On the expressive power of the Lambek calculus extended with a structural modality
–Abstract–

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Introduction

We consider **EL**, the product-free Lambek calculus [3] extended with a structural modality à la Girard [2], which obeys the following rules:

$$\frac{\Gamma, A, \Delta \vdash B}{\Gamma, !A, \Delta \vdash B}\ (!\text{ left}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!\text{ right}) \qquad \frac{!A, !A, \Gamma \vdash B}{!A, \Gamma \vdash B}\ (\text{contraction})$$

$$\frac{\Gamma \vdash B}{!A, \Gamma \vdash B}\ (\text{weakening}) \qquad \frac{\Gamma, B, !A, \Delta \vdash C}{\Gamma, !A, B, \Delta \vdash C}\ (\text{exchange}_1) \qquad \frac{\Gamma, !B, A, \Delta \vdash C}{\Gamma, A, !B, \Delta \vdash C}\ (\text{exchange}_2)$$

We show that any recursively enumerable language can be described by a categorial grammar based on **EL**. As an immediate corollary, we get the undecidability of **EL**.

Similar results may be found in the litterature. In [4], Lincoln et al. establish the undecidability of circular multiplicative linear logic extended with the same kind of modality, by encoding semi-thue systems in a straightforward way. Nevertheless, this encoding makes an essential use of the non-commutative product. Consequently, our result is a refinement of theirs since we are working in a product-free calculus. In [1], Buszkowski shows that the product-free Lambek calculus provided with proper axioms allows any recursively enumerable language to be described. This result is almost equivalent to ours. However, our proof is much simpler than Buszkowski's because the encoding we use is a substantial simplification of his. In addition, we use a transparent semantic argument in order to establish the faithfulness of our encoding while both [4] and [1] use proof-theoretic machinery.

Encoding phrase-structure grammars in EL

We consider a phrase-structure grammar $G = \langle N, T, P, S \rangle$ whose production rules have one of the following two forms:

$$A_0 A_1 \dots A_n \to B_0 B_1 \dots B_m \quad (1), \qquad A \to a \quad (2)$$

where $A, A_0, A_1, \dots, A_n, B_0, B_1, \dots, B_m \in N$ and $a \in T$.

We associate a categorial grammars to this phrase-structure grammars as follows. To each $p \in P$ of form (1), we associate the formula

$$F_p = !(((S/A_n) \cdots /A_0) \setminus ((S/B_m) \cdots /B_0)).$$

To each production of form (2), we associate the type assignment $(a : A)$. To each $A \in N \setminus \{S\}$, we associate a new atomic symbol $A^\bullet$—and we use $N^\bullet$ to denote the alphabet made of these new symbols—together with the following two formulas:

$$F_A^1 = !((S/A) \setminus (A^\bullet \setminus S)), \quad F_A^2 = !((S/A)/(A^\bullet \setminus S)).$$

Let $\{p_0, ..., p_{n_P}\}$ be the set of productions of form (1), and let $N = \{A_0, ..., A_{n_N}\}$. We define $!\Sigma$ to be the sequence of formulas $F_{p_0}, \cdots, F_{p_{n_P}}, F^1_{A_0}, F^2_{A_0}, \cdots, F^1_{A_{n_P}}, F^2_{A_{n_P}}$. Using the above definitions, we can simulate the rewriting relation of the phrase structure grammar as explained by the two lemmas that follow.

Lemma 1 *If* $!\Sigma, \Gamma, A \vdash S$ *then* $!\Sigma, A^\bullet, \Gamma \vdash S$*, and conversely.* □

Lemma 2 *Let* $p \in P$ *be any production of form (1), i.e.,* $p = A_0 A_1 \ldots A_n \to B_0 B_1 \ldots B_m$*. If* $!\Sigma, \Gamma, A_0, \ldots, A_n, \Delta \vdash S$ *then* $!\Sigma, \Gamma, B_0, \ldots, B_m, \Delta \vdash S$

Proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!\Sigma, \Gamma, A_0, \ldots, A_n, \Delta \vdash S}{!\Sigma, \Delta^\bullet, \Gamma, A_0, \ldots, A_n \vdash S}}{!\Sigma, \Delta^\bullet, \Gamma \vdash (S/A_n)\cdots/A_0} \quad \dfrac{\dfrac{\dfrac{B_m \vdash B_m \quad S \vdash S}{S/B_m, B_m \vdash S}}{\vdots \atop (S/B_m)\cdots/B_1, B_1, \ldots, B_m \vdash S} \quad B_0 \vdash B_0}{(S/B_m)\cdots/B_0, B_0, \ldots, B_m \vdash S}}{!\Sigma, \Delta^\bullet, \Gamma, ((S/A_n)\cdots/A_0) \setminus ((S/B_m)\cdots/B_0), B_0, \ldots, B_m \vdash S}}{!\Sigma, \Delta^\bullet, \Gamma, !(((S/A_n)\cdots/A_0) \setminus ((S/B_m)\cdots/B_0)), B_0, \ldots, B_m \vdash S}}{!\Sigma, \Delta^\bullet, \Gamma, B_0, \ldots, B_m \vdash S}}{!\Sigma, \Gamma, B_0, \ldots, B_m, \Delta \vdash S}$$

□

We obtain Proposition 3 as immediate corollary of Lemma 2.

Proposition 3 *Let* $\omega \in T^*$ *be a word of terminal symbols, and let* $\Omega \in N^*$ *be the sequence of non-terminal symbols assigned to* ω *by the typing assignment associated to the productions of form (2). If* $\omega \in L(G)$*, then* $\Omega \vdash (!\Sigma) \setminus S$*.* □

Faithfulness of the encoding

Define $\Rightarrow^*$ to be the least rewriting relation on the free monoid $(N \cup N^\bullet)^*$ that contains $\to^*$ (i.e., the rewriting relation of G) and that is closed under the following rule: $S \Rightarrow^* A^\bullet\Omega$ *whenever* $S \Rightarrow^* \Omega A$*, and conversely.*

Lemma 4 *Let* $\Omega \in N^*$*. If* $S \Rightarrow^* \Omega$ *then* $S \to^* \Omega$*.* □

We construct an (non-commutative) intuitionistic phase space [5] over the free monoid $(N \cup N^\bullet)^*$. The facts of this phase space are defined to be the sets of words closed under $\Rightarrow^*$. Let $[\![\cdot]\!]$ denote the semantic interpretation induced by the following valuation:

$$\begin{aligned} \eta(A) &= \{\Omega \mid A \Rightarrow^* \Omega\}, \text{ for } A \in N \\ \eta(A^\bullet) &= \{A^\bullet\}, \text{ for } A^\bullet \in N^\bullet. \end{aligned}$$

Lemma 5 *Let* $p \in P$ *be any production of form (1), and let* $A \in N$ *be any non-terminal symbol.* $[\![F_p]\!] = [\![F_A^1]\!] = [\![F_A^2]\!] = \{\mathbf{1}\}$*, where* $\mathbf{1}$ *is the empty word, i.e., the unit of the monoid.* □

Proposition 6 *Let* $\omega \in T^*$ *be a word of terminal symbols, and let* $\Omega \in N^*$ *be the sequence of non-terminal symbols assigned to* ω *by the typing assignment associated to the productions of form (2). If* $\Omega \vdash (!\Sigma) \setminus S$*, then* $\omega \in L(G)$*.*

Proof: We have that $[\![\Omega]\!] \subset [\![(!\Sigma) \setminus S]\!]$, which implies that $[\![!\Sigma]\!] \cdot [\![\Omega]\!] \subset [\![S]\!]$. This implies, by Lemma 5, that $[\![\Omega]\!] \subset [\![S]\!]$. Therefore, $S \Rightarrow^* \Omega$ and, by Lemma 4, $S \to^* \Omega$. Consequently, $S \to^* \omega$. □

References

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