5. Pushdown automata
   - Definition
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   - From PDA to CFG
A *pushdown automaton* is a 7-tuple \( P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle \), where

- \( Q \) is an alphabet of *states*;
- \( \Sigma \) is an alphabet of *input symbols*;
- \( \Gamma \) is an alphabet of *stack symbols*;
- \( \delta \in \mathcal{P}_{\text{fin}}(Q \times \Gamma^*) \times (\Sigma \cup \{\epsilon\}) \times \Gamma \) is the *transition function*;
- \( q_0 \in Q \) is the *initial state*;
- \( Z_0 \in Q \) is the *initial stack symbol*;
- \( F \subset Q \) is the set of *final states*. 
Move relation

Instantaneous description (ID): $(q, \alpha, \beta) \in Q \times \Sigma^* \times \Gamma^*$

Initial ID: $(q_0, \alpha, Z_0) \in \{q_0\} \times \Sigma^* \times \{Z_0\}$

Move relation: $(q, a, \alpha, Z, \beta) \vdash (r, \alpha, \gamma, Z, \beta)$ where:

$a \in \Sigma \cup \{\epsilon\}$

$(r, \gamma) \in \delta(q, a, Z)$. 

$\vdash*$ denotes the reflexive, transitive closure of $\vdash$. 
Move relation

Instantaneous description (ID):

$$(q, \alpha, \beta) \in Q \times \Sigma^* \times \Gamma^*$$
Move relation

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\[(q, \alpha, \beta) \in Q \times \Sigma^* \times \Gamma^*\]

Initial ID:

\[(q_0, \alpha, Z_0) \in \{q_0\} \times \Sigma^* \times \{Z_0\}\]
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Move relation:

\[(q, a\alpha, Z\beta) \vdash (r, \alpha, \gamma\beta)\]

where:

- \(a \in \Sigma \cup \{\epsilon\}\);
- \((r, \gamma) \in \delta(q, a, Z)\).
Move relation

Instantaneous description (ID):

\[(q, \alpha, \beta) \in Q \times \Sigma^* \times \Gamma^*\]

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where:

- \(a \in \Sigma \cup \{\epsilon\}\);
- \((r, \gamma) \in \delta(q, a, Z)\).

\(\vdash^*\) denotes the reflexive, transitive closure of \(\vdash\)
Language accepted by a PDA:

\[ L(P) = \{ \alpha \in \Sigma^* : \exists q \in F, \beta \in \Gamma^*. (q_0, \alpha, Z_0) \vdash^* (q, \epsilon, \beta) \} \]
Language accepted by a PDA by the empty stack:

\[ N(P) = \{ \alpha \in \Sigma^* : \exists q \in Q. (q_0, \alpha, Z_0) \vdash^* (q, \epsilon, \epsilon) \} \]
Acceptance by empty stack

Let $P_N = \langle Q_N, \Sigma_N, \Gamma_N, \delta_N, q_{N0}, Z_{N0}, F_N \rangle$ be a PDA. Define another PDA $P_F = \langle Q_F, \Sigma_F, \Gamma_F, \delta_F, q_{F0}, Z_{F0}, F_F \rangle$ as follows:

- $Q_F = Q_N \cup \{p_0, p_f\}$, where $p_0$ and $p_f$ are fresh symbols;
- $\Sigma_F = \Sigma_N$;
- $\Gamma_F = \Gamma_N \cup \{-\}$, where $-\$ is a fresh symbol;
- $\delta_F$ is such that:
  \[
  \delta_F(q, a, Z) = \delta_N(q, a, Z) \quad \text{for} \quad (q, a, Z) \in Q_N \times (\Sigma_N \cup \{\epsilon\}) \times \Gamma_N
  \]
  \[
  \delta_F(p_0, \epsilon, -\$) = \{(q_{N0}, Z_{N0} -\$)\}
  \]
  \[
  \delta_F(q, \epsilon, -\$) = \{(p_f, \epsilon)\} \quad \text{for every} \quad q \in Q_N
  \]
- $q_{F0} = p_0$;
- $Z_{F0} = -\$;
- $F_F = \{p_f\}$.
Proposition $L(P_F) = N(P_N)$. 
Let $P_F = \langle Q_F, \Sigma_F, \Gamma_F, \delta_F, q_{F0}, Z_{F0}, F_F \rangle$ be a PDA. Define another PDA $P_N = \langle Q_N, \Sigma_N, \Gamma_N, \delta_N, q_{N0}, Z_{N0}, F_N \rangle$ as follows:

- $Q_N = Q_F \cup \{p_0, p_f\}$, where $p_0$ and $p_f$ are fresh symbols;
- $\Sigma_N = \Sigma_F$;
- $\Gamma_N = \Gamma_F \cup \{-\}$, where $-$ is a fresh symbol;
- $\delta_N$ is such that:
  - $\delta_N(q, a, Z) = \delta_F(q, a, Z)$ for $(q, a, Z) \in Q_F \times (\Sigma_F \cup \{\epsilon\}) \times \Gamma_F$
  - $\delta_N(p_0, \epsilon, -) = \{(q_{N0}, Z_{N0} -)\}$
  - $\delta_N(q, \epsilon, Z) = \{(p_f, \epsilon)\}$ for every $q \in F_F$ and every $Z \in \Gamma_N$
  - $\delta_N(p_f, \epsilon, Z) = \{(p_f, \epsilon)\}$ for every $Z \in \Gamma_N$

- $q_{N0} = p_0$;
- $Z_{N0} = -$;
- $F_N = \{p_f\}$. 

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Proposition $N(P_N) = L(P_F)$. 
From CFG to PDA

Let \( G = \langle N, \Sigma, P, S \rangle \) be a context-free grammar. Define a PDA \( P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle \) as follows:

- \( Q = \{ q \} \);
- \( \Gamma = N \cup \Sigma \);
- \( \delta \) is such that:
  - \( \delta(q, \epsilon, A) = \{(q, \alpha) : (A \rightarrow \alpha) \in P\} \) for \( A \in N \)
  - \( \delta(q, a, a) = \{(q, \epsilon)\} \) for \( a \in \Sigma \)
  - \( \delta(q, -, -) = \emptyset \) in the other cases
- \( q_0 = q \);
- \( Z_0 = S \);
- \( F = \{ q \} \).
Proposition $N(P) = L(G)$. 
Let $P = \langle Q, \Sigma, \Gamma, \delta, q_0, Z_0, F \rangle$ be a PDA. Define a context-free grammar $G = \langle N, \Sigma, P, S \rangle$ as follows:

- $N = \{S\} \cup \{[pXq] : p, q \in Q \land X \in \Gamma\}$;

- $P$ contains the following rules:
  - for every $p \in Q$:
    $$S \rightarrow [q_0Z_0p]$$
  - for every $(r, Y_0Y_1 \ldots Y_{n-1}) \in \delta(q, a, Y)$:
    $$[qYr_{n-1}] \rightarrow a[rY_0r_0][r_0Y_1r_1] \ldots [r_{n-2}Y_{n-1}r_{n-1}]$$

where $r_0, r_1, \ldots r_{n-2}, r_{n-1} \in Q$
Proposition \( L(G) = N(P) \).