A complete axiomatisation for the inclusion of series-parallel partial orders

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Abstract. Series-parallel orders are defined as the least class of partial orders containing the one-element order and closed by ordinal sum and disjoint union. From this inductive definition, it is almost immediate that any series-parallel order may be represented by an algebraic expression, which is unique up to the associativity of ordinal sum and to the associativity and commutativity of disjoint union. In this paper, we introduce a rewrite system acting on these algebraic expressions that axiomatises completely the sub-ordering relation for the class of series-parallel orders.

1 Introduction

Among the several sorts of partial orders that are of interest in applied mathematics, much attention has been paid to *series-parallel partial orders* (SP-orders, for short).

The class of SP-orders is defined as the smallest class of partial orders containing the one-element order and closed by disjoint union and ordinal sum. For this reason, SP-orders are an appropriate abstraction of several kinds of structures that arise naturally in different applied and theoretical problems. These different classes of structures share the property of being generated from atomic elements by two composition operations: one of these operations, called parallel composition, is associative and commutative; the other one, called series composition, is associative and non-commutative.

In electrical engineering, for instance, SP-orders correspond to the line graphs of electrical networks. In operations research, they arise as time constraints in scheduling problems. In graph theory, they appear as both *vertex series-parallel digraphs* and *edge series-parallel multigraphs.* In computer science, they play an important role in the theory of parallelism where the two composition operations correspond naturally to the sequential and parallel composition of processes.

The ubiquity of SP-orders has lead to the independent discovery of the main concepts and results (see [3] for a comprehensive survey, including applications). More recently, SP-orders attracted the attention of the authors of the present paper when they were defining two calculi that both extend linear logic with non-commutative operators [1, 4, 6].

In Retoré's pomset logic [4, 6], the multiple conclusions of a proof are provided with a partial order. Then, when trying to formulate pomset logic by means of a sequent calculus, it is natural to restrict one's attention to sequents that may be interpreted as single formulas, which is possible if and only if the partial order on the conclusions is series-parallel. Similarly, in de Groote's partially commutative logic [1], sequents are built by means of two meta-operators that correspond to commutative and non-commutative conjunctions. Then, the interesting semantic interpretation of this syntactic construction is in terms of SP-orders.

The two calculi may be seen as logics dealing with ordering constraints and they both include rules that allow these constraints to be relaxed. Trying to axiomatise such rules gave rise to the following natural question. *Let t_1 and t_2 be two algebraic terms corresponding to two SP-orders, O_1 and O_2, on the same carrier (these are terms sharing the same atoms and built using two operators: the series and the parallel composition). Is there a rewriting system that would allow t_1 to be rewritten into t_2 if and only if O_2 is a sub-order of O_1?*

The present paper, which gives a positive solution to the above problem, is organised as follows. The next section is devoted to the basic concepts that will be needed in the sequel. Section 3 discusses the problem and introduces our solution. In Section 4, we prove that our axiomatisation is complete, while we explain in Section 5 how to extend our result to similar classes of relations.

2 Basic concepts and definitions

Let U be some denumerable universe. We will define SP-orders (over U) as simple digraphs $R = (\mathrm{V}_R, \mathrm{E}_R)$, where $\mathrm{V}_R \subset U$ is a finite set (called the *carrier*), and $\mathrm{E}_R \subset \mathrm{V}_R \times \mathrm{V}_R$ is an irreflexive relation over V_R. We define ONE to be the class of digraphs $(\{x\}, \varnothing)$, for $x \in U$. Given a digraph R and some non empty set $E \subset U$, $R_{|E}$ denotes the restriction of R to E and is defined as $R_{|E} = (\mathrm{V}_R \cap E, \mathrm{E}_R \cap E^2)$.

As explained in the introduction, SP-orders are built from atomic elements by means of two composition operations. These operations are formally defined as follows.

Definition 2.1 *Let $R = (\mathrm{V}_R, \mathrm{E}_R)$ and $S = (\mathrm{V}_S, \mathrm{E}_S)$ be two digraphs such that $\mathrm{V}_R \cap \mathrm{V}_S = \varnothing$. We define the parallel composition (or disjoint union) of R and S to be the digraph $R \oplus S = (\mathrm{V}_{R\oplus S}, \mathrm{E}_{R\oplus S})$ such that:*

$$\mathrm{V}_{R\oplus S} = \mathrm{V}_R \cup \mathrm{V}_S \quad \textit{and} \quad \mathrm{E}_{R\oplus S} = \mathrm{E}_R \cup \mathrm{E}_S.$$

We define the series composition (or ordinal sum) of R and S to be the digraph $R \oslash S = (\mathrm{V}_{R\oslash S}, \mathrm{E}_{R\oslash S})$ such that:

$$\mathrm{V}_{R\oslash S} = \mathrm{V}_R \cup \mathrm{V}_S \quad \textit{and} \quad \mathrm{E}_{R\oslash S} = \mathrm{E}_R \cup \mathrm{E}_S \cup (\mathrm{V}_R \times \mathrm{V}_S).$$

■

It is a simple exercice to establish that "$\oplus$" is associative and commutative and that "$\otimes$" is associative. Then, the above definition allows the class of SP-orders to be formally defined.

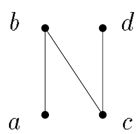
Definition 2.2 *The class of SP-orders is the smallest class of digraphs containing* ONE, *and closed by series and parallel compositions.* ■

Notice that if R is an SP-order then E_R is transitive and antisymmetric, which justifies the name *order* a posteriori.

It is immediate from definition 2.2 that any SP-order may be represented as an algebraic expression built from constants, corresponding to elements of U, by means of two function symbols denoting "$\oplus$" and "$\otimes$". Since series and parallel composition are defined only on disjoint carriers, these algebraic terms are sub-linear in the sense that each element of U appears at most once in any expression. We use $\mathcal{T}$ to denote this set of sub-linear terms over U. Given $t \in \mathcal{T}$, we write $\vartheta(t)$ for the set of constants occurring in t (hence, for any $t \in \mathcal{T}$, we have $\vartheta(t) \subset U$). In fact, the algebraic representation of any SP-order is unique modulo the associativity of $\otimes, \oplus$ and the commutativity of $\oplus$. We write $\mathcal{T}_{/\sim}$ for the corresponding quotient set, which is in bijection with the class of SP-orders.

We end this section by giving a well-known characterisation of SP-orders in terms of forbidden configurations [8] (see [4] for an alternative proof).

Definition 2.3 *A digraph* $R = (\mathrm{V}_R, \mathrm{E}_R)$ *is said to be* N*-free whenever its restriction to any four element set* $E = \{a, b, c, d\} \subset \mathrm{V}_R$ *is different from* $(E, \{(a, b), (c, d), (c, b)\})$, *i.e.,*



■

Proposition 2.4 *The class of series-parallel order is exactly the class of finite* N*-free orders.* □

From this one easily obtains that the class of SP-orders is stable under restriction.

Proposition 2.5 *Let* R *be an SP-order, and* E *be a subset of* V_R. *The restriction* $R_{|E}$ *is an SP-order.* □

3 Inclusion of series-parallel partial orders

Let R and S be two SP-orders. We say that R is a sub-order of S (and, by a slight abuse of notation, we write $R \subset S$) whenever $\mathrm{V}_R = \mathrm{V}_S$ and $\mathrm{E}_R \subset \mathrm{E}_S$. Let us illustrate the possible meaning of this relation by a practical example. Consider the following CCS expression:

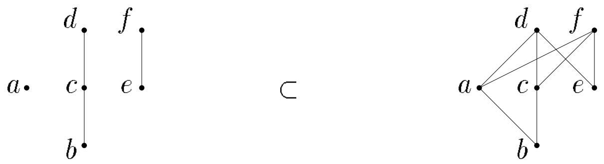
$$a.b \mid c.d \tag{1}$$

If we accept to simulate parallelism by interleaving, each of the following expression is a refinement of (1).

$$a.b.c.d \quad a.c.b.d \quad a.c.d.b \quad c.a.b.d \quad c.a.d.b \quad c.d.a.b \tag{2}$$

Now, if one consider the above expressions as SP-orders, it turns out that (1) is a sub-order of every expression in (2).

As we have seen in the previous section, any SP-order may be written as an algebraic expression, which is unique up to the associativity and the associativity/commutativity of the composition operators. Therefore, it makes sense to seek an axiomatisation of the sub-order relation at this algebraic level. More precisely, let R and S be two SP-orders, and t_R and t_S be algebraic terms representing them respectively. We are going to introduce a rewriting system such that $t_R \to t_S$ if and only if $S \subset R$. This problem is not as simple as it seems because, in general, the structure of the term t_S may be quite different from the structure of the term t_R. This is illustrated, for instance, by the following orders



whose algebraic representations are respectively:

$$a \oplus (b \olessthan c \olessthan d) \oplus (e \olessthan f) \quad \text{and} \quad ((b \olessthan (a \oplus c)) \oplus e) \olessthan (d \oplus f).$$

Our axiomatisation of the sub-order relation is given by the following definition.

Definition 3.1 *The rewriting relation "$\to$" is defined over $\mathcal{T}_{/\sim}$ by the following formal system:*

(a) $s \olessthan t \to s \oplus t$,

(b) $s \olessthan (t \oplus t') \to (s \olessthan t) \oplus t'$,

(c) $(s \oplus s') \olessthan t \to s \oplus (s' \olessthan t)$,

(d) $(s \oplus s') \olessthan (t \oplus t') \to (s \olessthan t) \oplus (s' \olessthan t')$,

(e) $t \to t$,

(f) $\dfrac{s \to s' \quad t \to t'}{s \olessthan t \to s' \olessthan t'}$, (g) $\dfrac{s \to s' \quad t \to t'}{s \oplus t \to s' \oplus t'}$, (h) $\dfrac{s \to t \quad t \to u}{s \to u}$.

■

The most important law in the above definition is Rule (d). Indeed, when enriching the algebra with an identity element ϵ (which would correspond to the empty SP-order on the empty carrier) together with the laws $\epsilon \oslash t = t \oslash \epsilon = t = \epsilon \oplus t$, Rules (a), (b), and (c) appear as particular cases of Rule (d).

Let us demonstrate how our system works on our last example:

$$
\begin{aligned}
((b \oslash (a \oplus c)) \oplus e) \oslash (d \oplus f) &\to ((b \oslash (a \oplus c)) \oslash d) \oplus (e \oslash f) && \text{by Rule (d)} \\
&\sim ((b \oslash (c \oplus a)) \oslash d) \oplus (e \oslash f) \\
&\to (((b \oslash c) \oplus a) \oslash d) \oplus (e \oslash f) && \text{by Rule (b)} \\
&\sim ((a \oplus (b \oslash c)) \oslash d) \oplus (e \oslash f) \\
&\to (a \oplus ((b \oslash c) \oslash d)) \oplus (e \oslash f) && \text{by Rule (c)}
\end{aligned}
$$

It is almost a routine exercise to check that the system of Definition 3.1 is a consistent axiomatisation of a sub-ordering relation for the class of SP-orders. More precisely, the following proposition holds.

Proposition 3.2 *Let R, S be SP-orders, and let t_R and t_S be their respective algebraic representations. If $t_R \to t_S$ then $R \supset S$.* □

The converse of this proposition, which is far from trivial, occupies the next section.

4 Completeness

This section contains our main result, namely the completeness of the axiomatisation given in Definition 3.1. More precisely, we intend to prove the following.

Proposition 4.1 *Let R, S be SP-orders, and let r and s be their respective algebraic representations. If $R \supset S$ then $r \to s$.* □

The difficulty in establishing this proposition is that a simple induction on the inductive structures of R and S does not seem to work. Indeed, these structures may be rather dissimilar as we have seen in the previous section. For this reason, our proof works by induction on the number of points in the common carrier of R and S. This allows us to apply induction hypotheses on restrictions of R and S, taking advantage of Proposition 2.5. To this end, however, we need the next definition together with a lemma whose proof is left to the reader.

Definition 4.2 *Let $t \in \mathcal{T}_{/\sim}$ and $E \subset U$ be such that $\vartheta(t) \cap E \neq \varnothing$. The syntactic restriction $t_{|E}$ of t to E is inductively defined as follows:*

(i)	$a_{\|E} = a$	*if* $a \in E$
(ii)	$(s \circ t)_{\|E} = s_{\|E} \circ t_{\|E}$	*if* $E \cap \vartheta(s) \neq \varnothing$ *and* $E \cap \vartheta(t) \neq \varnothing$
(iii)	$(s \circ t)_{\|E} = s_{\|E}$	*if* $E \cap \vartheta(t) = \varnothing$
(iv)	$(s \circ t)_{\|E} = t_{\|E}$	*if* $E \cap \vartheta(s) = \varnothing$

where $\circ$ is either $\oslash$ or $\oplus$. ■

Lemma 4.3 *Let R be an SP-order and r be its algebraic representation. Let $E \subset U$ be such that $E \cap \mathrm{V}_R \neq \varnothing$. Then, $r_{|E}$ is the algebraic representation of $R_{|E}$.* □

In the course of the proof of Proposition 4.1, we also need the following two decomposition lemmas.

Lemma 4.4 *Let R, R', R'', and S be SP-orders such that:*

(a) $R = R' \oplus R''$,
(b) $S \subset R$.

Then there exist SP-orders S' and S'' such that:

(c) $S' \subset R'$ *and* $S'' \subset R''$.
(d) $S = S' \oplus S''$.

Proof. Take $S' = S_{|\mathrm{V}_{R'}}$ and $S'' = S_{|\mathrm{V}_{R''}}$. By lemma 2.5, both S' and S'' are SP-orders. Moreover, since $S \subset R$, we obviously have that $S' \subset R'$ and $S'' \subset R''$. This establishes (c).

Now, let $a \in \mathrm{V}_{R'}$ and $b \in \mathrm{V}_{R''}$. Since $R = R' \oplus R''$, we have that $(a, b), (b, a) \notin \mathrm{E}_R$. Then, because $S \subset R$, we have that $(a, b), (b, a) \notin \mathrm{E}_S$, which establishes (d). □

Lemma 4.5 *Let R, S, S', and S'' be SP-orders such that:*

(a) $S = S' \otimes S''$,
(b) $S \subset R$.

Then there exist SP-orders R' and R'' such that:

(c) $S' \subset R'$ *and* $S'' \subset R''$.
(d) $R = R' \otimes R''$.

Proof. Take $R' = R_{|\mathrm{V}_{S'}}$ and $R'' = R_{|\mathrm{V}_{S''}}$. By lemma 2.5, both R' and R'' are SP-orders. Moreover, since $S \subset R$, we must have that $S' \subset R'$ and $S'' \subset R''$. This establishes (c).

Now, let $a \in \mathrm{V}_{S'}$ and $b \in \mathrm{V}_{S''}$. Since $S = S' \otimes S''$, we have that $(a, b) \in \mathrm{E}_S$. Then, because $S \subset R$, we have that $(a, b) \in \mathrm{E}_R$, which establishes (d). □

We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1 The proof proceeds by induction on the number of elements in V_R. For the base case, when $\#\mathrm{V}_R = 1$, there is nothing to prove. For the inductive case, when $\#\mathrm{V}_R > 1$, we proceed by case analysis on the inductive structure of R.

Case 1: the last operation in the inductive definition of R is $\oplus$. Then, R may be written as $r = r' \oplus r''$. Because $S \subset R$, by Lemma 4.4, the last operation of

S is $\oplus$ and S may be written as $s = s' \oplus s''$ with $\vartheta(r') = \vartheta(s')$ and $\vartheta(r'') = \vartheta(s'')$ (we choose $s' = s_{|\vartheta(r')}$ and $s'' = s_{|\vartheta(r'')}$). Thus by induction we have $r' \to s'$, $r'' \to s''$ and thus $r = r' \oplus r'' \to s' \oplus s'' = s$.

Case 2: the last operation in the inductive definition of R is $\otimes$. We distinguish between two subcases according to the inductive structure of S.

Subcase 2.1: the last operation in the inductive definition of S is $\otimes$. S may be written as $s = s' \otimes s''$. Because $S \subset R$, by Lemma 4.5, R may be written as $r = r' \otimes r''$ with $\vartheta(r') = \vartheta(s')$ and $\vartheta(r'') = \vartheta(s'')$ (we can choose $r' = r_{|\vartheta(s')}$ and $r'' = r_{|\vartheta(s'')}$). Thus by induction we have $r' \to s'$, $r'' \to s''$ and thus $r = r' \otimes r'' \to s' \otimes s'' = s$.

Subcase 2.2: the last operation in the inductive definition of S is $\oplus$. R may be written as $r = r' \otimes r''$ and S as $s = s' \oplus s''$. Now let us consider the two partitions of V_R given by

$$\mathrm{V}_R = \vartheta(r) = \vartheta(r') \cup \vartheta(r'') = \vartheta(s) = \vartheta(s') \cup \vartheta(s'')$$

and the four sets

$$\begin{array}{ll} \vartheta(r') \cap \vartheta(s') & \vartheta(r') \cap \vartheta(s'') \\ \vartheta(r'') \cap \vartheta(s') & \vartheta(r'') \cap \vartheta(s'') \end{array}$$

Since $\vartheta(r')$, $\vartheta(r'')$, $\vartheta(s')$, $\vartheta(s'')$ are not empty, the four sets above give a partition of V_R into two, three or four parts (because several intersections may be empty). Since $\oplus$ is symmetrical, we can reduce the cases to the four following configurations:

1. the four sets are not empty and V_R is split into four parts.
2. $\vartheta(r') \cap \vartheta(s'')$ is empty and V_R is split into three parts.
3. $\vartheta(r'') \cap \vartheta(s')$ is empty and V_R is split into three parts.
4. $\vartheta(r') \cap \vartheta(s'')$ and $\vartheta(r'') \cap \vartheta(s')$ are empty and V_R is split in two parts.

The four configurations correspond to Rules (d), (c), (b), and (a) of Definition 3.1.

For Configuration 1, the four sets are not empty. We can prove that:

$$\begin{array}{rll} r = & r' \otimes r'' & \\ \to & s_{|\vartheta(r')} \otimes s_{|\vartheta(r'')} & (1) \\ = & (s'_{|\vartheta(r')} \oplus s''_{|\vartheta(r')}) \otimes (s'_{|\vartheta(r'')} \oplus s''_{|\vartheta(r'')}) & \\ = & (s_{|\vartheta(r')\cap\vartheta(s')} \oplus s_{|\vartheta(r')\cap\vartheta(s'')}) \otimes (s_{|\vartheta(r'')\cap\vartheta(s')} \oplus s_{|\vartheta(r'')\cap\vartheta(s'')}) & \\ \to & (s_{|\vartheta(r')\cap\vartheta(s')} \otimes s_{|\vartheta(r'')\cap\vartheta(s')}) \oplus (s_{|\vartheta(r')\cap\vartheta(s'')} \otimes s_{|\vartheta(r'')\cap\vartheta(s'')}) & (2) \\ = & (s'_{|\vartheta(r')} \otimes s'_{|\vartheta(r'')}) \oplus (s''_{|\vartheta(r')} \otimes s''_{|\vartheta(r'')}) & \\ \to & s' \oplus s'' & (3) \end{array}$$

The SP-order corresponding to $s_{|\vartheta(r')}$ is included in the one corresponding to r'. Thus by induction we have $r' \to s_{|\vartheta(r')}$. For the same reason $r'' \to s_{|\vartheta(r'')}$ and (1) follows. (2) is obtained by the application Rule (d) of Definition 3.1. The SP-order corresponding to s' is included in the one corresponding to $s'_{|\vartheta(r')} \otimes$

$s'_{|\vartheta(r'')}$. Thus, by induction we have $s'_{|\vartheta(r')} \oslash s'_{|\vartheta(r'')} \to s'$. For the same reason, $s''_{|\vartheta(r')} \oslash s''_{|\vartheta(r'')} \to s''$ and (3) follows.

The other configurations are very similar to the general case where V_R is split in four parts except that some of the sub-terms are omited. The steps for Configuration 2 are:

$$\begin{aligned} r = {} & r' \oslash r'' \\ & \to s_{|\vartheta(r')} \oslash s_{|\vartheta(r'')} && (1) \\ & = s'_{|\vartheta(r')} \oslash \left(s'_{|\vartheta(r'')} \oplus s''_{|\vartheta(r'')}\right) \\ & = s_{|\vartheta(r')\cap\vartheta(s')} \oslash \left(s_{|\vartheta(r'')\cap\vartheta(s')} \oplus s_{|\vartheta(r'')\cap\vartheta(s'')}\right) \\ & \to \left(s_{|\vartheta(r')\cap\vartheta(s')} \oslash s_{|\vartheta(r'')\cap\vartheta(s')}\right) \oplus s_{|\vartheta(r'')\cap\vartheta(s'')} && (2) \\ & = \left(s'_{|\vartheta(r')} \oslash s'_{|\vartheta(r'')}\right) \oplus s''_{|\vartheta(r'')} \\ & \to s' \oplus s'' && (3) \end{aligned}$$

(2) comes from Rule (b) of Definition 3.1. The third configuration is similar, using Rule (c) instead of Rule (b).

For Configuration 4:

$$\begin{aligned} r = {} & r' \oslash r'' \\ & \to s_{|\vartheta(r')} \oslash s_{|\vartheta(r'')} && (1) \\ & = s'_{|\vartheta(r')} \oslash s''_{|\vartheta(r'')} \\ & = s_{|\vartheta(r')\cap\vartheta(s')} \oslash s_{|\vartheta(r'')\cap\vartheta(s'')} \\ & \to s_{|\vartheta(r')\cap\vartheta(s')} \oplus s_{|\vartheta(r'')\cap\vartheta(s'')} && (2) \\ & = s'_{|\vartheta(r')} \oplus s''_{|\vartheta(r'')} \\ & \to s' \oplus s'' = s && (3) \end{aligned}$$

(2) comes from Rule (a) of Definition 3.1. □

5 Adapting the result to other classes of relations

The inductive principle underlying the construction of series-parallel orders is typical of another class of graphs called *cographs* [3, 2]. This class, also known as *series-parallel graphs*, may defined by replacing, in Definition 2.2, the ordinal sum by a symmetric series composition.

Definition 5.1 *Let $R = (\mathrm{V}_R, \mathrm{E}_R)$ and $S = (\mathrm{V}_S, \mathrm{E}_S)$ be two digraphs such that $\mathrm{V}_R \cap \mathrm{V}_S = \varnothing$. We define the symmetric series composition of R and S to be the digraph $R \otimes S = (\mathrm{V}_{R\otimes S}, \mathrm{E}_{R\otimes S})$ such that:*

$$\mathrm{V}_{R\otimes S} = \mathrm{V}_R \cup \mathrm{V}_S \quad \textit{and} \quad \mathrm{E}_{R\otimes S} = \mathrm{E}_R \cup \mathrm{E}_S \cup (\mathrm{V}_R \times \mathrm{V}_S) \cup (\mathrm{V}_S \times \mathrm{V}_R).$$

■

Then, the class of SP-graphs is defined as the smallest class of digraphs containing ONE, and closed under $\otimes$ and $\oplus$. The inclusion relation within this class may be axiomatised by the following system, which is completely similar to that of Definition 3.1 (up to the commutativity of $\otimes$).

Inclusion of SP-graphs
$s \otimes t \to s \oplus t$
$s \otimes (t \oplus t') \to (s \otimes t) \oplus t'$
$(s \oplus s') \otimes (t \oplus t') \to (s \otimes t) \oplus (s' \otimes t')$

Now, let us come back for a while to our linear logic motivations. Pomset logic is based on the three multiplicative connectives $\parr$, $<$ and $\otimes$, which correspond, from a proofnet theoretic point of view, to the operator $\oplus$, $\olessthan$ and $\otimes$ [5]. This explains that we are interested in the more general class of digraphs that is inductively defined by means of the three composition operations. We call the members of this class, which does not appear in the literature, SSP-relations.

Definition 5.2 *The class of SSP-relations is the smallest class of digraphs containing* ONE, *and closed by series* ($\olessthan$), *symmetric series* ($\otimes$), *and parallel* ($\oplus$) *compositions.* ■

The inclusion relation, for this class, is axiomatised by the following rewritting rules.

Inclusion of SSP-relations
$s \otimes t \to s \oplus t$
$s \otimes (t \oplus t') \to (s \otimes t) \oplus t'$
$(s \oplus s') \otimes (t \oplus t') \to (s \otimes t) \oplus (s' \otimes t')$
$s \otimes t \to s \olessthan t$
$s \otimes (t \olessthan t') \to (s \otimes t) \olessthan t'$
$(s \olessthan s') \otimes t \to s \olessthan (s' \otimes t)$
$(s \oplus s') \olessthan (t \oplus t') \to (s \olessthan t) \oplus (s' \olessthan t')$
$s \olessthan t \to s \oplus t$
$s \olessthan (t \oplus t') \to (s \olessthan t) \oplus t'$
$(s \oplus s') \olessthan t \to s \oplus (s' \olessthan t)$
$(s \oplus s') \olessthan (t \oplus t') \to (s \olessthan t) \oplus (s' \olessthan t')$

The reason why the proof of Proposition 4.1 may be carried over to the cases of SP-graphs and SSP-relations is twofold:

1. both classes are inductively defined by means of composition operations;
2. both classes are stable by restriction.

In the case of SP-orders, the stability by restriction is a direct consequence of their characterisation as N-free digraphs. Similarly, SP-graphs may be characterised as P_4-free graphs, i.e., graphs whose restriction to any four element set $E = \{a, b, c, d\}$ is never $(E, \{(a, b), (b, a), (b, c), (c, b), (c, d), (d, c)\})$ [3]. The class of SSP-relations may also be characterised in terms of forbidden configurations. We end this section by giving this supplementary result, which is original to the best of our knowledge.

Our characterisation of the SSP-relations in terms of forbidden subgraphs and structural properties is given by the following proposition.

Proposition 5.3 *A digraph $R = (\mathrm{V}_R, \mathrm{E}_R)$ is an SSP-relation if an only if it satifies the three following properties*

1. *The directed part $R^\uparrow$ of R defined by $\mathrm{V}_{R^\uparrow} = \mathrm{V}_R$ and $\mathrm{E}_{R^\uparrow} = \{(x,y) \in \mathrm{E}_R \mid (y,x) \notin \mathrm{E}_R\}$ is* N*-free*
2. *The symmetrical part $R^\updownarrow$ of R defined by $\mathrm{V}_{R^\updownarrow} = \mathrm{V}_R$ and $\mathrm{E}_{R^\updownarrow} = \{(x,y) \in \mathrm{E}_R \mid (y,x) \in \mathrm{E}_R\}$ is P_4-free*
3. *E_R is weakly transitive, i.e. $(x,y) \in \mathrm{E}_{R^\uparrow} \wedge (y,z) \in \mathrm{E}_R \Rightarrow (x,z) \in \mathrm{E}_R$ and $(x,y) \in \mathrm{E}_R \wedge (y,z) \in \mathrm{E}_{R^\uparrow} \Rightarrow (x,z) \in \mathrm{E}_R$.*

$\square$

Remark that all these properties are preserved under restriction (and complement). Moreover, $R^\updownarrow$ is an SP-graph (since $R^\updownarrow$ is symmetrical by definition) and, similarly, $R^\uparrow$ is an SP-order (because the weak transitivity of R implies the transitivity of $R^\uparrow$).

To establish the above proposition we first establish a lemma.

Lemma 5.4 *Let R be a digraph satisfying 1,2 and 3 of Proposition 5.3. Let A be the carrier of a connected component of $\mathrm{E}_{R^\updownarrow}$ and B be the carrier of a connected component of $\mathrm{E}_{R^\uparrow}$. Then $A \cap B = \varnothing$ or $A \subset B$ or $B \subset A$.*

Proof. Because A and B are the carriers of connected components of $\mathrm{E}_{R^\updownarrow}$ and $\mathrm{E}_{R^\uparrow}$ respectively, we have:

$(*)$ if $x \in A \setminus B$ and $y \in B \setminus A$ then $(x,y) \notin \mathrm{E}_R$ and $(y,x) \notin \mathrm{E}_R$.

We proceed by contradiction: assuming that $A \setminus B$, $A \cap B$ and $B \setminus A$ are all non empty, we will refute $(*)$.

Since $A \cap B$ and $B \setminus A$ are not empty, while B is a connected component of $\mathrm{E}_{R^\uparrow}$, there should exists one arc $(c,b) \in \mathrm{E}_{R^\uparrow}$ (or $(b,c) \in \mathrm{E}_{R^\uparrow}$, but this case is symmetrical) with $c \in A \cap B$ and $b \in B \setminus A$. Now consider some $a \in A \setminus B$. There are two cases.

Case 1: $(a,c) \in \mathrm{E}_{R^\updownarrow}$. Then, by weak transitivity we would have $(a,b) \in \mathrm{E}_R$, which conflicts with $(*)$.

Case 2: $(a,c) \notin \mathrm{E}_{R^\updownarrow}$. Because $R^\updownarrow$ is P_4-free and connected, the distance between any two vertices must be less than two. Consequently, there exists an $a' \in A$ such that $(a,a') \in \mathrm{E}_{R^\updownarrow}$ and $(a',c) \in \mathrm{E}_{R^\updownarrow}$

Subcase 2.1: $a' \in A \setminus B$. We must have, by weak transitivity, $(a',b) \in R$, which contradicts $(*)$.

Subcase 2.2: $a' \in A \cap B$. Then by weak transitivity we have $(a',b) \in R$. But then, as A is a connected component of $\mathrm{E}_{R^\updownarrow}$, and $b \notin A$, we have $(a',b) \in \mathrm{E}_{R^\uparrow}$ (i.e., $(b,a) \notin \mathrm{E}_R$). Now we have $(a,a') \in \mathrm{E}_{R^\updownarrow}$ and $(a',b) \in \mathrm{E}_{R^\uparrow}$, and thus, by weak transitivity, we should have $(a,b) \in R$, which conflicts with $(*)$. $\square$

Proof of Proposition 5.3 Let R be a digraph satisfying 1, 2 and 3. We will show that $R = R' \oplus R''$ or $R = R' \olessthan R''$ or $R = R' \otimes R''$ where R', R'' are digraphs that necessarily satisfy 1, 2 and 3 because $R' = R_{|\mathrm{V}_{R'}}$ and $R'' = R_{|\mathrm{V}_{R''}}$ (indeed, 1, 2, and 3 are preserved under restriction). This immediately entails the theorem, by induction on $\#\mathrm{V}_R$.

Consider the connected components of $\mathrm{E}_{R\uparrow}$ and $\mathrm{E}_{R\updownarrow}$, and call $(C_i)_{1 \leq i \leq n}$ the maximal ones w.r.t. inclusion. By Lemma 5.4, these maximal components do not overlap. Consequently, they define a partition of V_R.

Case 1: $n \geq 2$. Then $R = R_{|C_1} \oplus R_{|C_2 \cup \cdots \cup C_n}$. Indeed, if $x_i \in C_i$ and $x_j \in C_j$ (with $i \neq j$), then $(x_i, x_j) \notin \mathrm{E}_R$ and $(x_j, x_i) \notin \mathrm{E}_R$, because such an arc can neither be in $\mathrm{E}_{R\uparrow}$ nor in $\mathrm{E}_{R\updownarrow}$, C_i and C_j being connected components of $\mathrm{E}_{R\uparrow}$ or $\mathrm{E}_{R\updownarrow}$.

Case 2: $n = 1$. Three subcases may occur.

Subcase 2.1: $\#(\mathrm{V}_R) = 1$. Then $R \in$ one.

Subcase 2.2: $C_1 = \mathrm{V}_R$ is a connected component of $\mathrm{E}_{R\updownarrow}$, with $\#\mathrm{V}_R > 1$. Then there exist S and T such that $(\mathrm{V}_R, \mathrm{E}_{R\updownarrow}) = S \otimes T$, and we take $R = R_{|\mathrm{V}_S} \otimes R_{|\mathrm{V}_T}$.

Subcase 2.3: $C_1 = \mathrm{V}_R$ is a connected component of $\mathrm{E}_{R\uparrow}$ with $\#\mathrm{V}_R > 1$. Then there exist S and T such that $(\mathrm{V}_R, \mathrm{E}_{R\uparrow}) = S \olessthan T$ and we take $R = R_{|\mathrm{V}_S} \olessthan R_{|\mathrm{V}_T}$. □

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