n$, and $t_1, \ldots, t_n \in \mathcal{T}$, then $f(t_1, \ldots, t_n) \in \mathcal{T}$.

Let $\mathcal{M} = \langle D, I \rangle$ be a model, and $\rho : D^\mathcal{X}$ be a valuation. The interpretation of a term $t$ is inductively defined as follows:
1. \([x] \rho = \rho(x), \) for \(x \in \mathcal{X}\);
2. \([c] \rho = I(c), \) for \(c \in \mathcal{F}\) of arity 0;
3. \([f(t_1, \ldots, t_n)] \rho = I(f)([t_1] \rho, \ldots, [t_n] \rho), \) for \(f \in \mathcal{F}\) of arity \(n\).

Let \(\rho \in D^{\mathcal{X}}\) be a valuation, and let \(a \in D\) and \(x \in \mathcal{X}\). The valuation \(\rho[x:=a]\) is defined as follows:
\[
\rho[x:=a](y) = \begin{cases} 
  a & \text{if } x = y \\
  \rho(y) & \text{if } x \neq y
\end{cases}
\]

Finally, let \(t\) and \(u\) be terms, and \(x\) be a variable. The substitution of \(x\) by \(u\) in \(t\), in notation \(t[u/x]\), is inductively defined as follows:
1. \(x[u/x] = u\);
2. \(y[u/x] = y\), for \(y \in \mathcal{X}\) and \(y \neq x\);
3. \(c[u/x] = c\), for \(c \in \mathcal{F}\) of arity 0;
4. \(f(t_1, \ldots, t_n)[u/x] = f(t_1[u/x], \ldots, t_n[u/x])\), for \(f \in \mathcal{F}\) of arity \(n\).

Let \(x\) be a variable, \(u\) be a term, and \(\rho\) be a valuation. Show that for every term \(t\):
\[
[t[u/x]] \rho = [t] \rho[x:=u] \rho
\]

Hint: proceed by induction on the structure of \(t\).