Structures Informatiques et Logiques pour la Modélisation Linguistique
(MPRI 2.27.1 - second part)

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 Semantic representations

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2 Modal logic
   - Intension and extension
   - Possibility and necessity
   - Kripke semantics
   - Hybrid Logic

3 Higher-order logic
   - Simply typed λ-calculus
   - Church’s simple theory of types
   - Standard model
   - Inherent incompleteness
   - Henkin Models
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Semantics

- Semantics is the study of *meaning*.

- In this setting, the *logical* meaning of a declarative utterance is reduced to its truth conditions (truth conditional semantics).

- Model-theoretic semantics: the *logical* meaning of a declarative utterance is captured by the set of models that make valid the *interpretation* of this utterance.

- Proof-theoretic semantics: the *logical* meaning of a declarative utterance is captured by a logical formula.
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Example

- John eats a red apple.

\[ \exists x. \text{apple}(x) \land \text{red}(x) \land \text{eat}(j, x) \]
Example

- **John eats a red apple.**

\[ \exists x. \text{apple}(x) \land \text{red}(x) \land \text{eat}(j, x) \]
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## Modal logic

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   - Simply typed $\lambda$-calculus
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Sinn (sense)/Bedeutung (reference) — Frege
Intension/Extension — Carnap

According to Frege, the sense of an expression is its “mode of presentation”, while the reference or denotation of an expression is the object it refers to.

For instance, both expressions “1 + 1” and “2” have the same denotation but not the same sense.
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An intensional proposition is a proposition whose validity is not invariant under extensional substitution.

Frege gives the example of “the morning star” and “the evening star” which both refer to the planet Venus.

Compare “the morning star is the evening star” with “John does not know that the morning star is the evening star”.
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In a strict sense, modal logic is concerned with the study of statements and reasonings that involve the notions of necessity and possibility.

In a more general sense, modal logic is concerned with the study of statements and reasonings that involve expressions (modals) that qualify the validity of a judgement:

- Alethic logic: *It is necessary that... It is possible that...*
- Deontic logic: *It is mandatory that... It is allowed that...*
- Epistemic logic: *Bob knows that... Bob ignores that...*
- Temporal logic: *It will always be the case that... It will eventually be the case that...*
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Modal logic Possibility and necessity

Leibniz

G.W. von Leibniz
(1646–1716)

A proposition is necessarily true if it is true in all possible worlds.

A proposition is possibly true if it is true in at least one possible world.

Pangloss enseignait la métaphysico-théologo-cosmolo-nigologie. Il prouvait admirablement qu’il n’y a point d’effet sans cause, et que, dans ce meilleur des mondes possibles, le château de monseigneur le baron était le plus beau des châteaux et madame la meilleure des baronnes possibles.

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Voltaire (Candide)
Modal logic
Possibility and necessity

Formalization

Syntax:

\[ F ::= a \mid \neg F \mid F \lor F \mid \Box F \]

Define the other connectives in the usual way. Define \( \Diamond A \) as \( \neg \Box \neg A \).

\( \Box A \) stands for “necessarily A”. \( \Diamond A \) stands for “possibly A”.

Validity:

let \( \mathcal{M} = \langle W, P \rangle \), where \( W \) is a set of “possible worlds”, and \( P \) is a function that assigns to each atomic proposition a subset of \( W \).

- \( \mathcal{M}, s \models a \) iff \( s \in P(a) \).
- \( \mathcal{M}, s \models \neg A \) iff not \( \mathcal{M}, s \models A \).
- \( \mathcal{M}, s \models A \lor B \) iff either \( \mathcal{M}, s \models A \) or \( \mathcal{M}, s \models B \), or both.
- \( \mathcal{M}, s \models \Box A \) iff for every \( t \in W \), \( \mathcal{M}, t \models A \).
Formalization

Syntax:

\[ F ::= a | \neg F | F \lor F | \square F \]

Define the other connectives in the usual way. Define \( \diamond A \) as \( \neg \square \neg A \).

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- \( \mathcal{M}, s \models \square A \) iff for every \( t \in W \), \( \mathcal{M}, t \models A \).
System S5

(P) all propositional tautologies

(K) \( \Box (A \supset B) \supset (\Box A \supset \Box B) \)

(T) \( \Box A \supset A \)

(5) \( \Diamond A \supset \Box \Diamond A \)

Modus ponens:

\[
\frac{A \supset B \quad A}{B}
\]

Rule of necessitation:

\[
\frac{A}{\Box A}
\]
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Kripke Semantics

let $\mathcal{M} = \langle W, R, P \rangle$, where $W$ is a set of “possible worlds”, $R$ is a binary relation over $W$, and $P$ is a function that assigns to each atomic proposition a subset of $W$.

- $\mathcal{M}, s \models \Box A$ iff for every $t \in W$ such that $sRt$, $\mathcal{M}, t \models A$.
- $\mathcal{M}, s \models \Diamond A$ iff for some $t \in W$ such that $sRt$, $\mathcal{M}, t \models A$. 
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- $\mathcal{M}, s \models \Diamond A$ iff for some $t \in W$ such that $sRt$, $\mathcal{M}, t \models A$. 
System K

(P) all propositional tautologies

(K) \( \Box(A \supset B) \supset (\Box A \supset \Box B) \)

Modus ponens:

\[
\begin{array}{c}
A \supset B \\
A
\end{array} \\
\hline
B
\]

Rule of necessitation:

\[
\begin{array}{c}
A
\end{array} \\
\hline
\Box A
\]
The following theorems of S5 are not valid in the class of all Kripke models:

(D) $\square A \supset \diamond A$
(T) $\square A \supset A$
(B) $A \supset \square \diamond A$
(4) $\square A \supset \square \square A$
(5) $\diamond A \supset \square \diamond A$

A binary relation $R \subset W \times W$ is serial if and only if for every $s \in W$ there exists $t \in W$ such that $sRt$. 
The following theorems of S5 are not valid in the class of all Kripke models:

(D) $\Box A \supset \Diamond A$

(T) $\Box A \supset A$

(B) $A \supset \Box \Diamond A$

(4) $\Box A \supset \Box \Box A$

(5) $\Diamond A \supset \Box \Diamond A$

A binary relation $R \subset W \times W$ is serial if and only if for every $s \in W$ there exists $t \in W$ such that $sRt$. 
### Some well-known systems

<table>
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<th>System</th>
<th>Description</th>
<th>Properties</th>
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<tr>
<td>KD</td>
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<td>serial</td>
</tr>
<tr>
<td>KT</td>
<td>basic alethic logic</td>
<td>reflexive</td>
</tr>
<tr>
<td>KTB</td>
<td>Brouwersche system</td>
<td>reflexive, symmetric</td>
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<tr>
<td>KT4</td>
<td>Lewis’ S4</td>
<td>reflexive, transitive</td>
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<td>KT5</td>
<td>Lewis’ S5</td>
<td>reflexive, symmetric, transitive</td>
</tr>
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Syntax

Key idea: provide the formula language with explicit means of speaking about worlds!

Two sorts of atoms: usual atomic propositions \((a, b, c, \ldots)\), and nominals \((i, j, k, \ldots)\). Nominals will be used for naming worlds.

\[
F ::= \ a \mid i \mid \neg F \mid F \lor F \mid \square F \mid \downarrow i. F \mid @iF
\]

\(\downarrow\) is a binder: the free occurrences of \(i\) in \(A\) are bound in \(\downarrow i. F\). On the other hand, \(@\) is simply a binary connectives whose first term must be a nominal.

Intuition: \(\downarrow\) is used for naming the “here-and-now”. It allows a nominal to be bound to the current world. \(@iA\) asserts that proposition \(A\) holds at world \(i\).
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Modal logic

Hybrid Logic

Syntax

Key idea: provide the formula language with explicit means of speaking about worlds!

Two sorts of atoms: usual atomic propositions \((a, b, c, \ldots)\), and nominals \((i, j, k, \ldots)\). Nominals will be used for naming worlds.

\[
F ::= a | i | \neg F | F \lor F | \Box F | \downarrow i. F | @i F
\]

\(\downarrow\) is a binder: the free occurrences of \(i\) in \(A\) are bound in \(\downarrow i. F\). On the other hand, \(\@\) is simply a binary connectives whose first term must be a nominal.

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Let $\mathcal{M} = \langle W, R, P \rangle$ be a Kripke model, and let $\eta$ be a valuation that assigns to each nominal an element of $W$.

$\mathcal{M}, \eta, s \models a$ iff $s \in P(a)$.

$\mathcal{M}, \eta, s \models i$ iff $s = \eta(i)$.

$\mathcal{M}, \eta, s \models \neg A$ iff not $\mathcal{M}, \eta, s \models A$.

$\mathcal{M}, \eta, s \models A \lor B$ iff either $\mathcal{M}, \eta, s \models A$ or $\mathcal{M}, \eta, s \models B$, or both.

$\mathcal{M}, \eta, s \models \Box A$ iff for every $t \in W$ such that $sRt$, $\mathcal{M}, \eta, t \models A$.

$\mathcal{M}, \eta, s \models \downarrow i \cdot A$ iff $\mathcal{M}, \eta[i:=s], s \models A$.

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$\mathcal{M}, \eta, s \models @i A$ iff $\mathcal{M}, \eta, \eta(i) \models A$. 
Axiomatization

1. all propositional tautologies
2. $\downarrow i. (A \supset B) \supset (A \supset \downarrow i. B)$, where $i$ does not occur free in $A$
3. $\downarrow i. A \supset (j \supset A[i:=j])$
4. $\downarrow i. (i \supset A) \supset \downarrow i. A$
5. $\downarrow i. A \equiv \neg \downarrow i. \neg A$

6. $@i (A \supset B) \supset (@i A \supset @i B)$
7. $@i A \equiv \neg @i \neg A$
8. $i \land A \supset @i A$
Axiomatization

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2. \( \downarrow i. (A \supset B) \supset (A \supset \downarrow i. B) \), where \( i \) does not occur free in \( A \)
3. \( \downarrow i. A \supset (j \supset A[i := j]) \)
4. \( \downarrow i. (i \supset A) \supset \downarrow i. A \)
5. \( \downarrow i. A \equiv \neg \downarrow i. \neg A \)
6. \( @i(A \supset B) \supset (@iA \supset @iB) \)
7. \( @iA \equiv \neg @i\neg A \)
8. \( i \land A \supset @iA \)
Axiomatization

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3. $\downarrow i. A \supset (j \supset A[i:=j])$
4. $\downarrow i. (i \supset A) \supset \downarrow i. A$
5. $\downarrow i. A \equiv \neg \downarrow i. \neg A$

6. $\@ i (A \supset B) \supset (@ i A \supset @ i B)$
7. $@ i A \equiv \neg @ i \neg A$
8. $i \land A \supset @ i A$
9 \[ @i i \]

10 \[ @i j \supset (\@j A \supset @i A) \]

11 \[ @i j \equiv @j i \]

12 \[ @i @j A \equiv @j A \]

13 \[ \Diamond @i A \supseteq @i A \]

14 \[ \Diamond i \land @i A \supseteq \Diamond A \]

\[
\begin{array}{c}
A \supset B \\
B
\end{array} \quad \begin{array}{c}
A \\
\Box A
\end{array}
\]

\[
\begin{array}{c}
A \\
\Downarrow i.A
\end{array} \quad \begin{array}{c}
A \\
@i A
\end{array} \quad \begin{array}{c}
@i (j \land A) \supset B \\
@i A \supset B
\end{array} \quad \begin{array}{c}
@i \Diamond (j \land A) \supset B \\
@i \Diamond A \supset B
\end{array}
\]

\[ (*) \]

\( j \) is distinct from \( i \) and does not occur free in \( A \) or \( B \).
9 \[ \@_i i \]
10 \[ \@_i j \supset (@_j A \supset @_i A) \]
11 \[ \@_i j \equiv @_j i \]
12 \[ \@_i @_j A \equiv @_j A \]
13 \[ \Diamond @_i A \supset @_i A \]
14 \[ \Diamond i \wedge @_i A \supset \Diamond A \]

\[
\begin{array}{c}
A \supset B & A & A \\
\hline
& B & \Box A
\end{array}
\]

\[
\begin{array}{c}
A & A \\
\hline
\downarrow i. A & @_i A
\end{array}\]

\[\begin{array}{c}
@_i (j \wedge A) \supset B \\
@_i A \supset B \\
\hline
(*) \quad @_i \Diamond (j \wedge A) \supset B
\end{array}\]

\[\begin{array}{c}
@_i \Diamond A \supset B \\
\hline
(*) \quad @_i \Diamond A \supset B
\end{array}\]

\((*) \) \( j \) is distinct from \( i \) and does not occur free in \( A \) or \( B \).
9  \[ \@_i i \]
10  \[ \@_i j \supset (\@_j A \supset \@_i A) \]
11  \[ \@_i j \equiv \@_j i \]
12  \[ \@_i \@_j A \equiv \@_j A \]
13  \[ \lozenge \@_i A \supset \@_i A \]
14  \[ \lozenge i \land \@_i A \supset \lozenge A \]

\[
\begin{align*}
A \supset B & \quad A & \quad A \\
\hline
B & \quad \square A
\end{align*}
\]

\[
\begin{align*}
A & \quad A \\
\downarrow i.A & \quad \@_i A \\
\hline
\@_i (j \land A) \supset B & \quad \@_i (j \land A) \supset B \\
\@_i A \supset B & \quad \@_i \lozenge A \supset B
\end{align*}
\]

\((*) j \) is distinct from \(i \) and does not occur free in \(A \) or \(B \).
9. $\@_i i$

10. $\@_i j \supset (@_j A \supset @_i A)$

11. $\@_i j \equiv @_j i$

12. $\@_i @_j A \equiv @_j A$

13. $\Diamond @_i A \supset @_i A$

14. $\Diamond i \land @_i A \supset \Diamond A$

\[
\frac{A \supset B}{B} \quad \frac{A}{\Box A}
\]

\[
\frac{A}{\downarrow i. A} \quad \frac{A}{@_i A} \quad \frac{@_i (j \land A) \supset B}{@_i A \supset B} \quad \frac{@_i \Diamond (j \land A) \supset B}{@_i \Diamond A \supset B}
\]

(* $j$ is distinct from $i$ and does not occur free in $A$ or $B$. )
The binary operator of temporal logic:

$$A \text{ until } B$$

may be defined as:

$$↓i. ◊↓j. @_i(◊(j ∧ B) ∧ □(◊j ⊃ A))$$
The binary operator of temporal logic:

\[ A \text{ until } B \]

may be defined as:

\[ \downarrow i. \lozenge \downarrow j. @i (\lozenge(j \land B) \land \Box(\lozenge j \supset A)) \]
Semantic representations

1. Introduction

2. Modal logic
   - Intension and extension
   - Possibility and necessity
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   - Hybrid Logic

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   - Simply typed \( \lambda \)-calculus
   - Church’s simple theory of types
   - Standard model
   - Inherent incompleteness
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What is lambda-calculus?

- An intensional theory of functions.
- A simple functional programming language.
- A theory of free- and bound-variables, of scope and substitution.
- The keystone of higher-order syntax and higher-order logic.
- The algebra of natural-deduction proofs.
Syntax

Terms:

\[ T ::= x \mid \lambda x. T \mid (TT) \]

\(\lambda\) is a binder: the free occurrences of \(x\) in \(t\) are bound in \(\lambda x. t\).

Warning: You should solve, once and for all, any problem you could have with the notions of free and bound occurrences of variables.

Reduction rule:

\[(\lambda x. t) u \rightarrow_\beta t[x:=u]\]

Church-Rosser Theorem: For all \(\lambda\)-terms \(t, u,\) and \(v\) such that:

\[ t \rightarrow_\beta u \quad \text{and} \quad t \rightarrow_\beta v \]

there exists a \(\lambda\)-term \(w\) such that:

\[ u \rightarrow_\beta w \quad \text{and} \quad v \rightarrow_\beta w \]
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\[ T ::= x | \lambda x. T | (TT) \]

\( \lambda \) is a binder: the free occurrences of \( x \) in \( t \) are bound in \( \lambda x. t \).

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there exists a \( \lambda \)-term \( w \) such that:

\[ u \rightarrow_\beta w \quad \text{and} \quad v \rightarrow_\beta w \]
Typing rules

\[
\begin{align*}
\Gamma, \ x : A & \vdash \ x : A \\
\Gamma \ & \vdash \ \lambda x. \ t : A \rightarrow B \\
\Gamma \ & \vdash \ t : A \rightarrow B \quad \Gamma \ & \vdash \ u : A \\
\Gamma \ & \vdash \ (t \ u) : B
\end{align*}
\]

Strong-Normalisation Theorem: There is no infinite reduction sequence.
Typing rules

\[ \Gamma, x : A \vdash x : A \]

\[ \frac{x : A, \Gamma \vdash t : B}{\Gamma \vdash \lambda x. t : A \to B} \quad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash u : A}{\Gamma \vdash (tu) : B} \]

**Strong-Normalisation Theorem**: There is no infinite reduction sequence.
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Higher-order logic

Church’s simple theory of types

Syntax

Two atomic types: $\iota, o$

Logical constants:

- $\bot : o$
- $\supset : o \to o \to o$
- $\forall_\alpha : (\alpha \to o) \to o$ (at each type $\alpha$)

$\iota$ is the type of individuals and $o$ is the type of propositions.

Formulas are defined to be well-typed $\lambda$-terms of type $o$. We write $P \supset Q$ and $\forall x. P$ for $\supset P Q$ and $\forall_\alpha (\lambda x. P)$, respectively. Similarly for the other connectives ($\neg, \land, \lor, \equiv, \exists$), which are defined in the usual way.

$t = u$ is defined as $\forall P. P t \supset P u$. 
Syntax

Two atomic types: $\iota, \omicron$

Logical constants:

$\bot : \omicron$

$\supset : \omicron \to \omicron \to \omicron$

$\forall_\alpha : (\alpha \to \omicron) \to \omicron$ (at each type $\alpha$)

$\iota$ is the type of individuals and $\omicron$ is the type of propositions.

Formulas are defined to be well-typed $\lambda$-terms of type $\omicron$. We write $P \supset Q$ and $\forall x. P$ for $\supset P Q$ and $\forall_\alpha (\lambda x. P)$, respectively. Similarly for the other connectives ($\neg, \land, \lor, \equiv, \exists$), which are defined in the usual way.

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\[
\begin{align*}
\bot & : o \\
\supset & : o \to o \to o \\
\forall_\alpha & : (\alpha \to o) \to o \quad \text{(at each type } \alpha) \\
\end{align*}
\]

$\iota$ is the type of individuals and $o$ is the type of propositions.

Formulas are defined to be well-typed $\lambda$-terms of type $o$. We write $P \supset Q$ and $\forall x. P$ for $\supset P Q$ and $\forall_\alpha (\lambda x. P)$, respectively. Similarly for the other connectives ($\neg, \land, \lor, \equiv, \exists$), which are defined in the usual way.

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Syntax

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Logical constants:

\[
\begin{align*}
\bot & : o \\
\supset & : o \rightarrow o \rightarrow o \\
\forall_\alpha & : (\alpha \rightarrow o) \rightarrow o \quad \text{(at each type } \alpha) \\
\end{align*}
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\( \iota \) is the type of individuals and \( o \) is the type of propositions.

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\begin{align*}
\bot & : o \\
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\( \iota \) is the type of individuals and \( o \) is the type of propositions.

Formulas are defined to be well-typed \( \lambda \)-terms of type \( o \). We write \( P \supset Q \) and \( \forall x. P \) for \( \supset P \cdot P \) and \( \forall_\alpha (\lambda x. P) \), respectively. Similarly for the other connectives \( (\neg, \land, \lor, \equiv, \exists) \), which are defined in the usual way.

\( t = u \) is defined as \( \forall P. P \cdot t \supset P \cdot u \).
Deductive system

Logical rules:

$\Gamma, A \vdash A$

$\Gamma, A \vdash B$

$\Gamma \vdash A \supset B$

$\Gamma, A \vdash B$

$\Gamma \vdash A \supset B$

$\Gamma \vdash A$

$\Gamma \vdash B$

$\Gamma \vdash \forall \alpha (\lambda x_\alpha . A)$

$x$ of type $\alpha$, $x \not\in \text{FV}(\Gamma)$

$\Gamma \vdash \forall \alpha A$

$\Gamma \vdash A B$

$\Gamma, \neg A \vdash \bot$

$\Gamma \vdash A$
Deductive system

Conversion rule:

\[
\frac{\Gamma \vdash A}{\Gamma \vdash B} \quad \text{where} \ A =_\beta B
\]

Extensionality axioms:

\[
\Gamma \vdash (\forall x. A x = B x) \supset (A = B)
\]

\[
\Gamma \vdash (A \equiv B) \supset (A = B)
\]
Deductive system

Conversion rule:

$$
\frac{\Gamma \vdash A}{\Gamma \vdash B} \text{ where } A = \beta B
$$

Extensionality axioms:

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\Gamma \vdash (\forall x. A x = B x) \supset (A = B)
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### Semantic representations

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2. **Modal logic**
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   - Simply typed $\lambda$-calculus
   - Church’s simple theory of types
   - **Standard model**
     - Inherent incompleteness
     - Henkin Models
Interpretation of the types and the terms

- \( M = \langle (D_a)_{a \in T}, \mathcal{I} \rangle \)

- \( D_l \) is given.
- \( D_o = \{0, 1\} \)
- \( D_{A \rightarrow B} = D_B^{D_A} \)

\[
\begin{align*}
[c]_\eta &= \mathcal{I}(c) \\
[x]_\eta &= \eta(x) \\
[\lambda x. t]_\eta &= a \mapsto [t]_\eta[x := a] \\
[t u]_\eta &= [t]_\eta([u]_\eta)
\end{align*}
\]

With the expected interpretations for the logical constants.
Interpretation of the types and the terms

- $M = \langle (D_a)_{a \in T}, \mathcal{I} \rangle$
- $D_I$ is given.
- $D_o = \{0, 1\}$
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- $\llbracket c \rrbracket_\eta = \mathcal{I}(c)$
- $\llbracket x \rrbracket_\eta = \eta(x)$
- $\llbracket \lambda x. t \rrbracket_\eta = a \mapsto \llbracket t \rrbracket_\eta[x:=a]$
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With the expected interpretations for the logical constants.
Interpretation of the types and the terms

- \( M = \langle (D_a)_{a \in T}, \mathcal{I} \rangle \)
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- \( [c]_\eta = \mathcal{I}(c) \)
- \( [x]_\eta = \eta(x) \)
- \( [\lambda x. t]_\eta = a \mapsto [t]_\eta[x:=a] \)
- \( [t u]_\eta = [t]_\eta([u]_\eta) \)

With the expected interpretations for the logical constants.
Higher-order logic as a set theory

- Sets as characteristic functions, i.e., sets of “elements” of type $\alpha$ as terms of type $\alpha \to o$.

- $\{x \mid P\}$ as $\lambda x. P$
- $t \in A$ as $At$
Higher-order logic as a set theory

- Sets as characteristic functions, i.e., sets of “elements” of type $\alpha$ as terms of type $\alpha \rightarrow o$.
  
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Higher-order logic

Inherent incompleteness

The set of natural numbers

\[ S \triangleq (\forall x. s \ x \neq 0) \land (\forall xy. s \ x = s \ y \supset x = y) \]

\[ N \triangleq \lambda x. (\forall R. R \ 0 \land (\forall y. R \ y \supset R (s \ y)) \supset R \ x) \]

The only model of \( S \land \forall x. N \ x \) is the set of natural numbers.
Higher-order logic

The set of natural numbers

\[ S \triangleq (\forall x. sx \neq 0) \land (\forall xy. sx = sy \supset x = y) \]

\[ N \triangleq \lambda x. (\forall R. R 0 \land (\forall y. Ry \supset R(sy)) \supset Rx) \]

The only model of \( S \land \forall x. N x \) is the set of natural numbers.
Incompleteness

Let $\phi$ be a formula of Peano’s Arithmetic, and define $\phi^N$ as follows:

- $\phi^N = \phi$, for $\phi$ an atomic formula,
- $(\neg \phi)^N = \neg \phi^N$,
- $(\phi * \psi)^N = \phi^N * \psi^N$, for $* \in \{\wedge, \vee, \supset, \equiv\}$,
- $(\forall x. \phi)^N = \forall x.(N \ x \supset \phi^N)$,
- $(\exists x. \phi)^N = \exists x.(N \ x \land \phi^N)$.

Let $D$ be the conjunction of the universal closures of the defining equations for addition and multiplication, and let $PA$ be $S \land \forall x. N \ x \land D$.

Then, the formula $PA \supset \phi^N$ is valid if and only if $\phi$ is true in the standard model of Peano’s arithmetic.

Corollary: incompleteness of higher-order logic.
Incompleteness

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Let $D$ be the conjunction of the universal closures of the defining equations for addition and multiplication, and let $PA$ be $S \land \forall x. N x \land D$.

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- $(\forall x. \phi)^N = \forall x. (\mathbb{N} x \supset \phi^N)$,
- $(\exists x. \phi)^N = \exists x. (\mathbb{N} x \land \phi^N)$.

Let $D$ be the conjunction of the universal closures of the defining equations for addition and multiplication, and let $\text{PA}$ be $S \land \forall x. \mathbb{N} x \land D$.

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Relaxing the interpretation

- $M = \langle (D_a)_{a \in T}, \mathcal{I} \rangle$

- $D_l$ is given.
- $D_o = \{0, 1\}$
- $D_{A \rightarrow B} \subset D_B^{D_A}$

- $[c]_\eta = \mathcal{I}(c)$
- $[x]_\eta = \eta(x)$
- $[\lambda x. t]_\eta = a \mapsto [t]_\eta[x:=a]$
- $[t \ u]_\eta = [t]_\eta([u]_\eta)$

With domains that are rich enough to interpret $\lambda$-abstraction, equality, and the logical constants.
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Higher-order logic

Henkin Models

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- $[[c]]_\eta = \mathcal{I}(c)$
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With domains that are rich enough to interpret $\lambda$-abstraction, equality, and the logical constants.
Relaxing the interpretation

- \( M = \langle (D_a)_{a \in T}, \mathcal{I} \rangle \)
- \( D_l \) is given.
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- \( \llbracket c \rrbracket_\eta = \mathcal{I}(c) \)
- \( \llbracket x \rrbracket_\eta = \eta(x) \)
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