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On the expressive power of the Lambek calculus extended with a structural modality

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We consider **EL**, the product-free associative Lambek calculus (Lambek, 1958) extended with a structural modality à la Girard (Girard, 1987), which allows the left strucrural rules (weakening, contraction, and exchange) to be performed in a controlled way. We show that any recursively enumerable language can be described by a categorial grammar based on **EL**. As an immediate corollary, we get the undecidability of **EL**.

1.1 Introduction

Lambek's calculus of syntactic types, introduced in the late fifties (Lambek, 1958), remains one of the core components of modern categorial grammars. Nevertheless, it appeared in the eighties that the Lambek calculus was not powerful enough to handle several linguistic phenomena. This was theoretically confirmed by Pentus (1993), who proved that the Lambek grammars are context-free in their weak generative capacity. Consequently, several extensions or variants of the Lambek calculus have been proposed in the litterature, among which Moortgat's multimodal categorial logic is maybe the most prominent (Moortgat, 1997).

Typically, these extensions introduce new connectives that feature some forms of structural rules. In this paper, we study such an ex-

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tension, which consists of the product-free associative Lambek calculus provided with the *of-course* exponential of linear logic (Girard, 1987). This modal connective allows Gentzen's structural rules (weakening, exchange, and contraction) to be performed. The resulting calculus, which we call **EL**, is such that any recursively enumerable language can be described by an **EL**-categorial grammar. Consequently, we get that **EL** is undecidable.

Similar results may be found in the litterature. Lincoln et al. (1992) establish the undecidability of circular multiplicative linear logic extended with the same kind of modality, by encoding semi-thue systems in a straightforward way. Nevertheless, this encoding makes an essential use of the non-commutative product. Consequently, our result is a refinement of theirs since we are working in a product-free calculus. Buszkowski (1982) shows that the product-free Lambek calculus provided with proper axioms allows any recursively enumerable language to be described. This result may be considered equivalent to ours (using the generalised cut elimination theorem of Lincoln et al. (1992)). However, our proof is much simpler than Buszkowski's because the encoding we use is a substantial simplification of his (where Buszkowski uses $3q^3n^5 + q^3n^4 + (2q^3 + 3q^2)n^3 + 4q^2n^2 + p$ proper axioms, we use $2n + p + q$ modal formulas—for n, p, q being constants characterising the size of the initial phrase structure grammar to be encoded). In addition, we use a transparent semantic argument in order to establish the faithfulness of our encoding while both Lincoln et al. (1992) and Buszkowski (1982) use proof-theoretic machinery.

EL may be easily defined using Moortgat's multimodal categorial logic. Consequently, we also get as a corollary the turing-completeness of multimodal categorial grammars. This result has been independently proved by Carpenter (1999) and Moot (2002). Nevertheless, our result is slightly different from theirs. Indeed, both Carpenter's and Moot's proofs use an unbounded number of modes and structural rules, while only two modes and a finite number of structural rules are needed in order to define **EL** as a multimodal categorial system.

The paper is organized as follows. Section 2 introduces **EL** and the notion of **EL**-grammar. In Section 3, we define a notion of phase semantics for **EL**, and we prove its soundness in Section 4. In Section 5 we show how to encode any phrase structure grammar as an **EL**-grammar. Finally, in Section 6, we prove the faithfulness of this encoding by means of the notion of phase semantics developed in Section 3.

1.2 Lambek calculus and categorial grammars

This section introduces the notion of **EL**-grammar, which is a form of categorial grammars based on an extension of the associative Lambek calculus (Lambek, 1958). This extension, which we call **EL** (for *Exponential Lambek calculus*), amounts to the product-free associative Lambek calculus provided with a modality that enables the structural rules of weakening, contraction, and exchange.

Let $\mathcal{A}$ be a set of atomic formulas. The formulas of **EL**, built upon the set $\mathcal{A}$, obey the following syntax:

$$\mathcal{F} \ ::= \ \mathcal{A} \mid \mathcal{F} \setminus \mathcal{F} \mid \mathcal{F}/\mathcal{F} \mid \mathbf{!}\mathcal{F}$$

The formulas of of the form $A \backslash B$ correspond to direct implications (i.e., A *implies* B), and the formulas of the form A/B to retro-implication (i.e., A *is implied by* B). The formulas of the form $\mathbf{!}A$ are called exponential formulas. We let Roman uppercase letters range over formulas, and Greek uppercase letters over sequences of formulas. Given a set of atomic formulas $\mathcal{A}$, We write $\mathcal{F}_{\mathcal{A}}$ to denote the set of formulas built upon $\mathcal{A}$. The elements of $\mathcal{F}_{\mathcal{A}}$ will also be called *types* or *syntactic categories* in order to stress the proof-theoretic or the categorial nature of **EL**.

The deduction relation of **EL** is specified by means of the following intutionistic sequent calculus.

Identity rules

$$A \vdash A \quad (\text{ident}) \qquad\qquad \frac{\Gamma \vdash A \quad \Delta_1, A, \Delta_2 \vdash B}{\Delta_1, \Gamma, \Delta_2 \vdash B} \quad (\text{cut})$$

Logical rules

$$\frac{\Gamma \vdash A \quad \Delta_1, B, \Delta_2 \vdash C}{\Delta_1, \Gamma, A \setminus B, \Delta_2 \vdash C} \quad (\setminus \text{left}) \qquad\qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \setminus B} \quad (\setminus \text{right})$$

$$\frac{\Gamma \vdash A \quad \Delta_1, B, \Delta_2 \vdash C}{\Delta_1, B/A, \Gamma, \Delta_2 \vdash C} \quad (/ \text{left}) \qquad\qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash B/A} \quad (/ \text{right})$$

$$\frac{\Gamma, A, \Delta \vdash B}{\Gamma, \mathbf{!}A, \Delta \vdash B} \quad (\mathbf{!} \text{ left}) \qquad\qquad \frac{\mathbf{!}\Gamma \vdash A}{\mathbf{!}\Gamma \vdash \mathbf{!}A} \quad (\mathbf{!} \text{ right})$$

where, in Rule (**!** right), $\mathbf{!}\Gamma$ stands for a sequence of exponential formulas.

Structural rules

$$\frac{\Gamma \vdash B}{!A, \Gamma \vdash B}\ \text{(weakening)} \qquad \frac{!A, !A, \Gamma \vdash B}{!A, \Gamma \vdash B}\ \text{(contraction)}$$

$$\frac{\Gamma, B, !A, \Delta \vdash C}{\Gamma, !A, B, \Delta \vdash C}\ \text{(exchange}_1\text{)} \qquad \frac{\Gamma, !B, A, \Delta \vdash C}{\Gamma, A, !B, \Delta \vdash C}\ \text{(exchange}_2\text{)}$$

When a sequent $\Gamma \vdash A$ is derivable according to the above sequent calculus, we write

$$\vdash (\Gamma \vdash A).$$

We end this section by giving the definition of an **EL**-grammar.

Definition 1 An **EL**-grammar $G = \langle \Sigma, \mathcal{A}, \mathcal{L}, S \rangle$ is a quadruple where:

1. Σ is a finite set of terminal symbols;
2. $\mathcal{A}$ is a finite set of atomic types;
3. $\mathcal{L} : \Sigma \longrightarrow 2^{\mathcal{F}_{\mathcal{A}}}$ is a *lexicon* that assigns to each terminal symbol a finite set of types built upon $\mathcal{A}$;
4. $S \in \mathcal{F}_{\mathcal{A}}$ is a distinguished type, called the *initial type* of the grammar.

A word $a_0 a_1 \ldots a_n \in \Sigma^*$ belongs to the language generated by G if and only if there exist $A_0 \in \mathcal{L}(a_0), A_1 \in \mathcal{L}(a_1), \ldots A_n \in \mathcal{L}(a_n)$ such that

$$\vdash (A_0, A_1, \ldots A_n \vdash S).$$

1.3 Phase semantics

EL may be studied as a pure logical system. In particular, one may provide mathematical semantics for it. This is the purpose of this section, where we define a notion of model for **EL** by adapting Girard's phase semantics (Girard, 1987). We first define a notion of phase space.

Definition 2 (Phase space) A phase space $\mathbf{P} = \langle M, \cdot, \mathbf{1}, \mathcal{F} \rangle$ is a structure such that:

1. $\langle M, \cdot, \mathbf{1} \rangle$ is a monoid;
2. $\mathcal{F}$, which is called the set of facts, is a set of subsets of M;
3. the set of facts $\mathcal{F}$ is closed by arbitrary intersection;
4. $M \in \mathcal{F}$;
5. $\forall a \in M, \forall F \in \mathcal{F}$.
   (a) $\exists G \in \mathcal{F}, \forall b \in M, a \cdot b \in F$ iff $b \in G$,
   (b) $\exists G \in \mathcal{F}, \forall b \in M, b \cdot a \in F$ iff $b \in G$.

EL being an intuitionistic system, the above definition is inspired by Okada's notion of an intutionistic phase space (Okada, 1999). The main difference is that we do not require M to be a *commutative* monoid because **EL** is a non-commutative system.

Phase semantics interpret the formulas as subsets of M, or more precisely, as facts. The next definition introduces several operations on subsets of M, which will be useful when defining the interpretation of the formulas.

Definition 3 (Operations on subsets of M) Let $\mathbf{P} = \langle M, \cdot, \mathbf{1}, \mathcal{F}\rangle$ be a phase space. Let $A, B \subset M$. We define the following operations:

1. $A \cdot B \;=\; \{x \in M \mid \exists a \in A, \exists b \in B,\, x = a \cdot b\}$;
2. $A \to B \;=\; \{x \in M \mid \forall a \in A,\, a \cdot x \in B\}$;
3. $B \leftarrow A \;=\; \{x \in M \mid \forall a \in A,\, x \cdot a \in B\}$;
4. $\mathrm{C}_{\mathcal{F}}A \;=\; \bigcap\{F \in \mathcal{F} \mid A \subset F\}$.

The next two lemmas establish properties of the above operations.

Lemma 1 *Let* $\mathbf{P} = \langle M, \cdot, \mathbf{1}, \mathcal{F}\rangle$ *be a phase space and* $A, B, C, D \subset M$.

1. $A \cdot B \subset C$ *if and only if* $B \subset (A \to C)$.
2. $A \cdot B \subset C$ *if and only if* $A \subset (C \leftarrow B)$.
3. *if* $A \subset C$ *and* $B \subset D$ *then* $(A \cdot B) \subset (C \cdot D)$.
4. $A \in \mathcal{F}$ *if and only if* $\mathrm{C}_{\mathcal{F}}A \subset A$.

Proof.

1. $A \cdot B \subset C$ if and only if $\forall b \in B, \forall a \in A, a \cdot b \in C$ if and only if $\forall b \in B, b \in (A \to C)$ if and only if if $B \subset (A \to C)$.
2. The proof is similar to the one of Property 1.
3. Suppose that $\forall a \in A, a \in C$ and that $\forall b \in B, b \in D$. Obviously, $\forall a \in A, \forall b \in B, a \cdot b \in C \cdot D$.
4. If $A \in \mathcal{F}$, it is obvious that $\mathrm{C}_{\mathcal{F}}A = A$. Conversely, it is immediate that $\mathrm{C}_{\mathcal{F}}A \subset A$ implies $\mathrm{C}_{\mathcal{F}}A = A$. But then, since $\mathcal{F}$ is closed by arbitrary intersection, we have $A \in \mathcal{F}$.

□

Lemma 2 *Let* $\mathbf{P} = \langle M, \cdot, \mathbf{1}, \mathcal{F}\rangle$ *be a phase space. If* $A \subset M$, *and* $F \in \mathcal{F}$ *then:*

1. $(A \to F) \in \mathcal{F}$;
2. $(F \leftarrow A) \in \mathcal{F}$.

Proof. We prove Property 1, the proof of Property 2 being similar.

By Lemma 1, Property 4, we have to show that $\mathrm{C}_{\mathcal{F}}(A \to F) \subset (A \to F)$. Suppose that $x \in \mathrm{C}_{\mathcal{F}}(A \to F)$ or, equivalently, suppose that:

$$\forall H \in \mathcal{F},\ \text{if } (A \to F) \subset H \text{ then } x \in H \qquad \text{(a)}$$

Suppose also that $a \in A$, we have to show that $a \cdot x \in F$. Now, by Condition 5.a in Definition 2, there exists $G \in \mathcal{F}$ such that:

$$\forall y \in M, a \cdot y \in F \text{ iff } y \in G \qquad \text{(b)}$$

By instantiating H with G in (a), we obtain:

$$\text{if } (A \to F) \subset G \text{ then } x \in G$$

Then, using (b), we obtain:

$$\text{if } (A \to F) \subset G \text{ then } a \cdot x \in F,$$

and it remains to show that $(A \to F) \subset G$. To this end, suppose that $z \in (A \to F)$, i.e., $\forall a' \in A$, $a' \cdot z \in F$. Since $a \in A$, we have that $a \cdot z \in F$. Then, by (b), we have that $z \in G$. □

The next lemma shows that $\mathrm{C}_{\mathcal{F}}$ is a monoidal closure operator.

Lemma 3 *Let* $\mathbf{P} = \langle M, \cdot, \mathbf{1}, \mathcal{F}\rangle$ *be a phase space. If* $A, B \subset M$ *then:*

1. $A \subset \mathrm{C}_{\mathcal{F}}A$;
2. *if* $A \subset B$ *then* $\mathrm{C}_{\mathcal{F}}A \subset \mathrm{C}_{\mathcal{F}}B$;
3. $\mathrm{C}_{\mathcal{F}}\mathrm{C}_{\mathcal{F}}A \subset \mathrm{C}_{\mathcal{F}}A$;
4. $(\mathrm{C}_{\mathcal{F}}A \cdot \mathrm{C}_{\mathcal{F}}B) \subset \mathrm{C}_{\mathcal{F}}(A \cdot B)$.

Proof. Properties 1 to 3 are straightforward. Consequently, we only give the proof of Property 4.

By Property 1, we have that $(A \cdot B) \subset \mathrm{C}_{\mathcal{F}}(A \cdot B)$. Then, by Property 1 of Lemma 1, we have that $B \subset (A \to \mathrm{C}_{\mathcal{F}}(A \cdot B))$ and, using Property 2 of the present lemma, we obtain that $\mathrm{C}_{\mathcal{F}}B \subset \mathrm{C}_{\mathcal{F}}(A \to \mathrm{C}_{\mathcal{F}}(A \cdot B))$. Now, by Lemma 2, $(A \to \mathrm{C}_{\mathcal{F}}(A \cdot B))$ is a fact and, consequently, $\mathrm{C}_{\mathcal{F}}(A \to \mathrm{C}_{\mathcal{F}}(A \cdot B)) = (A \to \mathrm{C}_{\mathcal{F}}(A \cdot B))$. This implies that $\mathrm{C}_{\mathcal{F}}B \subset (A \to \mathrm{C}_{\mathcal{F}}(A \cdot B))$. Then, using again Property 1 of Lemma 1, we obtain $(A \cdot \mathrm{C}_{\mathcal{F}}B) \subset \mathrm{C}_{\mathcal{F}}(A \cdot B)$. From this, by a similar reasoning (using Property 2 of Lemma 1), we obtain that $(\mathrm{C}_{\mathcal{F}}A \cdot \mathrm{C}_{\mathcal{F}}B) \subset \mathrm{C}_{\mathcal{F}}(A \cdot B)$. □

As a consequence of Property 4, we have that $\mathrm{C}_{\mathcal{F}}(\mathrm{C}_{\mathcal{F}}A \cdot \mathrm{C}_{\mathcal{F}}B) = \mathrm{C}_{\mathcal{F}}(\mathrm{C}_{\mathcal{F}}A \cdot B) = \mathrm{C}_{\mathcal{F}}(A \cdot \mathrm{C}_{\mathcal{F}}B) = \mathrm{C}_{\mathcal{F}}(A \cdot B)$. Then, it is easy to show that $\mathrm{C}_{\mathcal{F}}(\mathrm{C}_{\mathcal{F}}(A \cdot B) \cdot C) = \mathrm{C}_{\mathcal{F}}(A \cdot \mathrm{C}_{\mathcal{F}}(B \cdot C))$. This establishes that the interpretation of an antecedent, which is given in the next definition, is invariant under the associativity of ",".

Let $\mathbf{P} = \langle M, \cdot, \mathbf{1}, \mathcal{F}\rangle$ be a phase space, and consider the set

$$J = \{j \in \mathrm{C}_{\mathcal{F}}\{\mathbf{1}\} \,|\, j \cdot j = j \text{ and } \forall a \in M,\ j \cdot a = a \cdot j\}.$$

J is a submonoid of M. Indeed it is easy to prove that $1 \in J$ and $\forall a, b \in J$, $a \cdot b \in J$. Let K be some submonoid of J. The structure $\langle M, \cdot, \mathbf{1}, \mathcal{F}, K\rangle$ is called an enriched phase space.

Definition 4 (Interpretation) Let $\mathbf{P} = \langle M, \cdot, \mathbf{1}, \mathcal{F}, K\rangle$ be an enriched phase space. Let $\eta : \mathcal{A} \longrightarrow \mathcal{F}$ be a valuation that assigns a fact to each atomic formula. The interpereation $[\![A]\!]\eta$ of a formula A is inductively defined as follows:

1. $[\![a]\!]\eta \quad = \quad \eta(a);$
2. $[\![A \setminus B]\!]\eta \quad = \quad [\![A]\!]\eta \rightarrow [\![B]\!]\eta;$
3. $[\![A/B]\!]\eta \quad = \quad [\![A]\!]\eta \leftarrow [\![B]\!]\eta;$
4. $[\![\mathbf{!}A]\!]\eta \quad = \quad \mathrm{C}_{\mathcal{F}}([\![A]\!]\eta \cap K).$

The interpereation $[\![\Gamma]\!]\eta$ of an antecedent Γ is defined similarly:

1. $[\![\Gamma]\!]\eta \quad = \quad \mathrm{C}_{\mathcal{F}}\{\mathbf{1}\}$ when Γ is empty;
2. $[\![\Gamma, A]\!]\eta \quad = \quad \mathrm{C}_{\mathcal{F}}([\![\Gamma]\!]\eta \cdot [\![A]\!]\eta).$

Lemma 2 ensures that the above definition is such that any formula is interpreted as a fact.

The notion of a valid sequent is as usual. We spell out the definition for the sake of completeness.

Definition 5 (Validity) Let $\mathbf{P} = \langle M, \cdot, \mathbf{1}, \mathcal{F}, K\rangle$ be an enriched phase space, and let $\eta : \mathcal{A} \longrightarrow \mathcal{F}$ be a valuation.

1. $\mathbf{P}, \eta \models (\Gamma \vdash A)$ if and only if $[\![\Gamma]\!]\eta \subset [\![A]\!]\eta$. ($\eta$ satifies the sequent $\Gamma \vdash A$ in $\mathbf{P}$.)
2. $\mathbf{P} \models (\Gamma \vdash A)$ if and only $\mathbf{P}, \rho \models (\Gamma \vdash A)$ for any valuation ρ. (The sequent $\Gamma \vdash A$ is valid in $\mathbf{P}$.)
3. $\models (\Gamma \vdash A)$ if and only $\mathbf{Q} \models (\Gamma \vdash A)$ for any enriched phase space $\mathbf{Q}$. (The sequent $\Gamma \vdash A$ is valid.)

1.4 Soundness

In this section we prove the soundness of **EL** with respect to the semantics given by Definition 5.

We first establish some easy properties of the interpretation.

Lemma 4 *The following properties hold:*

1. *if* $[\![\Gamma_1]\!]\eta \subset [\![\Gamma_2]\!]\eta$ *then* $[\![\Delta_1, \Gamma_1, \Delta_2]\!]\eta \subset [\![\Delta_1, \Gamma_2, \Delta_2]\!]\eta.$
2. $[\![A, A \setminus B]\!]\eta \subset [\![B]\!]\eta.$
3. $[\![A/B, B]\!]\eta \subset [\![A]\!]\eta.$
4. $[\![\mathbf{!}A]\!]\eta \subset [\![A]\!]\eta.$
5. $[\![\mathbf{!}A, \Gamma]\!]\eta \subset [\![\Gamma]\!]\eta.$
6. $[\![\mathbf{!}A]\!]\eta \subset [\![\mathbf{!}A, \mathbf{!}A]\!]\eta.$
7. $[\![\mathbf{!}A, B]\!]\eta = [\![B, \mathbf{!}A]\!]\eta.$

Proof.

1. By Lemma 1, Property 3, and Lemma 3, Property 2.
2. By Lemma 1, Property 1, we have $[\![A]\!]\eta \cdot ([\![A]\!]\eta \rightarrow B]\!]\eta) \subset [\![B]\!]\eta$. Then we are done, by Lemma 3, Property 2, and Lemma 1, Property 4.
3. Similarly to the previous property, using Lemma 1, Property 2.
4. We have $[\![A]\!]\eta \cap K \subset [\![A]\!]\eta$. We conclude by Lemma 3, Property 2, and Lemma 1, Property 4.
5. Since $K \subset \mathrm{C}_{\mathcal{F}}\{1\}$, we have $[\![A]\!]\eta \cap K \subset \mathrm{C}_{\mathcal{F}}\{1\}$. Then, by Lemma 3, Properties 2 and 3, $[\![\mathbf{!}A]\!]\eta \subset \mathrm{C}_{\mathcal{F}}\{1\}$. By Lemma 1, Property 3, $[\![\mathbf{!}A]\!]\eta \cdot [\![\Gamma]\!]\eta \subset \mathrm{C}_{\mathcal{F}}\{1\} \cdot [\![\Gamma]\!]\eta$, which implies, by Lemma 3, Property 2, $[\![\mathbf{!}A, \Gamma]\!]\eta \subset \mathrm{C}_{\mathcal{F}}(\mathrm{C}_{\mathcal{F}}\{1\} \cdot [\![\Gamma]\!]\eta)$. Hence, by Lemma 3, Property 4, $[\![\mathbf{!}A, \Gamma]\!]\eta \subset \mathrm{C}_{\mathcal{F}}(\{1\} \cdot [\![\Gamma]\!]\eta)$, which implies $[\![\mathbf{!}A, \Gamma]\!]\eta \subset \mathrm{C}_{\mathcal{F}}([\![\Gamma]\!]\eta)$. Finally, by Lemma 1, Property 4, $[\![\mathbf{!}A, \Gamma]\!]\eta \subset [\![\Gamma]\!]\eta)$.
6. The elements of K being idempotent, we have $[\![A]\!]\eta \cap K \subset ([\![A]\!]\eta \cap K) \cdot ([\![A]\!]\eta \cap K)$. We conclude by Lemma 3, Properties 2 and 4.
7. K being a submonoid of J, by definition of J, we have $([\![A]\!]\eta \cap K) \cdot [\![B]\!]\eta = [\![B]\!]\eta \cdot ([\![A]\!]\eta \cap K)$. Again, we conclude by Lemma 3, Properties 2 and 4.

□

We now establish the main proposition of this section.

Proposition 5 (Soundness) $\vdash (\Gamma \vdash A)$ *implies that* $\models (\Gamma \vdash A)$.

Proof. Let **P** be some enriched phase space, and let η be some valuation. We prove that $[\![\Gamma]\!]\eta \subset [\![A]\!]\eta$ whenever $\vdash (\Gamma \vdash A)$. The proof proceeds by induction on the derivation of $\Gamma \vdash A$.

(ident). We have $[\![A]\!]\eta \subset [\![A]\!]\eta$.

(cut). By induction hypothesis, using Lemma 4, Property 1.

(\ left). By induction hypothesis, using Lemma 4, Properties 2 and 1.

(\ right). By induction hypothesis, we have $[\![A, \Gamma]\!]\eta \subset [\![B]\!]\eta$. Now, suppose $x \in [\![\Gamma]\!]\eta$. For all $a \in [\![A]\!]\eta$, $a \cdot x \in [\![A]\!]\eta \cdot [\![\Gamma]\!]\eta$. This implies, by Lemma 3, Property 1, $a \cdot x \in [\![A, \Gamma]\!]\eta$. Then, using the induction hypothesis, $a \cdot x \in [\![B]\!]\eta$, which implies $x \in [\![A \setminus B]\!]\eta$.

(/ left). By induction hypothesis, using Lemma 4, Properties 3 and 1.

(/ right). Similarly to (\ right).

(**!** left). By induction hypothesis, using Lemma 4, Properties 4 and 1.

(**!** right). By induction hypothesis, we have $[\![\mathbf{!}A_1, \ldots, \mathbf{!}A_n]\!]\eta \subset [\![A]\!]\eta$. This implies, by Lemma 3, Properties 1 and 4, $([\![A_1]\!]\eta \cap K) \cdots ([\![A_n]\!]\eta \cap$

$K) \subset [\![A]\!]\eta$. On the other hand, K being a submomoid, we have $([\![A_1]\!]\eta \cap K) \cdots ([\![A_n]\!]\eta \cap K) \subset K$. Therefore, $([\![A_1]\!]\eta \cap K) \cdots ([\![A_n]\!]\eta \cap K) \subset [\![A]\!]\eta \cap K$. Then, we conclude using Lemma 3, Properties 2 and 4.

(weakening). By induction hypothesis, using Lemma 4, Properties 5 and 1.

(contraction). By induction hypothesis, using Lemma 4, Properties 6 and 1.

(Exchange$_1$). By induction hypothesis, using Lemma 4, Properties 7 and 1.

(Exchange$_2$). By induction hypothesis, using Lemma 4, Properties 7 and 1. □

1.5 Encoding phrase structure grammars in EL

In this section, we show how to transform any phrase structure grammar into a categorial **EL**-grammar. Then we show the completeness of the encoding in the sense that any word generated by the initial grammar is recognised by the corresponding **EL**-grammar.

Consider a phrase-structure grammar $G = \langle N, T, P, S \rangle$ whose production rules have one of the following two forms:

$$A \to a, \quad \text{for } A \in N, a \in T. \tag{1}$$
$$\alpha \to \beta, \quad \text{for } \alpha \in N^+, \beta \in (N \setminus \{S\})^*. \tag{2}$$

where the productions of form (1) induce a one-to-one correspondance between T and a subset of N.

The fact that S may not occur in the righthand sides of the productions of form (2) ensures that the grammar is not S-recursive, i.e., there do not exist words α and β such that $S \to^* \alpha S \beta$ (but for α and β both empty). This condtion will be used when proving the faithfulness of our encoding. Clearly, any phrase structure grammar may be easily transformed into another phrase structure grammar obeying the above requirements.

We construct an **EL**-grammar from the given phrase structure grammar as follows. To any production of form (1), we associate the type assignment $(a : A)$. To any production p of form (2), say,

$$A_0 \dots A_n \to B_0 \dots B_m,$$

we associate the formula

$$F_p = \mathbf{!}(((S/A_n) \cdots /A_0) \setminus ((S/B_m) \cdots /B_0)).$$

To each non-terminal symbol $A \in N \setminus \{S\}$, we associate a new atomic symbol $A^\bullet$—and we use $N^\bullet$ to denote the alphabet made of these new symbols—together with the following two formulas:

$$F^1_A = \mathbf{!}((S/A) \setminus (A^\bullet \setminus S)), \quad F^2_A = \mathbf{!}((S/A)/(A^\bullet \setminus S)).$$

Let $\{p_0, ..., p_{n_P}\}$ be the set of productions of form (2), and let $N \setminus \{S\} = \{A_0, ..., A_{n_N}\}$. We define $\mathbf{!}\Sigma$ to be the sequence of formulas $F_{p_0}, \cdots, F_{p_{n_P}}, F^1_{A_0}, F^2_{A_0}, \cdots, F^1_{A_{n_N}}, F^2_{A_{n_N}}$. Finally, we define the vocabulary of the **EL**-grammar to be T and its set of atomic types to be $N \cup N^\bullet$. Its lexicon is made of the type assignments corresponding to the productions of form (1), and its initial type is $\mathbf{!}\Sigma \setminus S$.

Using the above definition, we can simulate the rewriting relation of the phrase structure grammar as explained by the two lemmas that follow.

Lemma 6 *If* $\mathbf{!}\Sigma, \Gamma, A \vdash S$ *then* $\mathbf{!}\Sigma, A^\bullet, \Gamma \vdash S$*, and conversely.*

Proof. We prove the first part of the statement, the proof of its converse being similar. Consider the following derivation:

$$\begin{array}{c}
\dfrac{\dfrac{\dfrac{\mathbf{!}\Sigma, \Gamma, A \vdash S}{\mathbf{!}\Sigma, \Gamma \vdash S/A} \qquad \dfrac{A^\bullet \vdash A^\bullet \quad S \vdash S}{A^\bullet, A^\bullet \setminus S \vdash S}}{A^\bullet, \mathbf{!}\Sigma, \Gamma, (S/A) \setminus (A^\bullet \setminus S) \vdash S}}{A^\bullet, \mathbf{!}\Sigma, \Gamma, \mathbf{!}((S/A) \setminus (A^\bullet \setminus S)) \vdash S} \\
\hline\hline
A^\bullet, \mathbf{!}\Sigma, \Gamma \vdash S \quad (1) \\
\hline\hline
\mathbf{!}\Sigma, A^\bullet, \Gamma \vdash S \quad (2)
\end{array}$$

where (1) consists of several uses of Rule (exchange_1) followed by a contraction, and (2) of several uses of Rule (exchange_1). □

Lemma 7 *Let* $p = A_0 \dots A_n \to B_0 \dots B_m$ *be any production of form (2). If* $\mathbf{!}\Sigma, \Gamma, A_0, \dots, A_n, \Delta \vdash S$ *then* $\mathbf{!}\Sigma, \Gamma, B_0, \dots, B_m, \Delta \vdash S$

Proof.

$$\begin{array}{c}
\dfrac{
\begin{array}{c}
\mathbf{!}\Sigma, \Gamma, A_0, \dots, A_n, \Delta \vdash S \\
\hline\hline
\mathbf{!}\Sigma, \Delta^\bullet, \Gamma, A_0, \dots, A_n \vdash S \quad (1) \\
\hline\hline
\mathbf{!}\Sigma, \Delta^\bullet, \Gamma \vdash (S/A_n) \cdots /A_0 \quad (2)
\end{array}
\qquad
\dfrac{\begin{array}{c}\dfrac{B_m \vdash B_m \quad S \vdash S}{S/B_m, B_m \vdash S} \\ \vdots \end{array} \qquad B_0 \vdash B_0}{(S/B_m) \cdots /B_0, B_0, \dots, B_m \vdash S}
}{\dfrac{\mathbf{!}\Sigma, \Delta^\bullet, \Gamma, ((S/A_n) \cdots /A_0) \setminus ((S/B_m) \cdots /B_0), B_0, \dots, B_m \vdash S}{\mathbf{!}\Sigma, \Delta^\bullet, \Gamma, \mathbf{!}(((S/A_n) \cdots /A_0) \setminus ((S/B_m) \cdots /B_0)), B_0, \dots, B_m \vdash S}} \\
\hline\hline
\mathbf{!}\Sigma, \Delta^\bullet, \Gamma, B_0, \dots, B_m \vdash S \quad (3) \\
\hline\hline
\mathbf{!}\Sigma, \Gamma, B_0, \dots, B_m, \Delta \vdash S \quad (4)
\end{array}$$

where (1) and (4) consist in iterating Lemma 6, (2) corresponds to several uses of Rule (/ right), and (3) consists of several uses of Rule (exchange$_1$) followed by a contraction. □

We obtain Proposition 8 as an immediate corollary of Lemma 7.

Proposition 8 *Let $\omega \in T^*$ be a word of terminal symbols, and let $\Omega \in N^*$ be the sequence of non-terminal symbols assigned to ω by the typing assignment associated to the productions of form (1). If $\omega \in L(G)$, then $\Omega \vdash (!\Sigma) \setminus S$.*

1.6 Faithfulness of the encoding

In order to prove the converse of Proposition 8, we construct a specific model of **EL** that satisfies the sequence of formulas $!\Sigma$. To this end, we define $\Rightarrow^*$ to be the least rewriting relation on the free monoid $(N \cup N^\bullet)^*$ that contains $\rightarrow^*$ (i.e., the rewriting relation of G) and that is closed under the following rule: $S \Rightarrow^* A^\bullet\Omega$ *whenever* $S \Rightarrow^* \Omega A$*, and conversely.*

Definition 6 The relation $\Rightarrow^* \subset (N \cup N^\bullet)^* \times (N \cup N^\bullet)^*$ is the least relation that satifies the following formal system:

$$\alpha \Rightarrow^* \alpha \quad \text{(refl.)} \qquad\qquad \frac{(\alpha, \beta) \in P}{\alpha \Rightarrow^* \beta} \quad \text{(prod.)}$$

$$\frac{\alpha \Rightarrow^* \beta}{\gamma\alpha\delta \Rightarrow^* \gamma\beta\delta} \quad \text{(congr.)} \qquad\qquad \frac{\alpha \Rightarrow^* \beta \quad \beta \Rightarrow^* \gamma}{\alpha \Rightarrow^* \gamma} \quad \text{(trans.)}$$

$$\frac{S \Rightarrow^* \alpha A}{S \Rightarrow^* A^\bullet\alpha} \quad \text{(shift)} \qquad\qquad \frac{S \Rightarrow^* A^\bullet\alpha}{S \Rightarrow^* \alpha A} \quad \text{(shift}^{-1}\text{)}$$

where in Rule (congr.), either γ or δ is possibly empty.

The next step is to show that the relation $\Rightarrow^*$ is conservative over the relation $\rightarrow^*$. To this end, we first establish two easy technical lemmas, and we introduce the definition of a normal derivation from S.

Lemma 9 *If $\alpha \Rightarrow^* \beta$ and S occurs in β then S occurs in α.*

Proof. Since S does not occur in the righthand side of any production of the grammar, the only way of introducing an occurrence of S in β is by means of Axiom (refl.) or of Rule (congr.). In both cases, this will also introduce an occurrence of S in α. □

Lemma 10 *If $S \Rightarrow^* \alpha$ and $\alpha \neq S$ then S does not occur in α.*

Proof. The proof is by induction on the derivation of $S \Rightarrow^* \alpha$. The only case that is not straightforward is when the last rule of the derivation is Rule (trans.) Suppose that $S \Rightarrow^* \alpha$ is obtained from $S \Rightarrow^* \beta$ and $\beta \Rightarrow^* \alpha$, for some β. If $\beta = S$, we are done by induction hypothesis. Otherwise, by induction hypothesis, S does not occur in β. This implies, by Lemma 9, that S does not occur in α either. □

Definition 7 The notion of a normal derivation from S is inductively defined as follows.

1. $S \Rightarrow^* S$ is a normal derivation from S.
2. If Π is a normal derivation from S, so are:

$$\dfrac{\begin{array}{c}\Pi\\ \vdots\\ S \Rightarrow^* \alpha A\end{array}}{S \Rightarrow^* A^{\bullet}\alpha} \qquad \dfrac{\begin{array}{c}\Pi\\ \vdots\\ S \Rightarrow^* A^{\bullet}\alpha\end{array}}{S \Rightarrow^* \alpha A}$$

3. If Π is a normal derivation from S, so is:

$$\dfrac{\begin{array}{c}\Pi\\ \vdots\\ S \Rightarrow^* \delta\alpha\gamma\end{array} \quad \dfrac{\dfrac{(\alpha,\beta) \in P}{\alpha \Rightarrow^* \beta}}{\delta\alpha\gamma \Rightarrow^* \delta\beta\gamma}}{S \Rightarrow^* \delta\beta\gamma}$$

Let us write $trans(\Pi_1, \Pi_2)$ for a derivation whose conclusion is obtained from the two derivations Π_1 and Π_2 by means of Rule (trans.), and similarly for the other rules. We define the degree of a derivation Π (in notation, $\#\Pi$) as follows:

1. $\#refl = 0$,
2. $\#prod = 0$,
3. $\#congr(\Pi) = \#\Pi$,
4. $\#trans(\Pi_1, \Pi_2) = \begin{cases} \#\Pi_2 & \text{if } \Pi_1 \text{ is a normal derivation from S} \\ \#\Pi_1 + \#\Pi_2 + 1 & \text{otherwise,} \end{cases}$
5. $\#shift(\Pi) = \#\Pi + 1$,
6. $\#shift^{-1}(\Pi) = \#\Pi + 1$.

This definition allows us to prove that any derivation from S may be turned into a normal one.

Lemma 11 *If $S \Rightarrow^* \alpha$ is derivable then there exists a normal derivation from S whith the same conclusion.*

Proof. Without lost of generality we consider that there is no instance of Rule (congr.) immediately followed by another instance of Rule (congr) in the derivation of $S \Rightarrow^* \alpha$.

We proceed by induction on the degree of the derivation. If the derivation consists of axiom (refl.) then it is normal. If it consists of a single use of Rule (prod.), then it can be easily normalised as follows:

$$\dfrac{S \Rightarrow^* S \qquad \dfrac{(S,\alpha) \in P}{S \Rightarrow^* \alpha}}{S \Rightarrow^* \alpha}$$

The case where $S \Rightarrow^* \alpha$ would be obtained by means of Rule (congr.) is impossible, and the two cases correponding to Rules (shift) and (shift^{-1}) are straightforward. Therefore we focus on the remaining case, i.e., when the conclusion of the derivation is obtained by transitivity:

$$\dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ S \Rightarrow^* \beta\end{array} \qquad \begin{array}{c}\Pi_2\\ \vdots\\ \beta \Rightarrow^* \alpha\end{array}}{S \Rightarrow^* \alpha}$$

If $\beta = S$ then we are done by induction hypothesis on Π_2. Therefore suppose $\beta \neq S$ and let $\Pi_1^\star$ be the normal proof obtained by applying the induction hypothesis on Π_1. Suppose that $\mathit{trans}(\Pi_1^\star, \Pi_2)$ is not normal. Then it must be of the following shape—*possibly without the last occurrence of Rule* (congr.):

$$\dfrac{\begin{array}{c}\Pi_1^\star\\ \vdots\\ S \Rightarrow^* \alpha_1\beta_2\alpha_3\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\Pi_{21}\\ \vdots\\ \beta_2 \Rightarrow^* \gamma_2\end{array} \qquad \begin{array}{c}\Pi_{22}\\ \vdots\\ \gamma_2 \Rightarrow^* \alpha_2\end{array}}{\beta_2 \Rightarrow^* \alpha_2}}{\alpha_1\beta_2\alpha_3 \Rightarrow^* \alpha_1\alpha_2\alpha_3}}{S \Rightarrow^* \alpha_1\alpha_2\alpha_3} \tag{1}$$

because, by Lemma 10, S may not occur in β, and consequently there cannot be any occurrence of Rule (shift) or (shift^{-1}) in Π_2. Now, Derivation (1) may be transformed as follows:

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1^\star\\ \vdots\\ S \Rightarrow^* \alpha_1\beta_2\alpha_3\end{array} \qquad \dfrac{\begin{array}{c}\Pi_{21}\\ \vdots\\ \beta_2 \Rightarrow^* \gamma_2\end{array}}{\alpha_1\beta_2\alpha_3 \Rightarrow^* \alpha_1\gamma_2\alpha_3}}{S \Rightarrow^* \alpha_1\gamma_2\alpha_3} \qquad \dfrac{\begin{array}{c}\Pi_{22}\\ \vdots\\ \gamma_2 \Rightarrow^* \alpha_2\end{array}}{\alpha_1\gamma_2\alpha_3 \Rightarrow^* \alpha_1\alpha_2\alpha_3}}{S \Rightarrow^* \alpha_1\alpha_2\alpha_3} \tag{2}$$

and Derivation (2) may be normalised by applying twice the induction hypothesis. □

We are now in a position of proving the announced conservativity result.

Lemma 12 *If $S \Rightarrow^* A^\bullet\beta$ and $\beta \in N^*$ then $S \rightarrow^* \beta A$.*

Proof. According to Lemma 11, the proof proceeds by induction on the structure of the normal derivations from S. The base case is immediate, and the inductive cases corresponding to Clause 2 of Definition 7 are direct. Consequently, we concentrate on the case corresponding to clause 3:

$$\frac{\begin{array}{c}\vdots\\ S \Rightarrow^* A^\bullet\beta_1\gamma_2\beta_3\end{array} \quad \dfrac{\dfrac{(\gamma_2,\beta_2)\in P}{\gamma_2 \Rightarrow^* \beta_2}}{A^\bullet\beta_1\gamma_2\beta_3 \Rightarrow^* A^\bullet\beta_1\beta_2\beta_3}}{S \Rightarrow^* A^\bullet\beta_1\beta_2\beta_3} \tag{1}$$

By induction hypothesis, we have $S \rightarrow^* \beta_1\gamma_2\beta_3 A$, hence:

$$\frac{\begin{array}{c}\vdots\\ S \rightarrow^* \beta_1\gamma_2\beta_3 A\end{array} \quad \dfrac{\dfrac{(\gamma_2,\beta_2)\in P}{\gamma_2 \rightarrow^* \beta_2}}{\beta_1\gamma_2\beta_3 A \rightarrow^* \beta_1\beta_2\beta_3 A}}{S \rightarrow^* \beta_1\beta_2\beta_3 A}$$

□

We get Proposition 13 as an immediate corollary of Lemma 12.

Proposition 13 *If $S \Rightarrow^* \alpha$ and $\alpha \in N^*$ then $S \rightarrow^* \alpha$.*

We now consider the phase space whose underlying monoid is the freely generated monoid $(N \cup N^\bullet)^*$, and whose facts are defined to be the subsets of $(N \cup N^\bullet)^*$ that are closed under $\Rightarrow^*$. We first have to show that this construction is actually a phase space. Clearly, the set of facts is closed by intersection, and $(N \cup N^\bullet)^*$ is a fact. It remains to prove that Conditions 5.(a) and 5.(b) of Definition 2 are satisfied. We prove Condition 5.(a), the proof of 5.(b) being similar.

Let $\alpha \in (N \cup N^\bullet)^*$, and let $F \subset (N \cup N^\bullet)^*$ be closed under $\Rightarrow^*$. Take $G = \{\alpha\} \rightarrow F$. By Definition 3, we have that $\forall\beta \in (N \cup N^\bullet)^*$, $\alpha\beta \in F$ iff $\beta \in G$. It remains to prove that G is a fact. Suppose $\beta \in G$ and $\beta \Rightarrow^* \gamma$. By definition of G, we have $\alpha\beta \in F$. By rule (congr.), we have $\alpha\beta \Rightarrow^* \alpha\gamma$. Therefore, $\alpha\gamma \in F$, which implies $\gamma \in G$.

Let $[\![\cdot]\!]$ denote the semantic interpretation induced by the following valuation:

$$\begin{array}{lcl}\eta(A) & = & \{\alpha \,|\, A \Rightarrow^* \alpha\},\ \text{for } A \in N\\ \eta(A^\bullet) & = & \{A^\bullet\},\ \text{for } A^\bullet \in N^\bullet.\end{array}$$

As we announced at the beginning of this section, the above interpretation satisfies the sequence of formulas $!\Sigma$. This is established by the next two lemmas.

Lemma 14 $1 \in [\![F_p]\!]$, *for any production p of form (2).*

Proof. Let $p = A_0 \dots A_n \to B_0 \dots B_m$, it suffices to prove that

$$[\![((S/A_n) \cdots /A_0)]\!] \subset [\![((S/B_m) \cdots /B_0)]\!].$$

Suppose that σ is such that:

$$\forall x_0 \in [\![A_0]\!], \dots, \forall x_n \in A_n, \sigma x_0 \dots x_n \in [\![S]\!] \tag{1}$$

Suppose also $\beta_i \in [\![B_i]\!]$, for $0 \leq i \leq m$. we have to prove that $\sigma\beta_0 \dots \beta_m \in [\![S]\!]$.

By definition of the valuation η, we have $B_i \Rightarrow^* \beta_i$, for $0 \leq i \leq m$. Consequently, $\sigma A_0 \dots A_n \Rightarrow^* \sigma B_0 \dots B_m \Rightarrow^* \sigma\beta_0 \dots \beta_m$. Then, by instantiating (1), we obtain $\sigma A_0 \dots A_n \in [\![S]\!]$, which implies $\sigma\beta_0 \dots \beta_m \in [\![S]\!]$ by definition of η. □

Lemma 15 $1 \in [\![F_A^1]\!]$ *and* $1 \in [\![F_A^2]\!]$, *for any* $A \in N \setminus \{S\}$.

Proof. We show that $[\![S/A]\!] \subset [\![A^\bullet \setminus S]\!]$. Suppose $\sigma \in [\![S/A]\!]$, from which it follows that $\sigma A \in [\![S]\!]$. Hence, $S \Rightarrow^* \sigma A$ by definition of η, and $S \Rightarrow^* A^\bullet\sigma$ by Rule (shift). This implies that $\sigma \in [\![A^\bullet \setminus S]\!]$.

The proof that $1 \in [\![F_A^2]\!]$ is similar. □

Remark that 1 (i.e., the empty word) is the only element that belongs to the closure of $\{1\}$, that is idempotent, and that commutes with all the others elements of $(N \cup N^\bullet)^*$. Consequently, Lemmas 14 and 15 imply that $[\![!\Sigma]\!] = \{1\}$.

We now have all the ingredients needed to prove the converse of Proposition 8.

Proposition 16 *Let* $\omega \in T^*$ *be a word of terminal symbols, and let* $\Omega \in N^*$ *be the sequence of non-terminal symbols assigned to* ω *by the typing assignment associated to the productions of form (1). If* $\Omega \vdash (!\Sigma) \setminus S$, *then* $\omega \in L(G)$.

Proof. By proposition 5, $[\![\Omega]\!] \subset [\![(!\Sigma) \setminus S]\!]$ or, equivalently, $[\![!\Sigma, \Omega]\!] \subset [\![S]\!]$. Hence, by Lemmas 14 and 15, $[\![\Omega]\!] \subset [\![S]\!]$. Therefore $\Omega \in [\![S]\!]$, which implies $S \Rightarrow^* \Omega$. Then, by Proposition 13, $S \to^* \Omega$. Consequently, $S \to^* \omega$. □

References

Buszkowski, W. 1982. Some decision problems in the theory of syntactic categories. *Zeitschr. f. math Logik und Grundlagen d. Math.* 28:539–548.

Carpenter, B. 1999. The turing-completeness of multimodal categorial grammars. In J. Gerbrandy, M. Marx, M. de Rijke, and Y. Venema, eds., *JFAK. Essays dedicated to Johan van Benthem in honor of his 50th Birthday*. Amsterdam University Press.

Girard, J.-Y. 1987. Linear logic. *Theoretical Computer Science* 50:1–102.

Lambek, J. 1958. The mathematics of sentence structure. *Amer. Math. Monthly* 65:154–170.

Lincoln, P., J. Mitchell, A. Scedrov, and N. Shankar. 1992. Decision problems for propositional linear logic. *Annals of Pure ans Applied Logic* 56:239–311.

Moortgat, M. 1997. Categorial type logics. In J. van Benthem and A. ter Meulen, eds., *Handbook of Logic and Language*, chap. 2. Elsevier.

Moot, R. 2002. *Proof Nets for Linguistic Analysis*. Ph.D. thesis, Utrecht University.

Okada, M. 1999. Phase semantic cut-elimination and normalization proofs of first- and higher-order linear logic. *Theoretical Computer Science* 227(1–2):333–396.

Pentus, M. 1993. Lambek grammars are context free. In *Proceedings of the eigth annual IEEE symposium on logic in computer science*, pages 429–433.

1.7 Appendix: Phase semantics completeness

The proof of Proposition 16 does not require the completeness of the phase semantics. Soundness only is needed. Nevertheless, it is interesting to note that the phase semantics of Section 1.3 is indeed complete. We establish this result in this appendix.

For any formula A, define the following set:

$$[A] = \{\Gamma \mid \vdash (\Gamma \vdash A)\}$$

Then, construct the following syntactic phase space:

1. M is the set of antecedents, with concatenation as product, and the empty sequence as unit;
2. $\mathcal{F}$ is defined to be the least set such that:
   (a) $[A] \in \mathcal{F}$, for any formula A,
   (b) $M \in \mathcal{F}$,
   (c) $\mathcal{F}$ is closed by arbitrary intersection.

Clearly, Conditions 1–4 of Definition 2 are satisfied. We check condition 5.(a), the proof of Condition 5.(b) being similar.

Let $\Gamma = A_0, \ldots, A_n \in M$ and let $F \in \mathcal{F}$. If $F = M$ then we take $G = M$. Otherwise, there exists a family $\mathcal{T}$ of formulas such that $F = \bigcap_{A \in \mathcal{T}} [A]$. Define

$$G = \bigcap_{A \in \mathcal{T}} [A_n \setminus (\cdots (A_0 \setminus A) \cdots)].$$

For all $\Delta \in M$ and all $A \in \mathcal{T}$, $\Gamma, \Delta \in [A]$ if and only if $\vdash (\Gamma, \Delta \vdash A)$ if and only if $\vdash (\Delta \vdash A_n \setminus (\cdots (A_0 \setminus A) \cdots))$ if and only if $\Delta \in [A_n \setminus (\cdots (A_0 \setminus A) \cdots)]$.

Then, we define an enriched phase space by taking K to be the set of sequences of exponential formulas.

Lemma 17 *The following properties hold:*

1. $[A \setminus B] = \{\Gamma \mid \forall \Delta. \vdash (\Delta \vdash A)$ *implies* $\vdash (\Delta, \Gamma \vdash B)\}$.
2. $[A/B] = \{\Gamma \mid \forall \Delta. \vdash (\Delta \vdash A)$ *implies* $\vdash (\Gamma, \Delta \vdash B)\}$.
3. *Let A be a formula and F be a fact such that $A \in F$. If* $\vdash (\Gamma \vdash A)$ *then* $\Gamma \in F$.

Proof.

1. Suppose $\Gamma \in [A \setminus B]$, i.e, $\vdash (\Gamma \vdash A \setminus B)$. This implies that $\vdash (A, \Gamma \vdash B)$. Then, for all Δ such that $\vdash (\Delta \vdash A)$, we have $\vdash (\Delta, \Gamma \vdash B)$, by Rule (cut). Conversely, suppose that $\forall \Delta. \vdash (\Delta \vdash A)$ implies $\vdash (\Delta, \Gamma \vdash B)$. By instantiating Δ with A, we have $\vdash (A, \Gamma \vdash B)$. Then, $\vdash (\Gamma \vdash A \setminus B)$, by Rule ($\setminus$ right).
2. Similarly.
3. If $F = M$ then the property holds. Otherwise, there exists a family $\mathcal{T}$ of formulas such that $F = \bigcap_{B \in \mathcal{T}} [B]$. Hence, for all $B \in \mathcal{T}$, $\vdash (A \vdash B)$. Consequently, by Rule (cut), for all $B \in \mathcal{T}$, $\vdash (\Gamma \vdash B)$, i.e., $\Gamma \in [B]$.

□

Lemma 18 *Let η be the valuation such that $\eta(a) = [a]$. Then, for all formula A, $[\![A]\!]\eta = [A]$.*

Proof. The proof proceeds by induction on the structure of A.

A is atomic. We have $[\![A]\!]\eta = \eta(A) = [A]$.

$A = B \setminus C$. We have $[\![B \setminus C]\!]\eta = [\![B]\!]\eta \to [\![C]\!]\eta$. By induction hypothesis, $[\![B]\!]\eta \to [\![C]\!]\eta = [B] \to [C] = \{\Gamma \mid \forall \Delta. \vdash (\Delta \vdash B)$ implies $\vdash (\Delta, \Gamma \vdash C)\}$, and we conclude by Lemma 17, Property 1.

$A = B/C$. Similarly to the previous case, using Lemma 17, Property 2.

$A = \mathbf{!}B$. We have $[\![\mathbf{!}B]\!]\eta = \mathcal{C}_{\mathcal{F}}([\![B]\!]\eta \cap K) = \mathcal{C}_{\mathcal{F}}([B] \cap K)$, by induction hypothesis, i.e., $[\![\mathbf{!}B]\!]\eta = \mathcal{C}_{\mathcal{F}}(\{\mathbf{!}\Gamma \mid \vdash (\mathbf{!}\Gamma \vdash B)\})$. By Rule ($\mathbf{!}$ right), we have $\{\mathbf{!}\Gamma \mid \vdash (\mathbf{!}\Gamma \vdash B)\} \subset [\mathbf{!}B]$. Then, by Lemma 3, Property 2, and Lemma 1, Property 4, $\mathcal{C}_{\mathcal{F}}(\{\mathbf{!}\Gamma \mid \vdash (\mathbf{!}\Gamma \vdash B)\}) \subset [\mathbf{!}B]$. Conversely, suppose that $\Gamma \in [\mathbf{!}B]$. By Rule ($\mathbf{!}$ left), we have $\mathbf{!}B \in \mathcal{C}_{\mathcal{F}}(\{\mathbf{!}\Gamma \mid \vdash (\mathbf{!}\Gamma \vdash B)\})$. Hence, by Lemma 17, Property 3, $\Gamma \in \mathcal{C}_{\mathcal{F}}(\{\mathbf{!}\Gamma \mid \vdash (\mathbf{!}\Gamma \vdash B)\})$. □

Proposition 19 (Completeness) $\models (\Gamma \vdash A)$ *implies that* $\vdash (\Gamma \vdash A)$.

Proof. Since $\Gamma \vdash A$ is valid, in particular, it is satisfied in the syntactic enriched phase space with the valuation η of Lemma 18. Therefore, $[\![\Gamma]\!]\eta \subset [\![A]\!]\eta$, which implies $\Gamma \in [\![A]\!]\eta$. Then, by Lemma 18, $\Gamma \in [A]$, i.e., $\vdash (\Gamma \vdash A)$. □