Formal Languages

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Outline

3. The pumping lemma for regular languages
   - Pumping Lemma
   - Proving that a language is not regular

Pumping Lemma

Proposition Let $L \subset \Sigma^*$ be a regular language. Then there exists a constant $k \in \mathbb{N}$ such that for every word $\alpha \in L$ such that $|\alpha| \geq k$, there exist words $\beta, \gamma, \delta \in \Sigma^*$ such that:

(1) $\alpha = \beta\gamma\delta$

(2) $|\beta\gamma| \leq k$

(3) $\gamma \neq \epsilon$

(4) For all $n \in \mathbb{N}$, $\beta\gamma^n\delta \in L$.

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Take k to be the number of states of A, and consider any word $\alpha \in L$, such that $|\alpha| \geq k$.

Let $a_1, a_2, \ldots, a_p \in \Sigma$ be such that $\alpha = a_1 a_2 \ldots a_p$. Let q_0 be the initial state of A, and define $q_i = \hat{\delta}(q_0, a_1 \ldots a_i)$. (Remark that $p \geq k$ and that q_p is a final state of A.

Pumping Lemma

Since $q_0, q_1, \ldots, q_p$ is a sequence of at least $k+1$ states, where k is the number of states of A, there exist $i, j \in \mathbb{N}$ such that $0 \leq i < j \leq p$ and $q_i = q_j$. Then, take:

- $\beta = a_1 \ldots a_i$
- $\gamma = a_{i+1} \ldots a_j$
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- $\beta = a_1 \ldots a_i$
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(1) and (2) are clearly satisfied. (3) is also satisfied because $i < j$.

Pumping Lemma

Finally, since $q_i = q_j$, β, γ, and δ are such that:

- $\hat{\delta}(q_0, \beta) = q_i$
- $\hat{\delta}(q_i, \gamma) = q_i$
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Therefore, for every $n \in \mathbb{N}$, we have that:

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Therefore, for every $n \in \mathbb{N}$, we have that:

$$\hat{\delta}(q_0, \beta\gamma^n\delta) = q_p$$

It implies that $\beta\gamma^n\delta \in L(A)$ because q_p is a final state of A.

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Then, consider $\alpha = a^k b^k$. According to the pumping lemma, α can be factorized into three words $\beta\gamma\delta = \alpha$ such that $|\beta\gamma| \leq k$ and $\gamma \neq \epsilon$.

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The only possibility is to have $\beta = a^p$ and $\gamma = a^q$, for some $p, q \in \mathbb{N}$ such that $p + q \leq k$ and $q \neq 0$.

But then, according to the pumping lemma, we would have that $a^p b^k \in L$, which is not the case because $p < k$. Therefore L is not regular.