Proof-theoretic aspects of the Lambek-Grishin Calculus

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Abstract. We compare the Lambek-Grishin Calculus (**LG**) as defined by Moortgat [9, 10] with the non-associative classical Lambek calculus (**CNL**) introduced by de Groote and Lamarche [4]. We provide a translation of **LG** into **CNL**, which allows **CNL** to be seen as a non-conservative extension of **LG**. We then introduce a bimodal version of **CNL** that we call 2-**CNL**. This allows us to define a faithful translation of **LG** into 2-**CNL**. Finally, we show how to accomodate Grishin's interaction principles by using an appropriate notion of polarity. From this, we derive a new one-sided sequent calculus for **LG**.

1 Introduction

The Lambek-Grishin calculus [10] is obtained from the non-associative Lambek calculus [8] by adding a family of connectives ($\oplus$, $\oslash$, and $\oslash$) dual to the Lambek connectives ($\otimes$, $\backslash$, and $/$). A sequent calculus may be easily derived for this new system by adding rules that are the mirror images of Lambek's original rules.

Consider for instance the left and right introduction rules of Lambek's left division ($\backslash$):

$$\frac{\Gamma \vdash A \qquad \Delta[B] \vdash C}{\Delta[\Gamma, A \backslash B] \vdash C} \qquad\qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \backslash B}$$

From these, one derives the left and right introduction rules for the new connective $\oslash$:

$$\frac{B \vdash A, \Gamma}{A \oslash B \vdash \Gamma} \qquad\qquad \frac{C \vdash \Delta[B] \qquad A \vdash \Gamma}{C \vdash \Delta[\Gamma, A \oslash B]}$$

The sequent calculus one obtains this way is sound and complete (in the presence of the cut rule). Nevertheless, it suffers a defect: it does not satisfy the cut-elimination property. The problem is that the lefthand side and the righthand side of a sequent are made of non-associative structures that need some sort of communication. In the case of the calculus we have sketched, this communication is performed by the cut rule.

In order to recover the cut-elimination property, Moortgat has introduced a display logic for the Lambek-Grishin calculus [10]. In this system, the communication between the two sides of a sequent is performed by appropriate display rules. In this paper, we follow another path akin to [2]. In order to recover cut-elimination, we introduce a one-sided sequent calculus. To this end, we first investigate the relation existing between the Lambek-Grishin calculus and the classical non-associative Lambek calculus [4].

2 Lambek-Grishin calculus

Elaborating on the work of Grishin [6], Moortgat introduced the non-associative Lambek-Grishin calculus as the foundations of a new kind of symmetric categorial grammar [9, 10], which allows for the treatment of linguistic phenomena such as displacement or discontinuous dependencies. This calculus is obtained from the non-associative Lambek calculus [8] by the addition of a set of connectives (sum, and left and right differences) that are dual to the product, and the left and right divisions.

More formally, the formulas of the Lambek-Grishin calculus are built upon a set of atomic formulas by means of following formation rules:

$$\mathcal{F} ::= a \mid (\mathcal{F} \otimes \mathcal{F}) \mid (\mathcal{F} \setminus \mathcal{F}) \mid (\mathcal{F} / \mathcal{F}) \mid (\mathcal{F} \oplus \mathcal{F}) \mid (\mathcal{F} \obslash \mathcal{F}) \mid (\mathcal{F} \oslash \mathcal{F})$$

where a ranges over atomic formulas.

In addition to the algebraic laws of the original Lambek calculus, the Lambek-Grishin calculus satifies co-residuation laws that connect the sum ($\oplus$) with the left difference ($\obslash$) and the right difference ($\oslash$). Accordingly, the Lambek-Grishin calculus obeys the following sets of laws:

Preorder laws

$A \leq A$

if $A \leq B$ and $B \leq C$ then $A \leq C$

Residuation laws

$$B \leq A \setminus C \quad \text{iff} \quad A \otimes B \leq C \quad \text{iff} \quad A \leq C / B$$

Co-residuation laws

$$A \obslash C \leq B \quad \text{iff} \quad C \leq A \oplus B \quad \text{iff} \quad C \oslash B \leq A$$

The above algebraic presentation corresponds to the bare Lambek-Grishin calculus, where the two families of connectives coexist without interacting with one another.[1] This does not really provide any additional expressive power with respect to the original non-associative Lambek calculus. In order to get a more powerful calculus, one must consider additional postulates. Grishin proposes four classes of such postulates. Two of these classes (Class I and Class IV) correspond

[1] As pointed out by an anonymous referee, this can be readily seen at the semantic level of the relational models of Kurtonina and Moortgat [7], where the two families of connectives are interpreted through *distinct* ternary relations

to weak distributivity laws that preserve linearity and polarity. Moortgat call them the Grishin interaction principles [9]. There are several equivalent ways of specifying these interaction principles. The presentation given here below is borrowed from [3].

Grishin postulates: Type I

$$A \otimes (B \oplus C) \leq (A \otimes B) \oplus C$$
$$A \otimes (B \oplus C) \leq B \oplus (A \otimes C)$$
$$(A \oplus B) \otimes C \leq A \oplus (B \otimes C)$$
$$(A \oplus B) \otimes C \leq (A \otimes C) \oplus B$$

Grishin postulates: Type IV

$$A \obslash (B \,/\, C) \leq (A \obslash B) \,/\, C$$
$$A \obslash (B \backslash C) \leq B \backslash (A \obslash C)$$
$$(A \backslash B) \oslash C \leq A \backslash (B \oslash C)$$
$$(A \,/\, B) \oslash C \leq (A \oslash C) \,/\, B$$

The Grishin interaction principles can also be specified by means of the following inference rules (see [3] for a proof of the equivalence of the two presentations).

Grishin interactions: Type I

$$\frac{A \oslash B \leq C \backslash D}{C \otimes A \leq D \oplus B} \qquad \frac{A \obslash B \leq C \,/\, D}{B \otimes D \leq A \oplus C}$$

$$\frac{A \obslash B \leq C \backslash D}{C \otimes B \leq A \oplus D} \qquad \frac{A \oslash B \leq C \,/\, D}{A \otimes D \leq C \oplus B}$$

Grishin interactions: Type IV

$$\frac{A \otimes B \leq C \oplus D}{B \oslash D \leq A \backslash C} \qquad \frac{A \otimes B \leq C \oplus D}{C \obslash A \leq D \,/\, B}$$

$$\frac{A \otimes B \leq C \oplus D}{C \obslash B \leq A \backslash D} \qquad \frac{A \otimes B \leq C \oplus D}{A \oslash D \leq C \,/\, B}$$

In the sequel we will use $\mathbf{LG}_\varnothing$ to designate the bare Lambek-Grishin calculus, and we will use $\mathbf{LG}_{\mathrm{I}}$, $\mathbf{LG}_{\mathrm{IV}}$, and $\mathbf{LG}_{\mathrm{I+IV}}$ to designate the Lambek-Grishin calculus provided with the type I interaction principles, with the type IV interaction principles, and with both the type I and type IV interaction principles, respectively.

3 Classical non-associative Lambek calculus

As we have seen, the Lambek-Grishin calculus is motivated by the will of providing the non-associative Lambek calculus, **NL**, with a connective dual to the product. Another way of achieving this has been proposed by Lamarche and the author of the present paper [4]. It consists in providing **NL** with an involutive negation. The resulting system, **CNL**, is a non-associative variant of multiplicative linear logic [5], and may be seen as the classical version of **NL**.

CNL is defined by means of a one-sided sequent calculus. The notions of formula ($\mathcal{F}$), structure($\mathcal{S}$), and sequent ($\mathcal{Q}$) are defined by the following formation rules:

$$\begin{aligned} \mathcal{F} &::= a \mid a^{\perp} \mid (\mathcal{F} \parr \mathcal{F}) \mid (\mathcal{F} \otimes \mathcal{F}) \\ \mathcal{S} &::= \mathcal{F} \mid (\mathcal{S} \bullet \mathcal{S}) \\ \mathcal{Q} &::= \vdash \mathcal{S}, \mathcal{S} \end{aligned}$$

where a range over atomic formulas.

As it is usual in linear logic, non-atomic negation is defined using De Morgan's laws:

$$(A \parr B)^{\perp} = B^{\perp} \otimes A^{\perp}$$
$$(A \otimes B)^{\perp} = B^{\perp} \parr A^{\perp}$$

As for the sequent calculus, it consists of the following rules:

Identity rules

$$\vdash A^{\perp}, A \text{ (Id)} \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta} \text{ (Cut)}$$

Structural rules

$$\frac{\vdash \Gamma, \Delta}{\vdash \Delta, \Gamma} \text{ (Perm)} \qquad \frac{\vdash \Gamma \bullet \Delta, \Theta}{\vdash \Gamma, \Delta \bullet \Theta} \text{ (L-shift)} \qquad \frac{\vdash \Gamma, \Delta \bullet \Theta}{\vdash \Gamma \bullet \Delta, \Theta} \text{ (R-shift)}$$

Logical rules

$$\frac{\vdash \Gamma, A \bullet B}{\vdash \Gamma, A \parr B} \text{ ($\parr$-intro)} \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Delta \bullet \Gamma, A \otimes B} \text{ ($\otimes$-intro)}$$

CNL enjoys cut elimination. We end this section by stating this property that we will use in the sequel of this paper.

Proposition 1. *Let* $\vdash \Gamma, \Delta$ *be a derivable sequent of* ***CNL****. Then,* $\vdash \Gamma, \Delta$ *is derivable without using the Cut rule.*

Proof. See Appendix A. □

4 Translation of LG into CNL

Both **LG** and **CNL** are systems that extend **NL** by providing the Lambek product with a dual connective. It is therefore legitimate to investigate how these two systems are related one to the other. A translation of **NL** into **CNL** is defined in [4]. Extending this translation to **LG** is almost straightforward.

Positive translation	*Negative translation*
$a^+ = a$	$a^- = a^\perp$
$(A \otimes B)^+ = A^+ \otimes B^+$	$(A \otimes B)^- = B^- \parr A^-$
$(A \backslash B)^+ = A^- \parr B^+$	$(A \backslash B)^- = B^- \otimes A^+$
$(A / B)^+ = A^+ \parr B^-$	$(A / B)^- = B^+ \otimes A^-$
$(A \oplus B)^+ = A^+ \parr B^+$	$(A \oplus B)^- = B^- \otimes A^-$
$(A \obslash B)^+ = A^- \otimes B^+$	$(A \obslash B)^- = B^- \parr A^+$
$(A \oslash B)^+ = A^+ \otimes B^-$	$(A \oslash B)^- = B^+ \parr A^-$

The above translation is sound with respect to the algebraic laws of $\mathbf{LG}_\varnothing$. This is stated by the next proposition.

Proposition 2. *Let A and B be two formulas of the Lambek-Grishin calculus such that*

$$A \leq B$$

holds in $\boldsymbol{LG}_\varnothing$*. Then, the* ***CNL****-sequent*

$$\vdash A^-, B^+$$

is derivable.

Proof. See Appendix B. □

The converse of proposition 2 does not hold. Consider, for instance, the two following LG-formulas:

$$A / (B \backslash C) \quad \text{and} \quad A \oplus (C \obslash B)$$

They both translate into the same CNL-formula:

$$\begin{aligned} (A / (B \backslash C))^+ &= A^+ \parr (B \backslash C)^- \\ &= A^+ \parr (C^- \otimes B^+) \\ (A \oplus (C \obslash B))^+ &= A^+ \parr (C \obslash B)^+ \\ &= A^+ \parr (C^- \otimes B^+) \end{aligned}$$

Consequently the following sequent is derivable:

$$\vdash (A / (B \backslash C))^-, (A \oplus (C \obslash B))^+$$

It is not the case, however, that

$$A / (B \backslash C) \leq A \oplus (C \obslash B)$$

5 Multimodal classical non-assiociative Lambek calculus

In $\mathbf{LG}_{\varnothing}$, there is no interaction between the two families of connectives. In particular, $\oplus$ and $\otimes$ are not related through any kind of De Morgan's law. This contrasts with **CNL** where the connective $\parr$ is actually the dual of the connective $\otimes$. This essential difference between the two systems explains why **CNL** is not a conservative extension of $\mathbf{LG}_{\varnothing}$. In order to obtain the converse of proposition 2, we need a bimodal version of **CNL**, that is a system with two distinct families of connectives.

Defining a multimodal version of **CNL** is straightforward. Let I be a set of modes. The formation rules and the sequent calculus of **CNL** are adapted to the multimodal case as follows.

$$\begin{aligned}\mathcal{F} &::= a \mid a^{\perp} \mid (\mathcal{F} \parr_i \mathcal{F}) \mid (\mathcal{F} \otimes_i \mathcal{F}) \\ \mathcal{S} &::= \mathcal{F} \mid (\mathcal{S} \bullet_i \mathcal{S}) \\ \mathcal{Q} &::= \vdash \mathcal{S}, \mathcal{S}\end{aligned}$$

where a ranges over atomic formulas, and $i \in I$.

Identity rules

$$\vdash A, A^{\perp}\ (\mathrm{Id}) \qquad \frac{\vdash A, \Gamma \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{Cut})$$

Structural rules

$$\frac{\vdash \Gamma, \Delta}{\vdash \Delta, \Gamma}\ (\mathrm{Perm}) \qquad \frac{\vdash \Gamma \bullet_i \Delta, \Theta}{\vdash \Gamma, \Delta \bullet_i \Theta}\ (\text{L-shift}) \qquad \frac{\vdash \Gamma, \Delta \bullet_i \Theta}{\vdash \Gamma \bullet_i \Delta, \Theta}\ (\text{R-shift})$$

Logical rules

$$\frac{\vdash A \bullet_i B, \Gamma}{\vdash A \parr_i B, \Gamma}\ (\parr_i\text{-intro}) \qquad \frac{\vdash A, \Gamma \qquad \vdash B, \Delta}{\vdash A \otimes_i B, \Delta \bullet_i \Gamma}\ (\otimes_i\text{-intro})$$

In the sequel of this paper, we only consider the case where $I = \{1, 2\}$, and we use 2-**CNL** as a name for this bimodal version of **CNL**.

6 Translation of LG into 2-CNL

Using 2-**CNL** as a target, we may now modify the translation of **LG** as follows.

Positive translation	*Negative translation*
$a^+ = a$	$a^- = a^\perp$
$(A \otimes B)^+ = A^+ \otimes_1 B^+$	$(A \otimes B)^- = B^- \parr_1 A^-$
$(A \setminus B)^+ = A^- \parr_1 B^+$	$(A \setminus B)^- = B^- \otimes_1 A^+$
$(A / B)^+ = A^+ \parr_1 B^-$	$(A / B)^- = B^+ \otimes_1 A^-$
$(A \oplus B)^+ = A^+ \parr_2 B^+$	$(A \oplus B)^- = B^- \otimes_2 A^-$
$(A \obslash B)^+ = A^- \otimes_2 B^+$	$(A \obslash B)^- = B^- \parr_2 A^+$
$(A \oslash B)^+ = A^+ \otimes_2 B^-$	$(A \oslash B)^- = B^+ \parr_2 A^-$

In order to prove that 2-**CNL** may be seen as a conservative extension of $\mathbf{LG}_\varnothing$ we need to introduce a notion of polarizability. We say that a 2-**CNL**-formula A is positively polarizable (respectively, negatively polarizable) if there exists an **LG**-formula B such that $A = B^+$ (respectively, $A = B^-$).

Lemma 1. *Let A be a 2-**CNL**-formula. Then there exists at most one **LG**-formula B such that either $A = B^+$ or $A = B^-$.*

Proof. By a straightforward induction on the formula A. □

Lemma 1 allows an inverse translation to be defined for the polarizable formulas. Let A be a polarizable 2-**CNL**-formula. We define $[A]$ to be the unique **LG**-formula B such that either $A = B^+$ or $A = B^-$.

We need to extend this inverse translation to the structures and the sequents. To this end, we first extend the notion of polarizability as follows:

- A structure consisting of a single formula A is positively polarizable iff A is positively polarizable as a formula.
- A structure $\Gamma \bullet_1 \Delta$ is positively polarizable iff either Γ is positively polarizable and Δ is negatively polarizable or Γ is negatively polarizable and Δ is positively polarizable.
- A structure $\Gamma \bullet_2 \Delta$ is positively polarizable iff both Γ and Δ are positively polarizable.

- A structure consisting of a single formula A is negatively polarizable iff A is negatively polarizable as a formula.
- A structure $\Gamma \bullet_1 \Delta$ is negatively polarizable iff both Γ and Δ are negatively polarizable.
- A structure $\Gamma \bullet_2 \Delta$ is negatively polarizable iff either Γ is negatively polarizable and Δ is positively polarizable or Γ is positively polarizable and Δ is negatively polarizable.

- A sequent $\vdash \Gamma, \Delta$ is polarizable iff either Γ is negatively polarizable and Δ is positively polarizable or Γ is positively polarizable and Δ is negatively polarizable.

By Lemma 1, using a simple induction, we have that there is no structure which is both positively and negatively polarizable. As a consequence, if a structure is polarizable, it is polarizable in a unique way. This allows functions to be defined by induction on the notion of polarizability. Let Γ^+ and Δ^+ (respectively, Γ^- and Δ^-) range over positively polarizable (respectively, negatively polarizable) structures. The inverse translation is extended to the polarizable structures as follows:

Positive structures	*Negative structures*
$[\Gamma^+ \bullet_1 \Delta^-] = [\Gamma^+] \,/\, [\Delta^-]$	$[\Gamma^- \bullet_1 \Delta^-] = [\Delta^-] \otimes [\Gamma^-]$
$[\Gamma^- \bullet_1 \Delta^+] = [\Gamma^-] \setminus [\Delta^+]$	$[\Gamma^- \bullet_2 \Delta^+] = [\Delta^+] \obslash [\Gamma^-]$
$[\Gamma^+ \bullet_2 \Delta^+] = [\Gamma^+] \oplus [\Delta^+]$	$[\Gamma^+ \bullet_2 \Delta^-] = [\Delta^+] \oslash [\Gamma^+]$

Finally, the polarizable sequents are translated in **LG**-inequalities as follows:

$$[\vdash \Gamma^-, \Delta^+] \;=\; [\Gamma^-] \leq [\Delta^+] \qquad\qquad [\vdash \Gamma^+, \Delta^-] \;=\; [\Delta^-] \leq [\Gamma^+]$$

We are now in a position of proving that 2-**CNL** is conservative over $\mathbf{LG}_\varnothing$.

Lemma 2. *If the conclusion of an inference rule of 2-**CNL** is polarizable then there exists a polarization of its premise(s) such that the rule obtained by applying the inverse translation $[\cdot]$ to the conclusion and to the premise(s) is admissible for $\boldsymbol{LG}_\varnothing$.*

Proof. By case analysis. □

Proposition 3. *Let A and B be two formulas of the Lambek-Grishin calculus. Then,*

$$A \leq B$$

*holds in $\boldsymbol{LG}_\varnothing$ if and only if the 2-**CNL** sequent*

$$\vdash A^-, B^+$$

is derivable.

Proof. The proof of the "if" part is by induction on the derivation of $\vdash A^-, B^+$, using Lemma 2. The proof of the "only if" part is similar to the proof of proposition 3. □

7 Grishin interactions

We now try to incorporate the Grishin interaction principles to 2-**CNL**. Consider, for instance, the first interaction rule of type I:

$$\frac{A \oslash B \leq C \setminus D}{C \otimes A \leq D \oplus B}$$

By applying the translation of **LG** into 2-**CNL**, the above rule is transformed as follows:

$$\frac{\vdash\ B^+ \parr_2 A^-,\ C^- \parr_1 D^+}{\vdash\ A^- \parr_1 C^-,\ D^+ \parr_2 B^+}$$

This suggests that the following rule could allow for one of the Grishin interaction principles:

$$\frac{\vdash\ A \parr_1 B,\ C \parr_2 D}{\vdash\ D \parr_1 A,\ B \parr_2 C}\ (1)$$

Rule (1), however, is not quite satisfactory. Adding it to 2-**CNL** would destroy the subformula property. To circumvent this difficulty, we seek a rule equivalent to rule (1) which would work at the structural level. This is possible because the connectives $\parr_1$ and $\parr_2$ are logically equivalent to the structural nodes $\bullet_1$ and $\bullet_2$. Following this idea, we end up with the following rule:

$$\frac{\vdash\ \Gamma \bullet_1 \Delta,\ \Theta \bullet_2 \Lambda}{\vdash\ \Lambda \bullet_1 \Gamma,\ \Delta \bullet_2 \Theta}$$

Applying the same transformation to the other interaction rules of type I, we obtain the following set of rules:

$$\frac{\vdash\ \Gamma \bullet_1 \Delta,\ \Theta \bullet_2 \Lambda}{\vdash\ \Lambda \bullet_1 \Gamma,\ \Delta \bullet_2 \Theta} \qquad \frac{\vdash\ \Gamma \bullet_1 \Delta,\ \Theta \bullet_2 \Lambda}{\vdash\ \Delta \bullet_1 \Theta,\ \Lambda \bullet_2 \Gamma}$$

$$\frac{\vdash\ \Gamma \bullet_1 \Delta,\ \Theta \bullet_2 \Lambda}{\vdash\ \Theta \bullet_1 \Gamma,\ \Lambda \bullet_2 \Delta} \qquad \frac{\vdash\ \Gamma \bullet_1 \Delta,\ \Theta \bullet_2 \Lambda}{\vdash\ \Delta \bullet_1 \Lambda,\ \Gamma \bullet_2 \Theta}$$

Now, if we apply the same method to the interaction rules of type IV, we get again the above set of rules. In other words, without any further proviso, the interaction rules of type I and the interaction rules of type IV collapse into the same set of rules when translated in 2-**CNL**. This is because there are several ways of polarizing the above rules. Indeed, these different ways of polarizing a same rule correspond to different interaction principles at the level of **LG**. Therefore, in order to accomodate 2-**CNL** with the Grishin interaction principles, one must take the polarities into account. This is what we will do in the next section.

8 A sequent calculus for LG

We are finally in the position of defining a one-sided sequent calculus for **LG**. This calculus, which is based on 2-**CNL**, works directly at the level of the **LG**-formulas. The formulas are therefore defined as in **LG**:

$$\mathcal{F} ::= a \mid (\mathcal{F} \otimes \mathcal{F}) \mid (\mathcal{F} \backslash \mathcal{F}) \mid (\mathcal{F} / \mathcal{F}) \mid (\mathcal{F} \oplus \mathcal{F}) \mid (\mathcal{F} \obslash \mathcal{F}) \mid (\mathcal{F} \oslash \mathcal{F})$$

To every formula A, one associates a co-formula $\overline{A}$. Then, the positive structures ($\mathcal{S}^+$) and the negative structures ($\mathcal{S}^-$) are defined as follows:

$$\mathcal{S}^+ ::= \mathcal{F} \mid (\mathcal{S}^- \bullet \mathcal{S}^+) \mid (\mathcal{S}^+ \bullet \mathcal{S}^-) \mid (\mathcal{S}^+ \circ \mathcal{S}^+)$$
$$\mathcal{S}^- ::= \overline{\mathcal{F}} \mid (\mathcal{S}^- \bullet \mathcal{S}^-) \mid (\mathcal{S}^- \circ \mathcal{S}^+) \mid (\mathcal{S}^+ \circ \mathcal{S}^-)$$

As for the sequents, they consists of two structures of opposite polarities:

$$\mathcal{Q} ::= \vdash \mathcal{S}^+, \mathcal{S}^- \mid \vdash \mathcal{S}^-, \mathcal{S}^+$$

The rules of the sequent calculus, which are directly derived from the rules of 2-**CNL**, are the following:

Strucural rules

$$\frac{\vdash \Gamma, \Delta}{\vdash \Delta, \Gamma}\ (\text{Perm})$$

$$\frac{\vdash \Gamma \bullet \Delta, \Theta}{\vdash \Gamma, \Delta \bullet \Theta}\ (\bullet\text{-L-shift}) \qquad \frac{\vdash \Gamma, \Delta \bullet \Theta}{\vdash \Gamma \bullet \Delta, \Theta}\ (\bullet\text{-R-shift})$$

$$\frac{\vdash \Gamma, \Delta \circ \Theta}{\vdash \Gamma \circ \Delta, \Theta}\ (\circ\text{-L-shift}) \qquad \frac{\vdash \Gamma \circ \Delta, \Theta}{\vdash \Gamma, \Delta \circ \Theta}\ (\circ\text{-R-shift})$$

Identity rules

$$\vdash \overline{A}, A\ (\text{Id}) \qquad \frac{\vdash \Gamma, A \qquad \vdash \overline{A}, \Delta}{\vdash \Gamma, \Delta}\ (\text{Cut})$$

Logical rules

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Delta \bullet \Gamma, A \otimes B}\ \otimes\text{-intro}^+ \qquad \frac{\vdash \overline{B} \bullet \overline{A}, \Gamma}{\vdash \overline{A \otimes B}, \Gamma}\ \otimes\text{-intro}^-$$

$$\frac{\vdash \Gamma, \overline{A} \bullet B}{\vdash \Gamma, A \setminus B}\ \setminus\text{-intro}^+ \qquad \frac{\vdash \Delta, A \qquad \vdash \overline{B}, \Gamma}{\vdash \overline{A \setminus B}, \Delta \bullet \Gamma}\ \setminus\text{-intro}^-$$

$$\frac{\vdash \Gamma, A \bullet \overline{B}}{\vdash \Gamma, A / B}\ /\text{-intro}^+ \qquad \frac{\vdash \overline{A}, \Delta \qquad \vdash \Gamma, B}{\vdash \overline{A / B}, \Delta \bullet \Gamma}\ /\text{-intro}^-$$

$$\frac{\vdash \Gamma, A \circ B}{\vdash \Gamma, A \oplus B}\ \oplus\text{-intro}^+ \qquad \frac{\vdash \overline{A}, \Delta \qquad \vdash \overline{B}, \Gamma}{\vdash \overline{A \oplus B}, \Delta \circ \Gamma}\ \oplus\text{-intro}^-$$

$$\frac{\vdash \overline{A}, \Gamma \qquad \vdash \Delta, B}{\vdash \Delta \circ \Gamma, A \obslash B}\ \obslash\text{-intro}^+ \qquad \frac{\vdash \overline{B} \circ A, \Gamma}{\vdash \overline{A \obslash B}, \Gamma}\ \obslash\text{-intro}^-$$

$$\frac{\vdash \Gamma, A \qquad \vdash \overline{B}, \Delta}{\vdash \Delta \circ \Gamma, A \oslash B}\ \oslash\text{-intro}^+ \qquad \frac{\vdash B \circ \overline{A}, \Gamma}{\vdash \overline{A \oslash B}, \Gamma}\ \oslash\text{-intro}^-$$

Grishin interactions: Type I

$$\frac{\vdash \Gamma^- \bullet \Delta^+, \Theta^+ \circ \Lambda^-}{\vdash \Lambda^- \bullet \Gamma^-, \Delta^+ \circ \Theta^+} \qquad \frac{\vdash \Gamma^+ \bullet \Delta^-, \Theta^- \circ \Lambda^+}{\vdash \Delta^- \bullet \Theta^-, \Lambda^+ \circ \Gamma^+}$$

$$\frac{\vdash \Gamma^- \bullet \Delta^+, \Theta^- \circ \Lambda^+}{\vdash \Theta^- \bullet \Gamma^-, \Lambda^+ \circ \Delta^+} \qquad \frac{\vdash \Gamma^+ \bullet \Delta^-, \Theta^+ \circ \Lambda^-}{\vdash \Delta^- \bullet \Lambda^-, \Gamma^+ \circ \Theta^+}$$

Grishin interactions: Type IV

$$\frac{\vdash \Gamma^- \bullet \Delta^-, \Theta^+ \circ \Lambda^+}{\vdash \Delta^- \bullet \Theta^+, \Lambda^+ \circ \Gamma^-} \qquad \frac{\vdash \Gamma^- \bullet \Delta^-, \Theta^+ \circ \Lambda^+}{\vdash \Lambda^+ \bullet \Gamma^-, \Delta^- \circ \Theta^+}$$

$$\frac{\vdash \Gamma^- \bullet \Delta^-, \Theta^+ \circ \Lambda^+}{\vdash \Delta^- \bullet \Lambda^+, \Gamma^- \circ \Theta^+} \qquad \frac{\vdash \Gamma^- \bullet \Delta^-, \Theta^+ \circ \Lambda^+}{\vdash \Theta^+ \bullet \Gamma^-, \Lambda^+ \circ \Delta^-}$$

9 Conclusion

We have introduced a new sequent calculus for the Lambek-Grishin calculus. This new sequent calculus derives from the classical non-associative Lambek calculus as defined in [4]. Consequently, it inherits the properties of this later system. In particular, it enjoys cut elimination.

Our new sequent calculus presents also interesting similarities with Moortgat display calculus [10]. A translation of one into the other can be easily defined. Nevertheless, our calculus is more economical. In particular, we only use two structural nodes (as opposed to six in the case of the display calculus). We also need less structural rules.

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A Proof of proposition 1

We show that a derivation containing a single cut may be transformed in a cut-free derivation. Then, the general case follows by a simple induction on the number of cuts.

The proof proceeds by case analysis and by induction on the structure of the cut formula. One distinguishes four cases.

Case 1 : the cut formula in the left premise of the cut rule is introduced by an axiom.

In this case, the derivation may be transformed as follows:

$$\dfrac{\begin{array}{c}\vdash A^{\perp}, A \\ \vdots\ (1) \\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdots\ (2) \\ \vdash A^{\perp}, \Delta\end{array}}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) \quad \rightsquigarrow \quad \begin{array}{c}\vdots\ (2) \\ \vdash A^{\perp}, \Delta \\ \vdots\ (1') \\ \vdash \Gamma, \Delta\end{array}$$

where Derivation (1') is obtained from Derivation (1) by replacing each occurrence of the cut formula by the structure Δ.

Case 2 : the cut formula in the right premise of the cut rule is introduced by an axiom.

This case is symmetric to Case 1:

$$\dfrac{\begin{array}{c}\vdots\ (1) \\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\vdash A^{\perp}, A \\ \vdots\ (2) \\ \vdash A^{\perp}, \Delta\end{array}}{\vdash \Gamma, \Delta}\ (\mathrm{Cut}) \quad \rightsquigarrow \quad \begin{array}{c}\vdots\ (1) \\ \vdash \Gamma, A \\ \vdots\ (2') \\ \vdash \Gamma, \Delta\end{array}$$

Case 3 : the cut formula is of the form $A \parr B$, and is introduced by introduction rules in both the left and right premises of the cut rule.

This case corresponds to the following derivation schemes:

$$
\dfrac{
\begin{array}{c}
\vdots\,(1) \\
\dfrac{\vdash \Gamma_1,\, A \bullet B}{\vdash \Gamma_1,\, A \parr B}\ (\parr\text{-intro}) \\
\vdots\,(2) \\
\vdash \Gamma,\, A \parr B
\end{array}
\qquad
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\,(3)\\ \vdash B^{\perp},\, \Delta_1\end{array} \quad \begin{array}{c}\vdots\,(4)\\ \vdash A^{\perp},\, \Delta_2\end{array}}{\vdash B^{\perp} \otimes A^{\perp},\, \Delta_2 \bullet \Delta_1}\ (\otimes\text{-intro}) \\
\vdots\,(5) \\
\vdash B^{\perp} \otimes A^{\perp},\, \Delta
\end{array}
}{\vdash \Gamma,\, \Delta}\ (\text{Cut})
$$

It can be transformed into the following derivation:

$$
\begin{array}{c}
\dfrac{
\dfrac{
\dfrac{
\dfrac{\begin{array}{c}\vdots\,(1)\\ \vdash \Gamma_1,\, A \bullet B\end{array}}{\vdash \Gamma_1 \bullet A,\, B}\ (\text{R-shift})
\quad
\begin{array}{c}\vdots\,(3)\\ \vdash B^{\perp},\, \Delta_1\end{array}
}{\vdash \Gamma_1 \bullet A,\, \Delta_1}\ (\text{Cut})
}{
\dfrac{\vdash \Delta_1,\, \Gamma_1 \bullet A}{\vdash \Delta_1 \bullet \Gamma_1,\, A}\ (\text{R-shift})
}\ (\text{Perm})
\quad
\begin{array}{c}\vdots\,(4)\\ \vdash A^{\perp},\, \Delta_2\end{array}
}{
\dfrac{\vdash \Delta_1 \bullet \Gamma_1,\, \Delta_2}{\dfrac{\vdash \Delta_2,\, \Delta_1 \bullet \Gamma_1}{\dfrac{\vdash \Delta_2 \bullet \Delta_1,\, \Gamma_1}{\vdash \Gamma_1,\, \Delta_2 \bullet \Delta_1}\ (\text{Perm})}\ (\text{R-shift})}\ (\text{Perm})
}\ (\text{Cut}) \\
\vdots\,(2') \\
\vdash \Gamma,\, \Delta_2 \bullet \Delta_1 \\
\vdots\,(5') \\
\vdash \Gamma,\, \Delta
\end{array}
$$

where the two new cuts are eliminable by induction hypothesis.

Case 4 : the cut formula is of the form $A \otimes B$, and is introduced by introduction rules in both the left and right premises of the cut rule.

This case, which is symmetric to Case 3, corresponds to the following derivation scheme:

$$
\dfrac{
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\,(1)\\ \vdash \Gamma_1,\, A\end{array} \quad \begin{array}{c}\vdots\,(2)\\ \vdash \Gamma_2,\, B\end{array}}{\vdash \Gamma_2 \bullet \Gamma_1,\, A \otimes B}\ \otimes\text{-intro} \\
\vdots\,(3) \\
\vdash \Gamma,\, A \otimes B
\end{array}
\qquad
\begin{array}{c}
\vdots\,(4) \\
\dfrac{\vdash B^{\perp} \bullet A^{\perp},\, \Delta_1}{\vdash B^{\perp} \parr A^{\perp},\, \Delta_1}\ \parr\text{-intro} \\
\vdots\,(5) \\
\vdash B^{\perp} \parr A^{\perp},\, \Delta
\end{array}
}{\vdash \Gamma,\, \Delta}\ (\text{Cut})
$$

It can be transformed as follows:

$$
\dfrac{\begin{matrix}\vdots\,(1)\\ \vdash \Gamma_1,\, A\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,(2)\\ \vdash \Gamma_2,\, B\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\,(4)\\ \vdash B^{\perp} \bullet A^{\perp},\, \Delta_1\end{matrix}}{\vdash B^{\perp},\, A^{\perp} \bullet \Delta_1}\,(\text{L-shift})}{\vdash \Gamma_2,\, A^{\perp} \bullet \Delta_1}\,(\text{Cut})}{\vdash A^{\perp} \bullet \Delta_1,\, \Gamma_2}\,(\text{Perm})}{\vdash A^{\perp},\, \Delta_1 \bullet \Gamma_2}\,(\text{L-shift})}{\dfrac{\dfrac{\dfrac{\vdash \Gamma_1,\, \Delta_1 \bullet \Gamma_2}{\vdash \Delta_1 \bullet \Gamma_2,\, \Gamma_1}\,(\text{Perm})}{\vdash \Delta_1,\, \Gamma_2 \bullet \Gamma_1}\,(\text{L-shift})}{\begin{matrix}\vdash \Gamma_2 \bullet \Gamma_1,\, \Delta_1\\ \vdots\,(3')\\ \vdash \Gamma,\, \Delta_1\\ \vdots\,(5')\\ \vdash \Gamma,\, \Delta\end{matrix}}\,(\text{Perm})}\,(\text{Cut})
$$

B Proof of proposition 2

We show that the translations of the algebraic laws of $\mathbf{LG}_{\varnothing}$ hold in **CNL**.

Preorder laws

Let A be an **LG**-formula. It is easy to show that $A^{-} = (A^{+})^{\perp}$. Consequently, the translations of the preorder laws correspond to the identity rules (Id and Cut).

Residuation laws

The two following derivation schemes show that the first residuation law holds.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdash A^{-},\, A^{+} \qquad \vdash B^{-},\, B^{+}}{\vdash B^{-} \bullet A^{-},\, A^{+} \otimes B^{+}}\,(\otimes\text{-intro}) \qquad \vdash B^{-} \parr A^{-},\, C^{+}}{\vdash B^{-} \bullet A^{-},\, C^{+}}\,(\text{Cut})}{\vdash B^{-},\, A^{-} \bullet C^{+}}\,(\text{L-shift})}{\vdash B^{-},\, A^{-} \parr C^{+}}\,(\parr\text{-intro})
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash B^-,\, A^- \parr C^+ \qquad \dfrac{\dfrac{\dfrac{\vdash C^-,\, C^+}{\vdash C^+,\, C^-}\,(\text{Perm}) \qquad \vdash A^-,\, A^+}{\vdash A^- \bullet C^+,\, C^- \otimes A^+}\,(\otimes\text{-intro})}{\vdash C^- \otimes A^+,\, A^- \bullet C^+}\,(\text{Perm})}{\vdash B^-,\, A^- \bullet C^+}\,(\text{Cut})}{\vdash B^- \bullet A^-,\, C^+}\,(\text{R-shift})}{\vdash C^+,\, B^- \bullet A^-}\,(\text{Perm})}{\vdash C^+,\, B^- \parr A^-}\,(\parr\text{-intro})}{\vdash B^- \parr A^-,\, C^+}\,(\text{Perm})
$$

The case of the second residuation law is similar.

Co-residuation laws

This case is symmetric to the case of the residuation laws, and it is handled in a similar way.