Formal Languages

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Outline

5. Pushdown automata
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Definition

A *pushdown automaton* is a 7-tuple $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F\rangle$, where

- Q is an alphabet of *states*;
- Σ is an alphabet of *input symbols*;
- Γ is an alphabet of *stack symbols*;
- $\delta \in \mathcal{P}_{\mathrm{fin}}(Q \times \Gamma^*)^{Q\times(\Sigma\cup\{\epsilon\})\times\Gamma}$ is the *transition function*;
- $q_0 \in Q$ is the *initial state*;
- $Z_0 \in Q$ is the *initial stack symbol*;
- $F \subset Q$ is the set of *final states*.

Move relation

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$$(q, \alpha, \beta) \in Q \times \Sigma^* \times \Gamma^*$$

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Move relation:

$$(q, a\alpha, Z\beta) \vdash (r, \alpha, \gamma\beta)$$

where:

- $a \in \Sigma \cup \{\epsilon\}$;
- $(r, \gamma) \in \delta(q, a, Z)$.

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$\vdash^*$ denotes the reflexive, transitive closure of $\vdash$

Language accepted by a PDA

Language accepted by a PDA:

$$L(P) = \{\alpha \in \Sigma^* : \exists q \in F, \beta \in \Gamma^*. \, (q_0, \alpha, Z_0) \vdash^* (q, \epsilon, \beta)\}$$

Acceptance by empty stack

Language accepted by a PDA by the empty stack:

$$N(P) = \{\alpha \in \Sigma^* : \exists q \in Q. \, (q_0, \alpha, Z_0) \vdash^* (q, \epsilon, \epsilon)\}$$

Acceptance by empty stack

Let $P_N = \langle Q_N, \Sigma_N, \Gamma_N, \delta_N, q_{N0}, Z_{N0}, F_N \rangle$ be a PDA. Define another PDA $P_F = \langle Q_F, \Sigma_F, \Gamma_F, \delta_F, q_{F0}, Z_{F0}, F_F \rangle$ as follows:

- $Q_F = Q_N \cup \{p_0, p_f\}$, where p_0 and p_f are fresh symbols;
- $\Sigma_F = \Sigma_N$;
- $\Gamma_F = \Gamma_N \cup \{\dashv\}$, where $\dashv$ is a fresh symbol;
- δ_F is such that:

$$\begin{aligned} \delta_F(q, a, Z) &= \delta_N(q, a, Z) \text{ for } (q, a, Z) \in Q_N \times (\Sigma_N \cup \{\epsilon\}) \times \Gamma_N \\ \delta_F(p_0, \epsilon, \dashv) &= \{(q_{N0}, Z_{N0} \dashv)\} \\ \delta_F(q, \epsilon, \dashv) &= \{(p_f, \epsilon)\} \text{ for every } q \in Q_N \end{aligned}$$

- $q_{F0} = p_0$;
- $Z_{F0} = \dashv$;
- $F_F = \{p_f\}$.

Acceptance by empty stack

Proposition $L(P_F) = N(P_N)$.

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\delta_N(q, \epsilon, Z) &= \{(p_f, \epsilon)\} \text{ for every } q \in F_F \text{ and every } Z \in \Gamma_N \\
\delta_N(p_f, \epsilon, Z) &= \{(p_f, \epsilon)\} \text{ for every } Z \in \Gamma_N
\end{aligned}$$

- $q_{N0} = p_0$;
- $Z_{N0} = \dashv$;
- $F_N = \{p_f\}$.

Acceptance by empty stack

Proposition $N(P_N) = L(P_F)$.

From CFG to PDA

Let $G = \langle N, \Sigma, P, S \rangle$ be a context-free grammar. Define a PDA $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ as follows:

- $Q = \{q\}$;
- $\Gamma = N \cup \Sigma$;
- δ is such that:

$$
\begin{array}{l}
\delta(q, \epsilon, A) = \{(q, \alpha) : (A \rightarrow \alpha) \in P\} \text{ for } A \in N \\
\delta(q, a, a) = \{(q, \epsilon)\} \text{ for } a \in \Sigma \\
\delta(q, \_, \_) = \varnothing \text{ in the other cases}
\end{array}
$$

- $q_0 = q$;
- $Z_0 = S$;
- $F = \{q\}$.

From CFG to PDA

Proposition $N(P) = L(G)$.

From PDA to CFG

Let $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ be a PDA. Define a context-free grammar $G = \langle N, \Sigma, P, S \rangle$ as follows:

- $N = \{S\} \cup \{[pXq] : p, q \in Q \wedge X \in \Gamma\}$;
- P contains the following rules:
  - for every $p \in Q$:

$$S \rightarrow [q_0 Z_0 p]$$

  - for every $(r, Y_0 Y_1 \ldots Y_{n-1}) \in \delta(q, a, Y)$:

$$[qY r_{n-1}] \rightarrow a[rY_0 r_0][r_0 Y_1 r_1] \ldots [r_{n-2} Y_{n-1} r_{n-1}]$$

    where $r_0, r_1, \ldots r_{n-2}, r_{n-1} \in Q$

From PDA to CFG

Proposition $L(G) = N(P)$.