Structures Informatiques et Logiques pour la Modélisation Linguistique (MPRI 2.27.1 - second part)

Philippe de Groote

Inria

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Semantic representations

1. Introduction
2. Modal logic
   - Intension and extension
   - Possibility and necessity
   - Kripke semantics
   - Hybrid Logic
3. Higher-order logic
   - Simply typed λ-calculus
   - Church's simple theory of types
   - Standard model
   - Inherent incompleteness
   - Henkin Models

Semantic representations

Semantics

- Semantics is the study of *meaning*.
- In this setting, the *logical* meaning of a declarative utterance is reduced to its truth conditions (truth conditional semantics).
- Model-theoretic semantics: the *logical* meaning of a declarative utterance is captured by the set of models that make valid the *interpretation* of this utterance.
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Example

- *John eats a red apple.*

$$\exists x.\mathbf{apple}(x) \wedge \mathbf{red}(x) \wedge \mathrm{eat}(\mathbf{j}, x)$$

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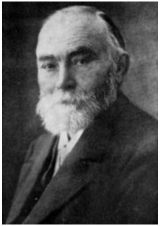

F.L.G. Frege
(1848–1925)

Sinn (sense)/Bedeutung (reference) — Frege
Intension/Extension — Carnap

According to Frege, the sense of an expression is its "mode of presentation", while the reference or denotation of an expression is the object it refers to.

For instance, both expressions "$1 + 1$" and "2" have the same denotation but not the same sense.

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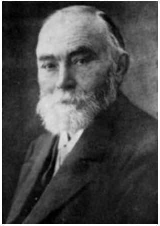

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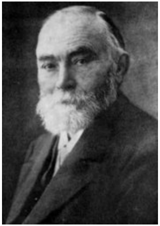

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Intensional proposition

An intensional proposition is a proposition whose validity is not invariant under extensional substitution.

Frege gives the example of “the morning star” and “the evening star” which both refer to the planet Venus.

Compare “the morning star is the evening star” with “John does not know that the morning star is the evening star”.

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Semantic representations

Modals

- In a strict sense, modal logic is concerned with the study of statements and reasonings that involve the notions of necessity and possiblity
- In a more general sense, modal logic is concerned with the study of statements and reasonings that involve expressions (modals) that qualify the validity of a judgement:
  - Alethic logic: *It is necessary that... It is possible that...*
  - Deontic logic: *It is mandatory that... It is allowed that...*
  - Epistemic logic: *Bob knows that... Bob ignores that...*
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G.W. von Leibniz
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A proposition is necessarily true if it is true in all possible worlds.

A proposition is possibly true if it is true in at least one possible world.

Pangloss enseignait la métaphysico-théologo-cosmolo-nigologie. Il prouvait admirablement qu'il n'y a point d'effet sans cause, et que, dans ce meilleur des mondes possibles, le château de monseigneur le baron était le plus beau des châteaux et madame la meilleure des baronnes possibles.

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Formalization

Syntax:

$$F \ ::= \ a \mid \neg F \mid F \vee F \mid \Box F$$

Define the other connectives in the usual way. Define $\Diamond A$ as $\neg\Box\neg A$.

$\Box A$ stands for "necessarily A". $\Diamond A$ stands for "possibly A".

Validity:

let $\mathcal{M} = \langle W, P \rangle$, where W is a set of "possible worlds", and P is a function that asigns to each atomic proposition a subset of W.

- $\mathcal{M}, s \models a$ iff $s \in P(a)$.
- $\mathcal{M}, s \models \neg A$ iff not $\mathcal{M}, s \models A$.
- $\mathcal{M}, s \models A \vee B$ iff either $\mathcal{M}, s \models A$ or $\mathcal{M}, s \models B$, or both.
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System S5

(P) all propositional tautologies

(K) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

(T) $\Box A \supset A$

(5) $\Diamond A \supset \Box\Diamond A$

Modus ponens:

$$\frac{A \supset B \quad A}{B}$$

Rule of necessitation:

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Semantic representations

Kripke Semantics

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The following theorems of S5 are not valid in the class of all Kripke models:

(D) $\Box A \supset \Diamond A$

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Some well-known systems

KD	basic deontic logic	serial
KT	basic alethic logic	reflexive
KTB	Brouwersche system	reflexive, symmetric
KT4	Lewis' S4	reflexive, transitive
KT5	Lewis' S5	reflexive, symmetric, transitive

Semantic representations

Syntax

Key idea: provide the formula language with explicit means of speaking about worlds!

Two sorts of atoms: usual atomic propositions $(a, b, c, \ldots)$, and nominals $(i, j, k, \ldots)$. Nominals will be used for naming worlds.

$$F ::= a \mid i \mid \neg F \mid F \vee F \mid \Box F \mid \downarrow i.\, F \mid @_i F$$

$\downarrow$ is a binder: the free occurrences of i in A are bound in $\downarrow i.\, F$. On the, other hand, $@$ is simply a binary connectives whose first term must be a nominal.

Intuition: $\downarrow$ is used for naming the "here-and-now". It allows a nominal to be bound to the current world. $@_i A$ asserts that proposition A holds at world i.

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$\mathcal{M}, \eta, s \models a$ iff $s \in P(a)$.

$\mathcal{M}, \eta, s \models i$ iff $s = \eta(i)$.

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Axiomatization

1. all propositional tautologies
2. $\downarrow i.\,(A \supset B) \supset (A \supset \downarrow i.\,B)$, where i does not occur free in A
3. $\downarrow i.\,A \supset (j \supset A[i{:=}j])$
4. $\downarrow i.\,(i \supset A) \supset \downarrow i.\,A$
5. $\downarrow i.\,A \equiv \neg\downarrow i.\,\neg A$

6. $@_i(A \supset B) \supset (@_iA \supset @_iB)$
7. $@_iA \equiv \neg @_i\neg A$
8. $i \wedge A \supset @_iA$

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9. $@_i i$
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13. $\Diamond @_i A \supset @_i A$
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$$\frac{A \supset B \quad A}{B} \qquad \frac{A}{\Box A}$$

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(*) j is distinct from i and does not occur free in A or B.

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Example

The binary operator of temporal logic:

$$A \ \mathrm{until}\ B$$

may be defined as:

$$\downarrow i.\, \Diamond \downarrow j. @_i(\Diamond(j \wedge B) \wedge \Box(\Diamond j \supset A))$$

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Semantic representations

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What is lambda-calculus?

- An intensional theory of functions.
- A simple functional programming language.
- A theory of free- and bound-variables, of scope and substitution.
- The keystone of higher-order syntax and higher-order logic.
- The algebra of natural-deduction proofs.

Syntax

Terms:

$$T \ ::= \ x \mid \lambda x.\,T \mid (T\,T)$$

λ is a binder: thre free occurrences of x in t are bound in $\lambda x.\,t$.

Warning: You should solve, once and for all, any problem you could have with the notions of free and bound occurrences of variables.

Reduction rule: $\quad (\lambda x.\,t)\,u \rightarrow_\beta t[x:=u]$

Church-Rosser Theorem: For all λ-terms t, u, and v such that:

$$t \twoheadrightarrow_\beta u \quad \text{and} \quad t \twoheadrightarrow_\beta v$$

there exists a λ-term w such that:

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Typing rules

$$\Gamma, x : A \vdash x : A$$

$$\frac{x : A, \Gamma \vdash t : B}{\Gamma \vdash \lambda x. t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash (t\, u) : B}$$

Strong-Normalisation Theorem: There is no infinite reduction sequence.

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Strong-Normalisation Theorem: There is no infinite reduction sequence.

Semantic representations

Syntax

A. Church
(1903–1995)

Two atomic types: ι, o

Logical constants:

$$
\begin{array}{lcll}
\perp & : & o & \\
\supset & : & o \rightarrow o \rightarrow o & \\
\forall_\alpha & : & (\alpha \rightarrow o) \rightarrow o & \text{(at each type } \alpha\text{)}
\end{array}
$$

ι is the type of individuals and o is the type of propositions.

Formulas are defined to be well-typed λ-terms of type o. We write $P \supset Q$ and $\forall x.\, P$ for $\supset P\, Q$ and $\forall_\alpha\, (\lambda x.\, P)$, respectively. Similarly for the other connectives $(\neg, \wedge, \vee, \equiv, \exists)$, which are defined in the usual way.

$t = u$ is defined as $\forall P. P\, t \supset P\, u$.

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Deductive system

Logical rules:

$$\Gamma, A \vdash A$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B} \qquad \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall_\alpha (\lambda x_\alpha . A)} \; x \text{ of type } \alpha,\ x \notin FV(\Gamma) \qquad \frac{\Gamma \vdash \forall_\alpha A}{\Gamma \vdash A\,B} \; B \text{ of type } \alpha$$

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}$$

Deductive system

Conversion rule:

$$\frac{\Gamma \vdash A}{\Gamma \vdash B} \quad \text{where } A =_\beta B$$

Extensionality axioms:

$$\Gamma \vdash (\forall_\alpha x. A\, x = B\, x) \supset (A = B)$$

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Interpretation of the types and the terms

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With the expected interpretations for the logical constants.

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Higher-order logic as a set theory

- Sets as characteristic functions, i.e., sets of “elements” of type α as terms of type $\alpha \to o$.

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The set of natural numbers

$$\mathrm{S} \triangleq (\forall x.\, s\,x \neq 0) \;\wedge\; (\forall xy.\, s\,x = s\,y \;\supset\; x = y)$$

$$\mathrm{N} \triangleq \lambda x.\, (\forall R.\, R\,0 \;\wedge\; (\forall y.\, R\,y \;\supset\; R\,(s\,y)) \;\supset\; R\,x)$$

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Incompleteness

Let ϕ be a formula of Peano's Arithmetic, and define ϕ^{N} as follows:

- $\phi^{\mathrm{N}} = \phi$, for ϕ an atomic formula,
- $(\neg\phi)^{\mathrm{N}} = \neg\, \phi^{\mathrm{N}}$,
- $(\phi * \psi)^{\mathrm{N}} = \phi^{\mathrm{N}} * \psi^{\mathrm{N}}$, for $* \in \{\wedge, \vee, \supset, \equiv\}$,
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- $(\exists x.\, \phi)^{\mathrm{N}} = \exists x.(\mathrm{N}\, x \wedge \phi^{\mathrm{N}})$.

Let D be the conjunction of the universal closures of the defining equations for addition and multiplication, and let PA be $\mathrm{S} \ \wedge\ \forall x.\mathrm{N}\, x \ \wedge\ \mathrm{D}$.

Then, the formula $\mathrm{PA} \supset \phi^{\mathrm{N}}$ is valid if and only if ϕ is true in the standard model of Peano's arithmetic.

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With domains that are rich enough to interpret λ-abstraction, equality, and the logical constants .

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