# Structures Informatiques et Logiques pour la Modélisation Linguistique (MPRI 2.27.1 - second part)

Philippe de Groote

Inria

2012-2013

# Semantic representations

1. Introduction

2. Modal logic
   - Possibility and necessity
   - Kripke semantics
   - Intension and extension

3. Higher-order logic
   - Simply typed $\lambda$-calculus
   - Church's simple theory of types
   - Standard model
   - Inherent incompleteness
   - Henkin Models

1. Introduction

2. Modal logic
   - Possibility and necessity
   - Kripke semantics
   - Intension and extension

3. Higher-order logic
   - Simply typed $\lambda$-calculus
   - Church's simple theory of types
   - Standard model
   - Inherent incompleteness
   - Henkin Models

# Semantics

- Semantics is the study of *meaning*.
- In this setting, the *logical* meaning of a declarative utterance is reduced to its truth conditions (truth conditional semantics).
- Model-theoretic semantics: the *logical* meaning of a declarative utterance is captured by the set of models that make valid the *interpretation* of this utterance.
- Proof-theoretic semantics: the *logical* meaning of a declarative utterance is captured by a logical formula.

# Example

- *John eats a red apple.*

$$\exists x.\text{apple}(x) \wedge \text{eat}(\mathbf{j}, x)$$

1 Introduction

2 Modal logic
- Possibility and necessity
- Kripke semantics
- Intension and extension

3 Higher-order logic
- Simply typed $\lambda$-calculus
- Church's simple theory of types
- Standard model
- Inherent incompleteness
- Henkin Models

# Modals

- In a strict sense, modal logic is concerned with the study of statements and reasonings that involve the notions of necessity and possiblity

- In a more general sense, modal logic is concerned with the study of statements and reasonings that involve expressions (modals) that qualify the validity of a judgement:
  - Alethic logic: *It is necessary that... It is possible that...*
  - Deontic logic: *It is mandatory that... It is allowed that...*
  - Epistemic logic: *Bob knows that... Bob ignores that...*
  - Temporal logic: *It will always be the case that... It will eventually be the case that...*

# Leibniz

G.W. von Leibniz

(1646–1716)

A proposition is necessarily true if it is true in all possible worlds.

A proposition is possibly true if it is true in at least one possible world.

*Pangloss enseignait la métaphysico-théologo-cosmolo-nigologie. Il prouvait admirablement qu'il n'y a point d'effet sans cause, et que, dans ce meilleur des mondes possibles, le château de monseigneur le baron était le plus beau des châteaux et madame la meilleure des baronnes possibles.*

Voltaire (Candide)

# Formalization

**Syntax**:

$$F \ ::= \ a \mid \neg F \mid F \vee F \mid \Box F$$

Define the other connectives in the usual way. Define $\Diamond A$ as $\neg\Box\neg A$.

$\Box A$ stands for "necessarily A". $\Diamond A$ stands for "possibly A".

**Validity**:

let $\mathcal{M} = \langle W, P \rangle$, where $W$ is a set of "possible worlds", and $P$ is a function that asigns to each atomic proposition a subset of $W$.

- $\mathcal{M}, s \models a$ iff $s \in P(a)$.
- $\mathcal{M}, s \models \neg A$ iff not $\mathcal{M}, s \models A$.
- $\mathcal{M}, s \models A \vee B$ iff either $\mathcal{M}, s \models A$ or $\mathcal{M}, s \models B$, or both.
- $\mathcal{M}, s \models \Box A$ iff for every $t \in W$, $\mathcal{M}, t \models A$.

# System S5

(P) all propositional tautologies

(K) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

(T) $\Box A \supset A$

(5) $\Diamond A \supset \Box \Diamond A$

Modus ponens:

$$\frac{A \supset B \quad A}{B}$$

Rule of necessitation:

$$\frac{A}{\Box A}$$

1 Introduction

2 Modal logic
- Possibility and necessity
- Kripke semantics
- Intension and extension

3 Higher-order logic
- Simply typed $\lambda$-calculus
- Church's simple theory of types
- Standard model
- Inherent incompleteness
- Henkin Models

# Kripke Semantics

let $\mathcal{M} = \langle W, R, P \rangle$, where $W$ is a set of "possible worlds", $R$ is a binary relation over $W$, and $P$ is a function that asigns to each atomic proposition a subset of $W$.

- $\mathcal{M}, s \models \Box A$ iff for every $t \in W$ such that $sRt$, $\mathcal{M}, t \models A$.
- $\mathcal{M}, s \models \Diamond A$ iff for some $t \in W$ such that $sRt$, $\mathcal{M}, t \models A$.

# System K

(P) all propositional tautologies

(K) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

Modus ponens:

$$\frac{A \supset B \quad A}{B}$$

Rule of necessitation:

$$\frac{A}{\Box A}$$

The following theorems of S5 are not valid in the class of all Kripke models:

(D) $\Box A \supset \Diamond A$

(T) $\Box A \supset A$

(B) $A \supset \Box\Diamond A$

(4) $\Box A \supset \Box\Box A$

(5) $\Diamond A \supset \Box\Diamond A$

A binary relation $R \subset W \times W$ is serial if and only if for every $s \in W$ there exists $t \in W$ such that $sRt$.

# Some well-known systems

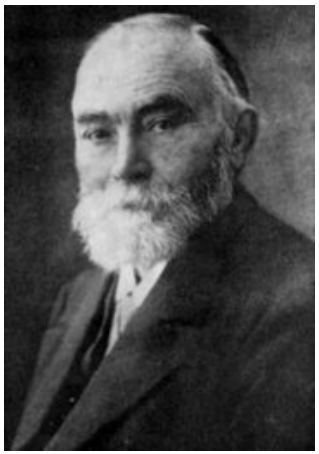
| | | |
|---|---|---|
| KD | basic deontic logic | serial |
| KT | basic alethic logic | reflexive |
| KTB | Brouwersche system | reflexive, symmetric |
| KT4 | Lewis' S4 | reflexive, transitive |
| KT5 | Lewis' S5 | reflexive, symmetric, transitive |

1 Introduction

2 Modal logic
- Possibility and necessity
- Kripke semantics
- Intension and extension

3 Higher-order logic
- Simply typed $\lambda$-calculus
- Church's simple theory of types
- Standard model
- Inherent incompleteness
- Henkin Models

# Sinn und bedeutung



F.L.G. Frege

(1848–1925)

Sinn (sense)/Bedeutung (reference) — Frege
Intension/Extension — Carnap

According to Frege, the sense of an expression is its "mode of presentation", while the reference or denotation of an expression is the object it refers to.

For instance, both expressions "$1 + 1$" and "2" have the same denotation but not the same sense.

# Intensional proposition

An intensional proposition is a proposition whose validity is not invariant under extensional substitution.

Frege gives the example of "the morning star" and "the evening star" which both refer to the planet Venus.

Compare "the morning star is the evening star" with "John does not know that the morning star is the evening star".

1 Introduction

2 Modal logic
- Possibility and necessity
- Kripke semantics
- Intension and extension

3 Higher-order logic
- Simply typed λ-calculus
- Church's simple theory of types
- Standard model
- Inherent incompleteness
- Henkin Models

# What is lambda-calculus?

- An intensional theory of functions.
- A simple functional programming language.
- A theory of free- and bound-variables, of scope and substitution.
- The keystone of higher-order syntax and higher-order logic.
- The algebra of natural-deduction proofs.

# Syntax

**Terms**:

$$T \ ::= \ x \mid \lambda x.\, T \mid (T\, T)$$

$\lambda$ is a binder: thre free occurrences of $x$ in $t$ are bound in $\lambda x.\, t$.

**Warning**: You should solve, once and for all, any problem you could have with the notions of free and bound occurrences of variables.

**Reduction rule**: $(\lambda x.\, t)\, u \ \rightarrow_\beta \ t[x{:=}u]$

**Church-Rosser Theorem**: For all $\lambda$-terms $t$, $u$, and $v$ such that:

$$t \twoheadrightarrow_\beta u \quad \text{and} \quad t \twoheadrightarrow_\beta v$$

there exists a $\lambda$-term $w$ such that:

$$u \twoheadrightarrow_\beta w \quad \text{and} \quad v \twoheadrightarrow_\beta w$$

# Typing rules

$$\Gamma, x : A \vdash x : A$$

$$\frac{x : A, \Gamma \vdash t : B}{\Gamma \vdash \lambda x.\, t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash (t\, u) : B}$$

**Strong-Normalisation Theorem**: There is no infinite reduction sequence.

1. Introduction

2. Modal logic
   - Possibility and necessity
   - Kripke semantics
   - Intension and extension

3. Higher-order logic
   - Simply typed $\lambda$-calculus
   - Church's simple theory of types
   - Standard model
   - Inherent incompleteness
   - Henkin Models

# Syntax

A. Church
(1903–1995)

Two atomic types: $\iota, o$

Logical constants:

$$\begin{array}{lcll} \bot & : & o & \\ \supset & : & o \to o \to o & \\ \forall_\alpha & : & (\alpha \to o) \to o & \text{(at each type } \alpha\text{)} \end{array}$$

$\iota$ is the type of individuals and $o$ is the type of propositions.

Formulas are defined to be well-typed $\lambda$-terms of type $o$. We write $P \supset Q$ and $\forall x.\, P$ for $\supset P\, Q$ and $\forall_\alpha\, (\lambda x.\, P)$, respectively. Similarly for the other connectives ($\neg, \wedge, \vee, \equiv, \exists$), which are defined in the usual way.

$t = u$ is defined as $\forall P. P\, t \supset P\, u$.

# Deductive system

**Logical rules:**

$$\Gamma, A \vdash A$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \qquad \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall_\alpha (\lambda x_\alpha . A)} \; x \text{ of type } \alpha,\ x \notin FV(\Gamma) \qquad \frac{\Gamma \vdash \forall_\alpha A}{\Gamma \vdash A\,B} \; B \text{ of type } \alpha$$

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}$$

# Deductive system

**Conversion rule:**

$$\frac{\Gamma \vdash A}{\Gamma \vdash B} \text{ where } A =_\beta B$$

**Extensionality axioms:**

$$\Gamma \vdash (\forall_\alpha x.A\,x = B\,x) \supset (A = B)$$

$$\Gamma \vdash (A \equiv B) \supset (A = B)$$

1 Introduction

2 Modal logic
- Possibility and necessity
- Kripke semantics
- Intension and extension

3 Higher-order logic
- Simply typed $\lambda$-calculus
- Church's simple theory of types
- Standard model
- Inherent incompleteness
- Henkin Models

# Interpretation of the types and the terms

- $M = \langle (D_a)_{a \in T}, \mathcal{I} \rangle$
- $D_\iota$ is given.
- $D_o = \{0, 1\}$
- $D_{A \to B} = {D_B}^{D_A}$
- $[\![c]\!]_\eta = \mathcal{I}(c)$
- $[\![x]\!]_\eta = \eta(x)$
- $[\![\lambda x.\, t]\!]_\eta = a \mapsto [\![t]\!]_{\eta[x:=a]}$
- $[\![t\, u]\!]_\eta = [\![t]\!]_\eta([\![u]\!]_\eta)$

With the expected interpretations for the logical constants.

# Higher-order logic as a set theory

- Sets as characteristic functions, i.e., sets of "elements" of type $\alpha$ as terms of type $\alpha \to o$.

- $\{x \mid P\}$ as $\lambda x.\, P$
- $t \in A$ as $A\, t$

1 Introduction

2 Modal logic
- Possibility and necessity
- Kripke semantics
- Intension and extension

3 Higher-order logic
- Simply typed $\lambda$-calculus
- Church's simple theory of types
- Standard model
- Inherent incompleteness
- Henkin Models

# The set of natural numbers

$$\mathrm{S} \triangleq (\forall x.\, s\, x \neq 0) \ \wedge \ (\forall xy.\, s\, x = s\, y \ \supset \ x = y)$$

$$\mathrm{N} \triangleq \lambda x.\, (\forall R.\, R\, 0 \ \wedge \ (\forall y.\, R\, y \ \supset \ R\, (s\, y)) \ \supset \ R\, x)$$

The only model of $\mathrm{S} \ \wedge \ \forall x.\, \mathrm{N}\, x$ is the set of natural numbers.

# Incompleteness

Let $\phi$ be a formula of Peano's Arithmetic, and define $\phi^{\mathrm{N}}$ as follows:

- $\phi^{\mathrm{N}} = \phi$, for $\phi$ an atomic formula,
- $(\neg\phi)^{\mathrm{N}} = \neg\,\phi^{\mathrm{N}}$,
- $(\phi * \psi)^{\mathrm{N}} = \phi^{\mathrm{N}} * \psi^{\mathrm{N}}$, for $* \in \{\wedge, \vee, \supset, \equiv\}$,
- $(\forall x.\,\phi)^{\mathrm{N}} = \forall x.(\mathrm{N}\,x \supset \phi^{\mathrm{N}})$,
- $(\exists x.\,\phi)^{\mathrm{N}} = \exists x.(\mathrm{N}\,x \wedge \phi^{\mathrm{N}})$.

Let D be the conjunction of the universal closures of the defining equations for addition and multiplication, and let PA be $\mathrm{S} \wedge \forall x.\,\mathrm{N}\,x \wedge \mathrm{D}$.

Then, the formula $\mathrm{PA} \supset \phi$ is valid if and only if $\phi$ is true in the standard model of Peano's arithmetic.

**Corollary**: incompleteness of higher-order logic.

1. Introduction

2. Modal logic
   - Possibility and necessity
   - Kripke semantics
   - Intension and extension

3. Higher-order logic
   - Simply typed $\lambda$-calculus
   - Church's simple theory of types
   - Standard model
   - Inherent incompleteness
   - Henkin Models

# Relaxing the interpretation

- $M = \langle (D_a)_{a \in T}, \mathcal{I} \rangle$
- $D_\iota$ is given.
- $D_o = \{0, 1\}$
- $D_{A \to B} \subset {D_B}^{D_A}$
- $[\![c]\!]_\eta = \mathcal{I}(c)$
- $[\![x]\!]_\eta = \eta(x)$
- $[\![\lambda x.\, t]\!]_\eta = a \mapsto [\![t]\!]_{\eta[x:=a]}$
- $[\![t\, u]\!]_\eta = [\![t]\!]_\eta([\![u]\!]_\eta)$

With domains that are rich enough to interpret $\lambda$-abstraction, equality, and the logical constants .