Towards a Montagovian Account of Dynamics

Philippe de Groote

LORIA & Inria-Lorraine

Introduction

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A man enters the room. He smiles.

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$$= \exists x.\mathbf{man}(x) \wedge \mathbf{enters\_the\_room}(x) \wedge \mathbf{smiles}(x).$$

A well known solution: DRT.

- The reference markers of DRT act as existential quantifiers.
- Nevertheless, from a technical point of view, they must be considered as free variables.

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- We will interpret a sentence according to both its left and right contexts.
- These two kinds of contexts will be abstracted over the meaning of the sentences.

Typing the left and the right contexts

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Montague semantics is based on Church's simple type theory, which provides a full hierarchy of functional types built upon two atomic types:

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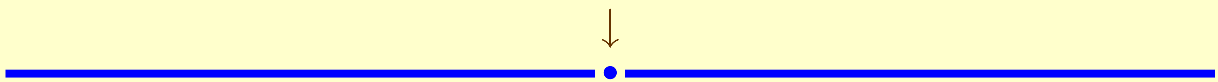

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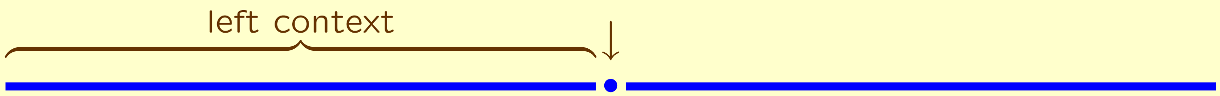

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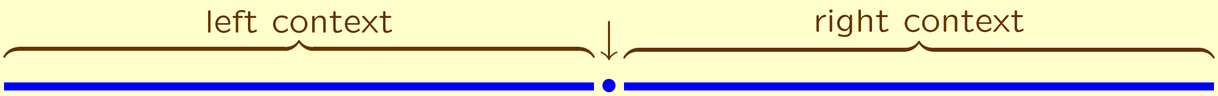

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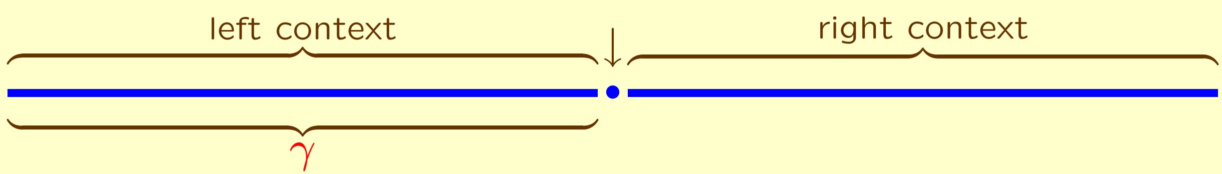

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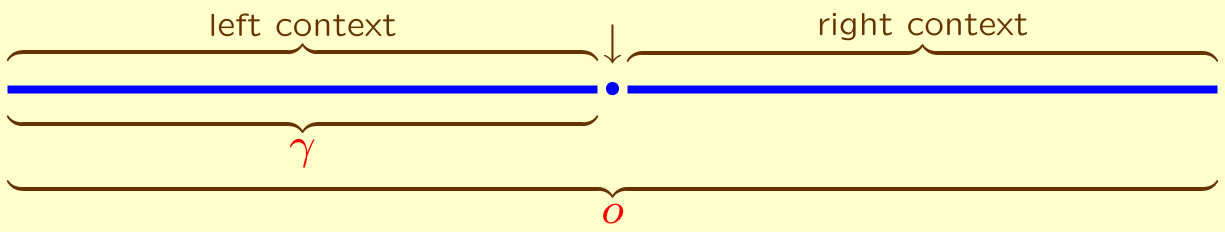

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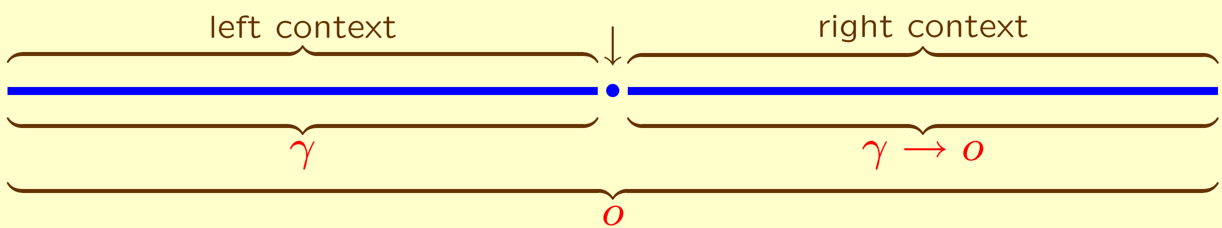

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Note that this operation is associative!

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To such a structure, corresponds the following λ-term of type $\gamma \rightarrow (\gamma \rightarrow o) \rightarrow o$:

$$\lambda e \phi. \exists x_1 \ldots x_n. C_1 \wedge \cdots \wedge C_m \wedge \phi\, e'$$

where e' is a context made of e and of the variables $x_1, \ldots, x_n$.

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$= \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \mathbf{smile}\, \mathbf{j}\, (\mathtt{sel}\, 2\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))) \wedge \phi\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))$

$$\lambda e\phi.\, [\![\text{John}^1 \text{ loves Mary}^2]\!]\, e\, (\lambda e'.\, [\![\text{He}_1 \text{ smiles at her}_2]\!]\, e'\, \phi)$$

$$= \lambda e\phi.\, (\lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \phi\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e)))\, e\, (\lambda e'.\, [\![\text{He}_1 \text{ smiles at her}_2]\!]\, e'\, \phi)$$

$$\rightarrow_\beta \lambda e\phi.\, (\lambda \phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \phi\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e)))\, (\lambda e'.\, [\![\text{He}_1 \text{ smiles at her}_2]\!]\, e'\, \phi)$$

$$\rightarrow_\beta \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge (\lambda e'.\, [\![\text{He}_1 \text{ smiles at her}_2]\!]\, e'\, \phi)\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))$$

$$\rightarrow_\beta \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge [\![\text{He}_1 \text{ smiles at her}_2]\!]\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))\, \phi$$

$$= \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge (\lambda e\phi.\, \mathbf{smile}\, (\mathtt{sel}\, 1\, e)\, (\mathtt{sel}\, 2\, e) \wedge \phi\, e)\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))\, \phi$$

$$\rightarrow_\beta \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge (\lambda \phi.\, \mathbf{smile}\, (\mathtt{sel}\, 1\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e)))\, (\mathtt{sel}\, 2\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))) \wedge \phi\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e)))\, \phi$$

$$\rightarrow_\beta \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \mathbf{smile}\, (\mathtt{sel}\, 1\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e)))\, (\mathtt{sel}\, 2\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))) \wedge \phi\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))$$

$$= \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \mathbf{smile}\, \mathbf{j}\, (\mathtt{sel}\, 2\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))) \wedge \phi\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))$$

$$= \lambda e\phi.\, \mathbf{love}\, \mathbf{j}\, \mathbf{m} \wedge \mathbf{smile}\, \mathbf{j}\, \mathbf{m} \wedge \phi\, (\mathtt{push}\, 2\, \mathbf{m}\, (\mathtt{push}\, 1\, \mathbf{j}\, e))$$

Assigning a semantics to the lexical entries

Assigning a semantics to the lexical entries

$$
\begin{aligned}
[\![s]\!] &= o \\
[\![n]\!] &= \iota \rightarrow o \\
[\![np]\!] &= (\iota \rightarrow o) \rightarrow o
\end{aligned}
$$

Assigning a semantics to the lexical entries

$$
\begin{aligned}
[\![s]\!] &= o \\
[\![n]\!] &= \iota \rightarrow o \\
[\![np]\!] &= (\iota \rightarrow o) \rightarrow o
\end{aligned}
$$

$$
\begin{aligned}
[\![s]\!] &= o && (1) \\
[\![n]\!] &= \iota \rightarrow [\![s]\!] && (2) \\
[\![np]\!] &= (\iota \rightarrow [\![s]\!]) \rightarrow [\![s]\!] && (3)
\end{aligned}
$$

Assigning a semantics to the lexical entries

$$
\begin{aligned}
[\![s]\!] &= o \\
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$$

$$
\begin{aligned}
[\![s]\!] &= o && (1) \\
[\![n]\!] &= \iota \rightarrow [\![s]\!] && (2) \\
[\![np]\!] &= (\iota \rightarrow [\![s]\!]) \rightarrow [\![s]\!] && (3)
\end{aligned}
$$

Replacing (1) with:

$$[\![s]\!] = \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o$$

Assigning a semantics to the lexical entries

$$
\begin{aligned}
[\![s]\!] &= o \\
[\![n]\!] &= \iota \rightarrow o \\
[\![np]\!] &= (\iota \rightarrow o) \rightarrow o
\end{aligned}
$$

$$
\begin{aligned}
[\![s]\!] &= o && (1) \\
[\![n]\!] &= \iota \rightarrow [\![s]\!] && (2) \\
[\![np]\!] &= (\iota \rightarrow [\![s]\!]) \rightarrow [\![s]\!] && (3)
\end{aligned}
$$

Replacing (1) with:

$$[\![s]\!] = \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o$$

we obtain:

$$
\begin{aligned}
[\![n]\!] &= \iota \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o \\
[\![np]\!] &= (\iota \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o) \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o
\end{aligned}
$$

Nouns

Nouns

$$[\![n]\!] = \iota \to \gamma \to (\gamma \to o) \to o$$

Nouns

$$[\![n]\!] = \iota \to \gamma \to (\gamma \to o) \to o$$

$$[\![\text{man}]\!] = \lambda xe\phi.\, \mathbf{man}\, x \wedge \phi\, e$$
$$[\![\text{woman}]\!] = \lambda xe\phi.\, \mathbf{woman}\, x \wedge \phi\, e$$
$$[\![\text{farmer}]\!] = \lambda xe\phi.\, \mathbf{farmer}\, x \wedge \phi\, e$$
$$[\![\text{donkey}]\!] = \lambda xe\phi.\, \mathbf{donkey}\, x \wedge \phi\, e$$

Noun phrases

Noun phrases

$$[\![np]\!] = (\iota \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o) \rightarrow \gamma \rightarrow (\gamma \rightarrow o) \rightarrow o$$

Noun phrases

$$[\![np]\!] = (\iota \to \gamma \to (\gamma \to o) \to o) \to \gamma \to (\gamma \to o) \to o$$

$$[\![\text{John}^i]\!] = \lambda \psi e \phi.\, \psi\, \mathbf{j}\, e\, (\lambda e.\, \phi\, (\mathtt{push}\, i\, \mathbf{j}\, e))$$
$$[\![\text{Mary}^i]\!] = \lambda \psi e \phi.\, \psi\, \mathbf{m}\, e\, (\lambda e.\, \phi\, (\mathtt{push}\, i\, \mathbf{m}\, e))$$
$$[\![\text{he}_i]\!] = \lambda \psi e \phi.\, \psi\, (\mathtt{sel}\, i\, e)\, e\, \phi$$
$$[\![\text{her}_i]\!] = \lambda \psi e \phi.\, \psi\, (\mathtt{sel}\, i\, e)\, e\, \phi$$
$$[\![\text{it}_i]\!] = \lambda \psi e \phi.\, \psi\, (\mathtt{sel}\, i\, e)\, e\, \phi$$

Determiners

Determiners

$$[\![det]\!] = [\![n]\!] \rightarrow [\![np]\!]$$

Determiners

$$[\![det]\!] = [\![n]\!] \rightarrow [\![np]\!]$$

$[\![\text{a}^i]\!] = \lambda n \psi e \phi.\, \exists x.\, n\, x\, e\, (\lambda e.\, \psi\, x\, (\texttt{push}\, i\, x\, e)\, \phi)$

$[\![\text{every}^i]\!] = \lambda n \psi e \phi.\, (\forall x.\, \neg(n\, x\, e\, (\lambda e.\, \neg(\psi\, x\, (\texttt{push}\, i\, x\, e)\, (\lambda e.\, \top))))) \wedge \phi\, e$

Transitive verbs

Transitive verbs

$$[\![tv]\!] = [\![np]\!] \longrightarrow [\![np]\!] \longrightarrow [\![s]\!]$$

Transitive verbs

$$[\![tv]\!] = [\![np]\!]\rightarrow[\![np]\!]\rightarrow[\![s]\!]$$

$$[\![\text{loves}]\!] = \lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{love}\, x\, y \wedge \phi\, e))$$
$$[\![\text{owns}]\!] = \lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{own}\, x\, y \wedge \phi\, e))$$
$$[\![\text{beats}]\!] = \lambda os.\, s\,(\lambda x.\, o\,(\lambda ye\phi.\, \mathbf{beat}\, x\, y \wedge \phi\, e))$$

Relative pronouns

Relative pronouns

$$[\![rel]\!] = ([\![np]\!] \rightarrow [\![s]\!]) \rightarrow [\![n]\!] \rightarrow [\![n]\!]$$

Relative pronouns

$$[\![rel]\!] = ([\![np]\!] \rightarrow [\![s]\!]) \rightarrow [\![n]\!] \rightarrow [\![n]\!]$$

$$[\![\text{who}]\!] = \lambda rnxe\phi.\, n\, x\, e\, (\lambda e.\, r\, (\lambda\psi.\, \psi\, x)\, e\, \phi)$$

$[\![\text{beats}]\!]\ [\![\text{it}_2]\!]\ ([\![\text{every}^1]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}^2]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!]))$

$[\![\text{beats}]\!]\ [\![\text{it}_2]\!]\ ([\![\text{every}^1]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}^2]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!]))$

$[\![\text{a}^2]\!]\ [\![\text{donkey}]\!]$

$[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$$[\![\text{a}^2]\!]\,[\![\text{donkey}]\!]$$
$$= \;(\lambda n\psi e\phi.\,\exists y.\,n\,y\,e\,(\lambda e.\,\psi\,y\,(\mathtt{push}\,2\,y\,e)\,\phi))\,[\![\text{donkey}]\!]$$

$[\![\text{beats}]\!]\, [\![\text{it}_2]\!]\, ([\![\text{every}^1]\!]\, ([\![\text{who}]\!]\, ([\![\text{owns}]\!]\, ([\![\text{a}^2]\!]\, [\![\text{donkey}]\!]))\, [\![\text{farmer}]\!]))$

$$\begin{aligned} &[\![\text{a}^2]\!]\, [\![\text{donkey}]\!] \\ &\quad = \quad (\lambda n \psi e \phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (\mathtt{push}\, 2\, y\, e)\, \phi))\, [\![\text{donkey}]\!] \\ &\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (\mathtt{push}\, 2\, y\, e)\, \phi) \end{aligned}$$

$[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}^2]\!]\,[\![\text{donkey}]\!] \\
&\quad = \; (\lambda n\psi e\phi.\, \exists y.\, n\,y\,e\,(\lambda e.\, \psi\,y\,(\mathtt{push}\,2\,y\,e)\,\phi))\,[\![\text{donkey}]\!] \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, [\![\text{donkey}]\!]\,y\,e\,(\lambda e.\, \psi\,y\,(\mathtt{push}\,2\,y\,e)\,\phi) \\
&\quad = \; \lambda\psi e\phi.\, \exists y.\, (\lambda x e\phi.\, \mathbf{donkey}\,x \wedge \phi\,e)\,y\,e\,(\lambda e.\, \psi\,y\,(\mathtt{push}\,2\,y\,e)\,\phi)
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$[\![\text{a}^2]\!]\,[\![\text{donkey}]\!]$

$$
\begin{aligned}
&= \ (\lambda n\psi e\phi.\,\exists y.\,n\,y\,e\,(\lambda e.\,\psi\,y\,(\mathtt{push}\,2\,y\,e)\,\phi))\,[\![\text{donkey}]\!] \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,[\![\text{donkey}]\!]\,y\,e\,(\lambda e.\,\psi\,y\,(\mathtt{push}\,2\,y\,e)\,\phi) \\
&= \ \lambda\psi e\phi.\,\exists y.\,(\lambda x e\phi.\,\mathbf{donkey}\,x \wedge \phi\,e)\,y\,e\,(\lambda e.\,\psi\,y\,(\mathtt{push}\,2\,y\,e)\,\phi) \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge (\lambda e.\,\psi\,y\,(\mathtt{push}\,2\,y\,e)\,\phi)\,e
\end{aligned}
$$

$[\![\text{beats}]\!]\ [\![\text{it}_2]\!]\ ([\![\text{every}^1]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}^2]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}^2]\!]\ [\![\text{donkey}]\!] \\
&\quad = \ (\lambda n\psi e\phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (\mathrm{push}\, 2\, y\, e)\, \phi))\ [\![\text{donkey}]\!] \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (\mathrm{push}\, 2\, y\, e)\, \phi) \\
&\quad = \ \lambda\psi e\phi.\, \exists y.\, (\lambda x e\phi.\, \mathbf{donkey}\, x \,\wedge\, \phi\, e)\, y\, e\, (\lambda e.\, \psi\, y\, (\mathrm{push}\, 2\, y\, e)\, \phi) \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \,\wedge\, (\lambda e.\, \psi\, y\, (\mathrm{push}\, 2\, y\, e)\, \phi)\, e \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \,\wedge\, \psi\, y\, (\mathrm{push}\, 2\, y\, e)\, \phi
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$[\![\text{a}^2]\!]\,[\![\text{donkey}]\!]$

$$
\begin{aligned}
&= \ (\lambda n\psi e\phi.\,\exists y.\,n\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi))\,[\![\text{donkey}]\!] \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,[\![\text{donkey}]\!]\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi) \\
&= \ \lambda\psi e\phi.\,\exists y.\,(\lambda xe\phi.\,\mathbf{donkey}\,x \wedge \phi\,e)\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi) \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge (\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)\,e \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi
\end{aligned}
$$

$[\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!])$

$[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$[\![\text{a}^2]\!]\,[\![\text{donkey}]\!]$

$$\begin{aligned}
&= (\lambda n\psi e\phi.\,\exists y.\,n\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi))\,[\![\text{donkey}]\!] \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,[\![\text{donkey}]\!]\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi) \\
&= \lambda\psi e\phi.\,\exists y.\,(\lambda x e\phi.\,\mathbf{donkey}\,x \wedge \phi\,e)\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi) \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge (\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)\,e \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi
\end{aligned}$$

$[\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!])$

$$= [\![\text{owns}]\!]\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)$$

$[\![\text{beats}]\!]\;[\![\text{it}_2]\!]\;([\![\text{every}^1]\!]\;([\![\text{who}]\!]\;([\![\text{owns}]\!]\;([\![\text{a}^2]\!]\;[\![\text{donkey}]\!]))\;[\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}^2]\!]\;[\![\text{donkey}]\!] \\
&\quad = \;(\lambda n\psi e\phi.\,\exists y.\,n\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi))\;[\![\text{donkey}]\!] \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,[\![\text{donkey}]\!]\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi) \\
&\quad = \;\lambda\psi e\phi.\,\exists y.\,(\lambda xe\phi.\,\mathbf{donkey}\,x\,\wedge\,\phi\,e)\,y\,e\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi) \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y\,\wedge\,(\lambda e.\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)\,e \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y\,\wedge\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{owns}]\!]\;([\![\text{a}^2]\!]\;[\![\text{donkey}]\!]) \\
&\quad = [\![\text{owns}]\!]\;(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y\,\wedge\,\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi) \\
&\quad = (\lambda os.\,s\,(\lambda x.\,o\,(\lambda ye\phi.\,\mathbf{own}\,x\,y\wedge\phi\,e)))\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y\wedge\psi\,y\,(\mathrm{push}\,2\,y\,e)\,\phi)
\end{aligned}
$$

$[\![\text{beats}]\!]\ [\![\text{it}_2]\!]\ ([\![\text{every}^1]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}^2]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}^2]\!]\ [\![\text{donkey}]\!] \\
&\quad = \ (\lambda n\psi e\phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi))\ [\![\text{donkey}]\!] \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi) \\
&\quad = \ \lambda\psi e\phi.\, \exists y.\, (\lambda x e\phi.\, \mathbf{donkey}\, x \wedge \phi\, e)\, y\, e\, (\lambda e.\, \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi) \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge (\lambda e.\, \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi)\, e \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{owns}]\!]\ ([\![\text{a}^2]\!]\ [\![\text{donkey}]\!]) \\
&\quad = [\![\text{owns}]\!]\ (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi) \\
&\quad = (\lambda o s.\, s\, (\lambda x.\, o\, (\lambda y e\phi.\, \mathbf{own}\, x\, y \wedge \phi\, e)))\ (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi) \\
&\quad \twoheadrightarrow_\beta \lambda s.\, s\, (\lambda x.\, (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi)\, (\lambda y e\phi.\, \mathbf{own}\, x\, y \wedge \phi\, e))
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{a}^2]\!]\,[\![\text{donkey}]\!]\\
&\quad = \;(\lambda n\psi e\phi.\,\exists y.\,n\,y\,e\,(\lambda e.\,\psi\,y\,(\text{push}\,2\,y\,e)\,\phi))\,[\![\text{donkey}]\!]\\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,[\![\text{donkey}]\!]\,y\,e\,(\lambda e.\,\psi\,y\,(\text{push}\,2\,y\,e)\,\phi)\\
&\quad = \;\lambda\psi e\phi.\,\exists y.\,(\lambda xe\phi.\,\mathbf{donkey}\,x \wedge \phi\,e)\,y\,e\,(\lambda e.\,\psi\,y\,(\text{push}\,2\,y\,e)\,\phi)\\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge (\lambda e.\,\psi\,y\,(\text{push}\,2\,y\,e)\,\phi)\,e\\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\text{push}\,2\,y\,e)\,\phi
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!])\\
&\quad = [\![\text{owns}]\!]\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\text{push}\,2\,y\,e)\,\phi)\\
&\quad = (\lambda os.\,s\,(\lambda x.\,o\,(\lambda ye\phi.\,\mathbf{own}\,x\,y \wedge \phi\,e)))\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\text{push}\,2\,y\,e)\,\phi)\\
&\quad \twoheadrightarrow_\beta \lambda s.\,s\,(\lambda x.\,(\lambda\psi e\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge \psi\,y\,(\text{push}\,2\,y\,e)\,\phi)\,(\lambda ye\phi.\,\mathbf{own}\,x\,y \wedge \phi\,e))\\
&\quad \twoheadrightarrow_\beta \lambda s.\,s\,(\lambda xe\phi.\,\exists y.\,\mathbf{donkey}\,y \wedge (\lambda ye\phi.\,\mathbf{own}\,x\,y \wedge \phi\,e)\,y\,(\text{push}\,2\,y\,e)\,\phi)
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))$

$[\![\text{a}^2]\!]\,[\![\text{donkey}]\!]$

$$\begin{aligned}
&= (\lambda n\psi e\phi.\, \exists y.\, n\, y\, e\, (\lambda e.\, \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi))\, [\![\text{donkey}]\!] \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, [\![\text{donkey}]\!]\, y\, e\, (\lambda e.\, \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi) \\
&= \lambda\psi e\phi.\, \exists y.\, (\lambda x e\phi.\, \mathbf{donkey}\, x \wedge \phi\, e)\, y\, e\, (\lambda e.\, \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi) \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge (\lambda e.\, \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi)\, e \\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi
\end{aligned}$$

$[\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!])$

$$\begin{aligned}
&= [\![\text{owns}]\!]\, (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi) \\
&= (\lambda o s.\, s\, (\lambda x.\, o\, (\lambda y e\phi.\, \mathbf{own}\, x\, y \wedge \phi\, e)))\, (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi) \\
&\twoheadrightarrow_\beta \lambda s.\, s\, (\lambda x.\, (\lambda\psi e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \psi\, y\, (\text{push}\, 2\, y\, e)\, \phi)\, (\lambda y e\phi.\, \mathbf{own}\, x\, y \wedge \phi\, e)) \\
&\twoheadrightarrow_\beta \lambda s.\, s\, (\lambda x e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge (\lambda y e\phi.\, \mathbf{own}\, x\, y \wedge \phi\, e)\, y\, (\text{push}\, 2\, y\, e)\, \phi) \\
&\twoheadrightarrow_\beta \lambda s.\, s\, (\lambda x e\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\, (\text{push}\, 2\, y\, e))
\end{aligned}$$

$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))$$
$$= \quad [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \quad [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\\
&\quad = \quad (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \ [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\\
&\quad = \ (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\qquad\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \ [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\\
&\quad = \ (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\qquad\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e))\, e\, \phi)
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \; [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\\
&\quad = \; (\lambda rnxe\phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad (\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\\
&\quad \twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\\
&\qquad\quad (\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e))\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\, (\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e))\, x\, e\, \phi)
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \ [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\\
&\quad = \ (\lambda rnxe\phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad (\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\\
&\quad \twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\\
&\qquad\quad (\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e))\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\, (\lambda xe\phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e))\, x\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e))
\end{aligned}
$$

$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))$

$$\begin{aligned}
&= [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\, (\text{push}\, 2\, y\, e)))\\
&= (\lambda rnxe\phi.\, n\, x\, e\, (\lambda e.\, r\, (\lambda\psi.\, \psi\, x)\, e\, \phi))\\
&\qquad (\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\, (\text{push}\, 2\, y\, e)))\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\, (\lambda e.\\
&\qquad (\lambda s.\, s\,(\lambda xe\phi.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\, (\text{push}\, 2\, y\, e)))\, (\lambda\psi.\, \psi\, x)\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\, (\lambda e.\, (\lambda\psi.\, \psi\, x)\, (\lambda xe\phi.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\, (\text{push}\, 2\, y\, e))\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\, (\lambda e.\, (\lambda xe\phi.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\, (\text{push}\, 2\, y\, e))\, x\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda nxe\phi.\, n\, x\, e\, (\lambda e.\, \exists y.\, \textbf{donkey}\, y \wedge \textbf{own}\, x\, y \wedge \phi\, (\text{push}\, 2\, y\, e))
\end{aligned}$$

$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \; [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\quad = \; (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\qquad\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, x\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]\\
&\quad = \; (\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\,[\![\text{farmer}]\!]
\end{aligned}
$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))$$

$$= \quad [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))$$

$$= \quad (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))$$
$$(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))$$

$$\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.$$
$$(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)$$

$$\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, e\, \phi)$$

$$\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, x\, e\, \phi)$$

$$\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]$$

$$= \quad (\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\,[\![\text{farmer}]\!]$$

$$\twoheadrightarrow_\beta \lambda x e \phi.\, [\![\text{farmer}]\!]\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \ [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\quad = \ (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\qquad\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, x\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]\\
&\quad = \ (\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\,[\![\text{farmer}]\!]\\
&\quad \twoheadrightarrow_\beta \lambda x e \phi.\, [\![\text{farmer}]\!]\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\\
&\quad = \ \lambda x e \phi.\, (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge \phi\, e)\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))
\end{aligned}
$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))$$

$$\begin{aligned}
&= [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&= (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, x\, e\, \phi)\\
&\twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))
\end{aligned}$$

$$[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]$$

$$\begin{aligned}
&= (\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\,[\![\text{farmer}]\!]\\
&\twoheadrightarrow_\beta \lambda x e \phi.\, [\![\text{farmer}]\!]\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\\
&= \lambda x e \phi.\, (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge \phi\, e)\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\\
&\twoheadrightarrow_\beta \lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, e
\end{aligned}$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\\
&\quad = \; [\![\text{who}]\!]\,(\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\quad = \; (\lambda r n x e \phi.\, n\, x\, e\,(\lambda e.\, r\,(\lambda \psi.\, \psi\, x)\, e\, \phi))\\
&\qquad\qquad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\\
&\qquad\quad (\lambda s.\, s\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\,(\lambda \psi.\, \psi\, x)\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda \psi.\, \psi\, x)\,(\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, (\lambda x e \phi.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, x\, e\, \phi)\\
&\quad \twoheadrightarrow_\beta \lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]\\
&\quad = \; (\lambda n x e \phi.\, n\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\,[\![\text{farmer}]\!]\\
&\quad \twoheadrightarrow_\beta \lambda x e \phi.\, [\![\text{farmer}]\!]\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\\
&\quad = \; \lambda x e \phi.\, (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge \phi\, e)\, x\, e\,(\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\\
&\quad \twoheadrightarrow_\beta \lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\lambda e.\, \exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))\, e\\
&\quad \twoheadrightarrow_\beta \lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e))
\end{aligned}
$$

$[\![\text{every}^1]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}^2]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!])$

$$[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])$$
$$= \quad [\![\text{every}^1]\!]\,(\lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))$$

$$
\begin{aligned}
&[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\quad = \ [\![\text{every}^1]\!]\,(\lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))\\
&\quad = \ (\lambda n \psi e \phi.\, (\forall x.\, \neg(n\, x\, e\, (\lambda e.\, \neg(\psi\, x\, (\mathtt{push}\, 1\, x\, e)\, (\lambda e.\, \top))))) \wedge \phi\, e)\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathtt{push}\, 2\, y\, e)))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\quad = \; [\![\text{every}^1]\!]\,(\lambda x e \phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
&\quad = \; (\lambda n \psi e \phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e)\\
&\qquad\qquad (\lambda x e \phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
&\quad \twoheadrightarrow_\beta \; \lambda \psi e \phi.\,(\forall x.\,\neg(\\
&\qquad\qquad (\lambda x e \phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
&\qquad\qquad\quad x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]) \\
&\quad = \quad [\![\text{every}^1]\!]\,(\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e))) \\
&\quad = \quad (\lambda n\psi e\phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e) \\
&\qquad\qquad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e))) \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\\
&\qquad\qquad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e))) \\
&\qquad\qquad\quad x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e \\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \\
&\qquad\qquad (\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top)))\,(\mathrm{push}\,2\,y\,e)))) \wedge \phi\,e
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\quad = \quad [\![\text{every}^1]\!]\,(\lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\quad = \quad (\lambda n \psi e \phi.\, (\forall x.\, \neg(n\, x\, e\, (\lambda e.\, \neg(\psi\, x\, (\mathrm{push}\, 1\, x\, e)\, (\lambda e.\, \top))))) \wedge \phi\, e)\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, (\forall x.\, \neg(\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\,(\mathrm{push}\, 2\, y\, e)))\\
&\qquad\qquad\quad x\, e\, (\lambda e.\, \neg(\psi\, x\, (\mathrm{push}\, 1\, x\, e)\, (\lambda e.\, \top))))) \wedge \phi\, e\\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge\\
&\qquad\qquad (\lambda e.\, \neg(\psi\, x\, (\mathrm{push}\, 1\, x\, e)\, (\lambda e.\, \top)))\, (\mathrm{push}\, 2\, y\, e)))) \wedge \phi\, e\\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge\\
&\qquad\qquad \neg(\psi\, x\, (\mathrm{push}\, 1\, x\, (\mathrm{push}\, 2\, y\, e))\, (\lambda e.\, \top))))) \wedge \phi\, e
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\quad= [\![\text{every}^1]\!]\,(\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e)))\\
&\quad= (\lambda n\psi e\phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e)\\
&\qquad\qquad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e)))\\
&\quad\twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\\
&\qquad\qquad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e)))\\
&\qquad\qquad\quad x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e\\
&\quad\twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad\qquad (\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top)))\,(\mathrm{push}\,2\,y\,e)))) \wedge \phi\,e\\
&\quad\twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad\qquad \neg(\psi\,x\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))\,(\lambda e.\,\top))))) \wedge \phi\,e
\end{aligned}
$$

$[\![\text{beats}]\!]\,[\![\text{it}_2]\!]$

$$
\begin{aligned}
&[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\quad = \; [\![\text{every}^1]\!]\,(\lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\, (\mathrm{push}\, 2\, y\, e)))\\
&\quad = \; (\lambda n \psi e \phi.\, (\forall x.\, \neg(n\, x\, e\, (\lambda e.\, \neg(\psi\, x\, (\mathrm{push}\, 1\, x\, e)\, (\lambda e.\, \top))))) \wedge \phi\, e)\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\, (\mathrm{push}\, 2\, y\, e)))\\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, (\forall x.\, \neg(\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \phi\, (\mathrm{push}\, 2\, y\, e)))\\
&\qquad\qquad\quad x\, e\, (\lambda e.\, \neg(\psi\, x\, (\mathrm{push}\, 1\, x\, e)\, (\lambda e.\, \top))))) \wedge \phi\, e\\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge\\
&\qquad\qquad (\lambda e.\, \neg(\psi\, x\, (\mathrm{push}\, 1\, x\, e)\, (\lambda e.\, \top)))\, (\mathrm{push}\, 2\, y\, e)))) \wedge \phi\, e\\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge\\
&\qquad\qquad \neg(\psi\, x\, (\mathrm{push}\, 1\, x\, (\mathrm{push}\, 2\, y\, e))\, (\lambda e.\, \top))))) \wedge \phi\, e
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\\
&\quad = \; (\lambda o s.\, s\, (\lambda x.\, o\, (\lambda y e \phi.\, \mathbf{beat}\, x\, y \wedge \phi\, e)))\,[\![\text{it}_2]\!]
\end{aligned}
$$

$[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])$

$$\begin{aligned}
&= [\![\text{every}^1]\!]\,(\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e)))\\
&= (\lambda n\psi e\phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e)\\
&\qquad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e)))\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\\
&\qquad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathrm{push}\,2\,y\,e)))\\
&\qquad\quad x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad (\lambda e.\,\neg(\psi\,x\,(\mathrm{push}\,1\,x\,e)\,(\lambda e.\,\top)))\,(\mathrm{push}\,2\,y\,e)))) \wedge \phi\,e\\
&\twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad \neg(\psi\,x\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))\,(\lambda e.\,\top))))) \wedge \phi\,e
\end{aligned}$$

$[\![\text{beats}]\!]\,[\![\text{it}_2]\!]$

$$\begin{aligned}
&= (\lambda os.\,s\,(\lambda x.\,o\,(\lambda ye\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e)))\,[\![\text{it}_2]\!]\\
&\twoheadrightarrow_\beta \lambda s.\,s\,(\lambda x.\,[\![\text{it}_2]\!]\,(\lambda ye\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e))
\end{aligned}$$

$$
\begin{aligned}
&[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\quad = \;[\![\text{every}^1]\!]\,(\lambda x e\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
&\quad = \;(\lambda n\psi e\phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e)\\
&\qquad\qquad (\lambda x e\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\\
&\qquad\qquad (\lambda x e\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
&\qquad\qquad\quad x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e\\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad\qquad (\lambda e.\,\neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\,\top)))\,(\mathtt{push}\,2\,y\,e)))) \wedge \phi\,e\\
&\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad\qquad \neg(\psi\,x\,(\mathtt{push}\,1\,x\,(\mathtt{push}\,2\,y\,e))\,(\lambda e.\,\top))))) \wedge \phi\,e
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\\
&\quad = \;(\lambda o s.\,s\,(\lambda x.\,o\,(\lambda y e\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e)))\,[\![\text{it}_2]\!]\\
&\quad \twoheadrightarrow_\beta \lambda s.\,s\,(\lambda x.\,[\![\text{it}_2]\!]\,(\lambda y e\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e))\\
&\quad = \;\lambda s.\,s\,(\lambda x.\,(\lambda\psi e\phi.\,\psi\,(\mathtt{sel}\,2\,e)\,e\,\phi)\,(\lambda y e\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\quad = \ [\![\text{every}^1]\!]\,(\lambda x e \phi.\, \mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
&\quad = \ (\lambda n \psi e \phi.\, (\forall x.\, \neg(n\,x\,e\,(\lambda e.\, \neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\, \top))))) \wedge \phi\,e)\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, (\forall x.\, \neg(\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
&\qquad\qquad\quad x\,e\,(\lambda e.\, \neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\, \top))))) \wedge \phi\,e\\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad\qquad (\lambda e.\, \neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\, \top)))\,(\mathtt{push}\,2\,y\,e)))) \wedge \phi\,e\\
&\quad \twoheadrightarrow_\beta \lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad\qquad \neg(\psi\,x\,(\mathtt{push}\,1\,x\,(\mathtt{push}\,2\,y\,e))\,(\lambda e.\, \top))))) \wedge \phi\,e
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\\
&\quad = \ (\lambda o s.\, s\,(\lambda x.\, o\,(\lambda y e \phi.\, \mathbf{beat}\,x\,y \wedge \phi\,e)))\,[\![\text{it}_2]\!]\\
&\quad \twoheadrightarrow_\beta \lambda s.\, s\,(\lambda x.\, [\![\text{it}_2]\!]\,(\lambda y e \phi.\, \mathbf{beat}\,x\,y \wedge \phi\,e))\\
&\quad = \ \lambda s.\, s\,(\lambda x.\, (\lambda \psi e \phi.\, \psi\,(\mathtt{sel}\,2\,e)\,e\,\phi)\,(\lambda y e \phi.\, \mathbf{beat}\,x\,y \wedge \phi\,e))\\
&\quad \twoheadrightarrow_\beta \lambda s.\, s\,(\lambda x e \phi.\, (\lambda y e \phi.\, \mathbf{beat}\,x\,y \wedge \phi\,e)\,(\mathtt{sel}\,2\,e)\,e\,\phi)
\end{aligned}
$$

$$
\begin{array}{l}
[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
\quad = \; [\![\text{every}^1]\!]\,(\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
\quad = \; (\lambda n\psi e\phi.\,(\forall x.\,\neg(n\,x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e)\\
\qquad\quad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\\
\qquad\quad (\lambda xe\phi.\,\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \phi\,(\mathtt{push}\,2\,y\,e)))\\
\qquad\qquad x\,e\,(\lambda e.\,\neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\,\top))))) \wedge \phi\,e\\
\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
\qquad\quad (\lambda e.\,\neg(\psi\,x\,(\mathtt{push}\,1\,x\,e)\,(\lambda e.\,\top)))\,(\mathtt{push}\,2\,y\,e)))) \wedge \phi\,e\\
\quad \twoheadrightarrow_\beta \lambda\psi e\phi.\,(\forall x.\,\neg(\mathbf{farmer}\,x \wedge (\exists y.\,\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
\qquad\quad \neg(\psi\,x\,(\mathtt{push}\,1\,x\,(\mathtt{push}\,2\,y\,e))\,(\lambda e.\,\top))))) \wedge \phi\,e
\end{array}
$$

$$
\begin{array}{l}
[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\\
\quad = \; (\lambda os.\,s\,(\lambda x.\,o\,(\lambda ye\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e)))\,[\![\text{it}_2]\!]\\
\quad \twoheadrightarrow_\beta \lambda s.\,s\,(\lambda x.\,[\![\text{it}_2]\!]\,(\lambda ye\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e))\\
\quad = \; \lambda s.\,s\,(\lambda x.\,(\lambda\psi e\phi.\,\psi\,(\mathtt{sel}\,2\,e)\,e\,\phi)\,(\lambda ye\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e))\\
\quad \twoheadrightarrow_\beta \lambda s.\,s\,(\lambda xe\phi.\,(\lambda ye\phi.\,\mathbf{beat}\,x\,y \wedge \phi\,e)\,(\mathtt{sel}\,2\,e)\,e\,\phi)\\
\quad \twoheadrightarrow_\beta \lambda s.\,s\,(\lambda xe\phi.\,\mathbf{beat}\,x\,(\mathtt{sel}\,2\,e) \wedge \phi\,e)
\end{array}
$$

$[\![\text{beats}]\!]\ [\![\text{it}_2]\!]\ ([\![\text{every}^1]\!]\ ([\![\text{who}]\!]\ ([\![\text{owns}]\!]\ ([\![\text{a}^2]\!]\ [\![\text{donkey}]\!]))\ [\![\text{farmer}]\!]))$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad = \;(\lambda s.\, s\,(\lambda xe\phi.\,\mathbf{beat}\,x\,(\texttt{sel}\,2\,e) \wedge \phi\,e))\\
&\qquad\qquad ([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad = \ (\lambda s.\, s\,(\lambda xe\phi.\, \mathbf{beat}\, x\,(\mathtt{sel}\, 2\, e) \wedge \phi\, e))\\
&\qquad\qquad ([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad \twoheadrightarrow_\beta [\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\qquad\qquad (\lambda xe\phi.\, \mathbf{beat}\, x\,(\mathtt{sel}\, 2\, e) \wedge \phi\, e)
\end{aligned}
$$

$$
\begin{array}{l}
[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
\quad = \;(\lambda s.\, s\,(\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e) \wedge \phi\,e))\\
\qquad\qquad ([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
\quad \twoheadrightarrow_\beta \;[\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e) \wedge \phi\,e)\\
\quad = \;(\lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
\qquad\qquad\quad \neg(\psi\,x\,(\mathtt{push}\,1\,x\,(\mathtt{push}\,2\,y\,e))\,(\lambda e.\, \top))))) \wedge \phi\,e)\\
\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e) \wedge \phi\,e)
\end{array}
$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad = \ (\lambda s.\, s\,(\lambda x e \phi.\, \mathbf{beat}\, x\, (\mathtt{sel}\, 2\, e) \wedge \phi\, e))\\
&\qquad\qquad ([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad \twoheadrightarrow_\beta [\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\, (\mathtt{sel}\, 2\, e) \wedge \phi\, e)\\
&\quad = \ (\lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge\\
&\qquad\qquad\quad \neg(\psi\, x\, (\mathtt{push}\, 1\, x\, (\mathtt{push}\, 2\, y\, e))\, (\lambda e.\, \top))))) \wedge \phi\, e)\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\, (\mathtt{sel}\, 2\, e) \wedge \phi\, e)\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge\\
&\qquad\qquad \neg((\lambda x e \phi.\, \mathbf{beat}\, x\, (\mathtt{sel}\, 2\, e) \wedge \phi\, e)\, x\, (\mathtt{push}\, 1\, x\, (\mathtt{push}\, 2\, y\, e))\, (\lambda e.\, \top))))) \wedge \phi\, e
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad = \;(\lambda s.\, s\,(\lambda x e \phi.\, \mathbf{beat}\, x\,(\mathtt{sel}\,2\,e)\,\wedge\,\phi\, e))\\
&\qquad\qquad ([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad \twoheadrightarrow_\beta [\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\,(\mathtt{sel}\,2\,e)\,\wedge\,\phi\, e)\\
&\quad = \;(\lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\,\wedge\,(\exists y.\, \mathbf{donkey}\, y\,\wedge\,\mathbf{own}\, x\, y\,\wedge\\
&\qquad\qquad\quad \neg(\psi\, x\,(\mathtt{push}\,1\,x\,(\mathtt{push}\,2\,y\,e))\,(\lambda e.\, \top)))))\,\wedge\,\phi\, e)\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\,(\mathtt{sel}\,2\,e)\,\wedge\,\phi\, e)\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\,\wedge\,(\exists y.\, \mathbf{donkey}\, y\,\wedge\,\mathbf{own}\, x\, y\,\wedge\\
&\qquad\qquad \neg((\lambda x e \phi.\, \mathbf{beat}\, x\,(\mathtt{sel}\,2\,e)\wedge\phi\, e)\, x\,(\mathtt{push}\,1\,x\,(\mathtt{push}\,2\,y\,e))\,(\lambda e.\, \top)))))\wedge\phi\, e\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\,\wedge\,(\exists y.\, \mathbf{donkey}\, y\,\wedge\,\mathbf{own}\, x\, y\,\wedge\\
&\qquad\qquad \neg(\mathbf{beat}\, x\,(\mathtt{sel}\,2\,(\mathtt{push}\,1\,x\,(\mathtt{push}\,2\,y\,e)))\wedge(\lambda e.\, \top)\,(\mathtt{push}\,1\,x\,(\mathtt{push}\,2\,y\,e))))))\\
&\qquad\quad \wedge\,\phi\, e
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad = \;(\lambda s.\, s\,(\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e)\;\wedge\;\phi\,e))\\
&\qquad\qquad ([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad \twoheadrightarrow_\beta [\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e)\;\wedge\;\phi\,e)\\
&\quad = \;(\lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x\;\wedge\;(\exists y.\, \mathbf{donkey}\,y\;\wedge\;\mathbf{own}\,x\,y\;\wedge\\
&\qquad\qquad\quad \neg(\psi\,x\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))\,(\lambda e.\,\top)))))\;\wedge\;\phi\,e)\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e)\;\wedge\;\phi\,e)\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x\;\wedge\;(\exists y.\, \mathbf{donkey}\,y\;\wedge\;\mathbf{own}\,x\,y\;\wedge\\
&\qquad\qquad \neg((\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e) \wedge \phi\,e)\,x\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))\,(\lambda e.\,\top))))) \wedge \phi\,e\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x\;\wedge\;(\exists y.\, \mathbf{donkey}\,y\;\wedge\;\mathbf{own}\,x\,y\;\wedge\\
&\qquad\qquad \neg(\mathbf{beat}\,x\,(\mathtt{sel}\,2\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))) \wedge (\lambda e.\,\top)\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))))))\\
&\qquad\quad \wedge\;\phi\,e\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x\;\wedge\;(\exists y.\, \mathbf{donkey}\,y\;\wedge\;\mathbf{own}\,x\,y\;\wedge\\
&\qquad\qquad \neg(\mathbf{beat}\,x\,(\mathtt{sel}\,2\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e)))\;\wedge\;\top))))\;\wedge\;\phi\,e
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad = \ (\lambda s.\, s\,(\lambda x e \phi.\, \mathbf{beat}\, x\,(\texttt{sel}\,2\,e)\ \wedge\ \phi\, e))\\
&\qquad\qquad ([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad \twoheadrightarrow_\beta [\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\,(\texttt{sel}\,2\,e)\ \wedge\ \phi\, e)\\
&\quad = \ (\lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\ \wedge\ (\exists y.\, \mathbf{donkey}\, y\ \wedge\ \mathbf{own}\, x\, y\ \wedge\\
&\qquad\qquad\quad \neg(\psi\, x\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))\,(\lambda e.\, \top)))))\ \wedge\ \phi\, e)\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\, x\,(\texttt{sel}\,2\,e)\ \wedge\ \phi\, e)\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\ \wedge\ (\exists y.\, \mathbf{donkey}\, y\ \wedge\ \mathbf{own}\, x\, y\ \wedge\\
&\qquad\qquad \neg((\lambda x e \phi.\, \mathbf{beat}\, x\,(\texttt{sel}\,2\,e) \wedge \phi\, e)\, x\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))\,(\lambda e.\, \top))))) \wedge \phi\, e\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\ \wedge\ (\exists y.\, \mathbf{donkey}\, y\ \wedge\ \mathbf{own}\, x\, y\ \wedge\\
&\qquad\qquad \neg(\mathbf{beat}\, x\,(\texttt{sel}\,2\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))) \wedge (\lambda e.\, \top)\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e))))))\\
&\qquad\quad \wedge\ \phi\, e\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x\ \wedge\ (\exists y.\, \mathbf{donkey}\, y\ \wedge\ \mathbf{own}\, x\, y\ \wedge\\
&\qquad\qquad \neg(\mathbf{beat}\, x\,(\texttt{sel}\,2\,(\mathrm{push}\,1\,x\,(\mathrm{push}\,2\,y\,e)))\ \wedge\ \top))))\ \wedge\ \phi\, e\\
&\quad = \ \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\, x \wedge (\exists y.\, \mathbf{donkey}\, y \wedge \mathbf{own}\, x\, y \wedge \neg(\mathbf{beat}\, x\, y\ \wedge\ \top))))\ \wedge\ \phi\, e
\end{aligned}
$$

$$
\begin{aligned}
&[\![\text{beats}]\!]\,[\![\text{it}_2]\!]\,([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad = \ (\lambda s.\, s\,(\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e) \wedge \phi\,e))\\
&\qquad\qquad ([\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!]))\\
&\quad \twoheadrightarrow_\beta [\![\text{every}^1]\!]\,([\![\text{who}]\!]\,([\![\text{owns}]\!]\,([\![\text{a}^2]\!]\,[\![\text{donkey}]\!]))\,[\![\text{farmer}]\!])\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e) \wedge \phi\,e)\\
&\quad = \ (\lambda \psi e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad\qquad\quad \neg(\psi\,x\,(\mathbf{push}\,1\,x\,(\mathbf{push}\,2\,y\,e))\,(\lambda e.\,\top))))) \wedge \phi\,e)\\
&\qquad\qquad (\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e) \wedge \phi\,e)\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad\qquad \neg((\lambda x e \phi.\, \mathbf{beat}\,x\,(\mathtt{sel}\,2\,e) \wedge \phi\,e)\,x\,(\mathbf{push}\,1\,x\,(\mathbf{push}\,2\,y\,e))\,(\lambda e.\,\top))))) \wedge \phi\,e\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad\qquad \neg(\mathbf{beat}\,x\,(\mathtt{sel}\,2\,(\mathbf{push}\,1\,x\,(\mathbf{push}\,2\,y\,e))) \wedge (\lambda e.\,\top)\,(\mathbf{push}\,1\,x\,(\mathbf{push}\,2\,y\,e))))))\\
&\qquad\quad \wedge \phi\,e\\
&\quad \twoheadrightarrow_\beta \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge\\
&\qquad\qquad \neg(\mathbf{beat}\,x\,(\mathtt{sel}\,2\,(\mathbf{push}\,1\,x\,(\mathbf{push}\,2\,y\,e))) \wedge \top)))) \wedge \phi\,e\\
&\quad = \ \lambda e \phi.\, (\forall x.\, \neg(\mathbf{farmer}\,x \wedge (\exists y.\, \mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y \wedge \neg(\mathbf{beat}\,x\,y \wedge \top)))) \wedge \phi\,e\\
&\quad \equiv \ \lambda e \phi.\, (\forall x.\, \mathbf{farmer}\,x \supset (\forall y.\, (\mathbf{donkey}\,y \wedge \mathbf{own}\,x\,y) \supset \mathbf{beat}\,x\,y)) \wedge \phi\,e
\end{aligned}
$$